HL IB Physics

Simple Harmonic Motion

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Describing Oscillations

Properties of Oscillations

- An **oscillation** is defined as follows:

  The repetitive variation with time *t* of the displacement *x* of an object about the equilibrium position (x = 0)

- A particle undergoing an oscillation can be described using the following properties:
  - **Equilibrium position (x = 0)** is the position when there is no resultant force acting on an object
    - This is the **fixed central point** that the object oscillates around
  - **Displacement (x)** is the **horizontal or vertical distance** of a point on the wave from its equilibrium position
    - It is a vector quantity
    - It can be positive or negative depending on which side of the oscillation it is
    - It is measured in metres (m)
  - **Period (T)** or time period, is the **time interval** for one complete oscillation measured in seconds (s)
    - If the oscillations have a **constant period**, they are said to be **isochronous**

- **Amplitude (x_0)** is the **maximum value of the displacement** on either side of the equilibrium position
  - Amplitude is measured in metres (m)

- **Frequency (f)** is the **number of oscillations per second** and it is measured in hertz (Hz)
  - Hz has the SI units per second s^{-1} because $f = \frac{1}{T}$ see below

- **Angular frequency (ω)** is the rate of change of angular displacement with respect to time
  - It is measured in radians per second (rad s^{-1})

Calculating Time Period of an Oscillation

- This equation relates the frequency and the time period of an oscillation:
- **Angular frequency** (ω) can be calculated using the equation:

$$\omega = \frac{2\pi}{T} = 2\pi f$$

- Where:
  - ω = angular frequency (rad s^{-1})
  - 2π = circumference of a circle
  - T = time period (s)
  - f = frequency of oscillation (Hz)
- The **angular displacement** of objects in oscillation can be determined by matching the **displacement** to an object in **circular motion**:
  - After moving from one amplitude position ***x* = –*A*** to the equilibrium position ***x* = 0** the mass on the spring has moved an angular displacement of $\frac{1}{4}$ of a circle = $\frac{1}{4} \times 2\pi = \frac{\pi}{2}$ radians
  - Continuing the oscillation from the **equilibrium position** to the **other amplitude position** the angular displacement is also $\frac{\pi}{2}$ radians
  - Continuing the oscillation **back to the starting point** means the mass travels a further angular displacement of $\frac{\pi}{2} + \frac{\pi}{2} = \pi$ radians
  - Hence, the **total angular displacement** in **one oscillation** is $\pi + \pi = 2\pi$ radians

Worked example

A child on a swing performs 0.2 oscillations per second.

Calculate the time period of the oscillation.

Answer:

Step 1: Write down the known quantities

- Frequency, f = 0.2 Hz

Step 2: Write down the relationship between the period *T* and the frequency *f*

$$T = \frac{1}{f}$$

Step 3: Substitute the value of the frequency into the above equation and calculate the period

$$T = \frac{1}{0.2} = 5.0 \text{ s}$$

Simple Harmonic Motion (SHM)

Conditions for Simple Harmonic Motion

- **Simple harmonic motion (SHM)** is a specific type of oscillation where:
  - There is **repetitive movement** back and forth through an equilibrium, or central, position, so the **maximum horizontal or vertical displacement** on one side of this position is equal to the maximum horizontal or vertical displacement on the other
  - The **time** interval of each complete vibration is the **same** (periodic)
  - The **force** responsible for the motion (**restoring force**) is always **directed horizontally or vertically towards** the **equilibrium** position and is directly proportional to the distance from it

Examples of SHM

- Examples of oscillators that undergo SHM are:
  - The pendulum of a clock
  - A child on a swing
  - The vibrations of a bowl
  - A bungee jumper reaching the bottom of his fall
  - A mass on a spring
  - Guitar strings vibrating
  - A ruler vibrating off the end of a table
  - The electrons in alternating current flowing through a wire
  - The movement of a swing bridge when someone crosses
  - A marble dropped into a bowl

Modelling SHM

- Not all oscillations are as simple as SHM
  - This is a particularly simple kind
  - It is relatively easy to analyse mathematically
  - Many other types of oscillatory motion can be broken down into a combination of SHMs
- An oscillation is defined to be SHM when:
  - **The acceleration is proportional to the horizontal or vertical displacement**
  - **The acceleration is in the opposite direction to the displacement** (directed towards the equilibrium position)
- The time period of oscillation is independent of the amplitude of the oscillation, for small angles of oscillation
- So, for acceleration *a* and horizontal displacement *x*

$$a \propto -x$$

- You will be required to perform calculations on and explain **two models** of simple harmonic motion:
  - A **simple pendulum** oscillating from side to side attached to a fixed point above
  - A **mass-spring system** oscillating **vertically** up and down or **horizontally** back and forth

An Example of not SHM

- A person jumping on a trampoline is not an example of simple harmonic motion because:
  - The **restoring force** on the person is **not proportional** to their **displacement** from the equilibrium position and always acts down
  - When the person is **not in contact** with the trampoline, the restoring force is equal to their weight, which is constant
  - This **does not change**, even if they jump higher

Worked example

Explain why a person jumping on a trampoline is **not** an example of simple harmonic motion.

Answer:

Step 1: Recall the conditions for simple harmonic motion

- The conditions required for SHM:
  - The restoring force/acceleration is **proportional** to the displacement
  - The restoring force/acceleration is in the **opposite direction** to the displacement

Step 2: Consider the forces in the scenario given

- When the person is not in contact with the trampoline, the restoring force is equal to their weight, which is constant
- The value of their weight does not change, even if they jump higher (increase displacement)

Step 3: Write a concluding sentence

- The restoring force on the person is not proportional to their distance from the equilibrium position, therefore, this scenario does not fulfil the conditions for SHM

The Defining Equation of Simple Harmonic Motion

- The acceleration of an object oscillating in simple harmonic motion is given by the equation:

$$a = -\omega^2 x$$

- Where:
  - a = acceleration (m s^{-2})
  - ω = angular frequency (rad s^{-1})
  - x = displacement (m)

- The equation demonstrates:
  - Acceleration reaches its **maximum** value when the displacement is at a **maximum,** i.e. $x = x_0$ at its amplitude
  - The **minus** sign shows that when the object is displaced to the **right**, the direction of the acceleration is to the **left** and vice versa (a and x are always in opposite directions to each other)

- Consider the oscillation of a **simple pendulum**:
  - The bob **accelerates** as it moves towards the **midpoint**
  - **Velocity** is at a **maximum** when it passes through the **equilibrium position**
  - The **pendulum slows down** as it continues towards the other **extreme of oscillation**
  - **$v = 0$** at **x_0** as it **changes direction**
  - The pendulum then **reverses** and starts to **accelerate again** towards the midpoin

Graphical Representation of SHM

- The displacement, velocity and acceleration of an object in simple harmonic motion can be represented by graphs against time
- All undamped SHM graphs are represented by **periodic functions**
  - This means they can all be described by sine and cosine curves

- You need to know what each graph looks like and how it relates to the other graphs
- Remember that:
  - Velocity is the rate of change of displacement $v = \frac{s}{t}$
  - Acceleration is the rate of change of velocity $a = \frac{\Delta v}{t}$

Graphs that Start at the Equilibrium Position

- When oscillations start from the **equilibrium position,** then:
  - The **displacement-time** graph is a **sine curve**
  - The **velocity-time** graph is the **gradient** of the **displacement-time** graph, so a **cosine** graph and 90° out of phase with the displacement-time graph
  - The **acceleration-time** graph is the **gradient** of the **velocity-time graph**, so a **negative sine** graph and 90° out of phase with the velocity-time graph

Graphs that Start at the Amplitude Position

- When oscillations start from the **amplitude position,** then:
  - The **displacement-time** graph is a **cosine curve**
  - The **velocity-time** graph is the **gradient** of the **displacement-time** graph, so a **negative sine** graph and 90° out of phase with the displacement-time graph
  - The **acceleration-time** graph is the **gradient** of the **velocity-time graph**, so a **negative cosine** graph and 90° out of phase with the velocity-time graph

Relationship Between Graphs

- **Key features of the displacement-time graphs:**
  - The amplitude of oscillations *A* is the maximum value of *x*
  - The time period of oscillations *T* is the time taken for one full wavelength cycle
- **Key features of the velocity-time graphs:**
  - The velocity of an oscillator at any time can be determined from the **gradient of the displacement-time graph:**

$$v = \frac{\Delta x}{\Delta t}$$

- **Key features of the acceleration-time graph:**
  - The acceleration graph is a reflection of the displacement graph on the x-axis
  - This means when a mass has positive displacement (to the right), the acceleration is in the opposite direction (to the left) and vice versa (from $a = -\omega^2 x$)
  - The acceleration of an oscillator at any time can be determined from the **gradient of the velocity-time graph:**

$$a = \frac{\Delta v}{\Delta t}$$

Time Period of a Mass–Spring System

Time Period of a Mass–Spring System

- A **mass-spring** system consists of a mass attached to the end of a spring
- The equation for the **restoring force** (the force responsible for the SHM) is $F_H = -kx$
  - This is the same as the equation for **Hooke's Law**

- The time period of a mass-spring system is given by:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- Where:
  - T = time period (s)
  - m = mass on the end of the spring (kg)
  - k = spring constant ($N\,m^{-1}$)
- The higher the spring constant k, the stiffer the spring and the shorter the time period of the oscillation

Time Period of a Simple Pendulum

Time Period of a Simple Pendulum

- A simple pendulum consists of a **string** and a **bob** at the end
  - The **bob** is a weight, generally spherical and considered a point mass
  - The bob moves from **side to side**
  - The string is **light and inextensible** remaining in tension throughout the oscillations
  - The string is attached to a **fixed point** above the equilibrium position
- The **time period** of a simple pendulum for small angles of oscillation is given by:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

- Where:
  - T = time period (s)
  - L = length of string (from the pivot to the centre of mass of the bob) (m)
  - g = gravitational field strength (N kg^{-1})

- The time period of a **pendulum** depends on **gravitational field strength**
  - Therefore, the time for a pendulum to complete one oscillation would be different on the Earth and the Moon

Small Angle Approximation

- This formula for time period is limited to **small angles** ($\theta < 10°$) and therefore **small amplitudes** of oscillation from the equilibrium point
- The **restoring force** of a pendulum is equal to the component of **weight** acting along the **arc** of the circle towards the equilibrium position

- It is assumed to act at an angle θ to the **horizontal**
- Using the small angle approximation: **sin $\theta \cong \theta$**

Worked example

A swinging pendulum with a length of 80.0 cm has a maximum angle of displacement of 8°.

Determine the angular frequency of the oscillation.

Answer:

Step 1: List the known quantities

- Length of the pendulum, L = 80 cm = 0.8 m
- Acceleration due to gravity, g = 9.81 m s^{-2}

Step 2: Write down the relationship between angular frequency, ω, and period, T

$$T = \frac{2\pi}{\omega}$$

Step 3: Write down the equation for the time period of a simple pendulum

$$T = 2\pi\sqrt{\frac{L}{g}}$$

- This equation is valid for this scenario since the maximum angle of displacement is less than 10°

Step 4: Equate the two equations and rearrange for ω

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \Rightarrow \omega = \sqrt{\frac{g}{L}}$$

Step 5: Substitute the values to calculate ω

$$\omega = \sqrt{\frac{9.81}{0.8}} = 3.50 \text{ rad s}^{-1}$$

Angular frequency: $\omega = 3.5$ rad s^{-1}

- **Note:** angular frequency ω is also known as **angular speed** or **velocity**

Energy Changes in Simple Harmonic Motion (SHM)

Energy Changes in Simple Harmonic Motion

- Simple harmonic motion also involves an **interplay** between different types of **energy**: potential and kinetic
  - The swinging of a **pendulum** is an interplay between **gravitational potential energy** and **kinetic energy**
  - The **horizontal** oscillation of a **mass on a spring** is an interplay between **elastic potential energy** and **kinetic energy**

Energy of a Horizontal Mass-Spring System

- The system has the maximum amount of **elastic potential energy** when held so the string is stretched beyond its equilibrium position
- When the **mass is released,** it moves back towards the equilibrium position, accelerating as it goes so the **kinetic energy increases**
- At the equilibrium position, **kinetic energy** is at its **maximum** and **elastic potential energy** is at its **minimum**
- Once **past the equilibrium** position, the **kinetic energy decreases** and elastic **potential energy increases**

Energy of a Simple-Pendulum

- At the **amplitude** at the top of the swing, the pendulum has a **maximum** amount of **gravitational potential energy**
- When the pendulum is **released**, it moves back towards the equilibrium position, **accelerating** as it goes so the **kinetic energy increases**
- As the **height** of the pendulum **decreases,** the **gravitational potential energy** also **decreases**
- Once the mass has passed the equilibrium position, **kinetic energy decreases** and **gravitational potential energy increases**

Total Energy of an SHM System

- The **total energy** in the system remains constant, but the amount of energy in **one form goes up** while the amount in the **other form goes down**
  - This constant total energy shows how energy in a **closed system** is never created or destroyed; it is transferred from one store to another
  - This is the **law of conservation of energy**

The total energy of a simple harmonic system always remains constant and is equal to the sum of the kinetic and potential energy

- The **total energy** is calculated using the equation:

$$E = E_P + E_K$$

- Where:
  - E = total energy in joules (J)
  - E_P = potential energy in joules (J)
  - E_K = kinetic energy in joules (J)

- Remember the equations for potential and kinetic energy:
  - Gravitational potential energy: $E_p = mgh$
  - Elastic potential energy, $E_p = \frac{1}{2}kx^2$
  - Kinetic energy, $E_k = \frac{1}{2}mv^2$

Energy-Displacement Graph

- The **kinetic** and **potential energy transfers** go through **two** complete cycles during one **period** of oscillation
  - One complete oscillation reaches the maximum displacement **twice** (on both the positive and negative sides of the equilibrium position)
- You need to be familiar with the graph showing the total, potential and kinetic energy transfers in **half an SHM oscillation** (half a cycle)

- **The key features of the energy-displacement graph for half a period of oscillation are:**
  - Displacement is a vector, so, the graph has both **positive** and **negative** x values
  - The potential energy is always maximum at the amplitude positions $x = x_0$, and 0 at the equilibrium position $x = 0$
  - This is represented by a **'U' shaped curve**
- The kinetic energy is the opposite: it is 0 at the amplitude positions $x = x_0$, and maximum at the equilibrium position $x = 0$
  - This is represented by an **'n' shaped curve**

- The total energy is represented by a **horizontal straight line** above the curves

Energy-Time Graph for a Simple Pendulum

- You also need to be familiar with the graph showing the total, **gravitational potential** and **kinetic energy** transfers against time for **multiple cycles** of a **simple pendulum** oscillating in **simple harmonic motion**

Equations for Simple Harmonic Motion (SHM) (HL)

Equations for Simple Harmonic Motion

Summary of SHM Equations

- For a body that begins oscillating from its **equilibrium** position (i.e. $x = 0$ when $t = 0$), its displacement, velocity and acceleration can be described by the equations:

$$x = x_0 \sin \omega t$$

$$v = \omega x_0 \cos \omega t$$

$$a = -\omega^2 x_0 \sin \omega t$$

- For a body that begins oscillating from its **amplitude** position (i.e. $x = x_0$ when $t = 0$), its displacement, velocity and acceleration can be described by the equations:

$$x = x_0 \cos \omega t$$

$$v = -\omega x_0 \sin \omega t$$

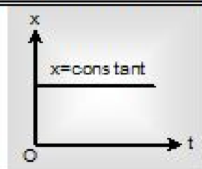
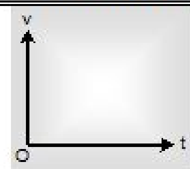
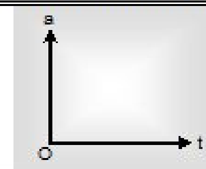
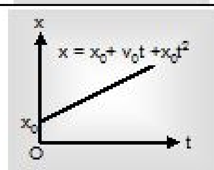
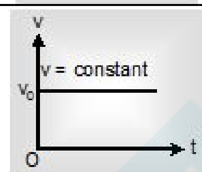
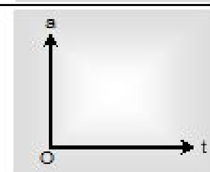
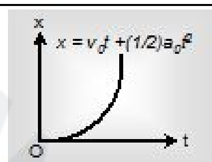
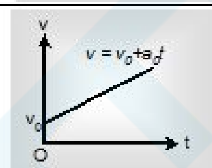
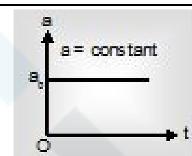
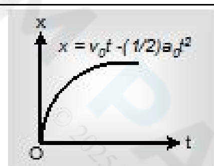
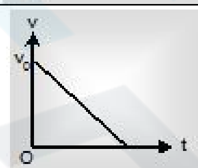
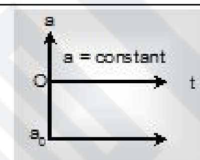
$$a = -\omega^2 x_0 \cos \omega t$$

- The variation of an oscillator's velocity with its displacement *x* is defined by:

$$v = \pm \omega \sqrt{\left(x_0{}^2 - x^2\right)}$$

- This equation shows that the larger the **amplitude** x_0 of an oscillation, the greater the **distance** it must travel in a **given time period**
  - Hence, the **faster** it travels, the closer it is to the equilibrium position
- In the above equations, the variables are as follows:
  - x = displacement of the oscillator (m)
  - x_0 = maximum displacement, or amplitude (m)
  - v = velocity of the oscillator (m s^{-1})
  - a = acceleration of the oscillator (m s^{-2})
  - ω = angular frequency (rad s^{-1})
  - t = time (s)

Summary table of equations and graphs for displacement, velocity and acceleration

	Displacement(x)	Velocity(v)	Acceleration (a)
a. At v=0;			
b. Motion with constant velocity			
c. Motion with constant acceleration			
d. Motion with constant deceleration			

The Origin of the Displacement Equations

- The **SHM displacement equation** for an object oscillating from its **equilibrium position** (x = 0 at t = 0) is:

$$x = x_0 \sin(\omega t + \Phi)$$

- Where:
  - Φ = phase difference (radians) = 0
- Because:
  - The graph of $x = \sin(t)$ starts from amplitude x_0 = 0 when the pendulum is in the equilibrium position at t = 0
  - The displacement is at its maximum when sin(ωt) equals 1 or –1, when $x = x_0$
- Use the IB revision notes on the graphs of trigonometric functions to aid your understanding of trigonometric graphs
- The **SHM displacement equation** for an object oscillating from its **amplitude position** ($x = x_0$ at t = 0) is:

$$x = x_0 \cos(\omega t + \Phi)$$

- The displacement will be at its maximum when cos(ωt) equals 1 or –1, when $x = x_0$
- This is because the cosine graph starts at a maximum, whereas the sine graph starts at 0

The Origin of the Velocity Equations in Trigonometric Form

- The trigonometric equation for the **velocity** of an object starting from its **equilibrium position** (x = 0 at t = 0) is:

$$v = \omega x_0 \cos(\omega t + \Phi)$$

- Where:
  - Φ = phase difference (radians) = 0
- This comes from the fact that velocity is the rate of change of displacement
  - It is the differential of the relevant displacement equation from above: $x = x_0 \sin(\omega t + \Phi)$
- The trigonometric equation for the **velocity** of an object starting from its **amplitude position** ($x = x_0$ at t = 0) is:

$$v = -\omega x_0 \sin(\omega t + \Phi)$$

- This is the differential of the relevant displacement equation from above: $x = x_0 \cos(\omega t + \Phi)$

The Origin of the Displacement-Velocity Relation

- The velocity of an object in simple harmonic motion varies as it oscillates back and forth and is given by the equation:

$$v = \pm\omega\sqrt{x_0^2 - x^2}$$

- ± = 'plus or minus'.
  - The value can be negative or positive
- This comes from the fact that acceleration is the **rate of change of velocity**
  - When the defining equation of simple harmonic motion is integrated using a differential equation the above equation for velocity is obtained
- This equation shows that when an oscillator has a greater amplitude x_0, it has to travel a greater distance in the same time and hence has greater speed v

Equations for Calculating Energy Changes in SHM

- The revision note on Calculating Energy Changes in SHM explains the origin of these two equations for calculating energy changes in simple harmonic motion
- Potential energy:

$$E_P = \frac{1}{2} m\omega^2 x^2$$

- Total energy at the amplitude of oscillation:

$$E_T = \frac{1}{2} m\omega^2 x_0^2$$

- Where:
  - m = mass (kg)
  - ω = angular frequency (rad s^{-1})
  - x_0 = amplitude (m)

Calculating Energy Changes in SHM (HL)

Calculating Energy Changes in Simple Harmonic Motion

Equations for Energy in SHM

- Potential energy:

$$E_P = \frac{1}{2} m\omega^2 x^2$$

- Total energy:

$$E_T = \frac{1}{2} m\omega^2 {x_0}^2$$

- The kinetic energy–displacement relation for SHM is:

$$E_K = \frac{1}{2} m\omega^2 \left({x_0}^2 - x^2\right)$$

- Where:
  - m = mass (kg)
  - ω = angular frequency (rad s^{-1})
  - x_0 = amplitude (m)

Calculating Total Energy in SHM

- Using the expression for the velocity v of a simple harmonic oscillator that begins oscillating from its **equilibrium** position:

$$v = \omega x_0 \cos(\omega t + \Phi)$$

- Where:
  - phase difference, $\Phi = 0$
  - v = velocity of oscillator (m s^{-1})
  - ω = angular frequency (rad s^{-1})
  - x_0 = amplitude (m)
  - t = time (s)
- The kinetic energy E_K of an oscillator can be written as:

$$E_K = \frac{1}{2} mv^2$$

$$E_K = \frac{1}{2} m\left(\omega x_0 \cos(\omega t)\right)^2$$

$$E_K = \frac{1}{2} m\omega^2 {x_0}^2 \cos^2(\omega t)$$

- Since the maximum value of $\sin(\omega t)$ or $\cos(\omega t)$ is 1, **maximum kinetic energy** is given by:

$$E_{K(max)} = \frac{1}{2} m\omega^2 {x_0}^2$$

- When the kinetic energy of the system is at a maximum, the potential energy is zero
  - Hence this represents the **total energy of the system**
- The total energy E_T of a system undergoing simple harmonic motion is, therefore, defined by:

$$E_T = \frac{1}{2} m\omega^2 {x_0}^2$$

- Where:
  - E_T = total energy of a simple harmonic system (J)
  - m = mass of the oscillator (kg)
  - ω = angular frequency (rad s^{-1})
  - x_0 = amplitude (m)
- **Note:** The same expression for total energy will be achieved if the other expression for velocity is used, for an object that begins oscillation at t = 0 from the amplitude position:

$$v = -\omega x_0 \sin(\omega t)$$

Calculating Potential Energy in SHM

- An expression for the potential energy of a simple harmonic oscillator can be derived using the expressions for velocity and displacement for an object starting its oscillations when t = 0 in the **equilibrium** position, so x = 0:

$$x = x_0 \sin(\omega t)$$

$$v = \omega x_0 \cos(\omega t)$$

- The key to deriving this expression is to use the trigonometric identity:

$$\sin^2(\omega t) + \cos^2(\omega t) = 1$$

- In a simple harmonic oscillation, the total energy of the system is equal to:

Total energy = Kinetic energy + Potential energy

$$E_T = E_K + E_P$$

- The **potential energy** of an oscillator can be written as:

$$E_P = E_T - E_K$$

$$E_P = \frac{1}{2}m\omega^2{x_0}^2 - \frac{1}{2}mv^2$$

- Substitute in for *v*:

$$E_P = \frac{1}{2}m\omega^2{x_0}^2 - \frac{1}{2}m\left(\omega x_0 \cos(\omega t)\right)^2$$

$$E_P = \frac{1}{2}m\omega^2{x_0}^2 - \frac{1}{2}m\omega^2{x_0}^2\cos^2(\omega t)$$

- Taking out a factor of $\frac{1}{2}m\omega^2{x_0}^2$ gives:

$$E_P = \frac{1}{2}m\omega^2{x_0}^2\left(1 - \cos^2(\omega t)\right)$$

$$E_P = \frac{1}{2}m\omega^2{x_0}^2\sin^2(\omega t)$$

$$E_P = \frac{1}{2}m\omega^2\left[x_0\sin(\omega t)\right]^2$$

- Since $x = x_0\sin(\omega t)$, the potential energy of the system can be written as:

$$E_P = \frac{1}{2}m\omega^2x^2$$

- Since the maximum potential energy occurs at the maximum displacement $\left(x = x_0\right)$ of the oscillation,

$$E_{P(max)} = \frac{1}{2}m\omega^2{x_0}^2$$

- Therefore, it can be seen that:

$$E_T = E_{K(max)} = E_{P(max)}$$

Kinetic Energy–Displacement Relation for SHM

- Using the displacement–velocity relation for SHM:

$$v = \pm\,\omega\sqrt{{x_0}^2 - x^2}$$

- Substituting into the equation for kinetic energy:

$$E_K = \frac{1}{2} m v^2$$

$$E_K = \frac{1}{2} m \left(\omega \sqrt{x_0{}^2 - x^2}\right)^2$$

- This leads to the kinetic energy-displacement relation for SHM:

$$E_K = \frac{1}{2} m \omega^2 \left(x_0{}^2 - x^2\right)$$

Phase Angles in Simple Harmonic Motion (SHM) (HL)

Phase Angles in Simple Harmonic Motion

- Two points on a sine wave, or on different waves, are in phase when they are at the **same point** in their wave cycle
  - The angle between their wave cycles is known as the **phase angle**
- If an oscillation does **not** start from the equilibrium position, then it will be out of phase by an angle of ϕ
  - This would be compared to an oscillation which does start from the equilibrium position
- The phase angle ϕ of an oscillation (in SHM) is defined as

  The difference in angular displacement compared to an oscillator which has a displacement of zero initially (i.e. $x = 0$ when $t = 0$)

- The phase angle can vary anywhere from 0 to 2π radians, i.e. one complete cycle
- With the inclusion of the phase angle ϕ, the displacement, velocity and acceleration SHM equations become:

$$x = x_0 \sin(\omega t + \phi)$$

$$v = \omega x_0 \cos(\omega t + \phi)$$

$$a = -\omega^2 x_0 \sin(\omega t + \phi)$$

- If two bodies in simple harmonic motion oscillate with the same frequency and amplitude, but are out of phase by $\frac{\pi}{2}$, then:
- The displacement of the oscillator starting from the equilibrium position is represented by the equation $x = x_0 \sin \omega t$
- The displacement of the oscillator which leads by $\frac{\pi}{2}$ is represented by the equation

$$x = x_0 \sin\left(\omega t - \frac{\pi}{2}\right)$$

Two oscillators which are out of phase by $\phi = \frac{\pi}{2}$. The blue-dotted wave represents an oscillator starting from the equilibrium position and the red wave represents an oscillator leading by $\frac{\pi}{2}$

- When a sine wave leads by a phase angle of $\frac{\pi}{2}$, this is equivalent to the cosine of the wave

$$x = x_0 \sin\left(\omega t - \frac{\pi}{2}\right) = x_0 \cos \omega t$$

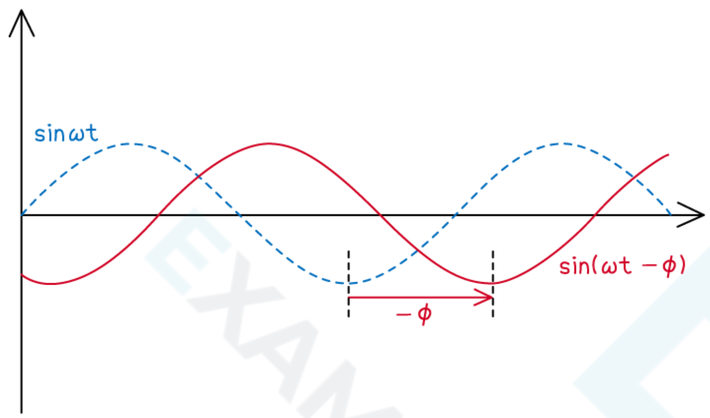
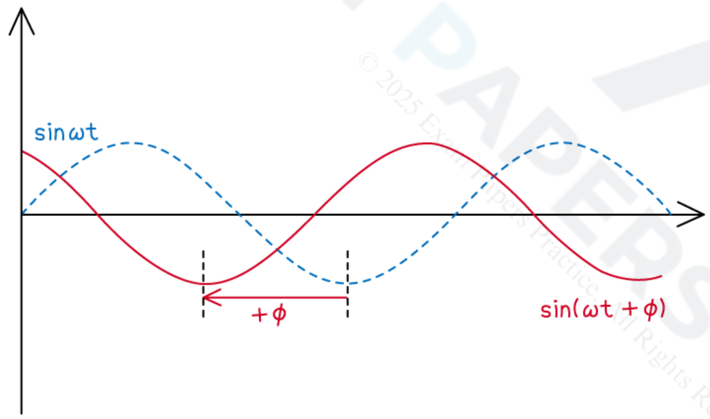
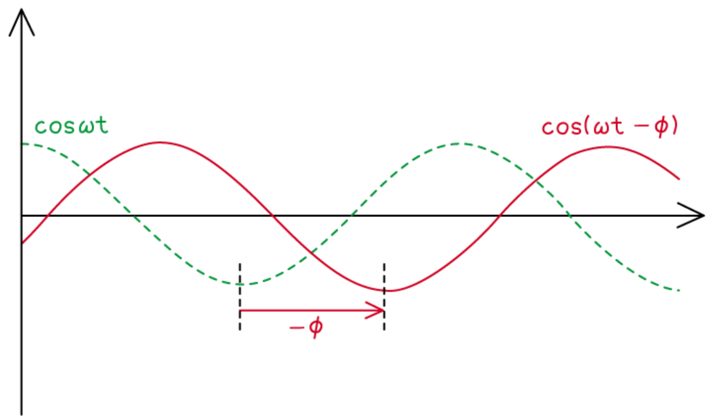
- Alternatively, a sine wave can be described as a cosine wave that lags by $\frac{\pi}{2}$

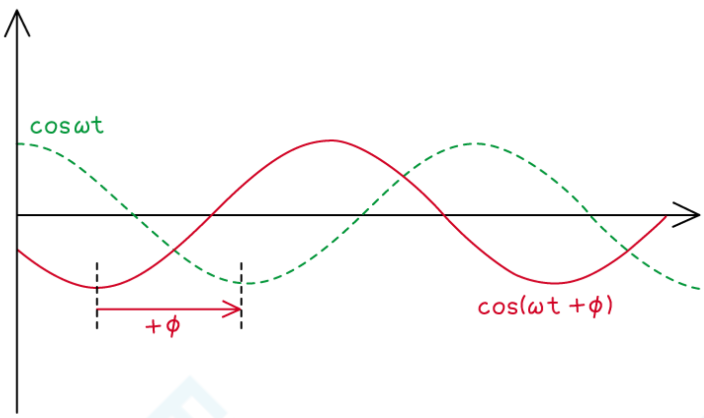
$$x = x_0 \cos\left(\omega t + \frac{\pi}{2}\right) = x_0 \sin \omega t$$

- **Notice:**
  - For a wave that **lags** the phase difference is $+\frac{\pi}{2}$
  - For a wave that **leads** the phase difference is $-\frac{\pi}{2}$
- This is the **opposite sign** to the one you might think.
  - To review this concept, use the notes on the Transformation of Trigonometric Functions

Sine and cosine functions are simply out of phase by $\frac{\pi}{2}$ radians

- The general rules for phase shifts of sine and cosine functions are shown in the table below

Graph	Equation	Phase shift
	$\sin(\omega t - \phi)$	Shifts by ϕ to the right (positive direction)
	$\sin(\omega t + \phi)$	Shifts by ϕ to the left (negative direction)
	$\cos(\omega t - \phi)$	Shifts by ϕ to the right (positive direction)

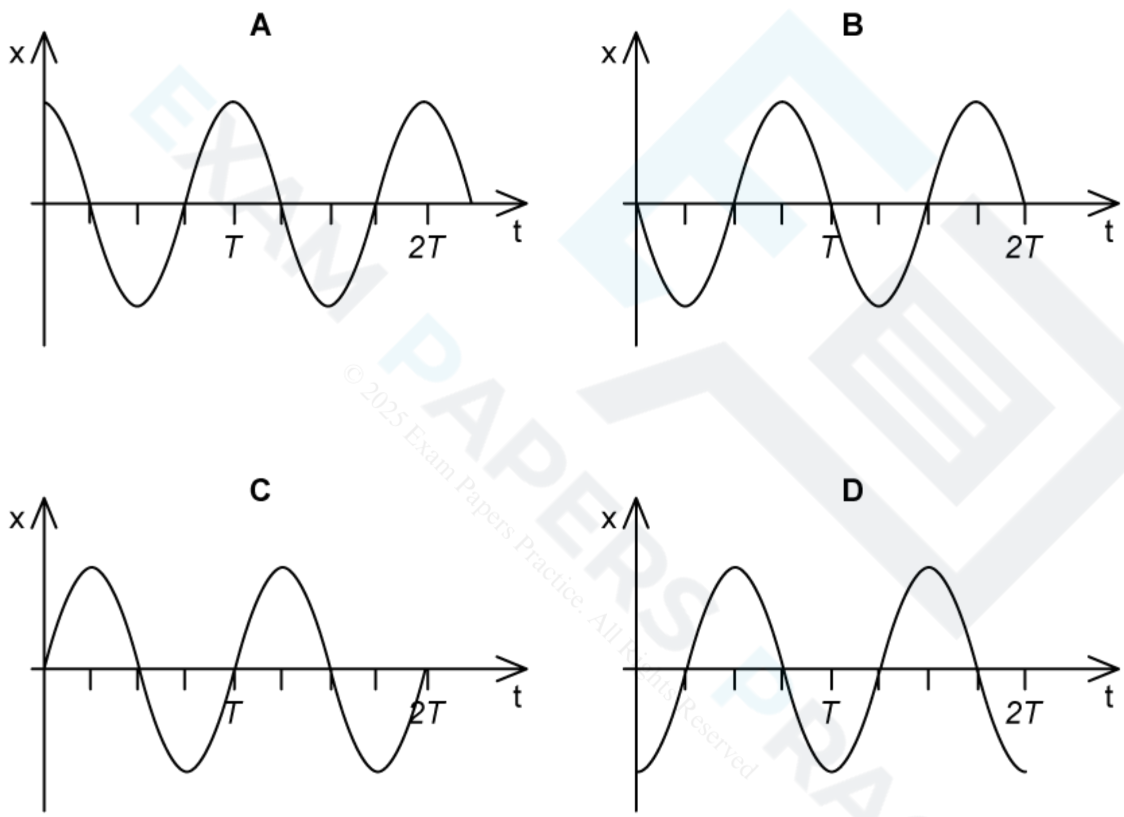
	$\cos(\omega t + \phi)$	Shifts by ϕ to the left (negative direction)

Worked example

An object oscillates with simple harmonic motion which can be described by the equation

$$x = x_0 \cos\left(\omega t - \frac{\pi}{2}\right)$$

Which of the following graphs correctly represents this equation?

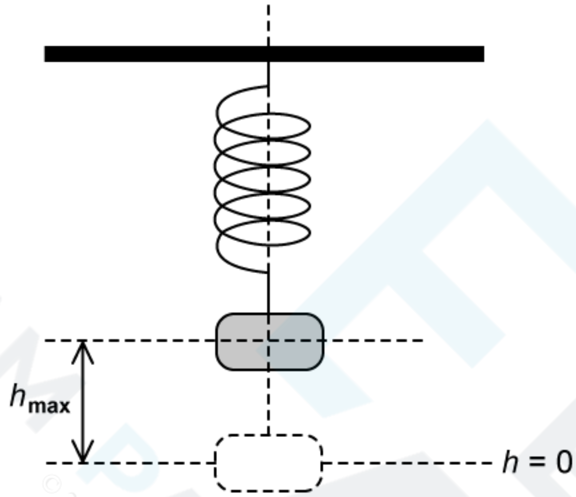


Answer: **C**

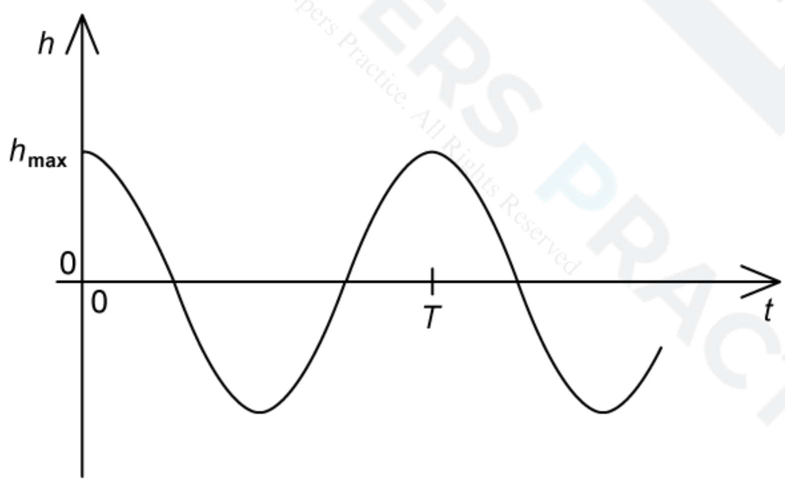
- The equation $x = x_0 \cos\left(\omega t - \frac{\pi}{2}\right)$ is equivalent to $x = x_0 \sin\left(\omega t\right)$
- This describes an oscillation where $x = 0$ when $t = 0$
  - Hence, options **A** & **D** are not correct
- When $\sin\left(\omega t\right)$ is positive, the oscillation will start moving in the $+x$ direction
  - Hence, option **B** is not correct

Worked example

A mass attached to a spring is released from a vertical height of h_{max} at time $t = 0$. The mass oscillates with a simple harmonic motion of period *T*.



The graph shows the variation of *h* with *t*.



(a) State the equation of motion for this oscillation.

(b) A second mass-spring system is set up and made to oscillate with the same frequency but with a phase angle of $\phi = -\frac{\pi}{4}$. On the graph, sketch the variation of *h* with *t* for the second mass-spring system.

Answer:

(a)

- The displacement-time equation for an oscillator released from a maximum displacement has the form

$$x = x_0 \cos \omega t$$

or

$$x = x_0 \sin\left(\omega t - \frac{\pi}{2}\right)$$

As the graph is leading a normal sine graph by $\frac{\pi}{2}$

- Where $x = h$ and $x_0 = h_{max}$
- Angular frequency ω is equal to

$$\omega = \frac{2\pi}{T}$$

- Therefore, the equation of motion for this oscillation is:

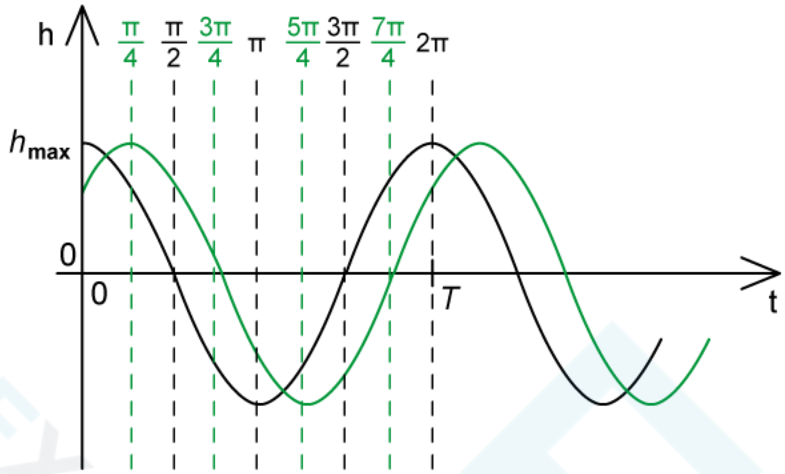
$$h = h_{max} \cos\left(\frac{2\pi}{T} t\right)$$

or

$$h = h_{max} \sin\left(\frac{2\pi}{T} t - \frac{\pi}{2}\right)$$

(b)

- One complete oscillation is equivalent to 2π rad



- A phase angle of $\phi = -\frac{\pi}{4}$ corresponds to a shift to the right (positive direction)