



Galilean & Special Relativity

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Reference Frames (HL)

Reference Frames

- The term **relative** is used often in Physics to make it clear which point of view we are referring to
 - For example, the velocity of a car **relative** to someone stationary is different to the velocity measured by another car travelling alongside the initial car at the same speed
- A reference frame, or a frame of reference, refers to the position of an object, it is defined as:
 A set of coordinates to record the position and time of events
- For example, you currently sitting on your chair at your desk is your current reference frame
 - You feel as if you are stationary, despite the fact the Earth is revolving on its axis and orbiting the sun
- It is the point of view where an object, at a specific co-ordinate, is at rest

Examples of Reference Frames

- An everyday example is the **direction** of an object from your point of view in comparison to someone else
- In this example, a car is driving down a road and two people are standing on opposite sides of that road
- Despite the car moving in one direction, each person will view its direction **relative to them** differently
 - The person on one side of the road would say the car is moving to the right, and the person on the other side of the road would say the car is moving to the left
 - Both are correct, but they are viewing the car's motion from different points of reference
- Another common example is of a train pulling out of a station, where Person A is on the platform and Person B is on the train
- As the train begins to move, Person A, on the platform, views Person B, on the train, moving to the right
 - Therefore, according to Person A, they, themselves, are stationary and Person B is moving to the right
- Things look a little different from Person B's perspective
- As the train begins to move (to the left from Person B's perspective), Person B, on the train, views Person A, on the platform, moving to the right
 - Therefore, according to Person B, they, themselves, are stationary (as they cannot feel the train moving to the left) and Person A is moving to the right

Diagram demonstrating different reference frames for a train leaving a station



Inertial Frames of Reference

- An inertial reference frame is
 - A reference frame that is non-accelerating
- Therefore, all inertial reference frames are moving at **constant velocity** with **respect** to each other
- There is no such thing as an absolute reference frame in our Universe
 - In other words, there is no place in the Universe that is completely stationary
 - Everything is always moving relative to everything else



A student is cycling to school with their friend who is also cycling exactly in line. As they cycle past a bus stop, they wave to their aunt who is stationary at the bus stop as she waits for her bus.

The student's aunt estimates the speed of the students to be 5 m s^{-1} .

At what speed would the friend measure the student to be travelling?

- **A** 5 m s⁻¹
- **B** -5 m s⁻¹
- **C** Oms⁻¹
- **D** 2 m s⁻¹

Answer:

The correct answer is **C** because:

- We must think about the friend's reference frame for this question, in which they are stationary (according to them)
- Since the friend is cycling in line with the student, this means they measure the student to be travelling at **0 m s⁻¹** relative to them



Galilean Relativity (HL)

Newton's Postulates of Time & Space

- We use inertial reference frames because Newton's laws of motion are the same in all of them This is known as Galilean Relativity
- For example, an object in an inertial reference frame will continue moving in a straight line with constant velocity unless acted upon by a force
 - This is in accordance with Newton's first law of motion
- This means that the same laws of Physics apply, regardless of one's frame of reference relative to another, as long as they are moving in a straight line at a constant velocity
- For an object moving with constant velocity in one reference frame, it will still have a constant (but different) velocity in another reference frame

Cartesian co-ordinates in 3D and 2D diagram

The Cartesian coordinate system is generally used for reference frames



Cartesian coordinates are used to represent a point in space

 Although there are an infinite number of inertial frames of reference in the Universe, there are ways to move between them



Galilean Relativity Equations

Galilean Transformation Equations

- Galilean transformation equations are used to convert between coordinates of space and time in one frame of reference to another for an event
 - This is because the velocity, position and time of an event appear differently from different reference frames
- For example, Person D is on a skateboard travelling at 4 m s⁻¹ when they throw a ball in a straight line at a constant velocity of 2 m s⁻¹. Person C is a stationary observer of the event.
 - In Person D's reference frame, the ball is travelling at 2 m s⁻¹
 - In Person C's reference frame, the ball is travelling at $4 + 2 = 6 \text{ m s}^{-1}$
- So what is the speed of the ball? Well, it depends!
- Mathematically, the position and time of an event in one reference frame, S, is represented by the coordinates (x, y, t)
- The position and time of an event in a second reference frame, moving relative to the first one, S', is represented by the co-ordinates (x', y', t')
 - Remember from the train example in Reference Frames, both Person A and Person B view the other as moving because they are viewing the event from two different frames of reference
- Now consider another example:
- Person C is stationary whilst Person D is moving away from Person C at velocity v
- Both Person C and Person D witness a balloon pop at some distance away
 - Person D, in reference frame S', measures the balloon pop at a distance **x'** away
 - Person C, in reference frame S, measures the balloon pop at a distance **x** = **x'** + **vt** away



In summary:

Table of the Galilean relativity equations for a stationary and moving reference frame

	Stationary reference frame	Moving reference frame	Transformation
	X	x'	x' = x - vt $x = x' + vt$
	у	у'	<i>y</i> = <i>y</i> '
	z	<i>z</i> '	<i>z</i> = <i>z</i> '
	t	ť'	t = t'

• Co-ordinates y and z are also the same in both reference frames because the relative motion is only in the x direction



A train travels through a station at a constant velocity of 15 m s^{-1} . One observer is sitting inside the train and another sits on the platform. As they pass each other, they start their stopwatches and watch a child on the train run at a constant speed in the same direction as the motion of the train. The observer on the train measures the speed of the child to be 2 m s^{-1} .

- (a) According to the observer on the train, how far has the child moved after 10 s?
- (b) According to the observer on the platform, how far has the child moved after 10 s?

Answer:

Draw a quick sketch of the situation

Label the stationary (x, t) and moving (x', t') reference frames



(a) Calculate the distance the child travels according to the observer on the train

- The observer measures the child to be travelling away at 2 m s⁻¹
- Therefore, using the equation

distance (x) = speed × time

• The distance, x' is:

(b) Calculate the distance the child travels according to the observer on the platform

• This observer is stationary in their reference frame, so we use the equation

$$x = x' + vt$$

x = 20 + (15 × 10) = 170 m

• v is the speed of the **whole** reference frame



Velocity Addition Equation

- Galilean transformations can also be used to transform velocities
 - This is known as **velocity addition**
- Velocity addition is used when there are **multiple velocities** in the scenario
- Let's go back to the example of Person D on a skateboard throwing a ball directly in **front** of them in a straight line
- In this example:
 - *u* is the speed of the ball measured in frame S (by Person C)
 - *u'* is the speed of the ball measured in frame S' (by Person D)
 - v is the speed of Person D
- Therefore:
 - Person D (frame S') measures the velocity of the ball to be **u'**
 - Whilst Person C (frame S) measures the velocity of the ball to be **u** = **v** + **u**'
- Hence, the velocity of the ball from Person D's reference frame in terms of speeds v and u is
 - u′ = u v
- Velocities are vectors, so their direction must be taken into account
- Let's say Person D now throws the ball directly behind them in a straight line at constant velocity
 - *u*, *u*' and *v* still refer to the same objects
- This time, since u' is in the **opposite** direction to v, it is now -u'
- Therefore:
 - Person D (frame S') measures the velocity of the ball to be -u'
 - Whilst Person C (frame S) measures the velocity of the ball to be **u** = **v u**'

Diagram showing velocity addition for two objects moving in opposite directions

In summary:

Table of the velocity addition equations for a stationary and moving reference frame

u u' u' u' = u - v	Stationary reference frame	Moving reference frame	Transformation
u u' u' u' = u – v			u = u' + v
	u	u'	u' = u – v



Two travellers X and Y walk past each other at an airport. Traveller X walks at 1.5 m s^{-1} whilst Traveller Y walks at 2.1 m s^{-1} .

Calculate the velocity of:

- (a) Traveller Y relative to Traveller X
- (b) Traveller X relative to Traveller Y

Traveller X now walks onto a travelator walkway in the airport at the same speed. The travelator moves at 0.3 m s^{-1} . Traveller Y decides to take a seat facing the travelator.

Calculate the velocity of:

- (c) Traveller X relative to Traveller Y when they are walking in the same direction as the travelator
- (d) Traveller X relative to Traveller Y when they are walking in the opposite direction to the travelator (just for fun)

Answer:

(a) The velocity of Traveller Y relative to Traveller X is:



- X is the reference frame at rest
- Y is the reference frame that is moving
- Therefore, from Traveller X's perspective, Traveller Y is moving at:
 21, 15, 0, 6 m o⁻¹
 - $2.1 1.5 = 0.6 \,\mathrm{m\,s^{-1}}$
- From Traveller X's frame of reference, Traveller Y is moving at a speed of 0.6 m s⁻¹
- Traveller Y is moving faster than Traveller X by 0.6 m s⁻¹

(b) The velocity of Traveller X relative to Traveller Y is:

- Y is the reference frame at rest
- X is the reference frame that is moving



• Therefore, from Traveller Y's perspective, Traveller X is moving at:

$$1.5 - 2.1 = -0.6 \,\mathrm{m\,s^{-1}}$$

- From Traveller Y's frame of reference, Traveller X is moving at a speed of -0.6 m s⁻¹
- Traveller X is moving slower than Traveller Y by 0.6 m s⁻¹

(c) The velocity of Traveller X relative to Traveller Y when walking in the same direction as the travelator





- X is the reference frame that is moving **against** the travelator
 - *u* = velocity that Y measures X to be
 - u' = velocity of the travelator
 - -v = velocity of X
- Therefore, they are moving at:

 $u = u' - v = 0.3 - 1.5 = -1.2 \text{ m s}^{-1}$





Postulates of Special Relativity (HL)

The Postulates of Special Relativity

- Galilean relativity states that Newton's laws of motion are the same in all inertial reference frames
- Newton then treated space and time as fixed and absolute
 - This means the time interval between two events in one frame (x, y) is the same as the time interval between another frame (x', y')
- However, this is **not** what happens when we are close to the speed of light
 - Space and time become relative, meaning, the length of an object or a time interval depends on the frame of reference
- Velocity addition works with speeds much lower than the speed of light (c)
- It doesn't work for objects travelling closer to the speed of light
 - According to Galilean relativity, if a rocket ship travels at 0.7c and releases a probe directly in front of it at 0.5c, a stationary observer would view this at 0.7c + 0.5c = 1.2c
 - However, we know that nothing can travel faster than the speed of light, so this is not possible
- Einstein's two **postulates of special relativity** are:

First Postulate

The laws of physics are the same in all inertial frames of reference

- In our own reference frame, we are always stationary
- This means in practice, we should **not** be able to tell whether we are moving or not
 - Someone conducting a physics experiment on a moving train versus on a stationary platform should produce the exact same results

Second Postulate

The speed of light, c, in a vacuum, is the same in all inertial frames of reference

- Two different observers will always measure the speed of light to be the same value, c in their reference frame
 - It makes no difference whether they are travelling or not. If it did, you would know whether you are moving, which counteracts the first postulate
- For example, a runner holding a flashlight in front of them will measure the speed of the light as **c**
 - However, someone stationary observing the runner will also see the speed of light as c and not c + the velocity of the runner
- This only works for the **speed of light**, not any other speed



Lorentz Transformations (HL)

Lorentz Transformation Equations

- To relate measurements from one reference frame to another in Galilean relativity, we used Galilean transformations
- However, Galilean transformations can no longer be used to describe distances, times and speeds for objects travelling close to the speed of light
- Einstein's postulates of special relativity lead to the Lorentz transformation equations for the coordinates of an event in two inertial reference frames

The Lorentz Factor

• Lorentz transformations are a **correction** of the Galilean transformations for speeds close to the speed of light, by multiplying by a scaling factor called the **Lorentz factor**, γ

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$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- As v will always be less than c (since nothing can travel faster than the speed of light), this means that γ will always be greater than 1
- This is especially important for time dilation and length contraction

Lorentz Transformation Equations

- Again we have the reference frame S measuring with co-ordinates (x, y, t), and S' with co-ordinates (x', y', t')
- Person F is moving away from Person E in their rocket ship at speed v which is close to the speed of light c
 - Person E is a stationary observer on Earth
- Both Person E and Person F witness a loud bang some distance away
 - Person F measures the loud bang to be a distance **x'** away
 - Person E measures the loud bang to be a distance x = y(x' + vt') away



Lorentz (v≈c)	Galilean (v << c)	
$x' = \gamma(x - vt)$	x' = x - vt	
$x = \gamma(x' + vt')$	x = x' + vt	
$t' = \gamma \left(t - \frac{VX}{c^2} \right)$	t' = t	
$t = \gamma \left(t' + \frac{vx'}{c^2} \right)$	t = t'	
y = y'	y = y'	
z = z'	z = z'	

Galilean vs. Lorentz Transformations

 Notice that time is different between reference frames t and t' for objects travelling close to the speed of light, whilst in Galilean transformations, time was absolute (it doesn't change) between reference frames



A rocket of proper length 150 m moves to the right with speed 0.91c relative to the ground.



Ground

A probe is released from the back of the rocket at speed 0.45c relative to the rocket.

Determine the time it takes the probe to reach the front of the rocket according to an observer

- (a) At rest in the rocket.
- (b) At rest on the ground.

Answer:

(a)

Step 1: List the known quantities

- Length of the rocket, *l* = 150 m
- Speed of the probe, v' = 0.40c

Step 2: Analyse the situation

- In the reference frame of an observer at rest in the rocket, they are **stationary**
- Therefore, the probe travels at a constant speed 0.45c across the full length of the rocket of 150 m

$$t' = \frac{1}{v'} = \frac{150}{0.45c} = \frac{150}{0.45 \times (3 \times 10^8)}$$
$$t' = 1.11 \times 10^{-6} \,\mathrm{s}$$

• t' and v' are used because they are the times and velocity of the **moving** object **in the reference frame** of the observer at **rest**



(b)

Step 1: List the known quantities

- Length of the rocket, I = 150 m
- Speed of the rocket, v = 0.91c
- Time taken for the probe to reach the front of the rocket, $t' = 1.11 \times 10^{-6}$ s

Step 2: Analyse the situation

- In reference to an observer at rest on the ground, they will see the probe taking **longer** to reach the front of the ship
- Since object in question, the probe, is **moving** in both reference frames, we need to use a Lorentz transformation

Step 3: Calculate the gamma factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(0.91c)^2}{c^2}}}$$
$$\gamma = \frac{1}{\sqrt{1 - 0.91^2}} = 2.412$$

Step 3: Substitute values into the Lorentz transformation

• Since the observer is at **rest**, the Lorentz equation for time t must be used

$$t = \gamma \left(t' + \frac{vx'}{c^2} \right)$$

$$t = 2.412 \left((1.11 \times 10^{-6}) + \frac{(0.91c)(150)}{c^2} \right) = 2.412 \left((1.11 \times 10^{-6}) + \frac{(0.91)(150)}{3 \times 10^8} \right)$$

$$t = 3.8 \times 10^{-6} \text{ s}$$



Velocity Addition Transformations (HL)

Velocity Addition Transformations

- Similar to velocity addition to Galilean transformations, the Lorentz transformation equations lead to relativistic velocity addition equations
- These are again used when there are multiple velocities in the scenario but now some are close to the speed of light
- Let's go back to the example of Person F in the rocket ship. They now release a missile in front of them
- In this example:
 - *u* is the speed of the missile measured in frame S (by Person E)
 - *u'* is the speed of the missile measured in frame S' (by Person F)
 - v is the speed of frame S' (Person F)
- In Galilean velocity addition, when v << c, these were:</p>
 - The speed of the missile as measured by Person E: u = u' + v
 - Or, u' = u v
- If v and u' are close to the speed of light, we have to use Lorentz velocity addition transformations instead
- These equations are:

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$
$$u = \frac{u' + v}{u'v}$$

- Where:
 - *u* = the velocity of an object measured from the **stationary** reference frame
 - *u*' = the velocity of an object measured from a **moving** reference frame
 - v = the velocity of the moving reference frame
 - c = the speed of light



A rocket moves to the right with speed 0.60c relative to the ground.



A probe is released from the back of the rocket at speed 0.82c relative to the rocket.

Calculate the speed of the probe relative to the ground.

Answer:

Step 1: List the known quantities

- Speed of the rocket, v = 0.60c
- Speed of the probe relative to the rocket, u' = 0.82c

Step 2: Analyse the situation

- We have multiple velocities in this scenario in terms of c, so we need to use the **Lorentz** velocity addition equations
- The probe is travelling in the **opposite** direction to the rocket, so its velocity is -0.82c
- We want the speed relative to the ground, which is a reference frame **at rest**, so this is u

Step 3: Substitute values into the equation

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$
82*c* + 0.60*c* (-0

$$u = \frac{-0.82c + 0.60c}{1 + \frac{(-0.82c)(0.60c)}{c^2}} = \frac{(-0.82 + 0.60)c}{1 + (-0.82)(0.60)}$$
$$u = -0.43c$$



Space-Time Interval (HL)

Space-Time Interval

- Einstein discovered that time and distance **changes** when moving from one inertial reference frame to another when travelling at speeds close to the speed of light
 - In other words, these reference frames are **not absolute**
 - However, some quantities **are** the same in all inertial frames. These are called **invariant**
- These are:
 - Propertime, t_0
 - Proper length, L₀
 - Space-time interval, ∆s
- These are a product of Einstein's second postulate
- In Galilean relativity:
 - Space and time are the same in all reference frames, i.e. $\Delta t = \Delta t'$ and $\Delta x = \Delta x'$
- In special relativity:
 - These are replaced with a space-time interval, as space and time are connected together as 4 coordinates (x, y, z, t) for an event
 - Motion can be represented as spanning both space and time using this coordinate system
- The diagram below shows a person moving in both space x and z and in time t
 - They can also move in the y direction, but 4 dimensions are not possible to draw accurately here (in 3-dimensional space)
- An interval in space-time is an **invariant** quantity in all inertial reference frames and is defined as:

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$$

- Where:
 - Δt = time interval / separation (s)
 - C = the speed of light
 - ΔX = spacial separation (m)
 - ΔS = space-time interval (m)
- This means that in two inertial reference frames, although Δt and Δx will be different in both frames,
- Δs will be the same
- These will be used in space-time diagrams



An inertial reference frame S' moves relative to S with a speed close to the speed of light. When clocks in both frames show zero the origins of the two frames coincide.

An event P has coordinates x = 2 m and ct = 0 in frame S, and x = 2.3 m in frame S'. Show that the time coordinate of event P in frame S' is -1.1 m.

Answer:

Step 1: List the known quantities:

- Spacial separation in frame S, $\Delta X = 2$ m
- Time separation in frame S, $C\Delta t = 0$
- Spacial separation in frame S', $\Delta x' = 2.3$ m

Step 2: Calculate the space-time interval in frame S

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$$

 $(\Delta s)^2 = (0) - (2)^2 = -4$

Step 3: Substitute values into the space-time interval for S'

• ΔS is the same (invariant) in both reference frames

$$(\Delta s)^2 = (c\Delta t')^2 - (\Delta x')^2$$
$$(c\Delta t')^2 = (\Delta s)^2 + (\Delta x')^2$$
$$(c\Delta t')^2 = -4 + (2.3)^2 = 1.29$$
$$c\Delta t' = \sqrt{1.29} = \pm 1.14$$



Proper Time & Length

- In special relativity, we have found that distances and times are relative
- This means the length of an object and the time interval of an event **changes** when observed in frames that are **moving relative** to that **object**
 - More on this is explored in length contraction and time dilation
- We need to define the difference mathematically between the time and lengths measured between each frame, to know which one is being referred to

What is Proper Time and Proper Length?

• **Proper time interval**, Δt₀ is defined as:

The time interval between two events measured from within the reference frame in which the two events occur at the same place

• **Proper length**, *L*₀ is defined as:

The length measured in a reference frame where the object is at rest (relative to the observer)

- These can be measured either in the moving frame S' or a rest frame S
 - This depends on the reference frame you are calculating the length and time from
- For example, if a person in moving frame S' (e.g. on a train) measures the length of a book, they are at rest relative to the book
 - They measure the distance between points x_2' and x_1'
 - This is the proper length, L₀ although they are technically moving (but they don't know this otherwise it would go against Einstein's first postulate)
- However, for an observer in frame S, at rest (e.g. on a platform
 - XI IKIIIIS Reserved • They will measure the distance between points x_2 and x_1
 - This a shortened length, L



Time Dilation (HL)

Time Dilation

- When objects travel close to the speed of light, to an observer **moving relative** to that object, it looks as if the object has **slowed down**
 - This is best demonstrated by clocks
- Observer H is in a rocket moving close to the speed of light
 - They will see their clock ticking at a regular pace, say, it is reading 15:00
- Observer G at rest on Earth, with remarkable eyesight, will measure the clock as ticking **slower**
 - They will observe that time has **slowed down** in the spaceship from **their** reference frame i.e. they may see the time as 14:45 instead of 15:00
- However, the same occurs the other way around
- For observer H on the rocket, it is observer G that is **moving relative** to them
 - Therefore, observer H will measure observer G's clock as ticking slower i.e. they see time slow down on Earth from their reference frame



Time Dilation Equation

- Consider a light clock. This consists of two mirrors facing each other with a beam of light travelling up and down between them
- An observer is at rest relative to the clock on a train platform and watches the light reflect between the mirrors
- The distance between the mirrors is L and the light travels at a speed c
- Therefore, the time interval for the light to travel from the top mirror back down to the bottom is:



- Another light clock is on a moving train, **relative** to the initial observer on the platform, travelling at constant velocity *v*
- The stationary observer on the platform sees the light clock on the train and watches the reflection of the rays between the mirrors
- It appears that the light rays travel to the right at an angle to the direction of motion



- The observer on the train platform sees the mirror at:
 - Position 1, when the light leaves the bottom mirror
 - Position 2, when the light returns to it



- The length of the light path as seen by the stationary observer is not *L*, they see the longer path, *D*
- The distance travelled by the light ray is now 2D, and the time observed between the reflections is now

$$\Delta t = \frac{2D}{c}$$

- The apparent distance horizontally travelled by the mirror is $V\Delta t$ where v is the speed of the train
- Notice that this is part of a right-angled triangle, so using Pythagoras' theorem we can see that:

$$D^2 = L^2 + \left(\frac{v\,\Delta t}{2}\right)^2$$

- Where: $D = \frac{c\Delta t}{2}$
- We want to find Δt , the time taken for the light ray to travel up and down in the reference frame of the stationary observer on the train platform, who is moving **relative** to the light clock on the train
 - Remember, although it is the train that is moving, in the reference frame of an observer on the train it is the observer on the **platform** that is moving

$$\Delta t = \frac{\frac{2L}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

• Remember the gamma factor $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ is from the Lorentz transformation equations

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Therefore:

$$t = \gamma \Delta t_0$$

- Where:
 - Δ t = the time interval measured from an observer moving relative to the time interval being measured (s)
 - Δt_0 = the **proper** time interval (s)
- As $\gamma > 1$, this means that the $\Delta t > \Delta t_0$
 - In other words, a clock observed from a reference frame moving **relative** to it will be measured to tick slower than a clock that is at rest in its frame of reference
- The observer on the **platform** will view the **train** clock as moving **slower**
- The observer on the train will view the platform as moving slower
- You may be wondering why it's the time that slows down for the light beam, and not the light beam just speeding up to hit each mirror at the same frequency



- This is due to Einstein's second postulate
- Both Observers G and H **must** measure the speed of light to be c, so it doesn't slow down or speed up according to either reference frame
- It is important to note that the time has been measured at the same **position**
 - In other words, the time interval is the position at which the light leaves the first mirror and at which it returns to the second mirror in the reference frame of the mirror



Alex's spacecraft is on a journey to a star travelling at 0.7 c. Emma is on a space station on Earth at rest. According to Emma, the distance from the base station to the star is 14.2 ly.

Show that Alex measures the time taken for her to travel from the base station to the star to be about 14.5 years.

Answer:

Step 1: List the known quantities:

- Distance of space station according to Emma = 14.2 ly
- Speed of Alex's spacecraft, v = 0.7c

Step 2: Analyse the situation

- We are trying to find the time that Alex measures for **her** travel i.e. the time she would measure on her own clock in the spaceship which **she is stationary relative to**
- This is the proper time, Δt_0

Step 3: Calculate the time taken according to Emma, Δt

- 1 ly (light year) is the **distance**, s, light (at speed c) travels in a year
- Therefore it takes the light 14.2 years (time) to travel the distance at speed c

$$s = speed \times time = 14.2 c m$$

• Therefore, the time taken according to Emma is:

$$\Delta t = \frac{s}{v} = \frac{14.2c}{0.7c} = \frac{14.2}{0.7} = 20.29$$
 years

Step 4: Substitute values into the time dilation equation

$$\Delta t = \gamma \Delta t_0 \quad \Rightarrow \quad \Delta t_0 = \frac{\Delta}{\gamma}$$

$$\Delta t_0 = \frac{20.29}{\frac{1}{\sqrt{1 - \frac{(0.7c)^2}{c^2}}}} = \frac{20.29}{\frac{1}{\sqrt{1 - (0.7)^2}}} = 14.5 \text{ years}$$

Step 5: Check whether your answer makes sense

- Since Emma (who is stationary) is viewing Alex's clock (which is moving) she would measure a **longer** time for Alex to reach the star than Alex will
- As Emma records 20.29 years, but Alex only records 14.5 years, this time makes sense



Length Contraction (HL)

Length Contraction

- When objects travel close to the speed of light, to an observer moving relative to that object, it looks as if the object has become shorter
 - This is best demonstrated using **rulers**
- Observer H, in their rocket moving close to the speed of light, measures the length of their pencil to be 14 cm
- Observer G, at rest on Earth, would measure (with remarkable eyesight) the length of the pencil to be shorter
 - They will see lengths **contracted** in the spaceship from **their** reference frame, e.g. the length may appear to be 10 cm instead of 14 cm
- However, the same occurs the other way around
- For observer H on the rocket, it is observer G that is **moving relative** to them
 - Therefore, observer H would measure the length of observer G's pencil as **shorter** i.e observer H, on the rocket, sees lengths **contracted** on Earth from **their** reference frame







Length Contraction Equation

- The length of an object is the difference in the position of its ends
- Consider the observers G and H measuring the length of a pencil, which is stationary in reference frame S' (for observer H)
- As observer H is in the moving frame S', they measure the length of the ruler as:

$$\Delta x' = x_2' - x_1' = L_0$$

- This is the proper length, L₀ as the pencil is not moving relative to observer H
 - Both the pencil and observer H are, however, moving relative to observer G
- Observer G needs to measure the length of the pencil by measuring the position of its ends at the same time (just like observer H did)
- They measure the length of the ruler to be:

$$\Delta x = x_2 - x_1 = L$$

- This is the observed length, L as the pencil is moving relative to observer G
 - Lorentz transformations tell us how the x and x' are related
- We want to find ΔX , the length measured in the reference frame of the stationary observer on Earth (G), who is moving **relative** to the observer on the rocket (H)
- Transforming these distances gives:

$$x_1' = \gamma(x_1 - vt)$$
$$x_2' = \gamma(x_2 - vt)$$

• These are then substituted into the equation for the proper length, *L*₀:

$$x_{2}' - x_{1}' = \gamma(x_{2} - vt) - \gamma(x_{1} - vt) = \gamma(x_{2} - x_{1})$$



Therefore:

$$L_0 = \gamma L$$

$$L = \frac{L_0}{\gamma}$$

- Where:
 - L = the length measured by an observer **moving relative** to the length being measured (m)
 - $L_0 = \text{the proper length (m)}$
- As γ > 1, this means that the $L < L_0$
 - In other words, lengths measured from a reference frame moving **relative** to the object will be measured as shorter than the lengths measured at rest from within their frame of reference
- Similar to time dilation, length contraction is also due to Einstein's second postulate
 - Both observers G and H **must** measure the speed of light to be c
 - Since the time for observer H will run slower, according to observer G (i.e. t increases), then for c to stay the same, the length of the object, L must decrease
- It is important to note that the length has been measured at the same time
 - This length is the difference between the ends of the pencil, with both ends measured at the same time
- The ruler used in both reference frames is **stationary** in their **own** reference frame
 - Otherwise, observer G would see the ruler on observer H's rocket contracting as well and wouldn't measure any difference in length



A spacecraft leaves Earth and moves towards a planet.

The spacecraft moves at a speed of 0.75c relative to the Earth. The planet is a distance of 15 ly away according to the observer on Earth.



The spacecraft passes a space station that is at rest relative to the Earth. The proper length of the space station is 482 m.

Calculate the length of the space station according to the observer in the spacecraft.

Answer:

Step 1: List the known quantities

- Speed of the spacecraft, v = 0.75c
- Proper length of the space station, $L_0 = 482$ m

Step 2: Analyse the situation

- We are trying to find the length of the space station in the reference frame of the observer in the **spacecraft**
- In this observer's reference frame, it is the **space station** that is moving away from them at 0.75c
- Therefore, we a measuring a length in the moving reference frame (relative to the spacecraft) this is the length, *L*

Step 3: Substitute values into the length contraction equation





Step 4: Check whether your answer makes sense

- As the observer in the spacecraft is stationary, the length of the space station they measure should be **shorter** than the proper length
- As the length recorded from the spacecraft is 319 years, and the proper length is 482 m, this length makes sense



Simultaneity in Special Relativity (HL)

Simultaneity in Special Relativity

- The term 'simultaneous' means to occur at the same time
- The relativity of simultaneity states that

Whether two spatially simultaneous events happen at the same time is not absolute, but depends on the observer's reference frame

- This means that in one reference frame, two events that occur at different points in space seem to happen at the same time, whilst in another reference frame moving relative to the first the events seem to happen one after another
 - This was not the case in Galilean relativity, where simultaneity was absolute
- This is best shown in the following example
- Person B is in a train carriage moving to the right at constant velocity
- They switch on a lamp above them and they observe that the light from the lamp reaches the two ends of the carriage, points X and Y, at the same time
- Meanwhile, Person A is stationary on a platform and observes the train travel past
- Person A will still see the light from the lamp move to both ends of the carriage at the same speed (c)
 - This is in line with Einstein's second postulate
- However, Person A will see the light reach point X **before** it reaches point Y
 - This is because the whole carriage is moving to the right (relative to Person A), so the left side of the carriage is moving towards the light ray and the right side of the carriage is moving away from the light ray
 - This means that Person A will see the light ray reach point Y slightly later



Space-Time Diagrams (HL)

Space-Time Diagrams

- Spacetime (or Minkowski) diagrams represent an object's motion in spacetime
- They help to visualise
 - Time dilation
 - Length contraction
 - Simultaneity
- Since 4D (x, y, z, t) diagrams cannot be drawn on a 2D page, we collapse 3D space (x, y, z) into 1 spacial dimension and keep time as its own dimension
- This gives a spacetime diagram
 - Lines drawn on a spacetime diagram are called world lines
- Instead of the usual distance-time graphs, we plot
 - The horiztonal axis as x
 - The vertical axis as **ct**
- Note that both axes have dimensions of length
 - This makes it easy to compare values on one axis and another
- This means the worldlines have a gradient of

$$\frac{c\Delta t}{\Delta x} = \frac{c}{V}$$
 where the velocity is $V = \frac{\Delta x}{\Delta t}$

This means:

- The steeper the gradient, the slower the object is moving
- The shallower the gradient, the faster the object is moving
- ct was first seen when the spacetime interval was introduced, and c is chosen deliberately so our diagram is oriented around the speed of light
 - ct is a sort of 'distance in time'



Multiple Reference Frames

- Every point on a spacetime diagram represents an event, for example, an object moving
- More than one inertial reference frame can be represented on a spacetime diagram
 - ct and x represent the co-ordinate axes for an observer in frame S
 - ct' and x' represent the co-ordinate axes for an observer in frame S' (moving at speed v with respect to frame S)
- Therefore, we can combine two separate spacetime diagrams for different inertial reference frames moving at constant speed relative to each other
 - The axes for the ct' and x' are at an angle
- The worldline T shows the equivalent worldline of P, but now in the S' reference frame
 This represents an object at rest in its own co-ordinate system
- The axes are tilted because the ct' reference frame is travelling at speed v relative to the ct reference frame
 - The x' axis must also be tilted in order for the speed of light (the dashed line in the middle) to be the **same** in **both** reference frames
- The scales on the time axes ct and ct' and on the space axes x and x' of two inertial reference frames moving relative to one another are not the same and are defined by lines of constant spacetime interval
- If an event occurs (such as a flash of light), both reference frames will measure a different time and different position with respect to each other
 - This can be seen on a spacetime diagram

Simultaneity

- We can now see that simultaneous events in one frame are not simultaneous in another moving inertial reference frame
- Let's go back to Observers A and B in Simultaneity in Special Relativity
 - We can see that Observer B sees the light reach points X and Y at the same time, whilst Observer A (in the ct'-x' co-ordinate system) sees the light from the lamp reach point X before point Y on a spacetime diagram



Time Dilation

- Consider two flashes of light at x = 0 in the S reference frame that occur one after another
- When these flashes are observed in the S' frame, we can see the time between the flashes is longer
 The time between them has increased (dilated)
- Another difference is that in the S' reference frame, the first flash now occurs on the -x axis
- This just means it takes place to the **left** of the observer

Length Contraction

- Consider a rod measured in the S reference frame where the rod moving relative to S
- The rod has the same speed as it does in the S' reference frame (which is also moving relative to S)



The spacetime diagram shows the axes of an inertial reference frame S and the axes of a second inertial reference frame S' that moves relative to S with speed 0.6432c. When clocks in both frames show zero the origins of the two frames coincide.



Event E has co-ordinates x = 1.5 m and ct = 0 in frame S.

(a) Label, on the diagram,

(i) the space co-ordinate of event E in the S' frame. Label this event with the letter Q.

(ii) the event that has co-ordinates x' = 1.5 m and ct' = 0. Label this event with the letter R.

(b) A rod at rest in frame S has a proper length of 1.5 m. At t = 0, the left-hand end of the rod is at x = 0 and the right-hand end is at x = 1.5 m.

Using the spacetime diagram, outline without calculation, why observers in frame S' measure the length of the rod to be less than 1.5 m.

Answer:

(a)

(i) Draw a line parallel to the ct' axis





(ii)

Step 1: List the known quantities

- Speed of the spacecraft, v = 0.6432c
- Position of event in frame S, x = 1.5 m

Step 1: Calculate the x' co-ordinate of point Q

• To convert between a position (or time) from one co-ordinate system and another, we can use **Lorentz** transformations

$$x' = \gamma(x - vt)$$

$$x' = \frac{1}{\sqrt{1 - \frac{(0.6432c)^2}{c^2}}} (1.5 - 0) = 1.959 = 2.0 \text{ m}$$

Since there are no other objects involved, speed v = 0

Step 2: Label this point on the axes as R

- The co-ordinates are x' = 1.5 m and ct' = 0
- Point R (at 1.5 m) is roughly $\frac{2}{3}$ of the distance of Q (at 2.0 m)





(b)

Step 1: Outline why observers in frame S' measure the length of the rod to be less than 1.5 m

- The ends of the rod must be recorded at the same time in frame S'
- This is shown on the spacetime diagram:











Velocity on a Space-Time Diagram

- The worldline for a moving particle on a spacetime diagram using the x-ct axis is a diagonal line
- The velocity of the particle can be calculated by the angle of the moving particle's worldline with the ct axis
- When we are using the *x*-*ct* axis, we can see that:

$$\tan\theta = \frac{opposite}{adjacent} = \frac{\Delta x}{c\Delta t}$$

From mechanics, we know that velocity is the rate of change of displacement:

$$\frac{\Delta x}{\Delta t} = x$$

Therefore:

$$\tan\theta = \frac{v}{c}$$

- Where:
 - θ = angle between the world line and the ct axis (°)
 - v = velocity of the object (m s⁻¹)
 - c = speed of light



A spaceship travels away from Earth in the direction of a nearby planet. A spacetime diagram for the Earth's reference frame shows the worldline of the spaceship. Assume the clock on the Earth, the clock on the planet, and the clock on the spaceship were all synchronized when ct = 0.



Show, using the spacetime diagram, that the speed of the spaceship relative to the Earth is 0.80c.

Answer:

Method 1: Using the gradient

Step 1: Choose a co-ordinate pair

• Choose any corresponding value of ct and x e.g. ct = 10, x = 8











Muon Lifetime Experiment (HL)

Muon Lifetime Experiment

- Muon decay experiments provide experimental evidence for time dilation and length contraction
- **Muons** are unstable, subatomic particles that are around 200 times heavier than an electron and are produced in the upper atmosphere as a result of pion decays produced by cosmic rays
- Muons travel at 0.98c and have a half-life of 1.6 µs (or mean lifetime of 2.2 µs)
 - The distance they travel in one half-life is around 470 m
- A considerable number of muons can be detected on the Earth's surface, which is about 10 km from the distance they are created
 - Therefore, according to Newtonian Physics, very few muons are expected to reach the surface as this is about 21 half-lives!
- The detection of the muons is a product of time dilation (or length contraction, depending on the viewpoint of the observer)

Muon Decay From Time Dilation

- According to the reference frame of an observer on Earth, time is dilated so the muon's half-life is longer
- We can see this from the time dilation equation

$$\Delta t = \gamma \Delta t$$

- Where:
 - The gamma factor, $\gamma = \frac{1}{\sqrt{1 (0.98)^2}} = 5$
 - Δt = the half-life measured by an observer on Earth
 - Δt_{0} = the proper time for the half-life measured in the muon's inertial frame
- Therefore, in the reference frame of an observer on Earth, the muons have a lifetime of

$$\Delta t = 5 \times 1.6 = 8 \ \mu s$$

• The time to travel 10 km at 0.98c is 33 µs or 4.1 half-lives, so a significant number of muons remain undecayed at the surface

Muon Decay From Length Contraction

- According to the muon's reference frame, length is contracted so the distance the need to travel is shorter
- We can see this from the length contraction equation



$$L = \frac{L_0}{\gamma}$$

- Where:
 - L_0 = the proper length for the distance measured in the muon's inertial frame
 - L = the distance measured by an observer on Earth
- Therefore, in the reference frame of the muons, they only have to travel a distance:

$$L = \frac{10\,000}{5} = 2000\,\mathrm{m}$$

• To travel this distance takes a time of $\frac{2000}{0.98c}$ = 6.8 µs which is about 4.3 half-lives again, so a

significant number of muons remain undecayed at the surface



Muons are created at a height of 4250 m above the Earth's surface. The muons move vertically downward at a speed of 0.980c relative to the Earth's surface. The gamma factor for this speed is 5.00. The half-life of a muon in its rest frame is 1.6 µs.

- (a) Estimate the fraction of the original number of muons that will reach the Earth's surface before decaying, from the Earth's frame of reference, according to:
 - (i) Newtonian mechanics
 - (ii) Special relativity
- (b) Demonstrate how an observer moving with the same velocity as the muons, accounts for the answer to (a)(ii).

Answer:

(a) (i)

Step 1: List the known quantities

- Height of muon creation above Earth's surface, h = 4250 m
- Speed of muons, v = 0.980c
- Lifetime of muon, $t = 1.6 \,\mu s = 1.6 \times 10^{-6} \,s$

Step 2: Calculate the time to travel for the muon

$$time = \frac{distance}{speed} = \frac{h}{v}$$

$$t = \frac{4250}{0.98 \times (3.0 \times 10^8)} = 1.45 \times 10^{-5} \,\mathrm{s}$$

Step 3: Calculate the number of half-lives

$$\frac{1.45 \times 10^{-5}}{1.6 \times 10^{-6}} = 9 \text{ half-lives}$$

Step 4: Calculate the fraction of the original muons that arrive

$$\frac{1}{2^9} \times 100 \% = 0.2 \%$$

(a) (ii)



Step 1: List the known quantities

• Time for the muon to travel, $\Delta t_0 = 1.45 \times 10$ s

Step 2: Calculate the time travelled in the muons rest frame

$$\Delta t = \gamma \Delta t_0$$
$$\Delta t = 5 \times (1.6 \times 10^{-6}) = 8 \times 10^{-6}$$

Step 3: Calculate the number of half-lives

$$\frac{1.45 \times 10^{-5}}{8 \times 10^{-6}} = 1.8 \text{ half-lives}$$

Step 4: Calculate the fraction of the original muons that arrive

$$\frac{1}{2^{1.8}} \times 100 \% = 29 \%$$

(b)

Step 1: Analyse the situation

• An observer moving with the same velocity as the muons will measure the **distance** to the surface

to be **shorter** by a factor of $\left(\frac{9}{1.8}\right) = 5$ OR length contraction occurs

Step 2: Calculate the distance travelled in the muon's rest frame

$$L = \frac{L_0}{\gamma}$$
$$L = \frac{4250}{5} = 850 \text{ m}$$

Step 3: Calculate the time to travel

time taken =
$$\frac{distance}{speed} = \frac{850}{0.98 \times (3 \times 10^8)} = 2.9 \times 10^{-6}$$

Step 4: Calculate the number of half-lives



$$\frac{2.9 \times 10^{-6}}{1.6 \times 10^{-6}} = 1.8 \text{ half-lives(same as(b))}$$

