Question 1

A plane Π contains the point A$(3, 9, -1)$ and has a normal vector $\begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$.

(a) Find the equation of the plane in its Cartesian form. [2]

A second point B has coordinates $(-4, 1, -3)$.

(b) Determine whether point B lies on the same plane. [2]

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

a) The components of the normal vector are the x-, y- and z-coefficients of the Cartesian form:

$$4x - 2y + 2z = d$$

And $(3, 9, -1)$ is on the plane, so:

$$4(3) - 2(9) + 2(-1) = d$$

$$12 - 18 - 2 = d \implies d = -8$$

Therefore

$$\boxed{4x - 2y + 2z = -8}$$

A plane Π contains the point A$(3, 9, -1)$ and has a normal vector $\begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$.

(a) Find the equation of the plane in its Cartesian form.

$\boxed{4x - 2y + 2z = -8}$

[2]

A second point B has coordinates $(-4, 1, -3)$.

(b) Determine whether point B lies on the same plane.

[2]

b) Test by putting coordinates into equation:

$4(-4) - 2(1) + 2(-3) = -24$

That is not equal to -8, so the point is not on the plane.

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

Question 2

A plane Π has equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$.

A line with equation $\boldsymbol{r} = \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ intersects Π at a point Q.

(a) Write down the equations of the line and the plane in their parametric forms.

[3]

(b) Given that the coordinates of Q are $(10, -2, 4)$, find the values for β, λ and μ at the point of intersection.

[5]

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

a)

plane: $\begin{cases} x = 3 - 2\lambda + 5\mu \\ y = 3 + 5\lambda + 2\mu \\ z = 2 + 3\lambda + 7\mu \end{cases}$

line: $\begin{cases} x = 6 + 4\beta \\ y = -2 \\ z = 1 + 3\beta \end{cases}$

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(a) Write down the equations of the line and the plane in their parametric forms. [3]

(b) Given that the coordinates of Q are $(10, -2, 4)$, find the values for β, λ and μ at the point of intersection. [5]

plane : $\begin{cases} x = 3 - 2\lambda + 5\mu \\ y = 3 + 5\lambda + 2\mu \\ z = 2 + 3\lambda + 7\mu \end{cases}$

line : $\begin{cases} x = 6 + 4\beta \\ y = -2 \\ z = 1 + 3\beta \end{cases}$

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

b) $6 + 4\beta = 10 \Rightarrow \beta = 1$ Use x-coordinate to find β

$1 + 3(1) = 4$ ✓ Check using z-coordinate

Now use x- and y-coordinates to find simultaneous equations for λ and μ :

$$\left.\begin{array}{l} 10 = 3 - 2\lambda + 5\mu \\ -2 = 3 + 5\lambda + 2\mu \end{array}\right\} \Rightarrow \begin{array}{l} 2\lambda - 5\mu = -7 \\ 5\lambda + 2\mu = -5 \end{array}$$

$\Rightarrow \lambda = -\frac{39}{29}, \ \mu = \frac{25}{29}$ Solve by hand or with GDC

$2 + 3\left(-\frac{39}{29}\right) + 7\left(\frac{25}{29}\right) = 4$ ✓ Check using z-coordinate

$$\boxed{\beta = 1 \quad \lambda = -\frac{39}{29} \quad \mu = \frac{25}{29}}$$

Question 3

Consider the two planes Π_1 and Π_2 which can be defined by the equations

$$\Pi_1: \ x + 2y - z = 5$$

$$\Pi_2: -3x - y + 8z = 1$$

(a) Write down expressions for the normal vectors of each of the two planes.

[2]

(b) Hence find the angle between the two planes. Give your answer in radians.

[3]

a) The components of the normal vector are the x-, y- and z-coefficients of the Cartesian form:

$$\boxed{\begin{array}{l} \Pi_1: \ \underline{n}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\ \\ \Pi_2: \ \underline{n}_2 = \begin{pmatrix} -3 \\ -1 \\ 8 \end{pmatrix} \end{array}}$$

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

Consider the two planes Π_1 and Π_2 which can be defined by the equations

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(a) Write down expressions for the normal vectors of each of the two planes. [2]

(b) Hence find the angle between the two planes. Give your answer in radians. [3]

$$\boxed{\Pi_1:\ \underline{n_1} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad \Pi_2:\ \underline{n_2} = \begin{pmatrix} -3 \\ -1 \\ 8 \end{pmatrix}}$$

Angle between two vectors	$\cos\theta = \dfrac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$

Scalar product	$\boldsymbol{v}\cdot\boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3$ $\quad \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},\ \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
	$\boldsymbol{v}\cdot\boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$

Magnitude of a vector	$\lvert \boldsymbol{v} \rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$ $\quad \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

b) The angle between the planes is equal to the angle between their normal vectors.

Find the angle between $\underline{n_1}$ and $\underline{n_2}$:

$$\lvert \underline{n_1} \rvert = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$$

$$\lvert \underline{n_2} \rvert = \sqrt{(-3)^2 + (-1)^2 + 8^2} = \sqrt{74}$$

We want the acute angle between the planes, so we have to use the absolute value here

$$\cos\theta = \frac{\lvert (1)(-3) + (2)(-1) + (-1)(8) \rvert}{\sqrt{6} \times \sqrt{74}} = \frac{\lvert -13 \rvert}{2\sqrt{111}} = \frac{13}{2\sqrt{111}}$$

$$\theta = \cos^{-1}\left(\frac{13}{2\sqrt{111}}\right) = 0.905931\ldots$$

$$\boxed{\theta = 0.906 \text{ radians } (3 \text{ s.f.})}$$

Question 4

The points A, B and C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, relative to the origin O.

The position vectors are given by

$$\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$$
$$\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$$
$$\boldsymbol{c} = \boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$$

(a) Find the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

[2]

Points A, B and C all lie on a single plane.

(b) Use the results from part (a) to write down the vector equation of the plane.

[2]

(c) Find the Cartesian equation of the plane.

[4]

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a}\cdot\boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underline{b} - \underline{a}$

$$\boxed{\overrightarrow{AB} = \begin{pmatrix}-1\\2\\2\end{pmatrix} - \begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}-3\\-1\\3\end{pmatrix}}$$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \underline{c} - \underline{a}$

$$\boxed{\overrightarrow{AC} = \begin{pmatrix}1\\-4\\3\end{pmatrix} - \begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}-1\\-7\\4\end{pmatrix}}$$

The points A, B and C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, relative to the origin O.

The position vectors are given by

$$\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$$
$$\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$$
$$\boldsymbol{c} = \boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$$

(a) Find the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\boxed{\overrightarrow{AB} = \begin{pmatrix}-3\\-1\\3\end{pmatrix}}$ $\boxed{\overrightarrow{AC} = \begin{pmatrix}-1\\-7\\4\end{pmatrix}}$

[2]

Points A, B and C all lie on a single plane.

(b) Use the results from part (a) to write down the vector equation of the plane.

[2]

(c) Find the Cartesian equation of the plane.

[4]

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a}\cdot\boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

b) A is a point on the plane, so use $\overrightarrow{OA}$ as '$\underline{a}$' in the vector equation of a plane formula.

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are non-parallel vectors in the plane, so use those as '$\underline{b}$' and '$\underline{c}$' in the formula.

$$\boxed{\underline{r} = \begin{pmatrix}2\\3\\-1\end{pmatrix} + \lambda\begin{pmatrix}-3\\-1\\3\end{pmatrix} + \mu\begin{pmatrix}-1\\-7\\4\end{pmatrix}}$$

Note: You could also use $\overrightarrow{OB}$ or $\overrightarrow{OC}$ as '$\underline{a}$' here.

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The points A, B and C have position vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ respectively, relative to the origin O.

The position vectors are given by

$$\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$$
$$\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$$
$$\boldsymbol{c} = \boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$$

(a) Find the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. $\boxed{\overrightarrow{AB} = \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix}}$ $\boxed{\overrightarrow{AC} = \begin{pmatrix} -1 \\ -7 \\ 4 \end{pmatrix}}$ [2]

Points A, B and C all lie on a single plane.

(b) Use the results from part (a) to write down the vector equation of the plane. [2]

(c) Find the Cartesian equation of the plane. [4]

Vector product	$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$,	$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
	$\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \sin\theta$	θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

c) Use vector product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to find a normal vector for the plane:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} -1 \\ -7 \\ 4 \end{pmatrix} = \begin{pmatrix} (-1)(4) - (3)(-7) \\ (3)(-1) - (-3)(4) \\ (-3)(-7) - (-1)(-1) \end{pmatrix} = \begin{pmatrix} 17 \\ 9 \\ 20 \end{pmatrix}$$

The components of the normal vector are the x-, y- and z-coefficients of the Cartesian form:

$$17x + 9y + 20z = d$$

And $A(2, 3, -1)$ is on the plane, so:

$$17(2) + 9(3) + 20(-1) = d \implies d = 41$$

$$\boxed{17x + 9y + 20z = 41}$$

Question 5

A plane lies parallel to the line with equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 9 \\ 1 \end{pmatrix}$ and contains the points P and X with coordinates $(5, 4, 5)$ and $(-2, 2, 0)$ respectively.

(a) Find the vector $\overrightarrow{PX}$. [2]

(b) By appropriate use of the vector product, find the normal to the plane. [2]

(c) Hence find the Cartesian equation of the plane. [2]

a) $\overrightarrow{PX} = \overrightarrow{OX} - \overrightarrow{OP}$

$$\boxed{\overrightarrow{PX} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix}}$$

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

A plane lies parallel to the line with equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 9 \\ 1 \end{pmatrix}$ and contains the points P and X with coordinates $(5, 4, 5)$ and $(-2, 2, 0)$ respectively.

(a) Find the vector $\overrightarrow{PX}$. [2]

$\boxed{\overrightarrow{PX} = \begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix}}$

(b) By appropriate use of the vector product, find the normal to the plane. [2]

(c) Hence find the Cartesian equation of the plane. [2]

Vector product	$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$,	$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
	$\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \sin\theta$	θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

b) Because the plane is parallel to the line, it is parallel to the line's direction vector $\begin{pmatrix} 3 \\ 9 \\ 1 \end{pmatrix}$.

Because P and X are in the plane, the plane is also parallel to $\overrightarrow{PX}$.

Therefore the vector product of $\overrightarrow{PX}$ and $\begin{pmatrix} 3 \\ 9 \\ 1 \end{pmatrix}$ will be a vector that is normal to the plane.

$$\begin{pmatrix} -7 \\ -2 \\ -5 \end{pmatrix} \times \begin{pmatrix} 3 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} (-2)(1) - (-5)(9) \\ (-5)(3) - (-7)(1) \\ (-7)(9) - (-2)(3) \end{pmatrix} = \boxed{\begin{pmatrix} 43 \\ -8 \\ -57 \end{pmatrix}}$$

<u>Note</u> : $\begin{pmatrix} -43 \\ 8 \\ 57 \end{pmatrix}$ is also fine here

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A plane lies parallel to the line with equation $\mathbf{r} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 9 \\ 1 \end{pmatrix}$ and contains the points P and X with coordinates $(5, 4, 5)$ and $(-2, 2, 0)$ respectively.

(a) Find the vector $\overrightarrow{PX}$.

[2]

(b) By appropriate use of the vector product, find the normal to the plane.

$\boxed{\begin{pmatrix} 43 \\ -8 \\ -57 \end{pmatrix}}$

[2]

(c) Hence find the Cartesian equation of the plane.

[2]

c) The components of the normal vector are the x-, y- and z-coefficients of the Cartesian form:

$43x - 8y - 57z = d$

And $(-2, 2, 0)$ is on the plane, so:

$43(-2) - 8(2) - 57(0) = d \implies d = -102$

$\boxed{43x - 8y - 57z = -102}$

Vector equation of a plane	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$
Equation of a plane (using the normal vector)	$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
Cartesian equation of a plane	$ax + by + cz = d$

Question 6

Consider the plane defined by the Cartesian equation $5x - 3y - z = 13$.

(a) Show that the line with equation $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}$ lies in the plane.

So plane's normal vector is $\begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$

[3]

(b) Show that the line with Cartesian equation $x - 2 = \frac{y-6}{2} = 2 - z$ is parallel to the plane but does not lie in the plane.

[3]

Remember, a scalar product of zero means that the vectors are perpendicular!

Scalar product	$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$	$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
	$\mathbf{v} \cdot \mathbf{w} = \lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert\cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$	

Vector equation of a plane	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$
Equation of a plane (using the normal vector)	$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
Cartesian equation of a plane	$ax + by + cz = d$

a) First show that the line's 'fixed point' $(3, 0, 2)$ is on the plane:

$5(3) - 3(0) - (2) = 15 - 2 = 13$, so the point $(3, 0, 2)$ is on the plane.

Then use the scalar product to show that the line's direction vector $\begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}$ is parallel to the plane:

$\begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix} = 5 - 12 + 7 = 0$

Therefore the line is perpendicular to the plane's normal vector, which means it is parallel to the plane.

Combining the above, we see that the line must lie in the plane.

Consider the plane defined by the Cartesian equation $5x - 3y - z = 13$.

(a) Show that the line with equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}$ lies in the plane.

So plane's normal vector is $\begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$

[3]

(b) Show that the line with Cartesian equation $x - 2 = \frac{y-6}{2} = 2 - z$ is parallel to the plane but does not lie in the plane.

[3]

Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
Parametric form of the equation of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$
Cartesian equations of a line	$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$

Remember, a scalar product of zero means that the vectors are perpendicular!

Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$	$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},\ \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
	$\boldsymbol{v} \cdot \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$	

b) Convert line equation to vector form:

$$x - 2 = \frac{y-6}{2} = 2 - z \implies \frac{x-2}{1} = \frac{y-6}{2} = \frac{z-2}{-1}$$

$$\implies x = 2 + \lambda,\ y = 6 + 2\lambda,\ z = 2 - \lambda \quad \text{parametric form}$$

$$\implies \underline{r} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \text{vector form}$$

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix} = 5 - 6 + 1 = 0 \quad \Big\} \text{ For the method here, see part (a)}$$

Therefore the line is perpendicular to the plane's normal vector, which means it is parallel to the plane.

$$5(2) - 3(6) - 2 = 10 - 18 - 2 = -10 \neq 13$$

Therefore the point $(2, 6, 2)$ is not on the plane, which means that the line does not lie in the plane.

Question 7

Consider the planes Π_1, Π_2 and Π_3, which are defined by the equations

$$\Pi_1:\ 3x - 5y + z = 27$$

$$\Pi_2:\ -4x + y + 2z = -10$$

$$\Pi_3:\ -2x - y - z = -1$$

(a) By solving the system of equations represented by the three planes show that the system of equations has a unique solution.

[3]

(b) Hence write down the coordinates of any point(s) where all three planes intersect.

[1]

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

a) Solve using GDC, or by row reduction:

$$\left\{\begin{array}{cccc} 3 & -5 & 1 & 27 \\ -4 & 1 & 2 & -10 \\ -2 & -1 & -1 & -1 \end{array}\right. \implies \left\{\begin{array}{cccc} 1 & -6 & 0 & 26 \\ -8 & -1 & 0 & -12 \\ -2 & -1 & -1 & -1 \end{array}\right. \begin{array}{l} ①+③ \\ ②+2\times③ \\ \end{array}$$

$$\implies \left\{\begin{array}{cccc} 49 & 0 & 0 & 98 \\ -8 & -1 & 0 & -12 \\ -2 & -1 & -1 & -1 \end{array}\right. \begin{array}{l} ①-6\times② \\ \\ \end{array} \implies \left\{\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 8 & 1 & 0 & 12 \\ 2 & 1 & 1 & 1 \end{array}\right. \begin{array}{l} ①\div 49 \\ ②\times(-1) \\ ③\times(-1) \end{array}$$

$$\implies \left\{\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 2 & 1 & 1 & 1 \end{array}\right. \begin{array}{l} \\ ②-8\times① \\ \end{array} \implies \left\{\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 1 & 1 & -3 \end{array}\right. \begin{array}{l} \\ \\ ③-2\times① \end{array}$$

$$\implies \left\{\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right. \begin{array}{l} \\ \\ ③-② \end{array} \implies \left\{\begin{array}{l} x = 2 \\ y = -4 \\ z = 1 \end{array}\right.$$

The unique solution is

$x = 2 \quad y = -4 \quad z = 1$

Consider the planes Π_1, Π_2 and Π_3, which are defined by the equations

$$\Pi_1:\ 3x - 5y + z = 27$$
$$\Pi_2:\ -4x + y + 2z = -10$$
$$\Pi_3:\ -2x - y - z = -1$$

(a) By solving the system of equations represented by the three planes show that the system of equations has a unique solution. [3]

The unique solution is $x = 2 \quad y = -4 \quad z = 1$

(b) Hence write down the coordinates of any point(s) where all three planes intersect. [1]

b) Each solution to the system of equations gives the coordinates of a point where all three planes intersect.

There is only one solution, therefore there is only one point of intersection.

The three planes intersect at the point $(2, -4, 1)$.

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a}\cdot\boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

Question 8

Consider the line L with vector equation $\boldsymbol{r} = (1-\lambda)\boldsymbol{i} + (\lambda - 2)\boldsymbol{j} + (3 + 2\lambda)\boldsymbol{k}$ and the plane Π with Cartesian equation $3x - 2y + z = 11$.

(a) Find the angle in radians between the line L and the normal to the plane Π. [4]

So plane's normal vector is $\begin{pmatrix}3\\-2\\1\end{pmatrix}$

(b) Hence find the angle in radians between the line L and the plane Π. [2]

a) Rewrite line equation in standard vector form:

$$\underline{r} = \begin{pmatrix}1-\lambda\\ \lambda-2\\ 3+2\lambda\end{pmatrix} = \begin{pmatrix}1-\lambda\\ -2+\lambda\\ 3+2\lambda\end{pmatrix} = \begin{pmatrix}1\\-2\\3\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\2\end{pmatrix}$$

The angle between the line and the normal vector is the same as the angle between the line's direction vector $\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and the normal vector.

$$\left|\begin{pmatrix}-1\\1\\2\end{pmatrix}\right| = \sqrt{(-1)^2 + 1^2 + 2^2} = \sqrt{6} \qquad \left|\begin{pmatrix}3\\-2\\1\end{pmatrix}\right| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$$

$$\begin{pmatrix}-1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\1\end{pmatrix} = (-1)(3) + (1)(-2) + (2)(1) = -3$$

We want the acute angle between the vectors, so we have to use the absolute value here

$$\cos\theta = \frac{|-3|}{\sqrt{6}\times\sqrt{14}} = \frac{3}{2\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right) = 1.237323\ldots$$

$$\theta = 1.24 \text{ radians (3 s.f.)}$$

Angle between two vectors	$\cos\theta = \dfrac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert}$	
Magnitude of a vector	$\lvert\boldsymbol{v}\rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$	$\boldsymbol{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$
Scalar product	$\boldsymbol{v}\cdot\boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3$	$\boldsymbol{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}$

$\boldsymbol{v}\cdot\boldsymbol{w} = \lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$

Consider the line L with vector equation $\mathbf{r} = (1-\lambda)\mathbf{i} + (\lambda - 2)\mathbf{j} + (3 + 2\lambda)\mathbf{k}$ and the plane Π with Cartesian equation $3x - 2y + z = 11$.

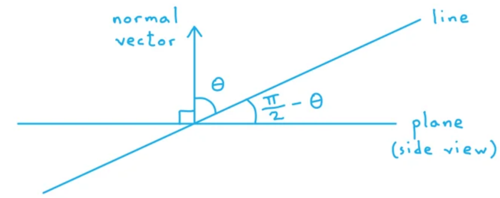
(a) Find the angle in radians between the line L and the normal to the plane Π.

$\boxed{\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)}$ [4]

(b) Hence find the angle in radians between the line L and the plane Π.

[2]

b)



Use exact value

$$\frac{\pi}{2} - \theta = \frac{\pi}{2} - \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right) = 0.333473\ldots$$

$\boxed{0.333 \text{ radians } (3 \text{ s.f.})}$

Vector equation of a plane	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$
Equation of a plane (using the normal vector)	$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
Cartesian equation of a plane	$ax + by + cz = d$

Question 9

Two planes Π_1 and Π_2 are defined by the equations

$$\Pi_1: 3x - 2y + 4z = 18$$
$$\Pi_2: -2x + y + 2z = 7$$

(a) Write down expressions for the normal vectors of each of the two planes.

[2]

(b) Find the cross product of the two normal vectors.

[2]

(c) Find the coordinates of a point that lies on both planes.

[3]

(d) Hence find a vector equation of the line of intersection of the two planes.

[2]

a) The components of the normal vector are the x-, y- and z-coefficients of the Cartesian form:

For Π_1 : $\underline{n}_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$

For Π_2 : $\underline{n}_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

Vector equation of a plane	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$
Equation of a plane (using the normal vector)	$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
Cartesian equation of a plane	$ax + by + cz = d$

Two planes Π_1 and Π_2 are defined by the equations

$$\Pi_1: 3x - 2y + 4z = 18$$

$$\Pi_2: -2x + y + 2z = 7$$

(a) Write down expressions for the normal vectors of each of the two planes.

$\boxed{\underline{n_1} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \quad \underline{n_2} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}}$ [2]

(b) Find the cross product of the two normal vectors.

[2]

(c) Find the coordinates of a point that lies on both planes.

[3]

(d) Hence find a vector equation of the line of intersection of the two planes.

[2]

Vector product	$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$,	$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
	$\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \sin\theta$	θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$

b) $\underline{n_1} \times \underline{n_2} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} (-2)(2) - (4)(1) \\ (4)(-2) - (3)(2) \\ (3)(1) - (-2)(-2) \end{pmatrix} = \boxed{\begin{pmatrix} -8 \\ -14 \\ -1 \end{pmatrix}}$

<u>Note</u> : $\underline{n_2} \times \underline{n_1} = \begin{pmatrix} 8 \\ 14 \\ 1 \end{pmatrix}$ is also okay here

Two planes Π_1 and Π_2 are defined by the equations

$$\Pi_1: 3x - 2y + 4z = 18$$

$$\Pi_2: -2x + y + 2z = 7$$

(a) Write down expressions for the normal vectors of each of the two planes.

[2]

(b) Find the cross product of the two normal vectors.

$\boxed{\begin{pmatrix} -8 \\ -14 \\ -1 \end{pmatrix}}$ [2]

(c) Find the coordinates of a point that lies on both planes.

[3]

(d) Hence find a vector equation of the line of intersection of the two planes.

[2]

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

c) Let $z = 0$, then solve simultaneous equations for x and y :

When $z = 0$,

$3x - 2y = 18$ from Π_1

and $-2x + y = 7$ from Π_2

This can be solved using row reduction :

$\begin{cases} 3 & -2 & 18 \\ -2 & 1 & 7 \end{cases} \Rightarrow \begin{cases} -1 & 0 & 32 \\ -2 & 1 & 7 \end{cases}$ ① + 2×②

$\Rightarrow \begin{cases} 1 & 0 & -32 \\ -2 & 1 & 7 \end{cases}$ (-1)×① $\Rightarrow \begin{cases} 1 & 0 & -32 \\ 0 & 1 & -57 \end{cases}$ ② + 2×①

$\Rightarrow \begin{cases} x = -32 \\ y = -57 \end{cases}$

$\boxed{(-32, -57, 0) \text{ is a point on both planes}}$

$(0, -1, 4)$ and $\left(\frac{4}{7}, 0, \frac{57}{14}\right)$ are also valid answers, found by setting $x = 0$ or $y = 0$

Two planes Π_1 and Π_2 are defined by the equations

$$\Pi_1 : 3x - 2y + 4z = 18$$

$$\Pi_2 : -2x + y + 2z = 7$$

(a) Write down expressions for the normal vectors of each of the two planes. [2]

(b) Find the cross product of the two normal vectors. $\boxed{\begin{pmatrix}-8\\-14\\-1\end{pmatrix}}$ [2]

(c) Find the coordinates of a point that lies on both planes. [3]

$\boxed{(-32, -57, 0)}$

(d) Hence find a vector equation of the line of intersection of the two planes. [2]

Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

d) The cross product of the two normals is perpendicular to both normals, therefore it is parallel to the line of intersection of the two planes. This provides the direction vector for the line equation.

$$\boxed{\begin{pmatrix}-32\\-57\\0\end{pmatrix} + \lambda\begin{pmatrix}-8\\-14\\-1\end{pmatrix}}$$

$\begin{pmatrix}8\\14\\1\end{pmatrix}$ is also fine here

$\begin{pmatrix}0\\-1\\4\end{pmatrix}$ or $\begin{pmatrix}4/7\\0\\57/14\end{pmatrix}$ could also be used here (see part (c))

Question 10

A line L_1 is defined by the Cartesian equation $\frac{x}{3d+1} = \frac{y-3}{4} = 5 - z$ and a plane Π is defined by the Cartesian equation $-x + dy - 4z = -29$, where d is a real constant.

So plane's normal vector is $\begin{pmatrix}-1\\d\\-4\end{pmatrix}$

The line L_1 lies in the plane Π.

(a) Use the fact that the line L_1 lies in the plane Π to find the value of the constant d.

[4]

Another line, L_2, passes through the origin and is perpendicular to the plane Π.

(b) Write down the equation of line L_2 in vector form.

[2]

(c) By considering the parametric form of the equation for L_2, or otherwise, determine the point of intersection between line L_2 and the plane Π.

[3]

(d) Hence determine the minimum distance between the plane Π and the origin.

[2]

Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$
Parametric form of the equation of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$
Cartesian equations of a line	$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$

Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
Equation of a plane (using the normal vector)	$\boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a}\cdot\boldsymbol{n}$
Cartesian equation of a plane	$ax + by + cz = d$

Remember, a scalar product of zero means that the vectors are perpendicular!

Scalar product	$\boldsymbol{v}\cdot\boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3$	$\boldsymbol{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix},\ \boldsymbol{w} = \begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}$
	$\boldsymbol{v}\cdot\boldsymbol{w} = \lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$	

a) Find vector form of line equation:

$$\frac{x}{3d+1} = \frac{y-3}{4} = 5-z \implies \frac{x}{3d+1} = \frac{y-3}{4} = \frac{z-5}{-1}$$

$$\implies x = (3d+1)\lambda,\quad y = 3+4\lambda,\quad z = 5-\lambda$$

$$\implies \underline{r} = \begin{pmatrix}0\\3\\5\end{pmatrix} + \lambda\begin{pmatrix}3d+1\\4\\-1\end{pmatrix}$$

For the line to lie in the plane, two things must be true:

① The line's 'fixed point' (0, 3, 5) must lie in the plane.

② The line must be parallel to the plane, which is equivalent to being perpendicular to the plane's normal vector.

We can use either one of those conditions to find the value of d.

Method 1: (0, 3, 5) must lie in plane, so

$$-(0) + d(3) - 4(5) = -29$$

$$\implies 3d - 20 = -29 \implies \boxed{d = -3}$$

Method 2: $\begin{pmatrix}3d+1\\4\\-1\end{pmatrix}$ must be perpendicular to $\begin{pmatrix}-1\\d\\-4\end{pmatrix}$

$$\begin{pmatrix}3d+1\\4\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\d\\-4\end{pmatrix} = 0$$

$$\implies (3d+1)(-1) + (4)(d) + (-1)(-4) = 0$$

$$\implies -3d - 1 + 4d + 4 = 0$$

$$\implies d + 3 = 0 \implies \boxed{d = -3}$$

A line L_1 is defined by the Cartesian equation $\frac{x}{3d+1} = \frac{y-3}{4} = 5 - z$ and a plane Π is defined by the Cartesian equation $-x + dy - 4z = -29$, where d is a real constant.

So plane's normal vector is $\begin{pmatrix} -1 \\ d \\ -4 \end{pmatrix}$

The line L_1 lies in the plane Π.

(a) Use the fact that the line L_1 lies in the plane Π to find the value of the constant d.

$\boxed{d = -3}$

[4]

Another line, L_2, passes through the origin and is perpendicular to the plane Π.

(b) Write down the equation of line L_2 in vector form.

[2]

(c) By considering the parametric form of the equation for L_2, or otherwise, determine the point of intersection between line L_2 and the plane Π.

[3]

(d) Hence determine the minimum distance between the plane Π and the origin.

[2]

Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$
Parametric form of the equation of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$
Cartesian equations of a line	$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$

b) 'Perpendicular to the plane' means parallel to the plane's normal vector.

passes through origin

$$\underline{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -3 \\ -4 \end{pmatrix}$$

perpendicular to the plane

$$\Rightarrow \boxed{\underline{r} = \lambda \begin{pmatrix} -1 \\ -3 \\ -4 \end{pmatrix}}$$

$\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ is also fine here

A line L_1 is defined by the Cartesian equation $\frac{x}{3d+1} = \frac{y-3}{4} = 5 - z$ and a plane Π is defined by the Cartesian equation $-x + dy - 4z = -29$, where d is a real constant.

The line L_1 lies in the plane Π.

(a) Use the fact that the line L_1 lies in the plane Π to find the value of the constant d.

$\boxed{d = -3}$

[4]

Another line, L_2, passes through the origin and is perpendicular to the plane Π.

(b) Write down the equation of line L_2 in vector form.

$\boxed{\underline{r} = \lambda \begin{pmatrix} -1 \\ -3 \\ -4 \end{pmatrix}}$

[2]

(c) By considering the parametric form of the equation for L_2, or otherwise, determine the point of intersection between line L_2 and the plane Π.

[3]

(d) Hence determine the minimum distance between the plane Π and the origin.

[2]

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Cartesian equations of a line	$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$

c) The parametric form is

$$x = -\lambda, \quad y = -3\lambda, \quad z = -4\lambda$$

So $(-\lambda, -3\lambda, -4\lambda)$ must lie on the plane for some value of λ:

$$-(-\lambda) - 3(-3\lambda) - 4(-4\lambda) = -29$$

$$\Rightarrow \lambda + 9\lambda + 16\lambda = -29 \Rightarrow \lambda = -\frac{29}{26}$$

The point of intersection is

$$\boxed{\left(\underset{-\lambda}{\frac{29}{26}}, \underset{-3\lambda}{\frac{87}{26}}, \underset{-4\lambda}{\frac{116}{26}}\right)}$$

$\frac{116}{26} = \frac{58}{13}$

A line L_1 is defined by the Cartesian equation $\frac{x}{3d+1} = \frac{y-3}{4} = 5 - z$ and a plane Π is defined by the Cartesian equation $-x + dy - 4z = -29$, where d is a real constant.

The line L_1 lies in the plane Π.

(a) Use the fact that the line L_1 lies in the plane Π to find the value of the constant d.

[4]

Another line, L_2, passes through the origin and is perpendicular to the plane Π.

(b) Write down the equation of line L_2 in vector form.

[2]

(c) By considering the parametric form of the equation for L_2, or otherwise, determine the point of intersection between line L_2 and the plane Π.

$\boxed{\left(\frac{29}{26}, \frac{87}{26}, \frac{116}{26}\right)}$

[3]

(d) Hence determine the minimum distance between the plane Π and the origin.

[2]

Distance between two points (x_1, y_1, z_1) & (x_2, y_2, z_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

d) The minimum distance is the perpendicular distance – i.e., the distance between the origin and the point of intersection from part (c).

$$d = \sqrt{\left(\frac{29}{26} - 0\right)^2 + \left(\frac{87}{26} - 0\right)^2 + \left(\frac{116}{26} - 0\right)^2}$$

$$\boxed{d = \frac{29\sqrt{26}}{26} \text{ units}} = 5.69 \text{ units (3 s.f.)}$$