Question 1

The points A and B are given by A(4, 2, −3) and B(0, 5, 1).

(a) Find a vector equation of the line L that passes through points A and B. [3]

(b) Determine if the point C(−1, 3, 2) lies on the line L. [3]

a) Find $\overrightarrow{AB}$

$$\overrightarrow{AB} = \begin{pmatrix} 0-4 \\ 5-2 \\ 1-(-3) \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 4 \end{pmatrix}$$

$\therefore$ Equation of L is

$$r = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 3 \\ 4 \end{pmatrix} \text{ or } r = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 3 \\ 4 \end{pmatrix}$$

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b) $\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

λ must be the same for each line

$4 - 4\lambda = -1 \quad \therefore \lambda = \frac{5}{4}$

$2 + 3\lambda = 3 \quad \therefore \lambda = \frac{1}{3}$

$-3 + 4\lambda = 2 \quad \therefore \lambda = \frac{5}{4}$

$\therefore$ C does not lie on L.

Question 2

Find the vector equation of a line that is parallel to the vector $\boldsymbol{a} = 3\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ and passes through the point $X(3, -2, 0)$.

[3]

Direction vector $= \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$ Point $X = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$

$$\boxed{\text{Vector equation : } r = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}}$$

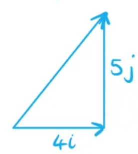
Question 3

Find the equation of the line that is perpendicular to the vector $4\boldsymbol{i} + 5\boldsymbol{j}$ and passes through the point $P(7, -1)$, leaving your answer in the form $ax + by + c = 0$, where a, b and $c \in \mathbb{Z}$.

[5]

Equations of a straight line	$y = mx + c$; $ax + by + d = 0$; $y - y_1 = m(x - x_1)$

Find gradient of the given vector



$m_1 = \frac{5}{4}$

Find gradient of perpendicular

$m_2 = -\frac{4}{5}$

Use point-gradient form of equation of a line

$$y - (-1) = -\frac{4}{5}(x - 7)$$

$$5y + 5 = -4x + 28$$

$$\boxed{5y + 4x - 23 = 0}$$

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Question 4

Consider the two lines l_1 and l_2 defined by the equations:

$$l_1: \boldsymbol{a} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$$

$$l_2: \boldsymbol{b} = \begin{pmatrix} 5 \\ -11 \\ 10 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 6 \\ 2 \end{pmatrix}.$$

(a) Find the scalar product of the direction vectors.

[2]

(b) Hence, find the angle, in radians, between the l_1 and l_2.

[4]

a) scalar product

$v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$ (in formula booklet)

scalar product $= (1)(-1) + (-3)(6) + (-5)(2)$

$\boxed{\text{scalar product} = -29}$

Consider the two lines l_1 and l_2 defined by the equations:

$$l_1: \boldsymbol{a} = \begin{pmatrix} 4 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$$

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(a) Find the scalar product of the direction vectors.

$\boxed{\text{scalar product} = -29}$

[2]

(b) Hence, find the angle, in radians, between the l_1 and l_2.

[4]

b) $v \cdot w = |v||w| \cos\theta$ (in formula booklet)

$|-29| = \sqrt{(1)^2 + (-3)^2 + (-5)^2}\sqrt{(-1)^2 + (6)^2 + (2)^2} \cos\theta$

$\therefore \theta = \cos^{-1}\left(\dfrac{29}{\sqrt{35}\sqrt{41}}\right) = 0.6989\ldots$

$\boxed{\theta = 0.699 \text{ radians}}$

Note: There is always an acute and an obtuse angle between two lines. In this question, we want the acute angle, that is why we use the absolute value of the scalar product.

Question 5

Consider the lines l_1 and l_2 defined by:

$$l_1: \begin{cases} x = 3 - \mu \\ y = -2 + 5\mu \\ z = 4 + 2\mu \end{cases}$$

$$l_2: \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}.$$

(a) Show that the lines are not parallel.

[2]

(b) Hence, show that the lines l_1 and l_2 do not intersect.

[5]

a) Use the direction vectors to show they are not parallel.

$$l_1: r = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$$

Direction vectors are not the same or colinear.

$\therefore$ they are not parallel.

Consider the lines l_1 and l_2 defined by:

$$l_1: \begin{cases} x = 3 - \mu \\ y = -2 + 5\mu \\ z = 4 + 2\mu \end{cases}$$

$$l_2: \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}.$$

(a) Show that the lines are not parallel.

[2]

(b) Hence, show that the lines l_1 and l_2 do not intersect.

[5]

b) To show the lines don't intersect, we must show inconsistent values for μ or λ.

$$l_1: \begin{cases} x = 3 - \mu \\ y = -2 + 5\mu \\ z = 4 + 2\mu \end{cases} \qquad l_2: \begin{cases} x = 3 + 4\lambda \\ y = -1 + 2\lambda \\ z = 2\lambda \end{cases}$$

$\therefore$ $x = 3 - \mu = 3 + 4\lambda$

$y = -2 + 5\mu = -1 + 2\lambda$

$z = 4 + 2\mu = 2\lambda \longrightarrow \lambda = 2 + \mu$

sub $\lambda = 2 + \mu$ into x and y lines.

$3 - \mu = 3 + 4(2 + \mu) = 11 + 4\mu \qquad \therefore \mu = -\frac{8}{5}$

$-2 + 5\mu = -1 + 2(2 + \mu) = 3 + 2\mu \qquad \therefore \mu = \frac{5}{3}$

Inconsistent values for μ, so the lines do not intersect.

$\therefore$ the lines are skew.

Question 6

Consider the line l which can be defined by both $\boldsymbol{r_1} = \begin{pmatrix} t \\ -2 \\ 5 \end{pmatrix} + \alpha \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{r_2} = \begin{pmatrix} -3 \\ 6 \\ 9 \end{pmatrix} + \beta \begin{pmatrix} 15 \\ 3k \\ -3 \end{pmatrix}$.

(a) Find the value of k. [2]

(b) Find the value of t. [4]

a) The direction vectors of r_1 and r_2 must be equal or have a constant scale factor, q.

$$q \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 3k \\ -3 \end{pmatrix} \longrightarrow \begin{matrix} -5q = 15 \\ 2q = 3k \\ q = -3 \end{matrix}$$

$\boxed{\therefore k = -2}$

Consider the line l which can be defined by both $\boldsymbol{r_1} = \begin{pmatrix} t \\ -2 \\ 5 \end{pmatrix} + \alpha \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{r_2} = \begin{pmatrix} -3 \\ 6 \\ 9 \end{pmatrix} + \beta \begin{pmatrix} 15 \\ 3k \\ -3 \end{pmatrix}$.

(a) Find the value of k. [2]

$\boxed{\therefore k = -2}$

(b) Find the value of t. [4]

b) $x = t - 5\alpha = -3 + 15\beta$

$y = -2 + 2\alpha = 6 - 6\beta \longrightarrow \alpha = 4 - 3\beta$

$z = 5 + \alpha = 9 - 3\beta \longrightarrow \alpha = 4 - 3\beta$

All variables should generate the same equation.

$$\alpha = \frac{-3 + 15\beta - t}{-5} = 4 - 3\beta$$

$$\therefore \frac{-3 - t}{-5} = 4 \longrightarrow -3 - t = -20$$

$\boxed{t = 17}$

Question 7

Consider the line l_1, which can be represented by the equation $\boldsymbol{r} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ and l_2, which can be represented by the equation $\boldsymbol{s} = (3 - \mu)i + (1 - \mu)j + (5 + 7\mu)k$.

(a) Write down the equation for l_2 in its vector form.

[2]

(b) Find vector product of the direction vectors of l_1 and l_2.

[2]

(c) Hence find the angle between l_1 and l_2.

[3]

a) $\boxed{l_2 : s = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix}}$

Consider the line l_1, which can be represented by the equation $\boldsymbol{r} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ and l_2, which can be represented by the equation $\boldsymbol{s} = (3 - \mu)i + (1 - \mu)j + (5 + 7\mu)k$.

(a) Write down the equation for l_2 in its vector form.

$\boxed{l_2 : s = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix}}$

[2]

(b) Find vector product of the direction vectors of l_1 and l_2.

[2]

(c) Hence find the angle between l_1 and l_2.

[3]

b) Vector product

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}||\boldsymbol{w}|\sin\theta \qquad \theta \text{ is the angle between } \boldsymbol{v} \text{ and } \boldsymbol{w}$$

$$r \times s = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} (4 \times 7) - (3 \times -1) \\ (3 \times -1) - (-1 \times 7) \\ (-1 \times -1) - (4 \times -1) \end{pmatrix}$$

$$\boxed{r \times s = \begin{pmatrix} 31 \\ 4 \\ 5 \end{pmatrix}}$$

Consider the line l_1, which can be represented by the equation $\boldsymbol{r} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ and l_2, which can be represented by the equation $\boldsymbol{s} = (3 - \mu)i + (1 - \mu)j + (5 + 7\mu)k$.

(a) Write down the equation for l_2 in its vector form.

[2]

(b) Find vector product of the direction vectors of l_1 and l_2.

$\boxed{r \times s = \begin{pmatrix} 31 \\ 4 \\ 5 \end{pmatrix}}$

[2]

(c) Hence find the angle between l_1 and l_2.

[3]

c) Vector product

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}||\boldsymbol{w}|\sin\theta \qquad \theta \text{ is the angle between } \boldsymbol{v} \text{ and } \boldsymbol{w}$$

$$|r \times s| = |r||s|\sin\theta \qquad \therefore \theta = \sin^{-1}\left(\frac{|r \times s|}{|r||s|}\right)$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{1002}}{\sqrt{51}\sqrt{26}}\right) = 60.3756\ldots$$

$$\boxed{\theta = 60.4^\circ}$$

Question 8

The lines l_1 and l_2 can be defined by:

$$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 2 \\ k \end{pmatrix}$$

$$l_2 : \mathbf{s} = \begin{pmatrix} -3 \\ -4 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -11 \\ -3 \\ 5 \end{pmatrix}$$

(a) Write down the parametric equations for l_1.

[2]

(b) Given that l_1 and l_2 intersect at point T,

(i) find the value of k.

(ii) determine the coordinates of the point of intersection, T.

[7]

a) Parametric form of a line (in formula booklet)

$$\begin{cases} x = x_0 + \lambda l \\ y = y_0 + \lambda m \\ z = z_0 + \lambda n \end{cases}$$

$$\boxed{l_1 : \begin{cases} x = 2 + 3\alpha \\ y = -5 + 2\alpha \\ z = 1 + k\alpha \end{cases}}$$

The lines l_1 and l_2 can be defined by:

$$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 2 \\ k \end{pmatrix}$$

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(a) Write down the parametric equations for l_1.

$$\boxed{l_1 : \begin{cases} x = 2 + 3\alpha \\ y = -5 + 2\alpha \\ z = 1 + k\alpha \end{cases}}$$

[2]

(b) Given that l_1 and l_2 intersect at point T,

(i) find the value of k.

(ii) determine the coordinates of the point of intersection, T.

[7]

b) $l_1 : \begin{cases} x = 2 + 3\alpha \\ y = -5 + 2\alpha \\ z = 1 + k\alpha \end{cases}$ $\quad l_2 : \begin{cases} x = -3 - 11\beta \\ y = -4 - 3\beta \\ z = 2 + 5\beta \end{cases}$

Equate the x, y and z lines

$x = 2 + 3\alpha = -3 - 11\beta$

$y = -5 + 2\alpha = -4 - 3\beta$

$z = 1 + k\alpha = 2 + 5\beta$

$2 + 3\alpha = -3 - 11\beta$ $\qquad$ $-5 + 2\alpha = -4 - 3\beta$

$\alpha = \dfrac{-5 - 11\beta}{3}$ $\qquad$ $\alpha = \dfrac{1 - 3\beta}{2}$

$\dfrac{-5 - 11\beta}{3} = \dfrac{1 - 3\beta}{2}$

cross multiply

$2(-5 - 11\beta) = 3(1 - 3\beta)$

$-10 - 22\beta = 3 - 9\beta \longrightarrow -13\beta = 13 \quad \therefore \beta = -1$

$\alpha = \dfrac{1 - 3(-1)}{2} = 2$

Question 9

Consider the triangle ABC. The points A, B and C have coordinates $(4, 0, -3)$, $(2, -2, -1)$ and $(8, 1, 5)$ respectively.

M is the midpoint of [AB].

(a) Find the coordinates of the midpoint M. [2]

(b) Hence, find a vector equation of the line, l, that passes through points C and M. [2]

(c) Show that the line l is perpendicular to [AB]. [3]

(d) Hence calculate the area of the triangle ABC. [3]

a) Midpoint, M (in formula booklet)

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

$$M = \left(\frac{4+2}{2}, \frac{0+(-2)}{2}, \frac{(-3)+(-1)}{2}\right)$$

$$\boxed{M = (3, -1, -2)}$$

Consider the triangle ABC. The points A, B and C have coordinates $(4, 0, -3)$, $(2, -2, -1)$ and $(8, 1, 5)$ respectively.

M is the midpoint of [AB].

(a) Find the coordinates of the midpoint M. $\boxed{M = (3, -1, -2)}$ [2]

(b) Hence, find a vector equation of the line, l, that passes through points C and M. [2]

(c) Show that the line l is perpendicular to [AB]. [3]

(d) Hence calculate the area of the triangle ABC. [3]

b) Find $\overrightarrow{CM}$ or $\overrightarrow{MC}$

$$\overrightarrow{CM} = \begin{pmatrix} 3-8 \\ -1-1 \\ -2-5 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ -7 \end{pmatrix} \quad \text{or} \quad \overrightarrow{MC} = \begin{pmatrix} 8-3 \\ 1-(-1) \\ 5-(-2) \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$$

There are 4 possible solutions.

$$\boxed{\begin{aligned} r &= \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -2 \\ -7 \end{pmatrix} \text{ or } r = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix} \text{ or} \\ r &= \begin{pmatrix} 8 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -2 \\ -7 \end{pmatrix} \text{ or } r = \begin{pmatrix} 8 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix} \end{aligned}}$$

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(a) Find the coordinates of the midpoint M.

[2]

(b) Hence, find a vector equation of the line, l, that passes through points C and M.

[2]

(c) Show that the line l is perpendicular to [AB].

[3]

(d) Hence calculate the area of the triangle ABC.

[3]

c) Scalar product for perpendicular lines equals zero

Direction vector for AB $= \begin{pmatrix} 2-4 \\ -2-0 \\ -1+3 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$

$\boxed{\text{Direction vector for AB} \cdot \text{Direction vector for CM} = (-2)(5) + (-2)(2) + (2)(7) = 0}$

Consider the triangle ABC. The points A, B and C have coordinates $(4, 0, -3)$, $(2, -2, -1)$ and $(8, 1, 5)$ respectively.

M is the midpoint of [AB].

(a) Find the coordinates of the midpoint M.

[2]

(b) Hence, find a vector equation of the line, l, that passes through points C and M.

[2]

(c) Show that the line l is perpendicular to [AB].

[3]

(d) Hence calculate the area of the triangle ABC.

[3]

d) Area of triangle $= \frac{1}{2}ab\sin C$ (in formula booklet)

Length AB $= \sqrt{(-2)^2 + (-2)^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$

Length CM $= \sqrt{5^2 + 2^2 + 7^2} = \sqrt{78}$

Area of triangle $= \frac{1}{2} \times 2\sqrt{3} \times \sqrt{78} = 3\sqrt{26}$

$\boxed{\text{Area of triangle} = 15.3 \text{ units}^2 \text{ (3 s.f.)}}$