Question 1

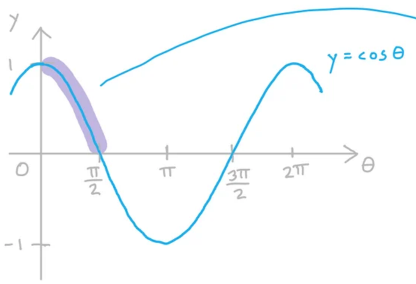
The value of $\sin\alpha = \frac{3}{7}$ for $0 \le \alpha \le \frac{\pi}{2}$. Find:

(i) $\cos\alpha$

(ii) $\sin 2\alpha$

(iii) $\cos 2\alpha$

(iv) $\tan 2\alpha$.



[6]

(i) $\cos^2\alpha + \left(\frac{3}{7}\right)^2 = 1$ Use Pythagorean identity

$\cos^2\alpha = 1 - \left(\frac{3}{7}\right)^2 = \frac{40}{49} \Rightarrow \cos\alpha = \pm\sqrt{\frac{40}{49}} = \pm\frac{2\sqrt{10}}{7}$

$\boxed{\cos\alpha = \frac{2\sqrt{10}}{7}}$ } $\cos\alpha \ge 0$ for $0 \le \alpha \le \pi/2$

(ii) $\sin 2\alpha = 2\left(\frac{3}{7}\right)\left(\frac{2\sqrt{10}}{7}\right)$ Use double angle identity

$\boxed{\sin 2\alpha = \frac{12\sqrt{10}}{49}}$

(iii) $\cos 2\alpha = 1 - 2\left(\frac{3}{7}\right)^2$ Use double angle identity

$\boxed{\cos 2\alpha = \frac{31}{49}}$ $1 - 2\sin^2\alpha$ is easiest here!

(iv) $\tan 2\alpha = \frac{\left(\frac{12\sqrt{10}}{49}\right)}{\left(\frac{31}{49}\right)}$ Use identity for tangent

$\boxed{\tan 2\alpha = \frac{12\sqrt{10}}{31}}$

$\cos^2\theta + \sin^2\theta = 1$ } Pythagorean identity

$\sin 2\theta = 2\sin\theta\cos\theta$ } Double angle identity

$$\left.\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}\right\}$$ Double angle identity

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ } Identity for $\tan\theta$

Question 2

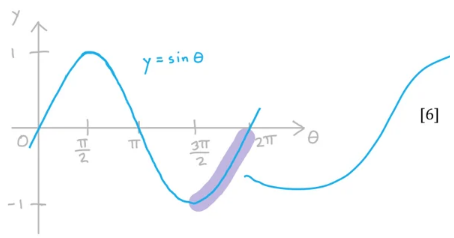
The value of $\cos B = \frac{1}{5}$, for $\frac{3\pi}{2} \le B \le 2\pi$. Find:

(i) $\cos 2B$

(ii) $\sin 2B$

(iii) $\tan 2B$.

[6]



$\cos^2\theta + \sin^2\theta = 1$ } Pythagorean identity

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$ } Double angle identity

$\sin 2\theta = 2\sin\theta\cos\theta$ } Double angle identity

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ } Identity for $\tan\theta$

$\left(\frac{1}{5}\right)^2 + \sin^2 B = 1$ Use Pythagorean identity to find $\sin B$

$\sin^2 B = 1 - \left(\frac{1}{5}\right)^2 = \frac{24}{25} \Longrightarrow \sin B = \pm\sqrt{\frac{24}{25}} = \pm\frac{2\sqrt{6}}{5}$

$\Longrightarrow \sin B = -\frac{2\sqrt{6}}{5}$ } $\sin B \le 0$ for $\frac{3\pi}{2} \le B \le 2\pi$

(i) $\cos 2B = 2\left(\frac{1}{5}\right)^2 - 1$ Use double angle identity

$2\cos^2 B - 1$ is easiest here!

$\boxed{\cos 2B = -\frac{23}{25}}$

(ii) $\sin 2B = 2\left(-\frac{2\sqrt{6}}{5}\right)\left(\frac{1}{5}\right)$ Use double angle identity

$\boxed{\sin 2B = -\frac{4\sqrt{6}}{25}}$

(iii) $\tan 2B = \frac{\left(-\frac{4\sqrt{6}}{25}\right)}{\left(-\frac{23}{25}\right)}$ Use identity for tangent

$\boxed{\tan 2B = \frac{4\sqrt{6}}{23}}$

Question 3

An angle M has the properties such that $\sin M = r$ and $\sin 2M = s$. Find, in terms of r and s, an expression for:

(i) $\cos M$

(ii) $\tan M$.

[4]

(i) $2 \sin M \cos M = \sin 2M$ Use double angle identity

$2r \cos M = s$

$\boxed{\cos M = \frac{s}{2r}}$ } This is valid as long as $r = \sin M \neq 0$

Note: $\cos^2 M + \sin^2 M = 1 \Rightarrow \cos^2 M + r^2 = 1$

$\Rightarrow \cos^2 M = 1 - r^2 \Rightarrow \cos M = \pm\sqrt{1-r^2}$

But we don't know if $\cos M$ is positive or negative, so we can't answer the question this way.

(ii) $\tan M = \frac{r}{\left(\frac{s}{2r}\right)}$ Use identity for tangent

$= r \div \frac{s}{2r} = r \times \frac{2r}{s}$

$\boxed{\tan M = \frac{2r^2}{s}}$

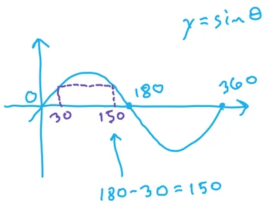
$\sin 2\theta = 2\sin\theta\cos\theta$ } Double angle identity

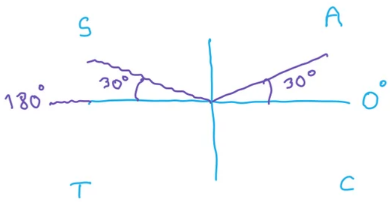
$\tan\theta = \frac{\sin\theta}{\cos\theta}$ } Identity for $\tan\theta$

Question 4

Solve the equation $2\sin 2\theta = 1$ for $0° \le \theta \le 360°$.

[3]





$0 \le \theta \le 360 \implies 0 \le 2\theta \le 720$ start by transforming the interval

Find solutions for 2θ in the transformed interval:

$2\sin 2\theta = 1 \implies \sin 2\theta = \tfrac{1}{2}$

$2\theta = \sin^{-1}(\tfrac{1}{2}) = 30°$ principal solution

$180 - 30 = 150$ use symmetry or CAST to find other solution between 0° and 360°

So $2\theta = 150°$ is also a solution

$30 + 360 = 390 \quad 150 + 360 = 510$ sine function repeats every 360°

So $2\theta = 390°$ or $510°$ are also solutions — solutions between 360° and 720°

As FINAL step, convert to solutions for θ:

$\frac{30}{2} = 15 \quad \frac{150}{2} = 75 \quad \frac{390}{2} = 195 \quad \frac{510}{2} = 255$

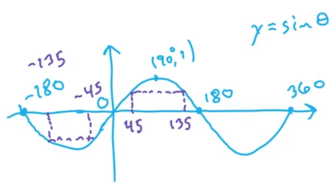
The solutions are

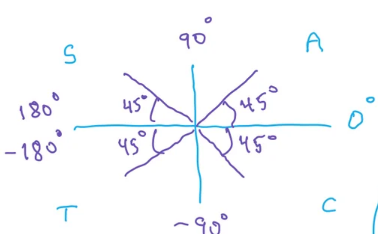
$\theta = 15°, 75°, 195°, 255°$

Question 5

Solve the equation $2\sin x = \dfrac{1}{\sin x}$ for $0° \le x \le 360°$.

[5]





$2\sin x = \frac{1}{\sin x}$

$2\sin^2 x = 1$

$\sin^2 x = \frac{1}{2} \implies \sin x = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

If $\sin x = \frac{1}{\sqrt{2}}$

$x = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$ principal solution

$180 - 45 = 135$ use symmetry or CAST to find other solution

So $x = 135°$ is another solution

If $\sin x = -\frac{1}{\sqrt{2}}$

$x = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -45°$ principal solution

$-180 + 45 = -135$ use symmetry or CAST to find other solution

So $x = -135°$ is another solution

Use fact that sine function repeats every 360° to find solutions in the interval $0° \le x \le 360°$

$-45 + 360 = 315 \quad -135 + 360 = 225$

So $x = 225°$ and $x = 315°$ are also solutions

The solutions are

$x = 45°, 135°, 225°, 315°$

Question 6

(a) Show that $(x+1)(x-2)(x-3) = x^3 - 4x^2 + x + 6$. [2]

(b) Use your result from part (a) to solve the equation

$$\tan^3 x - 4\tan^2 x + \tan x + 6 = 0$$

in the interval $0° \le x \le 360°$. [5]

a) $(x-2)(x-3) = x^2 - 3x - 2x + 6 = x^2 - 5x + 6$

Therefore

$$(x+1)(x-2)(x-3) = (x+1)(x^2-5x+6)$$
$$= x(x^2-5x+6) + 1(x^2-5x+6)$$
$$= x^3 - 5x^2 + 6x + x^2 - 5x + 6$$
$$\boxed{= x^3 - 4x^2 + x + 6}$$

(a) Show that $\underbrace{(x+1)(x-2)(x-3) = x^3 - 4x^2 + x + 6}$. [2]

Replacing x here with tan x gives

(b) Use your result from part (a) to solve the equation

$$\tan^3 x - 4\tan^2 x + \tan x + 6 = 0$$

in the interval $0° \le x \le 360°$. [5]

b) Use the result from part (a):

$\tan^3 x - 4\tan^2 x + \tan x + 6 = 0$

$\Rightarrow (\tan x + 1)(\tan x - 2)(\tan x - 3) = 0$

$\Rightarrow \tan x = -1, \tan x = 2$, or $\tan x = 3$

$-45°$ isn't in the solution interval

$x = \tan^{-1}(-1) = -45°$ principal value (from GDC)

or $-45 + 180 = 135°$
or $135 + 180 = 315°$
find other solutions in interval

$x = \tan^{-1}(2) = 63.434948... = 63.4°$ (1 d.p.) principal value (from GDC)

or $63.4 + 180 = 243.4°$ (1 d.p.) find other solutions in interval

$x = \tan^{-1}(3) = 71.565051... = 71.6°$ (1 d.p.) principal value (from GDC)

or $71.6 + 180 = 251.6°$ (1 d.p.) find other solutions in interval

$\tan(\theta \pm 180°) = \tan\theta$

[Property of tangent function]

You can see this in the graph of $y = \tan\theta$, which repeats every $180°$.

$$\boxed{x = 63.4°, 71.6°, 135°, 243.4°, 251.6°, 315° \text{ (to 1 d.p.)}}$$

Question 7

(a) Show that the equation $2\sin^2 x + 3\cos x = 0$ can be written in the form $a\cos^2 x + b\cos x + c = 0$, where a, b and c are integers to be found.

[2]

(b) Hence, or otherwise, solve the equation $2\sin^2 x + 3\cos x = 0$ for $-180° \le x \le 180°$.

[3]

$\sin^2 x + \cos^2 x \equiv 1 \Rightarrow \sin^2 x \equiv 1 - \cos^2 x$

a) $2\sin^2 x + 3\cos x = 0$

$2(1-\cos^2 x) + 3\cos x = 0$ substitute for $\sin^2 x$

$2 - 2\cos^2 x + 3\cos x = 0$ expand brackets

$$\boxed{\begin{gathered} 2\cos^2 x - 3\cos x - 2 = 0 \\ (a=2,\ b=-3,\ c=-2) \end{gathered}}$$

rearrange

Note that

$-2\cos^2 x + 3\cos x + 2 = 0$ $(a=-2,\ b=3,\ c=2)$

is also a valid answer. But the version in green will be more useful in part (b).

(a) Show that the equation $2\sin^2 x + 3\cos x = 0$ can be written in the form $a\cos^2 x + b\cos x + c = 0$, where a, b and c are integers to be found.

[2]

(b) Hence, or otherwise, solve the equation $2\sin^2 x + 3\cos x = 0$ for $-180° \le x \le 180°$.

[3]

b) $2\sin^2 x + 3\cos x = 0$

"hidden quadratic" → $2\cos^2 x - 3\cos x - 2 = 0$ from part (a)

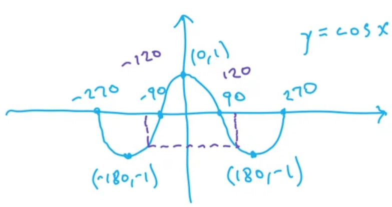
Let $y = \cos x$ rewrite quadratic in terms of y

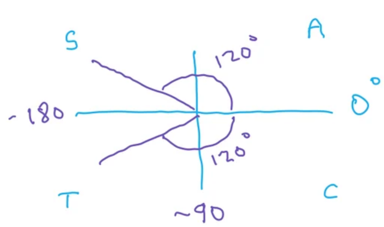
$$\left.\begin{aligned} 2y^2 - 3y - 2 &= 0 \\ (2y+1)(y-2) &= 0 \\ y = -\tfrac{1}{2} \quad &\text{or} \quad y = 2 \end{aligned}\right\} \text{solve quadratic}$$

So $\cos x = -\frac{1}{2}$ or $\cos x = 2$ → no solution — $\cos x$ is never greater than one!

$x = \cos^{-1}(-\frac{1}{2}) = 120°$ principal solution

or $x = -120°$ use symmetry or CAST to find other solution in range





$$\boxed{\begin{gathered} \text{The solutions are} \\ x = 120° \text{ or } -120° \end{gathered}}$$

Question 8

(a) Show that the equation

$$2\cos^2 x - \sin x = 1$$

can be written in the form

$$2\sin^2 x + \sin x - 1 = 0$$

[1]

(b) Hence, solve the equation $2\cos^2 x - \sin x = 1$, for $0 \le x \le 4\pi$

[5]

$\cos^2\theta + \sin^2\theta = 1$ } Pythagorean identity

a) Use Pythagorean identity

$\cos^2 x + \sin^2 x = 1 \implies \cos^2 x = 1 - \sin^2 x$

Therefore

$2\cos^2 x - \sin x = 1$

$\implies 2(1 - \sin^2 x) - \sin x = 1$

$2 - 2\sin^2 x - \sin x = 1$

$0 = 1 - 2 + 2\sin^2 x + \sin x$

$0 = 2\sin^2 + \sin x - 1$

$\boxed{2\sin^2 x + \sin x - 1 = 0}$

(a) Show that the equation

$$2\cos^2 x - \sin x = 1$$

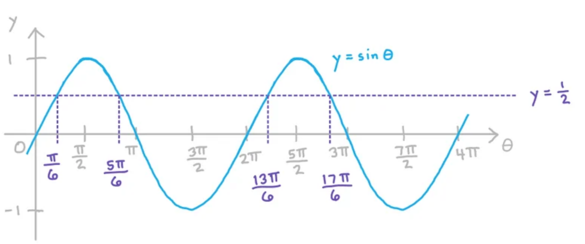
can be written in the form

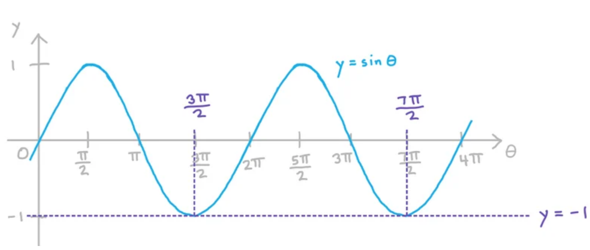
$$2\sin^2 x + \sin x - 1 = 0$$

[1]

(b) Hence, solve the equation $2\cos^2 x - \sin x = 1$, for $0 \le x \le 4\pi$

[5]





b) Use the result from part (a):

$2\cos^2 x - \sin x = 1$

$\implies 2\sin^2 x + \sin x - 1 = 0$ This is a 'hidden quadratic'

$(2\sin x - 1)(\sin x + 1) = 0$ factorise

$\sin x = \frac{1}{2}$ or $\sin x = -1$

Find primary solutions, then use symmetry of sine function to find other solutions in the interval:

$\sin x = \frac{1}{2}$: $x = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

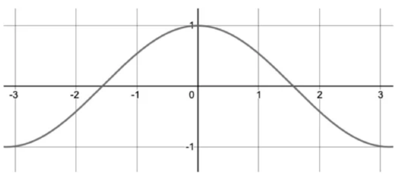
or $\frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$ or $\frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$

$\sin x = -1$: $x = \frac{3\pi}{2}$ or $\frac{3\pi}{2} + 2\pi = \frac{7\pi}{2}$

$$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{7\pi}{2}}$$

Question 9

The graph below shows the function $y = f(x)$ where $f(x) = \cos x$ for $-\pi \le x \le \pi$.



The function $g(x)$ is formed by translating the function $f(x)$ 1 unit vertically downwards.

The function $h(x)$ is formed by stretching the function $f(x)$ by a factor of $\frac{1}{2}$ in the y direction. The domain of $h(x)$ remains the same as $f(x)$.

(a) (i) Sketch the functions $y = h(x)$ and $y = g(x)$.

(ii) State the number of roots for $g(x)$.

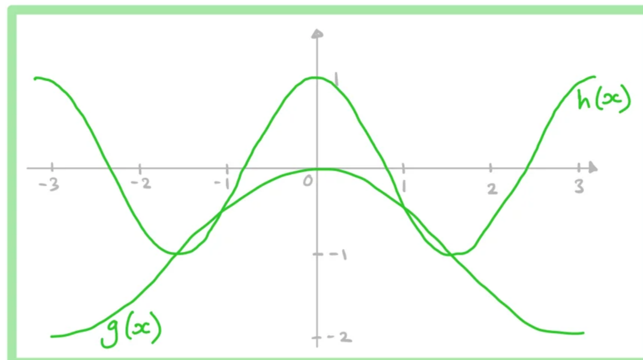
[4]

(b) Find the solutions to the equation $\cos 2x = \cos x - 1$, for $-\pi \le x \le \pi$, and label them clearly on the graph of $y = f(x)$ given above.

[4]

a) i) $g(x) = \cos(x) - 1$ SHIFT DOWN BY 1

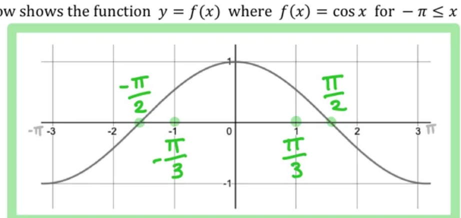
$h(x) = \cos(2x)$ MAKE HALF AS WIDE



ii) $g(x)$ TOUCHES x AXIS ONLY ONCE AT $(0,0)$

$\boxed{1 \text{ ROOT}}$

The graph below shows the function $y = f(x)$ where $f(x) = \cos x$ for $-\pi \le x \le \pi$.



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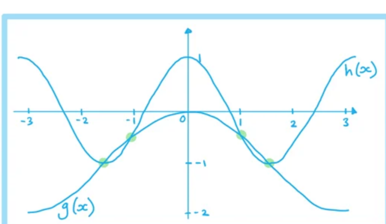

[4]

b) $\cos 2x = \begin{cases} \cos^2 x - \sin^2 x \\ 1 - 2\sin^2 x \\ 2\cos^2 x - 1 \end{cases}$

$2\cos^2 x \cancel{-1} = \cos x \cancel{-1}$

$2\cos^2 x - \cos x = 0$

$\cos x(2\cos x - 1) = 0$

$\cos x = 0 \qquad 2\cos x - 1 = 0$

$\cos\theta = \frac{1}{2}$

$\cos^{-1}(0) = \frac{\pi}{2} \qquad \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ EXACT VALUES

$-\pi \le x \le \pi$

$\boxed{x = -\frac{\pi}{2}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}}$

$\frac{\pi}{2} \approx 1.5\ldots \qquad \frac{\pi}{3} \approx 1.0\ldots$ LABEL ON GRAPH USE PART (a) GRAPH TO HELP