Question 1

It is known that in the town of Nikudy the weather displays the following patterns:

- If it rains on one day then there is a probability of 0.6 that it will rain on the following day
- If it does not rain on one day then there is only a probability of 0.2 that it will rain on the following day

(a) Represent this information as

(i) a transition state diagram

(ii) a transition matrix

[4]

Let $\boldsymbol{T}$ be the transition matrix found in part (a)(ii).

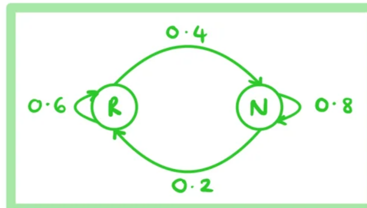
(b) Find $\boldsymbol{T}^2$.

[2]

(c) Hence find the probability that it will rain on Wednesday, given that it did not rain on the preceding Monday. Justify your answer.

[2]

(a) (i) R = Rain
N = No Rain



(ii)

$$\begin{array}{cc} & \text{Current state} \\ & \begin{array}{cc} R & N \end{array} \\ \text{Future state} \begin{array}{c} R \\ N \end{array} & \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix} \end{array} \quad \text{or} \quad \begin{array}{cc} & \text{Current state} \\ & \begin{array}{cc} N & R \end{array} \\ \text{Future state} \begin{array}{c} N \\ R \end{array} & \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix} \end{array}$$

$$\boxed{\begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}}$$

It is known that in the town of Nikudy the weather displays the following patterns:

- If it rains on one day then there is a probability of 0.6 that it will rain on the following day
- If it does not rain on one day then there is only a probability of 0.2 that it will rain on the following day

(a) Represent this information as

(i) a transition state diagram

(ii) a transition matrix

$$\boxed{\begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}}$$

[4]

Let $\boldsymbol{T}$ be the transition matrix found in part (a)(ii).

(b) Find $\boldsymbol{T}^2$.

[2]

(c) Hence find the probability that it will rain on Wednesday, given that it did not rain on the preceding Monday. Justify your answer.

[2]

(b) $T^2 = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}^2$

Find T^2 on your GDC

$$\boxed{T^2 = \begin{pmatrix} 0.44 & 0.28 \\ 0.56 & 0.72 \end{pmatrix}}$$

It is known that in the town of Nikudy the weather displays the following patterns:

- If it rains on one day then there is a probability of 0.6 that it will rain on the following day
- If it does not rain on one day then there is only a probability of 0.2 that it will rain on the following day

(a) Represent this information as

(i) a transition state diagram

(ii) a transition matrix

[4]

Let $\boldsymbol{T}$ be the transition matrix found in part (a)(ii).

(b) Find $\boldsymbol{T}^2$.

[2]

(c) Hence find the probability that it will rain on Wednesday, given that it did not rain on the preceding Monday. Justify your answer.

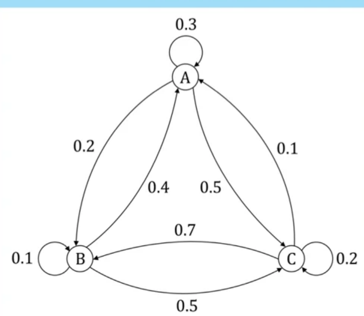
[2]

(c) The probability that it rains on Wednesday given that it did not rain on the preceeding Monday is 0.28.

If T gives the probabilities from one day to the next day, T^2 will give the probabilities from one day to two days later.

Question 2

In a robotics lab a robot is programmed to move randomly between three different locations, A, B and C, according to a fixed set of probabilities. At each 'step' of the robot's movement about the lab, the robot will either remain where it is or else move to another location according to the probabilities in the following transition state diagram:



(a) Write down the transition matrix $\boldsymbol{T}$ for this system of probabilities.

[2]

(b) Given that the robot begins at location C, find the probabilities that the robot will be at locations A, B or C three steps later.

[3]

(c) By considering the matrices T^{50} and T^{100}, determine the long-term probabilities of the robot being found at locations A, B or C. State whether or not these probabilities depend on the robot's starting position.

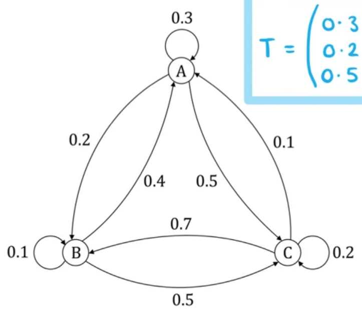
[4]

(a) Present state

$$T = \text{Future state} \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0.3 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.7 \\ 0.5 & 0.5 & 0.2 \end{pmatrix} \end{array}$$

$$T = \begin{pmatrix} 0.3 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.7 \\ 0.5 & 0.5 & 0.2 \end{pmatrix}$$

In a robotics lab a robot is programmed to move randomly between three different locations, A, B and C, according to a fixed set of probabilities. At each 'step' of the robot's movement about the lab, the robot will either remain where it is or else move to another location according to the probabilities in the following transition state diagram:



$$T = \begin{pmatrix} 0.3 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.7 \\ 0.5 & 0.5 & 0.2 \end{pmatrix}$$

(a) Write down the transition matrix $\boldsymbol{T}$ for this system of probabilities.

[2]

(b) Given that the robot begins at location C, find the probabilities that the robot will be at locations A, B or C three steps later.

[3]

(c) By considering the matrices T^{50} and T^{100}, determine the long-term probabilities of the robot being found at locations A, B or C. State whether or not these probabilities depend on the robot's starting position.

[4]

(b) Find T^3 using GDC

$$T^3 = \begin{pmatrix} 0.3 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.7 \\ 0.5 & 0.5 & 0.2 \end{pmatrix}^3$$

$$T^3 = \begin{pmatrix} 0.273 & 0.274 & 0.235 \\ 0.332 & 0.331 & 0.397 \\ 0.395 & 0.395 & 0.368 \end{pmatrix}$$

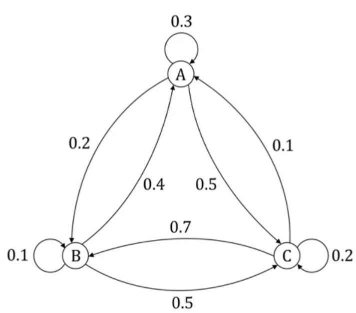
In 3 steps:

$P(C \to A) = 0.235$

$P(C \to B) = 0.397$

$P(C \to C) = 0.368$

In a robotics lab a robot is programmed to move randomly between three different locations, A, B and C, according to a fixed set of probabilities. At each 'step' of the robot's movement about the lab, the robot will either remain where it is or else move to another location according to the probabilities in the following transition state diagram:



(a) Write down the transition matrix $\boldsymbol{T}$ for this system of probabilities.

[2]

(b) Given that the robot begins at location C, find the probabilities that the robot will be at locations A, B or C three steps later.

[3]

(c) By considering the matrices T^{50} and T^{100}, determine the long-term probabilities of the robot being found at locations A, B or C. State whether or not these probabilities depend on the robot's starting position.

[4]

(c) Use the GDC to find T^{50} and T^{100}

$$T^{50} = T^{100} = \begin{pmatrix} 0.258741258 & 0.258741258 & 0.258741258 \\ 0.356643356 & 0.356643356 & 0.356643356 \\ 0.384615384 & 0.384615384 & 0.384615384 \end{pmatrix}$$

The probability of the robot ending up at a particular point is not dependent on its starting location

Longterm probabilities (rounded to 3sf)

$P(A) = 0.259$

$P(B) = 0.357$

$P(C) = 0.385$

Question 3

Two social media influencers, Mememe and EgoTiss, are in a constant struggle to steal each other's followers. No one who follows Mememe will ever follow EgoTiss at the same time, and no one who follows EgoTiss will ever follow Mememe at the same time. Each week, however, 15% of the people who follow Mememe switch to following EgoTiss and 20% of the people who follow EgoTiss switch to following Mememe. It may be assumed that there are no other gains or losses of followers by the two influencers.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the movement of followers between the two influencers in a particular week.

[2]

Initially Mememe and EgoTiss each have 7000 followers.

(b) (i) Write down the initial state vector $\boldsymbol{s}_0$ for the system.

(ii) Find the product $\boldsymbol{T}^5\boldsymbol{s}_0$.

(iii) Hence determine the number of followers that Mememe and EgoTiss will each have after five weeks.

[3]

(c) Find the number of followers that Mememe and EgoTiss will each have in the long term.

[3]

(d) Find the total number of followers per week in the long term who will change from following one influencer to following the other.

[2]

(a) M = Mememe $\quad E$ = EgoTiss

Present Influencer

$$T = \text{Future Influencer} \begin{array}{c|cc} & M & E \\ \hline M & 0.85 & 0.20 \\ E & 0.15 & 0.80 \end{array}$$

or

Present Influencer

$$T = \text{Future Influencer} \begin{array}{c|cc} & E & M \\ \hline E & 0.8 & 0.15 \\ M & 0.2 & 0.85 \end{array}$$

$$\boxed{T = \begin{pmatrix} 0.85 & 0.20 \\ 0.15 & 0.80 \end{pmatrix}}$$

Two social media influencers, Mememe and EgoTiss, are in a constant struggle to steal each other's followers. No one who follows Mememe will ever follow EgoTiss at the same time, and no one who follows EgoTiss will ever follow Mememe at the same time. Each week, however, 15% of the people who follow Mememe switch to following EgoTiss and 20% of the people who follow EgoTiss switch to following Mememe. It may be assumed that there are no other gains or losses of followers by the two influencers.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the movement of followers between the two influencers in a particular week.

$$\boxed{T = \begin{pmatrix} 0.85 & 0.20 \\ 0.15 & 0.80 \end{pmatrix}}$$

[2]

Initially Mememe and EgoTiss each have 7000 followers.

(b) (i) Write down the initial state vector $\boldsymbol{s}_0$ for the system.

(ii) Find the product $\boldsymbol{T}^5\boldsymbol{s}_0$.

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[3]

(c) Find the number of followers that Mememe and EgoTiss will each have in the long term.

[3]

(d) Find the total number of followers per week in the long term who will change from following one influencer to following the other.

[2]

(b) (i) $s_0 = \begin{pmatrix} M_0 \\ E_0 \end{pmatrix}$

$$\boxed{s_0 = \begin{pmatrix} 7000 \\ 7000 \end{pmatrix}}$$

(ii) $T^5 s_0 = \begin{pmatrix} 0.85 & 0.20 \\ 0.15 & 0.80 \end{pmatrix}^5 \begin{pmatrix} 7000 \\ 7000 \end{pmatrix}$

$$\boxed{T^5 s_0 = \begin{pmatrix} 7883.970938 \\ 6116.029063 \end{pmatrix}}$$

(iii) Round to the nearest person

Mememe = 7884 followers

EgoTiss = 6116 followers

Two social media influencers, Mememe and EegoTiss, are in a constant struggle to steal each other's followers. No one who follows Mememe will ever follow EegoTiss at the same time, and no one who follows EegoTiss will ever follow Mememe at the same time. Each week, however, 15% of the people who follow Mememe switch to following EegoTiss and 20% of the people who follow EegoTiss switch to following Mememe. It may be assumed that there are no other gains or losses of followers by the two influencers.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the movement of followers between the two influencers in a particular week.

$$\boxed{T = \begin{pmatrix} 0.85 & 0.20 \\ 0.15 & 0.80 \end{pmatrix}}$$

[2]

Initially Mememe and EegoTiss each have 7000 followers.

(b) (i) Write down the initial state vector $\boldsymbol{s}_0$ for the system.

$$\boxed{s_0 = \begin{pmatrix} 7000 \\ 7000 \end{pmatrix}}$$

(ii) Find the product $\boldsymbol{T}^5\boldsymbol{s}_0$.

(iii) Hence determine the number of followers that Mememe and EegoTiss will each have after five weeks.

[3]

(c) Find the number of followers that Mememe and EegoTiss will each have in the long term.

[3]

(d) Find the total number of followers per week in the long term who will change from following one influencer to following the other.

[2]

(c) Find the long term probabilities by looking at high powers of T

$$T^{50} = \begin{pmatrix} 0.85 & 0.20 \\ 0.15 & 0.80 \end{pmatrix}^{50}$$

$$= \begin{pmatrix} 0.57142857 & 0.57142857 \\ 0.42857143 & 0.42857143 \end{pmatrix}$$

The probabilities are consistent along the rows so are the long term probabilities

$$T^{50} s_0 = \begin{pmatrix} 0.85 & 0.20 \\ 0.15 & 0.80 \end{pmatrix}^{50} \begin{pmatrix} 7000 \\ 7000 \end{pmatrix}$$

$$= \begin{pmatrix} 8000 \\ 6000 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \text{Mememe} = 8000 \text{ followers} \\ \text{EegoTiss} = 6000 \text{ followers} \end{array}}$$

Two social media influencers, Mememe and EegoTiss, are in a constant struggle to steal each other's followers. No one who follows Mememe will ever follow EegoTiss at the same time, and no one who follows EegoTiss will ever follow Mememe at the same time. Each week, however, 15% of the people who follow Mememe switch to following EegoTiss and 20% of the people who follow EegoTiss switch to following Mememe. It may be assumed that there are no other gains or losses of followers by the two influencers.

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(ii) Find the product $\boldsymbol{T}^5\boldsymbol{s}_0$.

(iii) Hence determine the number of followers that Mememe and EegoTiss will each have after five weeks.

[3]

(c) Find the number of followers that Mememe and EegoTiss will each have in the long term.

$$\boxed{\begin{array}{l} \text{Mememe} = 8000 \text{ followers} \\ \text{EegoTiss} = 6000 \text{ followers} \end{array}}$$

[3]

(d) Find the total number of followers per week in the long term who will change from following one influencer to following the other.

[2]

(d) In the long term, the total number of followers for each influencer remains constant, which implies that the same number of followers are switching from Mememe to EegoTist is the same as the number of followers switching the other way round

Mememe → EegoTiss = $8000 \times 0.15 = 1200$

EegoTiss → Mememe = $6000 \times 0.20 = 1200$

$$\boxed{\text{Total number of followers switching each week: } 2400}$$

Question 4

In a videogame three mighty wizards – Eugenes (E), Ischyros (I) and Skleros (S) – are attempting to create armies of magical followers. They do this by magically changing members of the other armies into members of their own armies. This happens in the following ways:

- Eugenes' army is made up of unicorns. During each turn of the game he quietly turns 40% of Ischyros' myrmidons and 40% of Skleros' orcs into unicorns.
- Ischyros' army is made up of myrmidons. During each turn of the game he powerfully turns 20% of Eugenes' unicorns and 50% of Skleros' orcs into myrmidons.
- Skleros' army is made up of orcs. During each turn of the game he wickedly turns 20% of Eugenes' unicorns and 50% of Ischyros' myrmidons into orcs.

There is no other way for the numbers of creatures in each of the wizard's armies to increase or decrease.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the changes in the wizards' armies from turn to turn.

[3]

At the start of a particular game Eugenes has 10 unicorns, Ischyros has 80 myrmidons, and Skleros has 350 orcs.

(b) Find the number of creatures in each wizard's army after one turn.

[3]

(c) Find the number of creatures that each wizard can expect to have in his army if the game continues for a large number of turns.

[3]

(a) U = Unicorns
M = Myrmidons
O = Orcs

Current state

$$T = \begin{array}{c|c} & \begin{matrix} U & M & O \end{matrix} \\ \text{Future state} \; \begin{matrix} U \\ M \\ O \end{matrix} & \begin{pmatrix} \underline{0.6} & 0.4 & 0.4 \\ 0.2 & \underline{0.1} & 0.5 \\ 0.2 & 0.5 & \underline{0.1} \end{pmatrix} \end{array}$$

Fill in the probabilities for an army member not changing by remembering that the probabilities in each column sum to 1

$$\boxed{T = \begin{pmatrix} 0.6 & 0.4 & 0.4 \\ 0.2 & 0.1 & 0.5 \\ 0.2 & 0.5 & 0.1 \end{pmatrix}}$$

In a videogame three mighty wizards – Eugenes (E), Ischyros (I) and Skleros (S) – are attempting to create armies of magical followers. They do this by magically changing members of the other armies into members of their own armies. This happens in the following ways:

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- Skleros' army is made up of orcs. During each turn of the game he wickedly turns 20% of Eugenes' unicorns and 50% of Ischyros' myrmidons into orcs.

There is no other way for the numbers of creatures in each of the wizard's armies to increase or decrease.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the changes in the wizards' armies from turn to turn.

$$\boxed{T = \begin{pmatrix} 0.6 & 0.4 & 0.4 \\ 0.2 & 0.1 & 0.5 \\ 0.2 & 0.5 & 0.1 \end{pmatrix}}$$

[3]

At the start of a particular game Eugenes has 10 unicorns, Ischyros has 80 myrmidons, and Skleros has 350 orcs.

(b) Find the number of creatures in each wizard's army after one turn.

[3]

(c) Find the number of creatures that each wizard can expect to have in his army if the game continues for a large number of turns.

[3]

(b)
$$T s_0 = \begin{pmatrix} 0.6 & 0.4 & 0.4 \\ 0.2 & 0.1 & 0.5 \\ 0.2 & 0.5 & 0.1 \end{pmatrix} \begin{pmatrix} 10 \\ 80 \\ 350 \end{pmatrix} = \begin{pmatrix} 178 \\ 185 \\ 77 \end{pmatrix}$$

Use GDC to calculate

After 1 turn there are:
178 Unicorns
185 Myrmidons
77 Orcs

In a videogame three mighty wizards – Eugenes (E), Ischyros (I) and Skleros (S) – are attempting to create armies of magical followers. They do this by magically changing members of the other armies into members of their own armies. This happens in the following ways:

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There is no other way for the numbers of creatures in each of the wizard's armies to increase or decrease.

(a) Write down a transition matrix ***T*** representing the changes in the wizards' armies from turn to turn.

$$\boxed{T = \begin{pmatrix} 0.6 & 0.4 & 0.4 \\ 0.2 & 0.1 & 0.5 \\ 0.2 & 0.5 & 0.1 \end{pmatrix}}$$

[3]

At the start of a particular game Eugenes has 10 unicorns, Ischyros has 80 myrmidons, and Skleros has 350 orcs.

(b) Find the number of creatures in each wizard's army after one turn.

[3]

(c) Find the number of creatures that each wizard can expect to have in his army if the game continues for a large number of turns.

[3]

(c) Use your GDC to calculate the number of each type of army after a large number of turns, e.g. 100

It's a good idea to check that the probabilities for each row in T^{100} are consistent to know you have reached a steady state

$$T^{100}S_0 = \begin{pmatrix} 0.6 & 0.4 & 0.4 \\ 0.2 & 0.1 & 0.5 \\ 0.2 & 0.5 & 0.1 \end{pmatrix}^{100} \begin{pmatrix} 10 \\ 80 \\ 350 \end{pmatrix} = \begin{pmatrix} 220 \\ 110 \\ 110 \end{pmatrix}$$

After a large number of goes (and remaining constant thereafter):

220 Unicorns
110 Myrmidons
110 Orcs

Question 5

The marketing department of ShedHead brand shampoo ("It makes you look like you woke up in a garden shed!") is attempting to predict the percentage of potential customers who will purchase its product month by month in the future.

One marketing researcher believes that the probability of a potential customer buying ShedHead shampoo one month depends on what shampoo they bought the previous month, as well as what shampoo they bought the month before that.

(a) Explain why a Markov chain cannot be used to represent month by month sales for this marketing researcher's model.

[1]

(a) In a Markov chain the probability of each event depends only on the state of the event immediately preceeding it.
If the current state is dependent on both the event immediately preceeding it and the one before that it cannot be a Markhov chain.

Another marketing researcher believes that what shampoo a potential customer purchases one month depends only on what shampoo the customer purchased the previous month. Her research shows that if a customer buys ShedHead shampoo one month then there is a 93% chance they will buy it again the following month, while if a customer does not buy ShedHead shampoo one month then there is only a 5% chance that they will buy it the following month.

(b) Write down a transition matrix ***T*** representing customer behaviour according to this researcher's model.

[2]

Currently 20% of potential customers buy ShedHead shampoo.

(c) Find the probability that a randomly selected potential customer

(i) will purchase ShedHead shampoo next month

(ii) will purchase ShedHead shampoo in the long term.

[5]

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(b) Write down a transition matrix $\boldsymbol{T}$ representing customer behaviour according to this researcher's model.

[2]

Currently 20% of potential customers buy ShedHead shampoo.

(c) Find the probability that a randomly selected potential customer

(i) will purchase ShedHead shampoo next month

(ii) will purchase ShedHead shampoo in the long term.

[5]

(b) B = Buys ShedHead shampoo
N = Does not buy ShedHead shampoo

Current state

$$T = \text{Future state} \begin{array}{c|cc} & B & N \\ B & 0.93 & 0.05 \\ N & 0.07 & 0.95 \end{array} \quad \text{or} \quad T = \text{Future state} \begin{array}{c|cc} & N & B \\ N & 0.95 & 0.07 \\ B & 0.05 & 0.93 \end{array}$$

$$\boxed{T = \begin{pmatrix} 0.93 & 0.05 \\ 0.07 & 0.95 \end{pmatrix}}$$

The marketing department of ShedHead brand shampoo ("It makes you look like you woke up in a garden shed!") is attempting to predict the percentage of potential customers who will purchase its product month by month in the future.

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(b) Write down a transition matrix $\boldsymbol{T}$ representing customer behaviour according to this researcher's model.

[2]

Currently 20% of potential customers buy ShedHead shampoo.

(c) Find the probability that a randomly selected potential customer

(i) will purchase ShedHead shampoo next month

(ii) will purchase ShedHead shampoo in the long term.

[5]

(c) (i) Multiply the transition matrix by the initial state probability matrix $s_0 = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$

$$T s_0 = \begin{pmatrix} 0.93 & 0.05 \\ 0.07 & 0.95 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.226 \\ 0.774 \end{pmatrix}$$

There is a 22.6% chance that a randomly selected potential customer will buy ShedHead shampoo in the next month

(ii) Test T^n for high values of n to find the long term state probabilities

diverges from 3sf

$$T^{50} = \begin{pmatrix} 0.41\underline{7}64401 & 0.41\underline{5}96855 \\ 0.58\underline{2}35598 & 0.58\underline{4}03144 \end{pmatrix}$$

diverges from 3sf

$$T^{100} = \begin{pmatrix} 0.41666\underline{8}30 & 0.41666\underline{5}49 \\ 0.58333\underline{1}69 & 0.58333\underline{4}50 \end{pmatrix}$$

You can test higher powers but this should be good enough to be confident of the probabilities to 3sf

Multiply T^{100} by the initial state matrix to find the long term outcomes

$$T^{100} s_0 = \begin{pmatrix} 0.93 & 0.05 \\ 0.07 & 0.95 \end{pmatrix}^{100} \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.41666605 \\ 0.58333394 \end{pmatrix}$$

In the long term there is a 41.7% (3sf) probability that a randomly selected potential customer will buy ShedHead shampoo

Question 6

A delivery company operates a fleet of lorries serving three major cities, A, B and C. Past experience shows that if a lorry starts a week in city A there is a 70% chance that it will still be in city A at the start of the following week; otherwise there is a 20% chance that it will be in city B and a 10% chance that it will be in city C. If a lorry starts a week in city B there is an 80% chance that it will still be in city B at the start of the following week; otherwise it is equally likely to be in city A or city C. If a lorry starts a week in city C there is a 90% chance that it will still be in city C at the start of the following week; otherwise it will be in city A, with no chance of it being in city B.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the movement of the company's lorries from week to week according to the above information.

[2]

(b) By solving the system of linear equations represented by

$$\boldsymbol{T}\boldsymbol{p} = \boldsymbol{p}$$

determine a steady state vector $\boldsymbol{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ corresponding to matrix $\boldsymbol{T}$.

[3]

The company is about to replace its entire fleet of lorries with a fleet of 280 brand new lorries.

(c) Suggest how the company should initially distribute the new lorries between cities A, B and C. Be sure to justify your answer.

[2]

(a)

Current state

Future state

$$T = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0.1 & 0.9 \end{pmatrix} \end{array}$$

$$\boxed{T = \begin{pmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0.1 & 0.9 \end{pmatrix}}$$

A delivery company operates a fleet of lorries serving three major cities, A, B and C. Past experience shows that if a lorry starts a week in city A there is a 70% chance that it will still be in city A at the start of the following week; otherwise there is a 20% chance that it will be in city B and a 10% chance that it will be in city C. If a lorry starts a week in city B there is an 80% chance that it will still be in city B at the start of the following week; otherwise it is equally likely to be in city A or city C. If a lorry starts a week in city C there is a 90% chance that it will still be in city C at the start of the following week; otherwise it will be in city A, with no chance of it being in city B.

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$$\boxed{T = \begin{pmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0.1 & 0.9 \end{pmatrix}}$$

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(b) $$\begin{pmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0.1 & 0.9 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Set up a series of linear equations to solve

$0.7a + 0.1b + 0.1c = a \Rightarrow 3a - b - c = 0$

$0.2a + 0.8b = b \Rightarrow a = b$

$0.1a + 0.1b + 0.9c = c \Rightarrow c = a + b$

There is no unique solution, all we know is that $a = b$ and therefore $c = 2a$, so choose a value for one variable and use to determine the others

Let $a = 1 \Rightarrow b = 1,\ c = 2$

$$\boxed{p = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{(or any multiple of)}}$$

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(a) Write down a transition matrix $\boldsymbol{T}$ representing the movement of the company's lorries from week to week according to the above information.

[2]

(b) By solving the system of linear equations represented by

$$\boldsymbol{Tp} = \boldsymbol{p}$$

determine a steady state vector $\boldsymbol{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ corresponding to matrix $\boldsymbol{T}$.

$\boxed{p = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}$

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[2]

(c) For ease of planning the company would most likely want to have the number of lorries starting in each city each week, to be constant, therefore the lorries should be distributed in the ratio A : B : C = 1 : 1 : 2

A : 70 lorries
B : 70 lorries
C : 140 lorries

Question 7

In the town of Manh, all the residents belong to either one or the other of the town's two fitness clubs – Giang's House of Fitness (G) or Thu's Wonder Gym (T). Each year 30% of the members of G switch to T and 25% of the members of T switch to G. Any other losses or gains of members by the two fitness clubs may be ignored.

(a) Write down a transition matrix $\boldsymbol{T}$ representing the movement of members between the two clubs in a particular year.

[2]

(b) Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{T}$.

[4]

(c) Hence write down matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$.

[2]

Initially there are 2500 members of G and 800 members of T.

(d) Using the matrix power formula, show that the numbers of members of G and T after n years will be $\left(1500 + 1000(0.45^n)\right)$ and $\left(1800 - 1000(0.45^n)\right)$, respectively.

[6]

(e) Hence write down the number of customers that each of the fitness clubs can expect to have in the long term.

[2]

(a) Current state

$$T = \begin{array}{c} \\ \text{Future state} \end{array} \begin{array}{c} \\ G \\ T \end{array} \begin{array}{c} \begin{array}{cc} G & T \end{array} \\ \begin{pmatrix} 0.7 & 0.25 \\ 0.3 & 0.75 \end{pmatrix} \end{array}$$

$$\boxed{T = \begin{pmatrix} 0.7 & 0.25 \\ 0.3 & 0.75 \end{pmatrix}}$$

EXAM PAPERS PRACTICE

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(a) Write down a transition matrix $\boldsymbol{T}$ representing the movement of members between the two clubs in a particular year.

$$\boxed{T = \begin{pmatrix} 0.7 & 0.25 \\ 0.3 & 0.75 \end{pmatrix}}$$

[2]

(b) Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{T}$.

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(b) Find the characteristic polynomial of T and use it to find the eigenvalues

$$\begin{vmatrix} 0.7-\lambda & 0.25 \\ 0.3 & 0.75-\lambda \end{vmatrix} = (0.7-\lambda)(0.75-\lambda) - (0.3)(0.25)$$

$$= 0.525 - 1.45\lambda + \lambda^2 - 0.075$$

Set to 0 and solve $\quad 0 = \lambda^2 - 1.45\lambda + 0.45$

$$0 = (\lambda - 1)(\lambda - 0.45)$$

1 is always an eigenvalue of a transition matrix and the steady state solution is always the eigenvector that goes with $\lambda=1$

$$\boxed{\lambda_1 = 1 \;,\; \lambda_2 = 0.45}$$

Find eigenvectors by using $A\mathbf{x} = \lambda\mathbf{x}$ or $(A - \lambda I)\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\lambda_1 = 1 \quad \left(\begin{pmatrix} 0.7 & 0.25 \\ 0.3 & 0.75 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.3 & 0.25 \\ 0.3 & -0.25 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 6x = 5y \quad \Rightarrow \quad \boxed{\mathbf{x}_1 = \begin{pmatrix} 5 \\ 6 \end{pmatrix}}$$

$$\lambda_2 = 0.45 \quad \left(\begin{pmatrix} 0.7 & 0.25 \\ 0.3 & 0.75 \end{pmatrix} - 0.45\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0.25 & 0.25 \\ 0.3 & 0.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x = -y \quad \Rightarrow \quad \boxed{\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

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(b) Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{T}$.

$\boxed{\lambda_1 = 1 \quad x_1 = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \quad \lambda_2 = -0.45 \quad x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

[4]

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(c) $M^n = PD^nP^{-1}$ $\quad$ $\boldsymbol{P}$ is the matrix of eigenvectors and $\boldsymbol{D}$ is the diagonal matrix of eigenvalues $\leftarrow$ Formula booklet

$$\boxed{P = \begin{pmatrix} 5 & 1 \\ 6 & -1 \end{pmatrix}}$$

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$$P^{-1} = \frac{1}{(5)(-1)-(1)(6)}\begin{pmatrix} -1 & -1 \\ -6 & 5 \end{pmatrix} = \begin{pmatrix} \frac{1}{11} & \frac{1}{11} \\ \frac{6}{11} & -\frac{5}{11} \end{pmatrix}$$

$$\Rightarrow M^n = \begin{pmatrix} 5 & 1 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.45 \end{pmatrix}^n\begin{pmatrix} \frac{1}{11} & \frac{1}{11} \\ \frac{6}{11} & -\frac{5}{11} \end{pmatrix}$$

$$\begin{pmatrix} n_G \\ n_T \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.45^n \end{pmatrix}\begin{pmatrix} \frac{1}{11} & \frac{1}{11} \\ \frac{6}{11} & -\frac{5}{11} \end{pmatrix}\begin{pmatrix} 2500 \\ 800 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 1 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.45^n \end{pmatrix}\begin{pmatrix} 300 \\ 1000 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 1 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 300 \\ 1000(0.45^n) \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1500 + 1000(0.45^n) \\ 1800 - 1000(0.45^n) \end{pmatrix}}$$

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(e) As $n \to \infty$, $0.45^n \to 0$

Long term members:
G = 1500
T = 1800