

Transformations of Graphs

Mark Schemes

Question 1

The point $P(-1, 4)$ lies on the curve with equation $y = f(x)$.

State the coordinates of the image of point P on the curves with the following equations:

- (i) $y = f(x) + 3$
- (ii) $y = f(x + 3)$
- (iii) $y = 3f(x)$
- (iv) $y = f(3x)$

[4]

i) VERTICAL TRANSLATION y CHANGES $+3$

$$\boxed{(-1, 7)}$$

ii) HORIZONTAL TRANSLATION x CHANGES -3

$$\boxed{(-4, 4)}$$

iii) VERTICAL STRETCH y CHANGES $\times 3$

$$\boxed{(-1, 12)}$$

iv) HORIZONTAL STRETCH x CHANGES $\times \frac{1}{3}$
 $\div 3$

$$\boxed{\left(-\frac{1}{3}, 4\right)}$$

Question 2

The point $P(-3, -4)$ lies on the curve with equation $y = f(x)$.

State the coordinates of the image of point P on the curves with the following equations:

- (i) $y = f(-x)$
- (ii) $y = -f(x)$

[2]

i) HORIZONTAL REFLECTION y AXIS
 x CHANGES

$$\boxed{(3, -4)}$$

ii) VERTICAL REFLECTION x AXIS
 y CHANGES

$$\boxed{(-3, 4)}$$

Question 3

The point $P(3, 2)$ lies on the curve with equation $y = f(x)$.

- (i) On the graph of $y = f(x) + a$, where a is a constant, the point P is mapped to the point $(3, -5)$. Determine the value of a .
- (ii) On the graph of $y = f(x + b)$, where b is a constant, the point P is mapped to the point $(-1, 2)$. Determine the value of b .
- (iii) On the graph of $y = cf(x)$, where c is a constant, the point P is mapped to the point $(3, 1)$. Determine the value of c .
- (iv) On the graph of $y = f(dx)$, where d is a constant, the point P is mapped to the point $(1, 2)$. Determine the value of d .

[4]

i) VERTICAL TRANSLATION y CHANGED +a
 $(3, 2) \Rightarrow (3, -5)$ $a = -7$

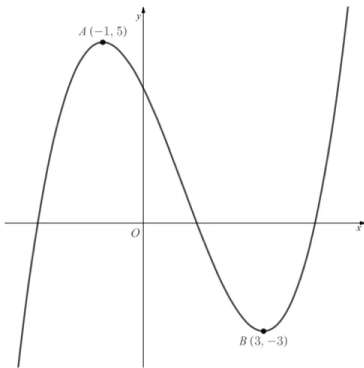
ii) HORIZONTAL TRANSLATION x CHANGED -b
 $(3, 2) \Rightarrow (-1, 2)$ $b = 4$

iii) VERTICAL STRETCH y CHANGED $\times c$
 $(3, 2) \Rightarrow (3, 1)$ $c = \frac{1}{2}$

iv) HORIZONTAL STRETCH x CHANGED $\times \frac{1}{d}$
 $(3, 2) \Rightarrow (1, 2)$ $d = 3$

Question 4

The diagram below shows the graph of $y = f(x)$. The two marked points $A(-1, 5)$ and $B(3, -3)$ lie on the graph.



(a) In separate diagrams, sketch the curves with equation **TRANSLATIONS**

- (i) $y = f(x - 1)$ HORIZONTAL x CHANGES +1
- (ii) $y = f(x) + 3$ VERTICAL y CHANGES +3

On each diagram, give the coordinates of the images of points A and B under the given transformation.

[4]

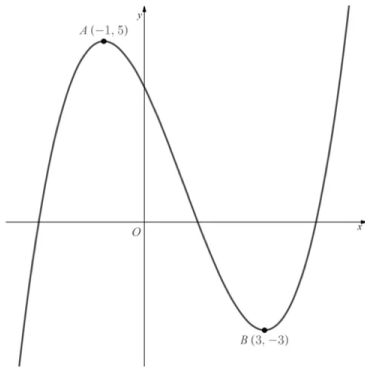
(b) On the graph of $y = f(x + a)$ the image of one of the two marked points has an x coordinate of 2. Find the two possible values of a .

[2]

a) i)

ii)

The diagram below shows the graph of $y = f(x)$. The two marked points $A(-1, 5)$ and $B(3, -3)$ lie on the graph.



(a) In separate diagrams, sketch the curves with equation

- (i) $y = f(x - 1)$
- (ii) $y = f(x) + 3$

On each diagram, give the coordinates of the images of points A and B under the given transformation.

[4]

(b) On the graph of $y = f(x + a)$ the image of one of the two marked points has an x coordinate of 2. Find the two possible values of a .

[2]

b) HORIZONTAL TRANSLATION x CHANGE $-a$

$$A = (-1, 5) \Rightarrow (2, 5) \quad a = -3$$

+3

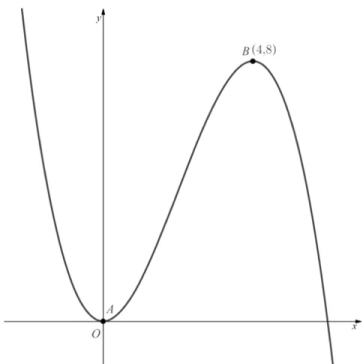
$$B = (3, -3) \Rightarrow (2, -3) \quad a = 1$$

-1

$a = -3 \text{ OR } a = 1$

Question 5

The diagram below shows the graph of $y = f(x)$. The marked point $B(4, 8)$ lies on the graph, and the graph meets the origin at the marked point A .



(a) In separate diagrams, sketch the curves with equation

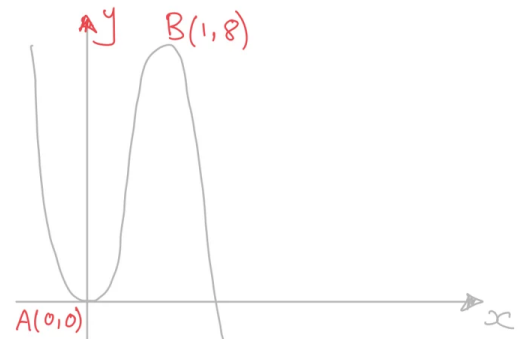
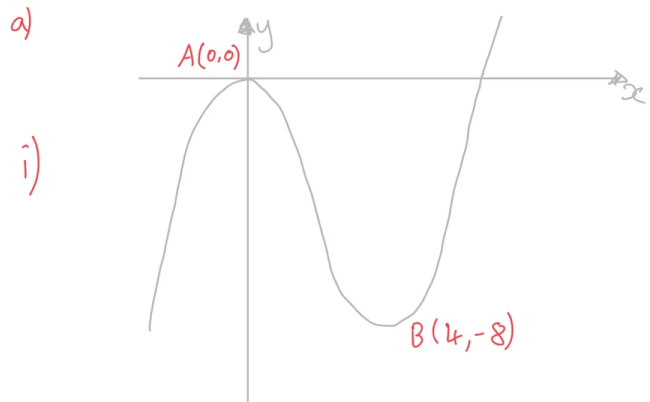
- (i) $y = -f(x)$ VERTICAL REFLECTION y CHANGES (x AXIS)
- (ii) $y = f(4x)$ HORIZONTAL STRETCH x CHANGES $\times \frac{1}{4}$

On each diagram, give the coordinates of the images of points A and B under the given transformation.

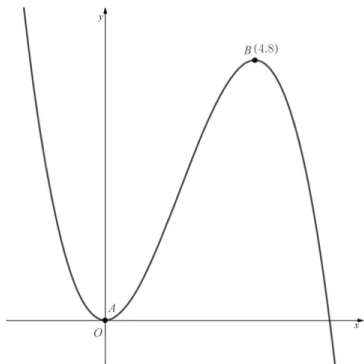
[4]

(b) On the graph of $y = af(x)$ the image of one of the two marked points has a y coordinate of 4. Find the value of a .

[2]



The diagram below shows the graph of $y = f(x)$. The marked point $B(4, 8)$ lies on the graph, and the graph meets the origin at the marked point A .



b) VERTICAL STRETCH y CHANGES $\times a$
 $A = (0, 0)$ WONT CHANGE
 $B = (4, 8) \Rightarrow (4, 4)$
 $\xrightarrow{\times a}$
 $a = \frac{1}{2}$

(a) In separate diagrams, sketch the curves with equation

- (i) $y = -f(x)$
- (ii) $y = f(4x)$

On each diagram, give the coordinates of the images of points A and B under the given transformation.

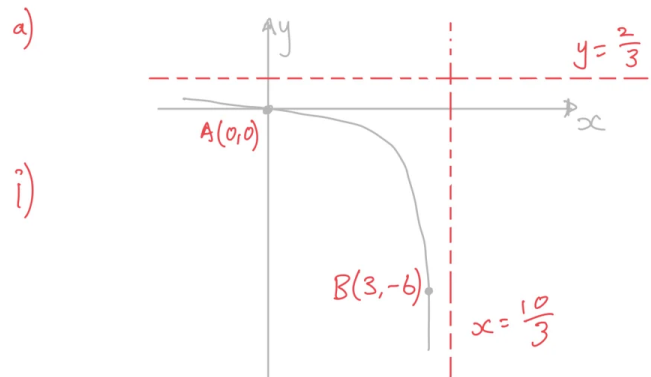
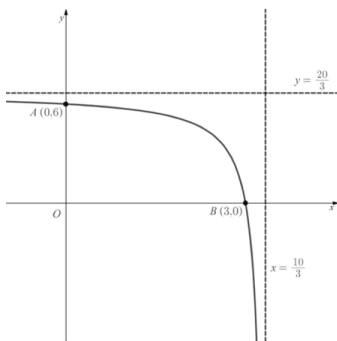
[4]

(b) On the graph of $y = af(x)$ the image of one of the two marked points has a y coordinate of 4. Find the value of a .

[2]

Question 6

The diagram below shows the graph of $y = f(x)$. The graph intersects the coordinate axes at the two marked points $A(0, 6)$ and $B(3, 0)$. The graph has two asymptotes as shown, with equations $y = \frac{20}{3}$ and $x = \frac{10}{3}$.



(a) In separate diagrams, sketch the curves with equation

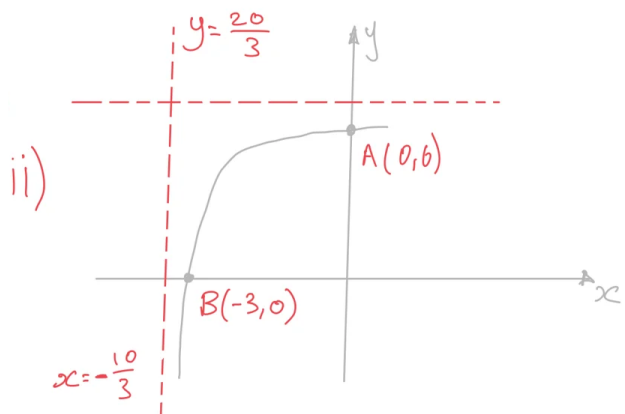
- (i) $y = f(x) - 6$ VERTICAL TRANSLATION y CHANGES -6
- (ii) $y = f(-x)$ HORIZONTAL REFLECTION x CHANGES (y AXIS)

On each diagram, give the coordinates of the images of points A and B under the given transformation, as well as stating the equations of the transformed asymptotes.

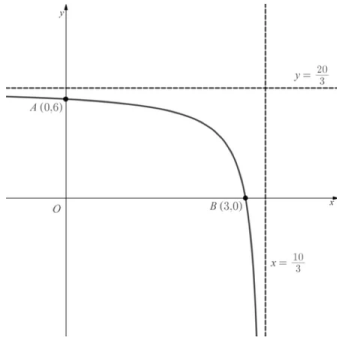
[6]

(b) The graph of $y = f(x + a)$ has an asymptote at one of the coordinate axes. Find the value of a .

[2]



The diagram below shows the graph of $y = f(x)$. The graph intersects the coordinate axes at the two marked points $A(0, 6)$ and $B(3, 0)$. The graph has two asymptotes as shown, with equations $y = \frac{20}{3}$ and $x = \frac{10}{3}$.



(a) In separate diagrams, sketch the curves with equation

- (i) $y = f(x) - 6$
- (ii) $y = f(-x)$

On each diagram, give the coordinates of the images of points A and B under the given transformation, as well as stating the equations of the transformed asymptotes.

[6]

(b) The graph of $y = f(x + a)$ has an asymptote at one of the coordinate axes. Find the value of a .

[2]

b) HORIZONTAL TRANSLATION CHANGES $-a$

$xc = \frac{10}{3} \Rightarrow x = 0$

$-\frac{10}{3}$

$a = \frac{10}{3}$

Question 7

Describe, in order, a sequence of transformations that maps the graph of $y = f(x)$ onto the following graphs:

- (i) $y = 3f(x + 2)$,
- (ii) $y = f(-x) - 1$.

[3]

DEAL WITH INSIDE BRACKETS FIRST
INSIDE = HORIZONTAL
OUTSIDE = VERTICAL

i) TRANSLATION BY $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$
VERTICAL STRETCH SF. 3

ii) REFLECTION IN y AXIS (OR $x = 0$)
TRANSLATION BY $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Question 8

Given that $f(x) = 3x^2 - 2x$ find an expression for $g(x)$, where $g(x)$ is obtained by applying the following sequence of transformations to $f(x)$.

1. Translation by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
2. Vertical stretch of scale factor 4
3. Translation by $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

[4]

$$\begin{aligned}
 1. \quad f(x) &\Rightarrow f(x-2) \\
 3x^2 - 2x &\Rightarrow 3(x-2)^2 - 2(x-2) \\
 &= 3(x^2 - 4x + 4) - 2x + 4 \\
 &= 3x^2 - 12x + 16
 \end{aligned}$$

$$\begin{aligned}
 2. \quad f(x-2) &\Rightarrow 4f(x-2) \\
 &= 4(3x^2 - 12x + 16) \\
 &= 12x^2 - 56x + 64
 \end{aligned}$$

$$\begin{aligned}
 3. \quad 4f(x-2) &\Rightarrow 4f(x-2) - 3 \\
 &= 12x^2 - 56x + 64 - 3
 \end{aligned}$$

$$\boxed{g(x) = 12x^2 - 56x + 61}$$

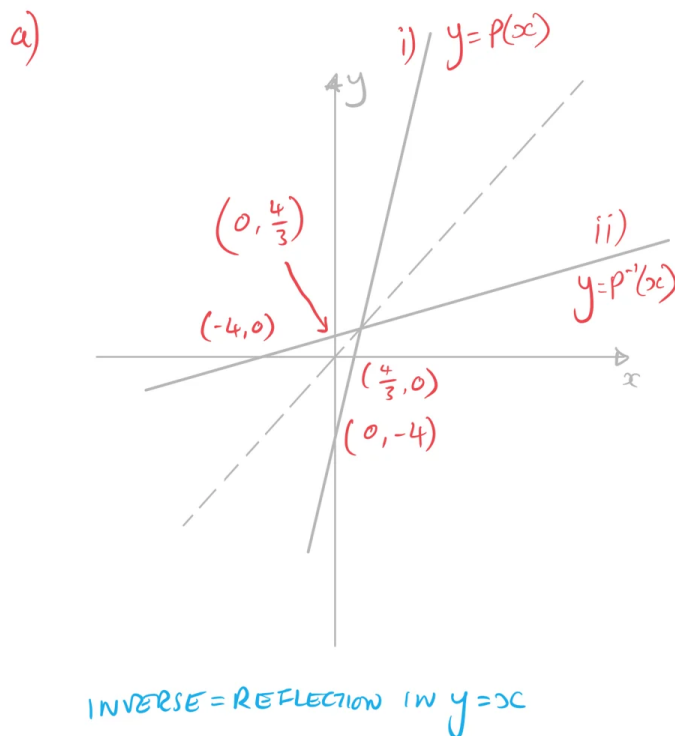
Question 9

- (a) (i) Sketch the graph of $y = p(x)$, where $p(x) = 3x - 4$.
 (ii) On the same set of axes, sketch the graph of $y = p^{-1}(x)$.
 Label the coordinates of the points where each graph crosses the coordinate axes.

[4]

- (b) (i) Find an expression for $p^{-1}(x)$.
 (ii) Find an expression for $\frac{1}{9}[p(x) + 16]$.
 (iii) What can you deduce about the sequence of transformations given by $\frac{1}{9}[p(x) + 16]$?

[4]



- (a) (i) Sketch the graph of $y = p(x)$, where $p(x) = 3x - 4$.
 (ii) On the same set of axes, sketch the graph of $y = p^{-1}(x)$.
 Label the coordinates of the points where each graph crosses the coordinate axes.

[4]

- (b) (i) Find an expression for $p^{-1}(x)$.
 (ii) Find an expression for $\frac{1}{9}[p(x) + 16]$.
 (iii) What can you deduce about the sequence of transformations given by $\frac{1}{9}[p(x) + 16]$?

[4]

b) i) LET $y = 3x - 4$ REARRANGE

$$y + 4 = 3x$$

$$\frac{y + 4}{3} = x$$

$$P^{-1}(x) = \frac{x + 4}{3} \text{ OR } P^{-1}(x) = \frac{1}{3}(x + 4)$$

ii)

$$\frac{1}{9}[(3x - 4) + 16]$$

$$\frac{1}{9}(3x + 12)$$

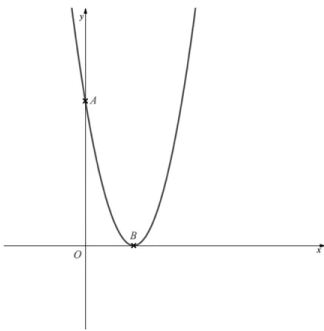
$$\frac{1}{3}x + \frac{4}{3} = \frac{1}{3}(x + 4)$$

iii)

$$\frac{1}{9}[p(x) + 16] = P^{-1}(x)$$

Question 10

The equation $y = f(x)$, where $f(x) = (x - a)^2$, with $a > 1$, is shown below.



The points A and B are the points where the graph intercepts the coordinate axes.

- (a) Write down, in terms of a , the coordinates of A and B .
 (b) Sketch the graph of $y = -f(-x)$, labelling the images of the points A and B and stating their coordinates in terms of a .
 (c) Write down the value of a such that the point A is three times as far from the origin as the point B .

[2]

[3]

[1]

a) BOTH POINTS ON AXES SO

$$y = 0 \text{ OR } x = 0$$

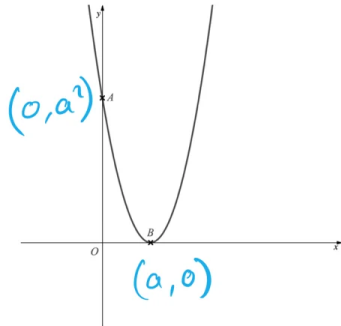
A $x = 0$ $f(x) = (0 - a)^2$
 $(-a)^2$
 $y = a^2$

$$A = (0, a^2)$$

B $y = 0$ $0 = (x - a)^2$
 $x - a = 0$
 $x = a$

$$B = (a, 0)$$

The equation $y = f(x)$, where $f(x) = (x - a)^2$, with $a > 1$, is shown below.



The points A and B are the points where the graph intercepts the coordinate axes.

(a) Write down, in terms of a , the coordinates of A and B .

[2]

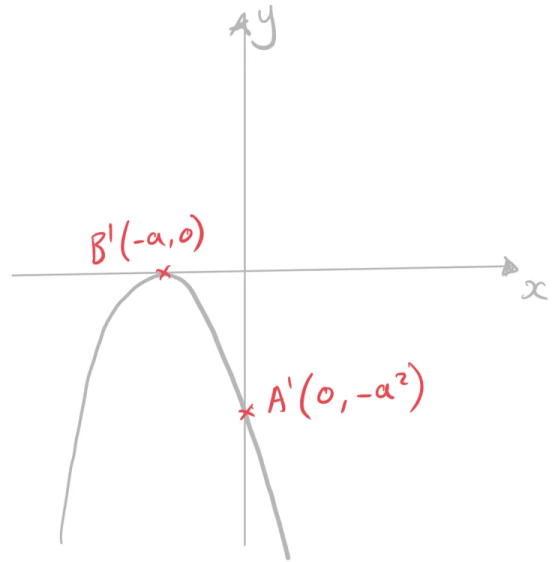
(b) Sketch the graph of $y = -f(-x)$, labelling the images of the points A and B and stating their coordinates in terms of a .

[3]

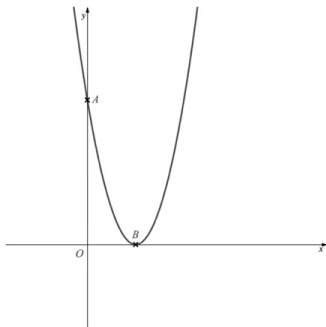
(c) Write down the value of a such that the point A is three times as far from the origin as the point B .

[1]

b) $f(-x)$ REFLECTING $-f(x)$ REFLECT IN X



The equation $y = f(x)$, where $f(x) = (x - a)^2$, with $a > 1$, is shown below.



The points A and B are the points where the graph intercepts the coordinate axes.

(a) Write down, in terms of a , the coordinates of A and B .

[2]

(b) Sketch the graph of $y = -f(-x)$, labelling the images of the points A and B and stating their coordinates in terms of a .

[3]

(c) Write down the value of a such that the point A is three times as far from the origin as the point B .

[1]

c) BOTH ON AXES SO HORIZONTAL AND VERTICAL DISTANCES FROM ORIGIN

$$A = 3B$$

$$(0, a^2) = (a, 0)$$

$$a^2 = 3a$$

$$a = 3$$

Question 11

The function $f(x)$ is to be transformed by a sequence of functions, in the order detailed below:

1. A horizontal stretch by scale factor 2
2. A reflection in the x -axis
3. A translation by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Write down an expression for the combined transformation in terms of $f(x)$.

[3]

1. HORIZONTAL = INSIDE BRACKETS
(SCALE FACTOR = $\frac{1}{a}$)

STRETCH = MULTIPLY

$$f\left(\frac{1}{2}x\right)$$

2. VERTICAL REFLECTION = OUTSIDE
(NEGATIVE)

$$-f\left(\frac{1}{2}x\right)$$

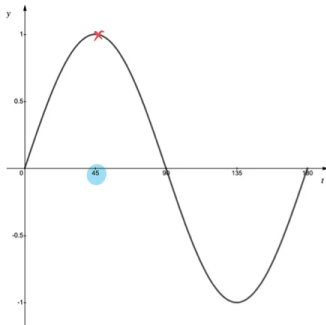
3. VERTICAL TRANSLATION = OUTSIDE

$$-f\left(\frac{1}{2}x\right) + 2$$

$$\boxed{2 - f\left(\frac{1}{2}x\right)}$$

Question 12

The diagram shows the graph of $y = f(t)$, where $f(t) = \sin 2t$, $0^\circ \leq t \leq 180^\circ$.



- (a) (i) Write down the maximum value of y when $y = 3f(t)$.
 (ii) Write down the first value of t for which this maximum occurs.

[2]

- (b) (i) Write down the minimum value of y when $y = 5f(t + 30^\circ)$.
 (ii) Write down the first value of t for which this minimum occurs.

[2]

- (c) Find, in terms of $f(t)$, the combination of transformations that would map the graph of $y = f(t)$ onto the graph of $y = 2 + \sin t$, $0^\circ \leq t \leq 180^\circ$.

[2]

a) i) VERTICAL STRETCH SF 3
 MAX OF $\sin 2t = 1$

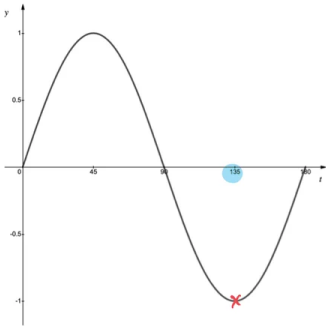
$$\text{MAX } 3(\sin 2t) = 3$$

$$\boxed{y = 3}$$

ii) VERTICAL STRETCH WON'T AFFECT
 HORIZONTAL VALUE

$$\boxed{t = 45^\circ}$$

The diagram shows the graph of $y = f(t)$, where $f(t) = \sin 2t$, $0^\circ \leq t \leq 180^\circ$.



- (a) (i) Write down the maximum value of y when $y = 3f(t)$.
 (ii) Write down the first value of t for which this maximum occurs.

[2]

- (b) (i) Write down the minimum value of y when $y = 5f(t + 30^\circ)$.
 (ii) Write down the first value of t for which this minimum occurs.

[2]

- (c) Find, in terms of $f(t)$, the combination of transformations that would map the graph of $y = f(t)$ onto the graph of $y = 2 + \sin t$, $0^\circ \leq t \leq 180^\circ$.

[2]

b) TRANSLATION $\begin{pmatrix} -30 \\ 0 \end{pmatrix}$ ONLY AFFECTS t
 VERTICAL STRETCH SFS ONLY AFFECTS y

i) MIN $\sin 2t = -1$
 MIN $5(\sin 2t + 30) = -5$

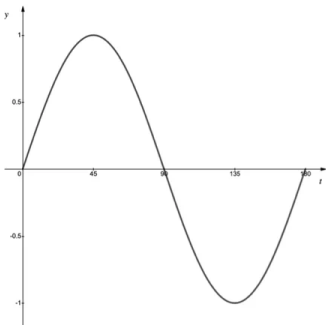
$$y = -5$$

ii) TRANSLATION $\begin{pmatrix} -30 \\ 0 \end{pmatrix}$

$$t = 135 - 30$$

$$t = 105^\circ$$

The diagram shows the graph of $y = f(t)$, where $f(t) = \sin 2t$, $0^\circ \leq t \leq 180^\circ$.



- (a) (i) Write down the maximum value of y when $y = 3f(t)$.
 (ii) Write down the first value of t for which this maximum occurs.

[2]

- (b) (i) Write down the minimum value of y when $y = 5f(t + 30^\circ)$.
 (ii) Write down the first value of t for which this minimum occurs.

[2]

- (c) Find, in terms of $f(t)$, the combination of transformations that would map the graph of $y = f(t)$ onto the graph of $y = 2 + \sin t$, $0^\circ \leq t \leq 180^\circ$.

[2]

c) $f(t) = \sin 2t$

$$f\left(\frac{1}{2}t\right) = \sin t$$

$$f\left(\frac{1}{2}t\right) + 2 = 2 + \sin t$$

$$f\left(\frac{1}{2}t\right) + 2$$

Question 13

Let $f(x) = 3x^2 + 18x + 27$.

(a) Write down the value of $f(-3)$.

The function f can be written in the form of $f(x) = a(x-h)^2 + k$.

(b) Find the values of a , h and k .

The graph of g is obtained from the graph of f by a reflection in the x -axis followed by a translation by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(c) Find $g(x)$, giving your answer in the form of $g(x) = rx^2 + sx + t$.

a) SUB IN $x = -3$

$$3(-3)^2 + 18(-3) + 27$$

$$27 - 54 + 27 = 0$$

$$f(-3) = 0$$

[3]

[4]

Let $f(x) = 3x^2 + 18x + 27$.

(a) Write down the value of $f(-3)$.

The function f can be written in the form of $f(x) = a(x-h)^2 + k$.

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(c) Find $g(x)$, giving your answer in the form of $g(x) = rx^2 + sx + t$.

b) FACTORISE 3 FROM x TERMS

$$f(x) = 3(x^2 + 6x) + 27$$

COMPLETE THE SQUARE ON $x^2 + 6x$

$$f(x) = 3[(x+3)^2 - 3^2] + 27$$

$$f(x) = 3[(x+3)^2 - 9] + 27$$

EXPAND AND SIMPLIFY INTO FORM $a(x-h)^2 + k$

$$f(x) = 3(x+3)^2 - 27 + 27$$

$$f(x) = 3(x+3)^2$$

$$a = 3 \quad h = -3 \quad k = 0$$

[1]

[3]

[4]

ALTERNATE METHOD EXPAND $a(x-h)^2 + k$

$$a(x-h)^2 + k = ax^2 - 2ahx + ah^2 + k$$

COMPARE COEFFICIENTS TO $3x^2 + 18x + 27$

$$ax^2 = 3x^2 \Rightarrow a = 3$$

$$-2ahx = 18x \Rightarrow -2(3)h = 18 \Rightarrow h = -3$$

$$ah^2 + k = 27 \Rightarrow 3(-3)^2 + k = 27 \Rightarrow k = 0$$

Let $f(x) = 3x^2 + 18x + 27$.

(a) Write down the value of $f(-3)$.

The function f can be written in the form of $f(x) = a(x-h)^2 + k$.

(b) Find the values of a, h and k .

The graph of g is obtained from the graph of f by a reflection in the x -axis followed by a translation by the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(c) Find $g(x)$, giving your answer in the form of $g(x) = rx^2 + sx + t$.

c) REFLECTION OF $f(x)$ IN x -AXIS : $-f(x)$

[1]

TRANSLATION BY $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$: $-f(x) + 1$

$$g(x) = -f(x) + 1$$

[3]

$$g(x) = -[3x^2 + 18x + 27] + 1$$

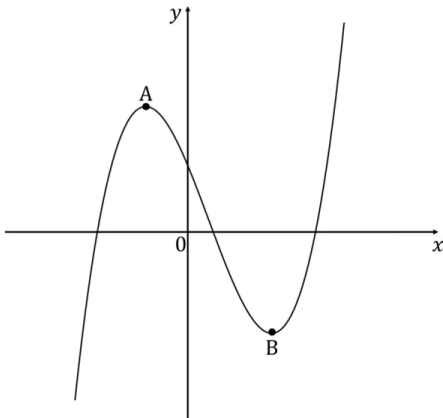
$$= -3x^2 - 18x - 27 + 1$$

[4]

$$g(x) = -3x^2 - 18x - 26$$

Question 14

The graph of f is shown below. The points $A(-2, 10)$ and $B(4, -10)$ lie on the curve.



(a) Sketch the graph of:

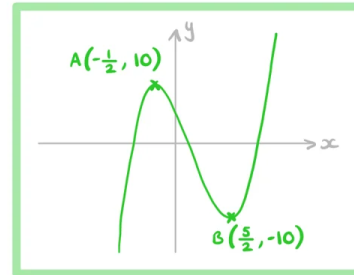
(i) $y = f(2x - 1)$,

(ii) $y = f(4 - x)$.

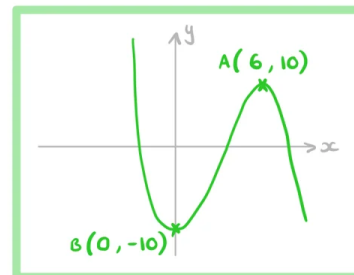
Clearly indicate the new coordinates of the images of the points A and B.

[7]

(a) (i) Translation $(1, 0)$ followed by horizontal stretch of factor $\frac{1}{2}$



(ii) Translation of $(-3, 0)$ followed by reflection in the y -axis



Question 15

Describe a sequence of transformations that map the graph of $y = \ln x$ onto the graph of $y = 5 + \ln\left(\frac{1}{2}x + 4\right)$.

[4]

$$\ln x \rightarrow 5 + \ln\left(\frac{1}{2}x + 4\right)$$

$$f(x) \rightarrow f\left(\frac{1}{2}x + 4\right) + 5$$

Translation of $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$ followed by horizontal stretch of factor 2

OR

Horizontal stretch of factor 2 followed by translation $\begin{pmatrix} -8 \\ 5 \end{pmatrix}$

Question 16

The function f is defined by

$$f(x) = \begin{cases} ax + 1 & \text{if } x \leq 7, \\ x^2 - 2x + 1 & \text{if } x > 7. \end{cases}$$

(a) Find the value of a such that the graph of f is continuous at $x = 7$.

[3]

The graph of the function g is obtained by translating the graph of f by the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, followed by a reflection in the x -axis.

(b) Find $g(x)$.

[4]

(a) $f(7)$ must have the same value for both parts of the piecewise function if it is to be continuous at $x=7$

$$a(7) + 1 = (7)^2 - 2(7) + 1$$

$$7a = 35$$

$$a = 5$$

The function f is defined by

$$f(x) = \begin{cases} ax + 1 & \text{if } x \leq 7, \\ x^2 - 2x + 1 & \text{if } x > 7. \end{cases}$$

(a) Find the value of a such that the graph of f is continuous at $x = 7$.

$$a = 5$$

[3]

The graph of the function g is obtained by translating the graph of f by the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, followed by a reflection in the x -axis.

(b) Find $g(x)$.

[4]

(b) Translation of $f(x)$ by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$f(x) \rightarrow f(x-1) - 1 \quad \text{Now 8 because of the translation}$$

$$f(x-1) - 1 = \begin{cases} 5(x-1) + 1 - 1 & x \leq 8 \\ (x-1)^2 - 2(x-1) + 1 - 1 & x > 8 \end{cases}$$

$$f(x-1) - 1 = \begin{cases} 5x - 5 & x \leq 8 \\ x^2 - 4x + 3 & x > 8 \end{cases}$$

Reflection in the x -axis

$$f(x-1) - 1 \rightarrow -(f(x-1) - 1)$$

$$g(x) = \begin{cases} -(5x - 5) & x \leq 8 \\ -(x^2 - 4x + 3) & x > 8 \end{cases}$$

$$g(x) = \begin{cases} 5 - 5x & x \leq 8 \\ -x^2 + 4x - 3 & x > 8 \end{cases}$$

Question 17

Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{x+1}{x-2}$.

Explain fully, the transformations of the graph of f to obtain the graph of g .

[5]

Rewrite $g(x)$ so that x appears just in the denominator

$$x + 1 = (x - 2) + 3$$

$$\begin{aligned} \Rightarrow \frac{x+1}{x-2} &= \frac{(x-2) + 3}{x-2} \\ &= \frac{\cancel{x-2} + 3}{\cancel{x-2}} \\ &= 1 + \frac{3}{x-2} \end{aligned}$$

$$\Rightarrow f(x) \rightarrow g(x)$$

$$\frac{1}{x} \rightarrow 1 + \frac{3}{x-2}$$

Translation of $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, then a stretch with SF=3 in y direction followed by a translation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$