## Question 1

Solve the following simultaneous equations.

(a)

$$5x - 3y = 19$$
$$2x + y = 1$$

[2]

(b)

$$a - 11b = 23$$
$$5a + 5b = -5$$

[2]

(c)

$$\frac{5}{4}m - \frac{3}{2}n = -\frac{9}{8}$$
$$\frac{1}{2}m + \frac{5}{3}n = \frac{11}{36}$$

[3]

(a) Multiply equation 2 by 3 and add to equation 1

$$5x - 3y = 19$$
$$6x + 3y = 3$$
$$11x = 22$$
$$\boxed{x = 2}$$

Substitute $x = 2$ into either equation to find $y$

$$2(2) + y = 1$$
$$y = 1 - 4$$
$$\boxed{y = -3}$$

Solve the following simultaneous equations.

(a)

$$5x - 3y = 19$$
$$2x + y = 1$$

[2]

(b)

$$a - 11b = 23$$
$$5a + 5b = -5$$

[2]

(c)

$$\frac{5}{4}m - \frac{3}{2}n = -\frac{9}{8}$$
$$\frac{1}{2}m + \frac{5}{3}n = \frac{11}{36}$$

[3]

(b) Rearrange equation 1 and substitute into 2

$$a - 11b = 23 \rightarrow a = 23 + 11b$$
$$5(23 + 11b) + 5b = -5$$
$$23 + 11b + b = -1$$
$$12b = -24$$
$$\boxed{b = -2}$$

Substitute $b = -2$ into rearranged equation to find $a$

$$a = 23 + 11(-2)$$
$$\boxed{a = 1}$$

Solve the following simultaneous equations.

(a)

$$5x - 3y = 19$$
$$2x + y = 1$$

[2]

(b)

$$a - 11b = 23$$
$$5a + 5b = -5$$

[2]

(c)

$$\frac{5}{4}m - \frac{3}{2}n = -\frac{9}{8}$$
$$\frac{1}{2}m + \frac{5}{3}n = \frac{11}{36}$$

[3]

(c) Rearrange equation 1 and substitute into equation 2

$$\frac{5}{4}m - \frac{3}{2}n = -\frac{9}{8} \rightarrow m = \frac{4}{5}\left(\frac{3}{2}n - \frac{9}{8}\right)$$

$$m = \frac{6}{5}n - \frac{9}{10}$$

$$\frac{1}{2}\left(\frac{6}{5}n - \frac{9}{10}\right) + \frac{5}{3}n = \frac{11}{36}$$

$$\frac{3}{5}n - \frac{9}{20} + \frac{5}{3}n = \frac{11}{36}$$

$$\frac{34}{15}n = \frac{\cancel{544}^{34}}{\cancel{720}_{45}}$$

$$n = \frac{\cancel{34}^{1}}{\cancel{45}_{3}} \times \frac{\cancel{15}^{1}}{\cancel{34}_{1}}$$

$$\boxed{n = \frac{1}{3}}$$

Substitute $n = \frac{1}{3}$ into rearranged equation to find $m$

$$m = \frac{6}{5}\left(\frac{1}{3}\right) - \frac{9}{10}$$

$$\boxed{m = -\frac{1}{2}}$$

## Question 2

Use the method of substitution to solve the following systems of linear equations.

(i)

$$\begin{aligned} ① \quad & x - y - z = 0 \\ ② \quad & 2x + y - 3z = 5 \\ ③ \quad & 2x - 3y + 4z = 4 \end{aligned}$$

(ii)

$$\begin{aligned} ⓐ \quad & 2x - y - 3z = 3 \\ ⓑ \quad & 3x + 2y - 2z = 12 \\ ⓒ \quad & 2x + y + 2z = -7 \end{aligned}$$

[8]

(i) Re-arrange equation ① to make $x$ the subject

$$x = y + z$$

Substitute into equation ②...

$$2(y+z) + y - 3z = 5$$

$$2y + 2z + y - 3z = 5$$

$$3y - z = 5 \quad \Rightarrow \quad z = 3y - 5$$

Re-arrange to make $z$ the subject

... and into equation ③...

$$2(y+z) - 3y + 4z = 4$$

$$2y + 2z - 3y + 4z = 4$$

$$6z - y = 4$$

Solve the results of those substitutions simultaneously by substituting the first into the second

$$6(3y - 5) - y = 4$$

$$18y - 30 - y = 4$$

$$17y = 34$$

$$\boxed{y = 2}$$

Substitute $y = 2$ into the other equations to find $x$ and $z$

$$6z - (2) = 4 \qquad\qquad x = (2) + (1)$$

$$\boxed{z = 1} \qquad\qquad \boxed{x = 3}$$

(ii) Re-arrange equation ⓐ to make $y$ the subject

$$y = 2x - 3z - 3$$

Substitute into equation ⓑ...

$$3x + 2(2x - 3z - 3) - 2z = 12$$

$$3x + 4x - 6z - 6 - 2z = 12$$

Re-arrange to make $z$ the subject

$$7x - 8z - 6 = 12 \Rightarrow z = \frac{7x - 18}{8}$$

... and into equation ⓒ...

$$2x + (2x - 3z - 3) + 2z = -7$$

$$4x - z - 3 = -7$$

Solve the results of those substitutions simultaneously by substituting the first into the second

$$4x - \left(\frac{7x - 18}{8}\right) - 3 = -7$$

$$32x - 7x + 18 - 24 = -56$$

$$25x = -50$$

$$\boxed{x = -2}$$

Substitute $x = -2$ into the other equations to find $y$ and $z$

$$z = \frac{7(-2) - 18}{8} \qquad y = 2(-2) - 3(-4) - 3$$

$$\boxed{z = -4} \qquad \boxed{y = 5}$$

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## Question 3

A festival charges $\$x$ USD for an adult ticket, $\$y$ USD for a child ticket and $\$z$ USD for a car parking pass.

Given that 4 adult tickets, 7 child tickets and 2 car passes cost \$540 USD, 2 adult tickets, 2 child tickets and 1 car pass cost \$210 USD and 7 adult tickets and 3 car passes cost \$450 USD,

(i) set up a system of linear equations in three unknowns,

(ii) find the values of $x$, $y$, and $z$.

[6]

(i)

$$\boxed{\begin{aligned} 4x + 7y + 2z &= 540 \\ 2x + 2y + z &= 210 \\ 7x + 3z &= 450 \end{aligned}}$$

(ii) Re-arrange the third equation to make $z$ the subject

$$z = \frac{450 - 7x}{3}$$

Substitute it into the first equation...

$$4x + 7y + 2\left(\frac{450-7x}{3}\right) = 540$$

$$12x + 21y + 900 - 14x = 1620$$

$$21y - 2x = 720$$

... and into the second equation

$$2x + 2y + \left(\frac{450-7x}{3}\right) = 210$$

$$6x + 6y + 450 - 7x = 630$$

Re-arrange to make $x$ the subject

$$6y - x = 180 \Rightarrow x = 6y - 180$$

Solve the results of those substitutions simultaneously by substituting the second into the first

$$21y - 2(6y - 180) = 720$$

$$21y - 12y + 360 = 720$$

$$9y = 360$$

$$\boxed{y = 40}$$

Substitute $y = 40$ into the other equations to find $x$ and $z$

$$x = 6(40) - 180$$

$$\boxed{x = 60}$$

$$z = \frac{450 - 7(60)}{3}$$

$$\boxed{z = 10}$$

## Question 4

Solve the following system of linear equations.

$$\begin{aligned} 3x + 2y - z &= 1 \\ x - y + 5z &= -2 \\ 2x + y &= 3 \end{aligned}$$

[6]

Use the Gaussian elimination method (row reduction)

$5R_1 + R_2 \rightarrow R_2$

$$\left\{\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 1 & -1 & 5 & -2 \\ 2 & 1 & 0 & 3 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 16 & 9 & 0 & 3 \\ 2 & 1 & 0 & 3 \end{array}\right.$$

$2R_3 - R_1 \rightarrow R_1$

$$\left\{\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 16 & 9 & 0 & 3 \\ 2 & 1 & 0 & 3 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 16 & 9 & 0 & 3 \\ 2 & 1 & 0 & 3 \end{array}\right.$$

$R_2 - 9R_3 \rightarrow R_2$

$$\left\{\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 16 & 9 & 0 & 3 \\ 2 & 1 & 0 & 3 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ -2 & 0 & 0 & -24 \\ 2 & 1 & 0 & 3 \end{array}\right.$$

$-\frac{1}{2}R_2 \rightarrow R_2$

$$\left\{\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ -2 & 0 & 0 & -24 \\ 2 & 1 & 0 & 3 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 1 & 0 & 0 & 12 \\ 2 & 1 & 0 & 3 \end{array}\right.$$

$R_1 - R_2 \rightarrow R_1$

$$\left\{\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 1 & 0 & 0 & 12 \\ 2 & 1 & 0 & 3 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 0 & 0 & 1 & -7 \\ 1 & 0 & 0 & 12 \\ 2 & 1 & 0 & 3 \end{array}\right.$$

$R_3 - 2R_2 \rightarrow R_3$

$$\left\{\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 1 & 0 & 0 & 12 \\ 2 & 1 & 0 & 3 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 0 & 0 & 1 & -7 \\ 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -21 \end{array}\right.$$

$$\boxed{x = 12 \quad y = -21 \quad z = -7}$$

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## Question 5

Solve the following the system of linear equations.

$$\begin{aligned} 2x + 2y - 3z &= -8 \\ 3x + 2y - z &= 0 \\ x - y + z &= 11 \end{aligned}$$

[6]

Use the Gaussian elimination method (row reduction)

$R_2 - R_1 \to R_2$

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -8 \\ 3 & 2 & -1 & 0 \\ 1 & -1 & 1 & 11 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & 2 & -3 & -8 \\ 1 & 0 & 2 & 8 \\ 1 & -1 & 1 & 11 \end{array}\right)$$

$2R_3 + R_1 \to R_1$

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -8 \\ 1 & 0 & 2 & 8 \\ 1 & -1 & 1 & 11 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 4 & 0 & -1 & 14 \\ 1 & 0 & 2 & 8 \\ 1 & -1 & 1 & 11 \end{array}\right)$$

$2R_1 + R_2 \to R_2$

$$\left(\begin{array}{ccc|c} 4 & 0 & -1 & 14 \\ 1 & 0 & 2 & 8 \\ 1 & -1 & 1 & 11 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 4 & 0 & -1 & 14 \\ 9 & 0 & 0 & 36 \\ 1 & -1 & 1 & 11 \end{array}\right)$$

$\frac{1}{9}R_2 \to R_2$

$$\left(\begin{array}{ccc|c} 4 & 0 & -1 & 14 \\ 9 & 0 & 0 & 36 \\ 1 & -1 & 1 & 11 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 4 & 0 & -1 & 14 \\ 1 & 0 & 0 & 4 \\ 1 & -1 & 1 & 11 \end{array}\right)$$

$R_1 + R_3 \to R_3$

$$\left(\begin{array}{ccc|c} 4 & 0 & -1 & 14 \\ 1 & 0 & 0 & 4 \\ 1 & -1 & 1 & 11 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 4 & 0 & -1 & 14 \\ 1 & 0 & 0 & 4 \\ 5 & -1 & 0 & 25 \end{array}\right)$$

$R_3 - 5R_2 \to R_3$

$$\left(\begin{array}{ccc|c} 4 & 0 & -1 & 14 \\ 1 & 0 & 0 & 4 \\ 5 & -1 & 0 & 25 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 4 & 0 & -1 & 14 \\ 1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 5 \end{array}\right)$$

$R_1 - 4R_2 \to R_1$

$$\left(\begin{array}{ccc|c} 4 & 0 & -1 & 14 \\ 1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & -1 & -2 \\ 1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 5 \end{array}\right)$$

$-R_1 \to R_1, \quad -R_3 \to R_3$

$$\left(\begin{array}{ccc|c} 0 & 0 & -1 & -2 \\ 1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \end{array}\right)$$

$\boxed{x = 4 \quad y = -5 \quad z = 2}$

## Question 6

Consider the system of equations

$$-6a + (k-3)b = 1$$
$$3ka - 5b = 4$$

(a) Find the values of the real parameter $k$ such that the system has a unique solution. [4]

(b) Find the unique solution in terms of $k$. [4]

(a) Eliminate $a$:

Multiply equation 1 by $k$ and equation 2 by 2

$$-6ka + k(k-3)b = k$$
$$6ka - 10b = 8 \quad +$$

$$\big(k(k-3) - 10\big)b = k+8$$
$$(k^2 - 3k - 10)b = k+8$$

If there is to be a unique solution, then the coefficient of $b$ cannot be 0

$$k^2 - 3k - 10 = (k-5)(k+2) \neq 0$$

$$\boxed{k \neq 5 \quad k \neq -2}$$

Consider the system of equations

$$-6a + (k-3)b = 1$$
$$3ka - 5b = 4$$

(a) Find the values of the real parameter $k$ such that the system has a unique solution. [4]

(b) Find the unique solution in terms of $k$. [4]

$$\boxed{(k^2 - 3k - 10)b = k+8}$$

(b) Express $b$ in terms of $k$

$$b = \frac{k+8}{k^2 - 3k - 10}$$

Substitute value for $b$ back into one of the equations to find $a$ in terms of $k$

$$3ka - 5\left(\frac{k+8}{k^2-3k-10}\right) = 4$$

$$3ka(k^2 - 3k - 10) - 5(k+8) = 4(k^2 - 3k - 10)$$

$$3ka = \frac{4k^2 - 12k - 40 + 5k + 40}{k^2 - 3k - 10}$$

$$3ka = \frac{4k^2 - 7k}{k^2 - 3k - 10}$$

$$a = \frac{4k^{\cancel{2}} - 7\cancel{k}}{3\cancel{k}(k^2 - 3k - 10)}$$

$$\boxed{\text{Unique solution: } \left(\frac{k+8}{k^2-3k-10}, \frac{4k-7}{3(k^2-3k-10)}\right)}$$

## Question 7

Solve the following system of equations using row operations.

$$\begin{aligned} 3x + 9y - 3z &= 45 \\ 6x + 3y + 3z &= 21 \\ 3x - 3y - 6z &= 0 \end{aligned}$$

[6]

$\frac{1}{3}R_1 \to R_1$ , $\frac{1}{3}R_2 \to R_2$ , $\frac{1}{3}R_3 \to R_3$

$$\left\{\begin{array}{ccc|c} 3 & 9 & -3 & 45 \\ 6 & 3 & 3 & 21 \\ 3 & -3 & -6 & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -2 & 0 \end{array}\right.$$

$R_1 + R_2 \to R_1$

$$\left\{\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -2 & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 3 & 4 & 0 & 22 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -2 & 0 \end{array}\right.$$

$2R_2 + R_3 \to R_3$

$$\left\{\begin{array}{ccc|c} 3 & 4 & 0 & 22 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -2 & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 3 & 4 & 0 & 22 \\ 2 & 1 & 1 & 7 \\ 5 & 1 & 0 & 14 \end{array}\right.$$

$-4R_3 + R_1 \to R_1$

$$\left\{\begin{array}{ccc|c} 3 & 4 & 0 & 22 \\ 2 & 1 & 1 & 7 \\ 5 & 1 & 0 & 14 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} -17 & 0 & 0 & -34 \\ 2 & 1 & 1 & 7 \\ 5 & 1 & 0 & 14 \end{array}\right.$$

$-\frac{1}{17}R_1 \to R_1$

$$\left\{\begin{array}{ccc|c} -17 & 0 & 0 & -34 \\ 2 & 1 & 1 & 7 \\ 5 & 1 & 0 & 14 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 7 \\ 5 & 1 & 0 & 14 \end{array}\right.$$

$R_3 - 5R_1 \to R_3$

$$\left\{\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 7 \\ 5 & 1 & 0 & 14 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 7 \\ 0 & 1 & 0 & 4 \end{array}\right.$$

$R_2 - 2R_1 \to R_2$

$$\left\{\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 7 \\ 0 & 1 & 0 & 4 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 4 \end{array}\right.$$

$R_2 - R_3 \to R_2$

$$\left\{\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 4 \end{array}\right. \Rightarrow \left\{\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 4 \end{array}\right.$$

$$\boxed{x = 2 \quad y = 4 \quad z = -1}$$

## Question 8

Consider the following system of equations

$$\begin{aligned} 2x + y - 3z &= -4 \\ x - y + 2z &= 2 \\ 4x + 2y - 6z &= k \end{aligned}$$

where $k \in \mathbb{R}$.

Show that the system has no unique solution for any value of $k$.

[6]

$2R_2 - R_1 \rightarrow R_2$

$$\left(\begin{array}{ccc|c} 2 & 1 & -3 & -4 \\ 1 & -1 & 2 & 2 \\ 4 & 2 & -6 & k \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & 1 & -3 & -4 \\ 0 & -3 & 7 & 8 \\ 4 & 2 & -6 & k \end{array}\right)$$

$R_3 - 2R_1 \rightarrow R_3$

$$\left(\begin{array}{ccc|c} 2 & 1 & -3 & -4 \\ 0 & -3 & 7 & 8 \\ 4 & 2 & -6 & k \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & 1 & -3 & -4 \\ 0 & -3 & 7 & 8 \\ 0 & 0 & 0 & k+8 \end{array}\right)$$

If $k \neq -8$, then the system is inconsistent and there are no solutions.

If $k = -8$, then $z$ is a free variable and the system either has 0 or infinite solutions.