OCR
Oxford Cambridge and RSA

Thursday 22 May 2025 – Afternoon

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Time allowed: 2 hours 40 minutes

1869096913

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **144**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

Section A (33 marks)

1 The complex number z satisfies the equation $z + 2\mathrm{i}z^* + 1 - 4\mathrm{i} = 0$.

You are given that $z = x + \mathrm{i}y$, where x and y are real numbers.

Determine the values of x and y. **[4]**

2 **In this question you must show detailed reasoning.**

Find the acute angle between the planes $2x - y + 2z = 5$ and $x + 2y + z = 8$. **[4]**

3 Using standard summation formulae, show that, for integers $n \geqslant 1$,

$$1 \times 3 + 2 \times 4 + \ldots + n \times (n+2) = \frac{1}{6}n(n+1)(an+b),$$

where a and b are integers to be determined. **[5]**

4 **(a)** You are given that **M** and **N** are non-singular 2×2 matrices.

Write down the product rule for the inverse matrices of **M**, **N** and **MN**. **[1]**

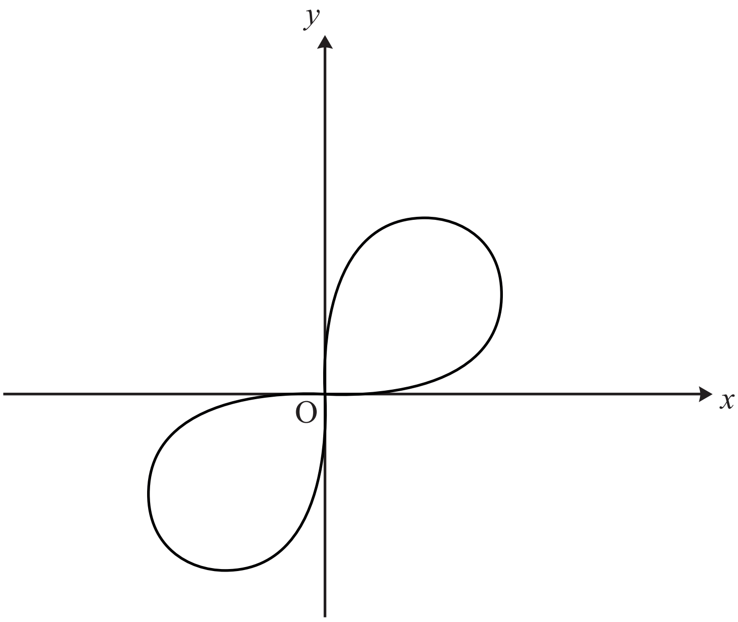
(b) Verify this rule for the matrices **M** and **N**, where

$\mathbf{M} = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix}$ and a and b are non-zero constants. **[6]**

5 The cubic equation $2x^3 - 3x + 4 = 0$ has roots α, β and γ.

Determine a cubic equation with integer coefficients whose roots are $\frac{1}{2}(\alpha+1)$, $\frac{1}{2}(\beta+1)$ and $\frac{1}{2}(\gamma+1)$. **[4]**

6 The figure below shows the curve with cartesian equation $(x^2 + y^2)^2 = xy$.



(a) Show that the polar equation of the curve is $r^2 = a\sin b\theta$, where a and b are positive constants to be determined. **[3]**

(b) Determine the exact maximum value of r. **[2]**

(c) Determine the area enclosed by **one** of the loops. **[4]**

Section B (111 marks)

7 In this question you must show detailed reasoning.

By first expressing $\frac{1}{x^2-4}$ in partial fractions, show that $\int_3^\infty \frac{1}{x^2-4}\,dx = \frac{1}{m}\ln n$, where m and n are integers to be determined. **[8]**

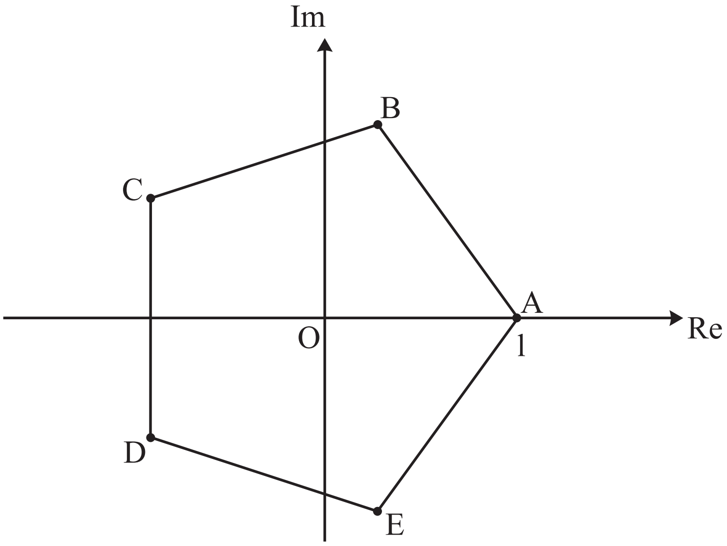
8 The function $f(x)$ is defined as $f(x) = \ln(1+x)$, for $x > -1$.

(a) Prove by mathematical induction that the nth derivative of $f(x)$, $f^{(n)}(x)$, for all $n \geqslant 1$, is given by $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$. **[4]**

(b) Hence prove that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... + \frac{(-1)^{n+1}x^n}{n} + ...$ for $-1 < x \leqslant 1$.

[You are not required to show this series for $\ln(1+x)$ converges for $-1 < x \leqslant 1$.] **[3]**

9 The figure below shows an Argand diagram with a **regular** pentagon ABCDE. The point A represents the real number 1. The point B represents the complex number w.



(a) (i) Write down, in terms of w, the complex numbers represented by the points C, D and E. **[1]**

(ii) Write down an equation whose roots are the complex numbers represented by the points A, B, C, D and E. **[1]**

(iii) Show that the sum of these roots is zero. **[2]**

(b) (i) Find w. Give your answer in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $\theta = k\pi$, where k is a positive constant to be found. **[1]**

(ii) By considering the line segment AB, show that the length of each side of the pentagon is $2\sin\frac{\pi}{5}$. **[5]**

10 **In this question you must show detailed reasoning.**

Evaluate $\int_0^{\frac{1}{2}} \frac{2}{x^2 - x + 1}\,\mathrm{d}x$. Give your answer in exact form. **[4]**

11 The lines l_1 and l_2 have equations

$$l_1: \frac{x-3}{a} = \frac{y+1}{b} = \frac{z-2}{1} \qquad l_2: \mathbf{r} = 2\mathbf{i}+\mathbf{j}+3\mathbf{k}+\lambda(-\mathbf{i}+c\mathbf{j}+2\mathbf{k})$$

where a, b and c are constants.

(a) In the case where $c = 0$ and l_1 and l_2 intersect at right angles, determine the coordinates of the point of intersection of the two lines. **[6]**

(b) Now consider the case where $c = 4$ and lines l_1 and l_2 are parallel.

(i) Find the values of a and b. **[3]**

(ii) Determine the distance between the two lines. **[5]**

12 **In this question you must show detailed reasoning.**

The roots of the equation $z^4 - z^3 + cz^2 + dz + 18 = 0$ are α, $\frac{2}{\alpha}$, β and $-\beta$.

Determine, in any order, the exact values of the following.

- The **four** roots of the equation
- The value of c
- The value of d **[8]**

13 The gradient of a curve $y = \mathrm{f}(x)$ satisfies the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 2 + x^2$.

(a) Show that the integrating factor for this differential equation is x^{-2}. **[3]**

(b) You are given that the curve passes through the point (1, 0).

By solving this differential equation, determine the exact x-coordinate of the stationary point on the curve $y = \mathrm{f}(x)$. **[8]**

14 **(a)** By using the definition of $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}, show that $\cosh^2 x + \sinh^2 x \equiv \cosh 2x$. **[2]**

(b) The transformation T of the plane has associated matrix **M**, where $\mathbf{M} = \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix}$ and $x > 0$.

Show that T transforms the unit square with coordinates (0, 0), (1, 0), (0, 1) and (1, 1) to a rhombus of unit area. **[6]**

(c) You are given that the length of each side of the rhombus is 2 units.

Determine the exact value of x. Give your answer in logarithmic form. **[2]**

15 **(a)** Show that $(3 - e^{4i\theta})(3 - e^{-4i\theta}) = a + b\cos 4\theta$, where a and b are integers to be determined. **[2]**

The infinite series C and S are defined as follows.

$$C = \cos\theta + \frac{1}{3}\cos 5\theta + \frac{1}{9}\cos 9\theta + \frac{1}{27}\cos 13\theta + \dots$$

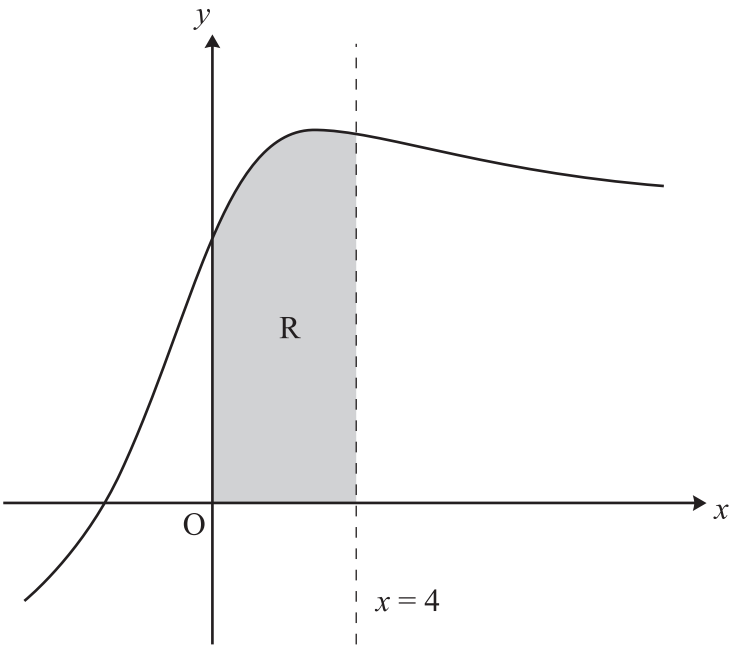
$$S = \sin\theta + \frac{1}{3}\sin 5\theta + \frac{1}{9}\sin 9\theta + \frac{1}{27}\sin 13\theta + \dots$$

(b) Show that $C + iS = \dfrac{3e^{i\theta}}{3 - e^{4i\theta}}$. **[4]**

(c) Hence show that $C = \dfrac{9\cos\theta - 3\cos 3\theta}{10 - 6\cos 4\theta}$. **[3]**

16 In this question you must show detailed reasoning.

The diagram shows the curve with equation $y = \dfrac{x+3}{\sqrt{x^2+9}}$.



The region R, shown shaded in the diagram, is bounded by the curve, the x-axis, the y-axis, and the line $x = 4$.

(a) Determine the area of R. Give your answer in the form $p + \ln q$ where p and q are integers to be determined. **[6]**

The region R is rotated through 2π radians about the x-axis.

(b) Determine the volume of the solid of revolution formed. Give your answer in the form $\pi\left(a + b\ln\left(\frac{c}{d}\right)\right)$ where a, b, c and d are integers to be determined. **[6]**

17 A researcher is modelling the height of a particular type of tree over its lifetime.

Data suggests that the maximum possible height of this type of tree over its lifetime is double the height of the tree 5 years after planting.

It is given that, t years after planting a seed for this type of tree, the corresponding height of the tree is h m, and that $h = 0$ when $t = 0$.

(a) The researcher first models the height of the tree by assuming that the rate of increase of h is proportional to $(20-h)$, with constant of proportionality 0.2.

(i) Write down the first order differential equation for this model. **[1]**

(ii) Show that this model predicts that the maximum possible height of the tree is 20 m. **[1]**

(iii) Show by integration that $h = 20(1-e^{-0.2t})$. **[4]**

(iv) Determine whether this model's prediction for the height of the tree 5 years after planting is consistent with the maximum possible height of the tree being 20 m. **[2]**

(b) The researcher refines the model for the height of the tree using the following second order differential equation.

$$\frac{d^2h}{dt^2} + 0.3\frac{dh}{dt} + 0.02h = 0.4$$

(i) Determine the general solution of this second order differential equation. **[4]**

(ii) Show that the refined model also predicts that the maximum possible height of the tree is 20 m. **[1]**

Further research determines that the initial rate of growth of this type of tree is 2.9 metres per year.

(iii) By applying the initial conditions to find the particular solution of this differential equation, determine whether the refined model's prediction for the height of the tree 5 years after planting is consistent with the maximum possible height of the tree being 20 m. **[5]**

END OF QUESTION PAPER

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