IB Maths: AI HL

Past Paper 3

Question Paper

These practice questions can be used by students and teachers and is Suitable for IB Maths AI HL Past Papers

Course	IB Maths
Section	Set A
Topic	Past Paper 3
Difficulty	Medium

Level: IB Maths

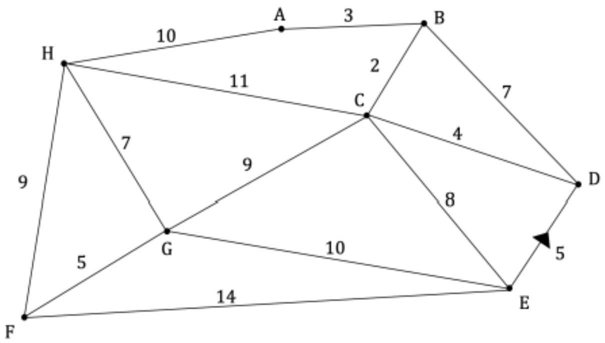
Subject: IB Maths AI HL

Board: IB Maths

Topic: Past Paper 3

Question 1

Jilly wants to encourage her university to compost the organic waste that is produced at various buildings on campus, and so she decides to conduct a feasibility study. Jilly starts by identifying all of the buildings on campus that produce organic waste and how they are connected by road. She constructs the graph *U* seen below, where the vertices represent the different university buildings that produce organic waste and the edges represent the time, in minutes, it takes to drive between the different buildings via the connecting roads. Due to construction work, the road connecting buildings D and E is temporarily one-way in the direction indicated.



a)
Write down a Hamiltonian cycle in *U*.

[1 mark]

A table for graph *U* showing the minimum travel times between each pair of buildings is given below:

	A	B	C	D	E	F	G	H
A		3	5	p	13	19	14	10
B	3		2	6	10	16	11	13
C	5	2		4	8	14	9	11
D	9	6	4		q	18	13	15
E	13	10	8	r		14	10	17
F	19	16	14	18	14		5	9
G	14	11	9	13	10	5		7
H	10	13	11	15	17	9	7	

b)
Find the values of

(i)
p

(ii)
q

(iii)
r.

[3 marks]

Jilly decides that the waste collecting route should start and finish at the same building.

She also wants to minimise the time taken to visit each building to collect the waste, and so decides to determine upper and lower bounds for the quickest waste collection route.

c)
Starting at vertex A and using the nearest neighbour algorithm, determine an upper bound on the minimum time required to visit each building.

[3 marks]

After the upper bound in part (c) has been calculated, but before any lower bound has been calculated, the construction work is finished and the one-way restriction between buildings D and E is removed.

d)
Explain how the removal of the one-way restriction will affect the upper bound found in part (c), given that the nearest neighbour algorithm is still used starting at vertex A.

[3 marks]

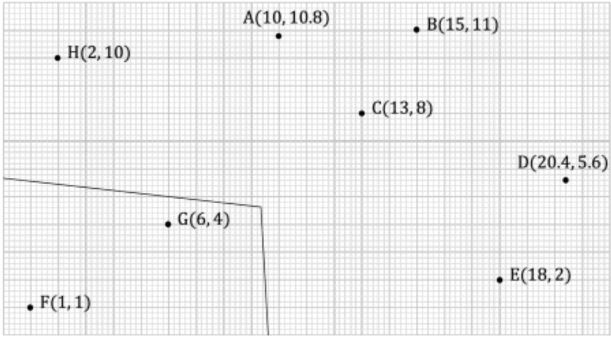
e)
(i)
By first deleting vertex A and its connecting edges, use Prim's algorithm starting at vertex B to find the weight of the minimum spanning tree of the remaining graph. You should indicate clearly the order in which the edges have been selected.

(ii)
Hence state a lower bound for the minimum time required to visit each building.

[6 marks]

Based on her estimates Jilly decides that the time taken to collect waste from all eight buildings will be too long to take place on a daily basis. She instead decides that waste will only be collected daily from the three buildings that produce the greatest volume of organic waste, buildings E, F and H. The organic waste from the each of the remaining buildings will need to be carried to the nearest of those three buildings to be collected.

The Voronoi diagram below shows the coordinates of each building on a grid relative to a fixed origin. It is partially completed with the perpendicular bisectors of HF and EF.



f)

(i)

Write down the equation of the missing perpendicular bisector. Give your answer in the form $y = mx + c$.

(ii)

Draw the perpendicular bisector on the diagram.

(iii)

Hence write down the collection points that buildings A, B, C, D and G should take their organic waste to.

[5 marks]

g)

Given that 5 minutes is allowed for loading the organic waste at each stop, find the time saved by collecting waste at points E, F and H only, rather than visiting all eight buildings. (You may disregard here the amount of time taken to carry the waste from the other five buildings to the collection points at buildings E, F and H.)

[4 marks]

Question 2

An infection is present in a population of 1000 people.

Let x be the number of susceptible people (those who have not caught the infection yet), let y be the number of infected people, and let z be the number of recovered people (who have had the infection but are now recovered and cannot catch it again). The time, t, in this question is to be measured in months.

You may assume that no one dies from the infection.

a)
Explain why, at any point in time,

$$x+y+z=1000$$

[1 mark]

The rate of change of **susceptible** people is modelled by the differential equation

$$\frac{dx}{dt}=bxy$$

where $b$is a positive constant.

b)
Explain what this differential equation predicts about the change in numbers of susceptible people over time.

[2 marks]

The rate at which the number of **recovered** people increases is equal to a constant (known as the recovery fraction) multiplied by the number of infected people.

c)
(i)
Use this information to write down a second differential equation.

(ii)
By first differentiating part (a) with respect to and rearranging, show that the following differential equation can be obtained from all the equations so far considered:

$$\frac{dy}{dt}=bxy-ky$$

[4 marks]

The constant b models the amount of contact an infected person has with uninfected people each month. A simplified model assumes that infected people are not allowed to be in contact with anyone (so that $b=0$), and that half of all infected people recover in the course of a month (so that $k=\frac{1}{2}$). This gives the simplified system

$$\frac{dx}{dt}=0 \qquad \frac{dy}{dt}=-\frac{1}{2}y \qquad \frac{dz}{dt}=\frac{1}{2}y$$

d)
Using the simplified model just given, and assuming that the population initially has 800 susceptible people, 200 infected people and 0 recovered people:

(i)
Explain why the number of susceptible people never changes.

(ii)
Use separation of variables to find the particular solution for the number of infected people at time t.

(iii)
Use your answer from part (d)(ii) to form a differential equation in z and t only, and hence find the particular solution for the number of recovered people at time t.

[7 marks]

It is determined that the model will be much more accurate if the amount of contact between infected and uninfected people is instead modelled by $b=0.001$, and if it is assumed that 70% of all infected people recover each month (so that $k=0.7$). Then the number of susceptible people, x, and the number of infected people, y, can be modelled by the system of coupled differential equations

$$\frac{dx}{dt}=-0.001xy$$

$$\frac{dy}{dt}=0.001xy-0.7y$$

e)
Identify the equilibrium points for this system of equations.

[3 marks]

The population initially has 800 susceptible people and 200 infected people. To see how these numbers change over time, Euler's method is used with a step size of 0.2, where is measured in months.

f)

(i)

Write down equations for x_{n+1}, y_{n+1} and t_{n+1} in terms of x_n, y_n and t_n.

(ii)

Use Euler's method to find an approximation for the numbers of susceptible, infected and recovered people after one month.

[4 marks]

g)

By considering Euler's method for the first **two** months, sketch the trajectory of the solution with the given initial conditions on x and y axes, over the ranges $500 < x < 800$ and $180 < y < 210$.

[3 marks]

h)

(i)

Use the chain rule to show that the system of two differential equations can be reduced to the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -1 + \frac{700}{x}$$

(ii)

Hence find the general solution in the form $y = f(x)$ for the set of all trajectory curves of the sort sketched in part (g).

(iii)

For the case where there are initially 800 susceptible people and 200 infected people, determine the number of people that the model predicts will never become infected.

[6 marks]