1. Number

CONTENTS

1. Number

1.1 ARITHMETIC

1.1.1 MULTIPLICATION (NON-CALC)

(Non-calculator) multiplication – why so many methods?

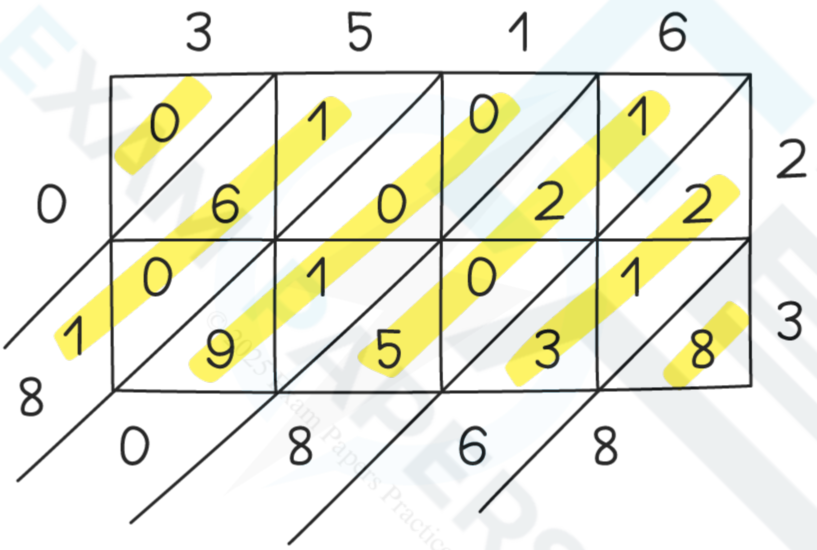
- Different methods work for different people, and some are better **depending** on the **size of number** you are dealing with
- We recommend the following **3 methods** depending on the size of number you are dealing with
  (If in doubt all methods will work for all numbers!)

1. Number

1. Lattice method

(Best for numbers with two or more digits)

- This method allows you to work with digits
- So in the number 3 516 you would only need to work with the digits 3, 5, 1 and 6
- So if you can multiply up to 9×9 you can’t go wrong!



So, 3516 × 23 = 80 868

1. Number

2. Partition method

(Best when one number has just one digit)

- This method keeps the value of the larger number intact
- So with 3 516 you would use 3000, 500, 10 and 6
- This method is **not suitable** for two larger numbers as you can end up with a lot of zero digits that are hard to keep track of

	3000	500	10	6
7	21 000	3500	70	42

$$\begin{array}{lr} & 21\,000 \\ & 3\,500 \\ & 70 \\ + & 42 \\ \hline & 24612 \end{array}$$

So, 3516 × 7 = 24 612

1. Number

3. Repeated addition method

(Best for smaller, simpler cases)

- You may have seen this called ‘chunking’
- It is a way of building up to the answer using simple multiplication facts that can be worked out easily

  eg. 13 × 23
  1 × 23 = 23
  2 × 23 = 46
  4 × 23 = 92
  8 × 23 =184

  So, 13 × 23 = 1 × 23 + 4 × 23 + 8 × 23 = 23 + 92 + 184 = 299

Decimals

- These 3 methods can easily be adapted for use with decimal numbers
- You ignore the decimal point whilst multiplying but put it back in the correct place in order to reach a final answer

  eg. 1.3 × 2.3
  Ignoring the decimals this is 13 × 23, which from above is 299
  There are two decimal places in total in the question, so there will be two decimal places in the answer

  So, 1.3 × 2.3 = 2.99

Exam Tip

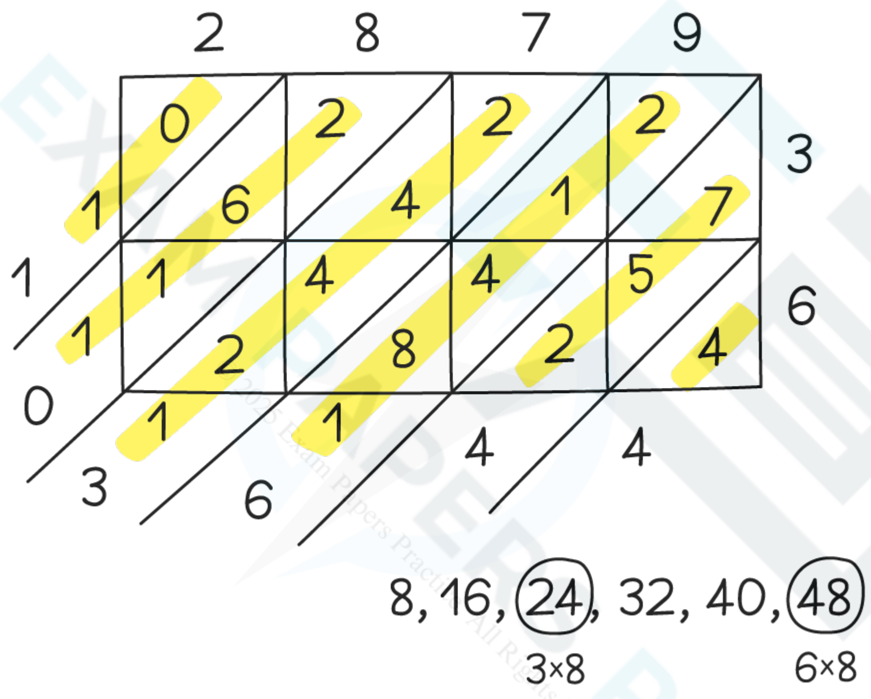
If you do forget your times tables then in the exam write a list out of the table you need as you do a question.
So for example, if you need to multiply by 8, and you’ve forgotten your 8 times tables, write it down: 8, 16, 24, 32, 40, 48, etc. as far as you need to.

1. Number

Worked Example

1. Multiply 2879 by 36

- As you have a 4-digit number multiplied by a 2-digit number then the lattice method (1) is the best choice
- Start with a 4×2 grid....



- Notice the use of listing the 8 times table at the bottom to help with any you may have forgotten
  2879 × 36 = 103 644
- Note that the method would still work if you had set it up as a 2×4 grid

1. Number

2. Pencils are sold in boxes. Each box costs £1.25 and each box contains 15 pencils.

Tyler buys 35 boxes of pencils.

(a) Work out how many pencils Tyler has in total.

(b) Work out the total cost for all the boxes Tyler buys.

(a)

This is a roundabout way of asking you to work out 15×35
As this is a simpl-ish case (3) you should use the repeated addition method

1 × 35 = 35
2 × 35 = 70
4 × 35 = 140
8 × 35 = 280
16 × 35 = 560

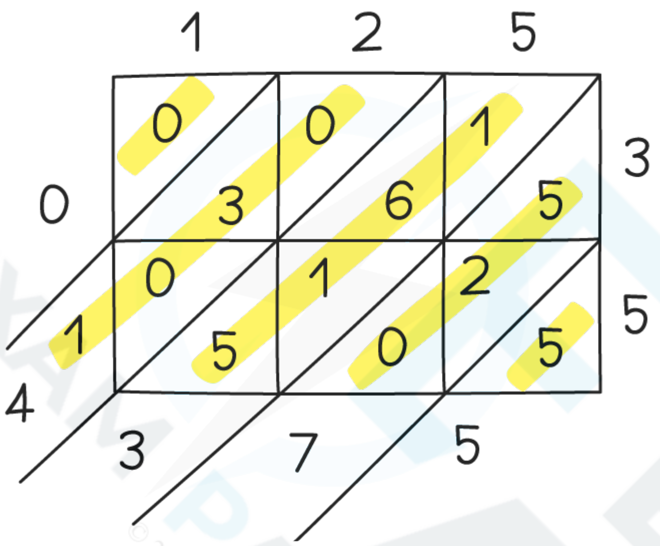
It doesn't matter if you go past 15 …

15 × 35 = 16 × 35 – 1 × 35 = 560 – 35
15 × 35 = 525

1. Number

(b)

This question is 1.25×35 so involves decimals (4)
Ignoring the decimals it becomes 125×35 and so the lattice method is best



125 × 35 = 4375

Now count the decimal places from the question and put the decimal point back in the correct place

£1.25 × 35 = £43.75

Exam Tip

Okay, getting the highlighter out during an exam may be a touch excessive! But do use your grid/diagram to help you answer the question – the highlighter in the example above makes it clear which digits to add up at each stage. You can do this in pen or pencil but do make sure you can still read the digits underneath as it is all part of your method/working.

1. Number

1.1.2 DIVISION (NON-CALC)

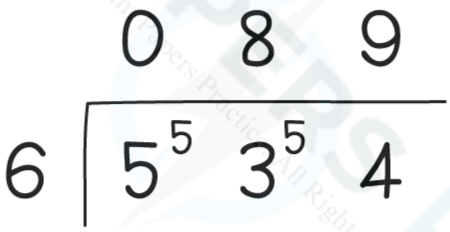
(Non-calculator) division – more methods

- Most students will have seen short division (bus stop method) and long division and there is often confusion between the two
- Fortunately, you only need one – so use short division
- While short division is best when dividing by a single digit, for bigger numbers you need a different approach
- You can use other areas of maths that you know to help – eg. cancelling fractions, “shortcuts” for dividing by 2 and 10, and the repeated addition (“chunking”) method covered in (Non-Calculator) Multiplication

1. Short division (bus stop method)

- Apart from where you can use shortcuts such as dividing by 2 or by 5, this method is best used when dividing by a single digit

eg. 534 ÷ 6



So, 534 ÷ 6 = 89

1. Number

2. Factoring & cancelling

- This involves treating division as you would if you were asked to cancel fractions
- You can use the fact that with division, most non-calculator questions will have only number answers
- The only thing to be aware of is that this might not be the case if you've been asked to write a fraction as a mixed number (but if you are asked to do that it should be obvious from the question)

  eg. 1008 ÷ 28

  1008 ÷ 28 = 504 ÷ 14 = 252 ÷ 7 = 36

- You may have spotted the first two values (1008 and 28) are both divisible by 4 which is fine but if not, divide top and bottom by any number you can
- To do the last part (252 ÷ 7) you can use the short division method above

3. Intelligent repeated addition

- This is virtually identical to the version for multiplication – the process stops when the number dividing into is reachedeg 1674 ÷ 27This is the same as saying ? × 27 = 1674So we can build up in "chunks" of 27 until we get to 16741 × 27 = 27
  10 × 27 = 270
  20 × 27 = 540
  40 × 27 = 1080
  60 × 27 = 1620 … by using the last two results added together.Now you are close we can add on 27 one at a time again.
  61 × 27 = 1647
  62 × 27 = 1674
  So 1674 ÷ 27 = 62

4. Dividing by 10, 100, 1000, … (Powers of 10)

- This is a case of moving digits (or decimal points) or knocking off zeros
  eg. 380 ÷ 10 = 38
  45 ÷ 100 = 0.45

5. Dividing by 2, 4, 8, 16, 32, … (Powers of 2)

- This time it is a matter of repeatedly halving
  eg. 280 ÷ 8 = 140 ÷ 4= 70 ÷ 2 = 35
  1504 ÷ 32 = 752 ÷ 16 = 376 ÷ 8 = 188 ÷ 4 = 94 ÷ 2 = 47

1. Number

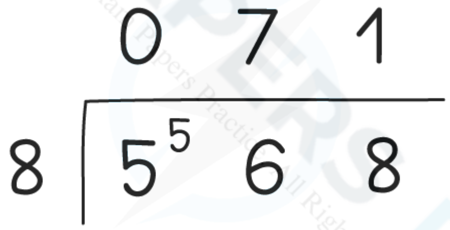
Exam Tip

On the non-calculator paper, division is very likely to have a whole number (exact) answer. So if, when using the repeated addition method, you do not reach this figure then it is likely you've made an error in your calculations somewhere.

Worked Example

1. After a fundraising event, the organiser wishes to split the £568 raised between 8 charities. How much will each charity get?

- This is division by a single digit so short division would be an appropriate method
- If you spot it though, 8 is also a power of 2 so you could just halve three times
- Method 1 – short division:



568 ÷ 8 = 71

- Method 2 – powers of 2:

568 ÷ 2 = 284
284 ÷ 2 = 142
142 ÷ 2 = 71
568 ÷ 8 = 71

568 ÷ 8 = 71

You know to halve three times since
2 × 2 × 2 = 8

1. Number

2. A robot packs tins of soup into boxes. Each box holds 24 cans of soup.
The robot has 1824 cans of soup to pack into boxes.
How many boxes will be produced by the robot?

1824 ÷ 24

Both numbers are large so intelligent repeated addition is the best approach

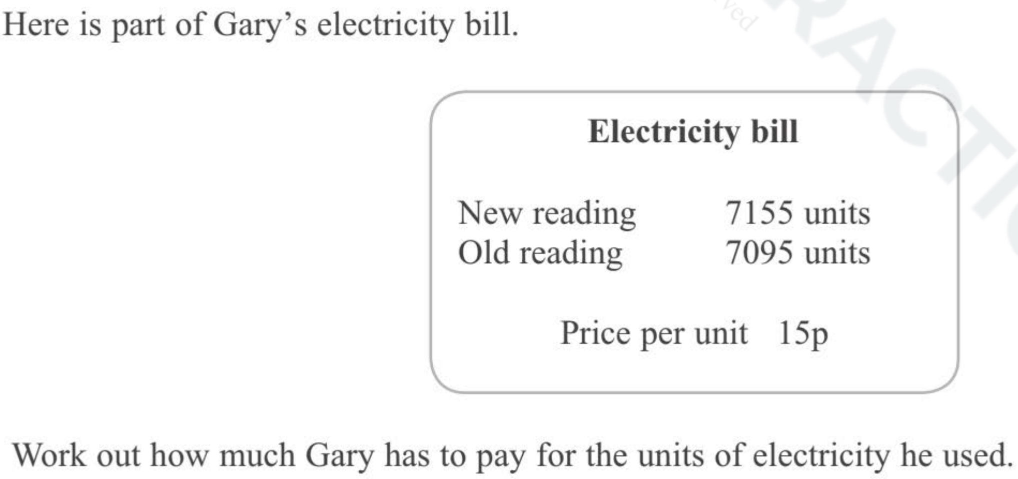
1 × 24 = 24
10 × 24 = 240
20 × 24 = 480
40 × 24 = 960
80 × 24 = 1920 … going too far doesn't matter as you can subtract …
79 × 24 = 1896
78 × 24 = 1872
77 × 24 = 1848
76 × 24 = 1824

- Although this may at first look like a trial and improvement method it is important to show logic throughout as you build the number up – that's why we call it INTELLIGENT repeated addition here at SME!

1824 ÷ 24 = 76

The robot will produce 76 boxes of cans of soup

Exam Question: Easy



Here is part of Gary's electricity bill.

Electricity bill

New reading	7155 units
Old reading	7095 units

Price per unit 15p

Work out how much Gary has to pay for the units of electricity he used.

1. Number

Exam Question: Medium

One sheet of paper is 9×10^{-3} cm thick.

Mark wants to put 500 sheets of paper into the paper tray of his printer.
The paper tray is 4 cm deep.

Is the paper tray deep enough for 500 sheets of paper?
You must explain your answer.

Exam Question: Hard

Each day a company posts some small letters and some large letters.

The company posts all the letters by first class post.

The tables show information about the cost of sending a small letter by first class post and the cost of sending a large letter by first class post.

Small Letter

Weight	First Class Post
0–100 g	60p

Large Letter

Weight	First Class Post
0–100 g	£1.00
101–250 g	£1.50
251–500 g	£1.70
501–750 g	£2.50

One day the company wants to post 200 letters.

The ratio of the number of small letters to the number of large letters is 3 : 2

70% of the large letters weigh 0–100 g.

The rest of the large letters weigh 101–250 g.

Work out the total cost of posting the 200 letters by first class post.

1. Number

1.2 FRACTIONS

1.2.1 MIXED NUMBERS & TOP HEAVY FRACTIONS

What are mixed numbers & top heavy fractions?

- A mixed number has a whole number (integer) part and a fraction
  eg. 3 3/4 means "three and three quarters"
- A top heavy fraction – also called an **improper fraction** – is one with the top (numerator) bigger than the bottom (denominator)
  eg. 15/4 means "fifteen quarters"

Turning mixed numbers into top heavy fractions

1. **Multiply** the big number by the bottom (denominator)
2. **Add** that to the top (numerator)
3. Write as **top heavy** fraction

Turning top heavy fractions into mixed numbers

- **Divide** the top by the bottom (to get a whole number and a remainder)
- The **whole number** is the big number
- The **remainder** goes over the bottom

Worked Example

1. *Write* $3\frac{3}{4}$ *as a top heavy fraction.*

$3 \times 4 = 12$ 1 – *multiply the big number by the bottom*

$12 + 3 = 15$ 2 – *add to the top*

$3\frac{3}{4} = \frac{15}{4}$ 3 – *your final answer should be top heavy*

2. *Write* $\frac{17}{5}$ *as a mixed number.*

$17 \div 5 = 3$ *remainder* 2 4 – *divide the top by the bottom*

$\frac{17}{5} = 3\frac{2}{5}$ 5 – *final answer is a mixed number*

1. Number

1.2.2 ADDING & SUBTRACTING FRACTIONS

Dealing with mixed numbers

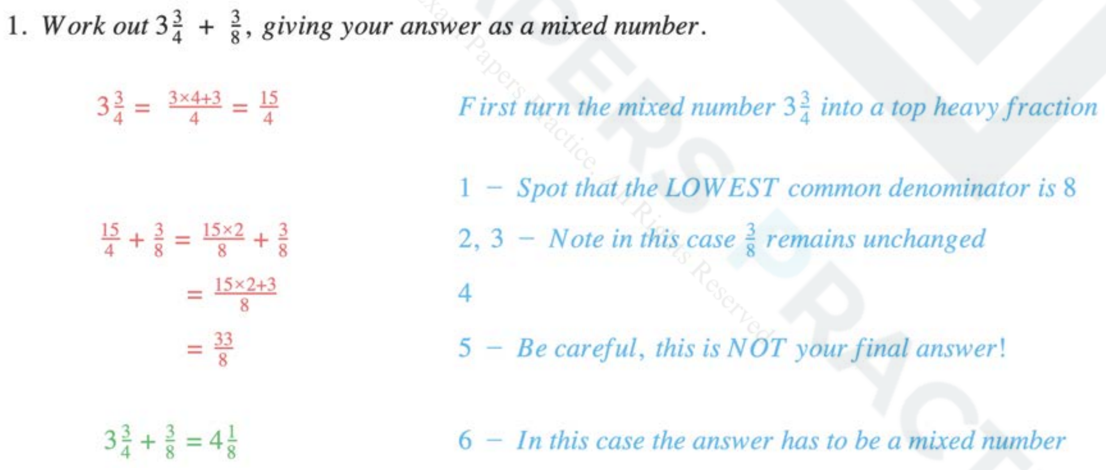
- Always turn Mixed Numbers into Top Heavy Fractions before doing calculations

Adding & Subtracting

- Adding and subtracting are treated in exactly the same way:

1. Find the **lowest** common bottom (denominator)
2. Write fractions with the **new** bottoms
3. **Multiply** tops by same as bottoms
4. Write as a **single** fraction (take care if subtracting)
5. **Simplify** the top
6. Turn Top Heavy Fractions back into Mixed Numbers (if necessary)

Worked Example



1. *Work out* $3\frac{3}{4} + \frac{3}{8}$, *giving your answer as a mixed number.*

$3\frac{3}{4} = \frac{3\times4+3}{4} = \frac{15}{4}$ — *First turn the mixed number* $3\frac{3}{4}$ *into a top heavy fraction*

1 – *Spot that the LOWEST common denominator is 8*

$\frac{15}{4} + \frac{3}{8} = \frac{15\times2}{8} + \frac{3}{8}$ — 2, 3 – *Note in this case* $\frac{3}{8}$ *remains unchanged*

$= \frac{15\times2+3}{8}$ — 4

$= \frac{33}{8}$ — 5 – *Be careful, this is NOT your final answer!*

$3\frac{3}{4} + \frac{3}{8} = 4\frac{1}{8}$ — 6 – *In this case the answer has to be a mixed number*

1. Number

1.2.3 MULTIPLYING & DIVIDING FRACTIONS

Dealing with mixed numbers

- Always turn Mixed Numbers into Top Heavy Fractions before doing calculations

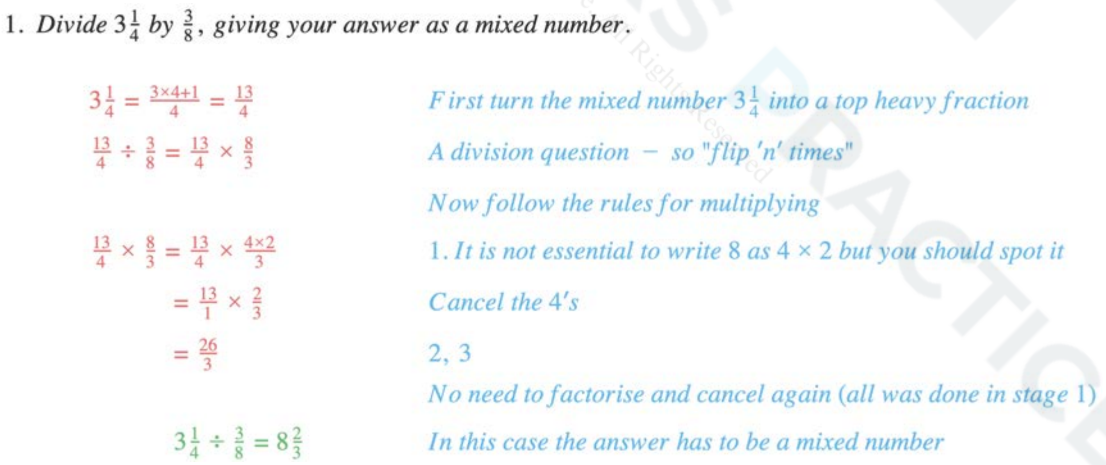
Dividing fractions

- Never try to divide fractions
- Instead “flip’n’times”
- So “**÷a/b**” becomes “**×b/a**”
- And follow the rules for multiplying…

Multiplying fractions

- **Simplify** by factorising and cancelling (ignore the × between the fractions)
- Multiply the **tops**
- Multiply the **bottoms**
- **Simplify** by factorising and cancelling (if you missed something earlier)
- Turn Top Heavy Fractions back into Mixed Numbers (if necessary)

Worked Example



1. *Divide* $3\frac{1}{4}$ *by* $\frac{3}{8}$*, giving your answer as a mixed number.*

$3\frac{1}{4} = \frac{3\times4+1}{4} = \frac{13}{4}$	*First turn the mixed number* $3\frac{1}{4}$ *into a top heavy fraction*
$\frac{13}{4} \div \frac{3}{8} = \frac{13}{4} \times \frac{8}{3}$	*A division question – so "flip 'n' times"*
	Now follow the rules for multiplying
$\frac{13}{4} \times \frac{8}{3} = \frac{13}{4} \times \frac{4\times2}{3}$	1. *It is not essential to write 8 as 4 × 2 but you should spot it*
$= \frac{13}{1} \times \frac{2}{3}$	*Cancel the 4's*
$= \frac{26}{3}$	2, 3
	No need to factorise and cancel again (all was done in stage 1)
$3\frac{1}{4} \div \frac{3}{8} = 8\frac{2}{3}$	*In this case the answer has to be a mixed number*

1. Number

Exam Question: Easy

(a) Work out $\frac{1}{7} \times \frac{2}{3}$

(b) Work out $\frac{3}{5} - \frac{1}{3}$

Exam Question: Medium

(a) Work out $1\frac{1}{5} \times 2\frac{1}{3}$

Give your answer as a mixed number in its simplest form.

(b) Work out $2\frac{7}{15} - 1\frac{2}{3}$

1. Number

Exam Question: Hard

(a) Work out $2\frac{1}{4} \times 3\frac{1}{3}$

Give your answer as a mixed number in its simplest form.

(b) Write the numbers 3, 4, 5 and 6 in the boxes to give the greatest possible total.
You may write each number only once.

$$\square\frac{1}{\square} + \square\frac{2}{\square}$$

1. Number

1.3 BASIC PERCENTAGES

1.3.1 BASIC PERCENTAGES

What is a percentage?

- “Per-cent” simply means “ ÷ 100” (or “out of 100”)
- You can think of a percentage as a standardised way of expressing a fraction – by always expressing is “out of 100”
- That means it is a useful way of comparing fractions.
- Eg. ½ = 50% (50% means 50 ÷ 100)
  ⅖ = 40%
  ¾ = 75%

Things to remember:

- Decimal equivalent = percentage ÷ 100
- Percentage = decimal equivalent × 100
- To find “a percentage of A”: multiply by the decimal equivalent
- To find “A as a percentage of B”: do A ÷ B to get decimal equivalent

Exam Tip

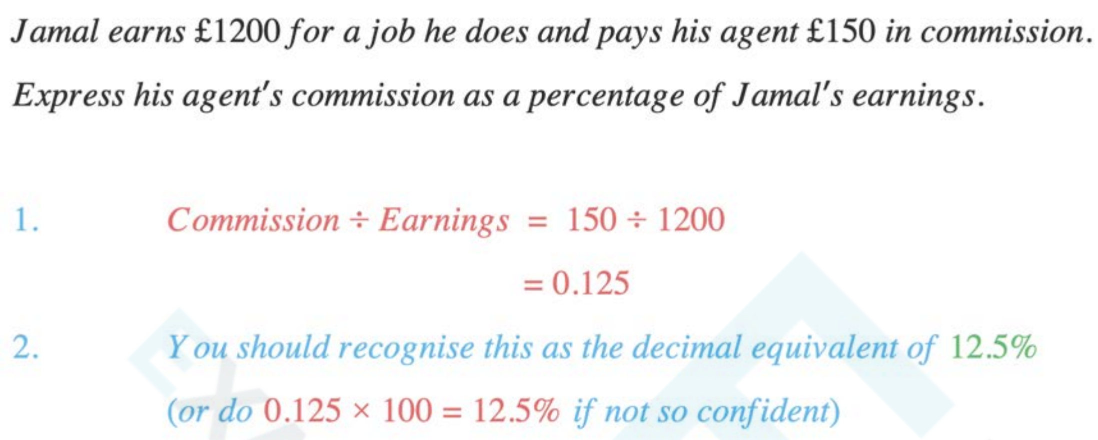
You can always use the decimal equivalent instead of doing a more traditional percentage calculation:

For example, to find 35% of 80 you can just do 80 × 0.35 = 28

(rather than doing the more complicated calulation 80 × 35 ÷ 100)

1. Number

Worked Example



Jamal earns £1200 *for a job he does and pays his agent* £150 *in commission.*

Express his agent's commission as a percentage of Jamal's earnings.

1. $\text{Commission} \div \text{Earnings} = 150 \div 1200$

   $= 0.125$

2. *You should recognise this as the decimal equivalent of* 12.5%

   (*or do* $0.125 \times 100 = 12.5\%$ *if not so confident*)

1. Number

1.3.2 PERCENTAGE INCREASES & DECREASES

How to increase or decrease by a percentage

- Identify “before” & “after” quantities
- Find percentage of “before” that we want:
    - Increase – add percentage to 100
    - Decrease – subtract percentage from 100
- Write down a statement connecting “before” and “after”:
    - “after is a percentage of before”
- Write down the statement as an equation using decimal equivalent
    - remember “is” means “=”
- Substitute and solve

Worked Example

Jennie earns £1200 per week in her job.

She is to receive a 5% pay rise.

Find her new weekly pay.

We can call the "before" Old Pay and the "after" New Pay

1. $\quad We\ want\ 100 + 5 = 105\%$

2. $\quad New\ Pay\ is\ 105\%\ of\ Old\ Pay$

$The\ decimal\ equivalent\ of\ 105\%\ is\ 1.05$

3. $\quad New\ Pay = 1.05 \times Old\ Pay$

4. $\quad New\ Pay = 1.05 \times 1200$

$New\ Pay = £1260\ per\ week$

1. Number

Exam Question: Easy

Bill's weight decreases from 64.8 kg to 59.3 kg.

Calculate the percentage decrease in Bill's weight.
Give your answer correct to 3 significant figures.

Exam Question: Medium

Railtickets and Cheaptrains are two websites selling train tickets.

Each of the websites adds a credit card charge and a booking fee to the ticket price.

Railtickets

Credit card charge: 2.25% of ticket price

Booking fee: 80 pence

Cheaptrains

Credit card charge: 1.5% of ticket price

Booking fee: £1.90

Nadia wants to buy a train ticket.
The ticket price is £60 on each website.
Nadia will pay by credit card.

Will it be cheaper for Nadia to buy the train ticket from Railtickets or from Cheaptrains?

1. Number

1.4 REVERSE PERCENTAGES

1.4.1 REVERSE PERCENTAGES

What is compound interest?

- Compound interest is where interest is paid on the interest from the year (or whatever time frame is being used) before as well as on the original amount
- This is different from **simple interest** where interest is only paid on the original amount

How do you work with compound interest?

- For COMPOUND changes (can be a decrease as well as an increase):
  - Keep multiplying by the decimal equivalent of the percentage you want
- Otherwise do the same as normal:
- Identify “before” & “after” quantities
- FIND percentage we want:
  - Increase – ADD percentage to 100
  - Decrease – SUBTRACT from 100
- Write down a STATEMENT connecting “before” and “after”:
  - “after is a percentage of before”
- Write down the statement as an EQUATION using decimal equivalent
  - remember “is” means “=”
- SUBSTITUTE and SOLVE

Exam Tip

This method works for any Compound Change – increase or decrease.

Remembering “$\times m \times m \times m = \times m^{3}$” can make life a lot quicker:

It is usually much easier to multiply by decimal equivalent raised to a power than to multiply by the decimal equivalent several times in a row.

1. Number

Worked Example

$Jasmina\ invests\ £1200\ in\ a\ Savings\ Account\ which\ pays\ Compound\ Interest\ at\ the\ rate\ of\ 2\%\ per\ year\ for\ 7\ years.$

$To\ the\ nearest\ pound,\ what\ is\ her\ investment\ worth\ at\ the\ end\ of\ the\ 7\ years?$

$We\ can\ call\ the\ "before"\ Investment\ and\ the\ "after"\ Final\ Value$

1. $We\ want\ 100 + 2 = 102\%\ EACH\ YEAR$

   $The\ decimal\ equivalent\ of\ 102\%\ is\ 1.02\ and\ we\ want\ to\ apply\ it\ 7\ times\ so\ our\ multiplier\ is\ 1.02^7$

2. $Final\ Value\ is\ 102\%\ (applied\ for\ 7\ years)\ of\ Investment$

3. $Final\ Value\ =\ 1.02^7 \times Investment$

4. $Final\ Value\ =\ 1.02^7 \times 1200\ =\ 1378.4228...$

   $Final\ Value\ =\ £1378\ (to\ the\ nearest\ £)$

What is a reverse percentage?

- A reverse percentage question is one where we are given **the value after a percentage increase or decrease** and asked to find the value before the change

How to do reverse percentage questions

- You should do these in exactly the same way as percentage increase & decrease questions!
- Identify “before” & “after” quantities
- FIND percentage we want:
  - Increase – ADD percentage to 100
  - Decrease – SUBTRACT from 100
- Write down a STATEMENT connecting “before” and “after”:
  - “after is a percentage of before”
- Write down the statement as an EQUATION using decimal equivalent
  - remember “is” means “=”
- SUBSTITUTE and SOLVE

1. Number

Worked Example

Jennie now earns £31500 per year in her job.

She has recently had a 5% pay rise.

Find her annual pay before the pay rise.

We can call the "before" Old Pay and the "after" New Pay

1. We want $100 + 5 = 105\%$

2. New Pay is 105% of Old Pay

   The decimal equivalent of 105% is 1.05

3. $\text{New Pay} = 1.05 \times \text{Old Pay}$

4. $31500 = 1.05 \times \text{Old Pay}$

Divide by 1.05 : $\text{Old Pay} = 31500 \div 1.05$

$\text{Old Pay} = £30000$ per year

Exam Question: Medium

The normal price of a television is reduced by 30% in a sale.

The sale price of the television is £350

Work out the normal price of the television.

1. Number

Exam Question: Hard

In a sale normal prices are reduced by 20%.

A washing machine has a sale price of £464

By how much money is the normal price of the washing machine reduced?

1. Number

1.6 LCM / HCF / PRIME FACTORS

1.6.1 PRIME FACTORS

What are prime factors?

- Factors are things that are multiplied together
- Prime numbers are numbers which can only be divided by themselves and 1
- The prime factors of a number are therefore all the prime numbers which multiply to give that number
- You should remember the first few prime numbers:
  2, 3, 5, 7, 11, 13, 17, 19, …

How to find prime factors

- Use a FACTOR TREE to find prime factors
- Write the prime factors IN ASCENDING ORDER with × between
- Write with POWERS if asked

Language

This is one of those topics where questions can use different phrases that all mean the same thing …

- Express … as the product of prime factors
- Find the prime factor decomposition of …
- Find the prime factorisation of …

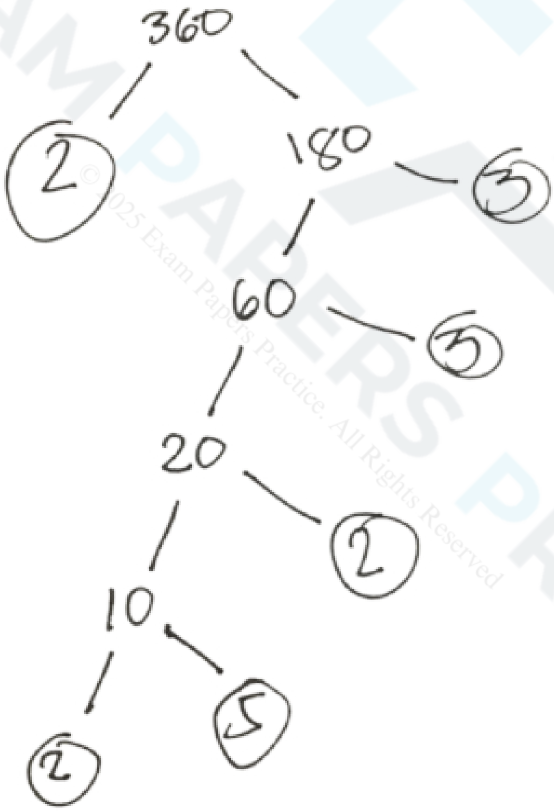
1. Number

Worked Example

1. *Find the prime factors of* 360.

Give your answer in the form $2^p \times 3^q \times 5^r$ *where* p, q *and* r *are integers to be found.*

1 − *For each number find any two numbers which are factors (not 1 ...) and write those as the next pair of numbers in the tree. If a number is prime, put a circle round it. When all the end numbers are circled you're done!*



$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$

2 − *Be careful to get them all, especially those repeated ones*

$360 = 2^3 \times 3^2 \times 5^1$

3 − *Write using powers as the question asks for this*

The 1 as a power of 5 isn't really necessary but the question asked for it...

(*So* $p = 3$, $q = 2$, $r = 1$)

1. Number

1.6.2 HCFS & LCMS

What are HCFs & LCMs?

- HCF is Highest Common Factor
- This is the biggest number which is a factor of (divides into) two numbers
- LCM is Lowest Common Multiple
- This is the smallest number which two numbers divide into (are factors of)
- You should remember the first few Prime Numbers:
  2, 3, 5, 7, 11, 13, 17, 19, ...

How to find HCFs & LCMs

1. Find PRIME FACTORS
2. Create a VENN DIAGRAM
3. $\cap$ is HIGHEST COMMON FACTOR (Overlap)
4. $\cup$ is LOWEST COMMON MULTIPLE (Union)

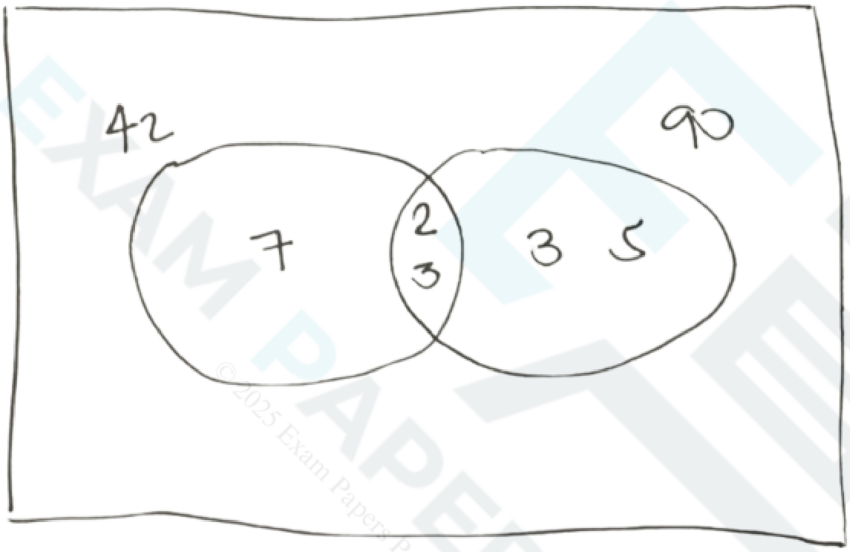
1. Number

Worked Example

1. $Find\ the\ HCF\ and\ LCM\ of\ 42\ and\ 90.$

$42 = 2 \times 3 \times 7$
$90 = 2 \times 3 \times 3 \times 5$

$1 - $ *See the separate notes for finding Prime Factors*



$2 - $ *Box is not essential for our purposes here*

$HCF = 2 \times 3$

$HCF = 6$

$3 - \cap,$ *intersection/overlap*

$LCM = 7 \times 2 \times 3 \times 3 \times 5$

$LCM = 630$

$4 - \cup,$ *union/'or'*

1. Number

Exam Question: Easy

Buses to Acton leave a bus station every 24 minutes.
Buses to Barton leave the same bus station every 20 minutes.

A bus to Acton and a bus to Barton both leave the bus station at 9 00 am.

When will a bus to Acton and a bus to Barton next leave the bus station at the same time?

Exam Question: Medium

Matt and Dan cycle around a cycle track.

Each lap Matt cycles takes him 50 seconds.
Each lap Dan cycles takes him 80 seconds.

Dan and Matt start cycling at the same time at the start line.

Work out how many laps they will each have cycled when they are next at the start line together.

1. Number

1.7 ROOTS & INDICES

1.7.1 ROOTS & INDICES - BASICS

What are indices?

- An Index (plural = indices) is just a power that a number (called the base) is raised to:

Laws of indices – what you need to know

- There are lots of very important laws (or rules)
- It is important that you know and can apply these
- Understanding the explanations will help you remember them:

1. Number

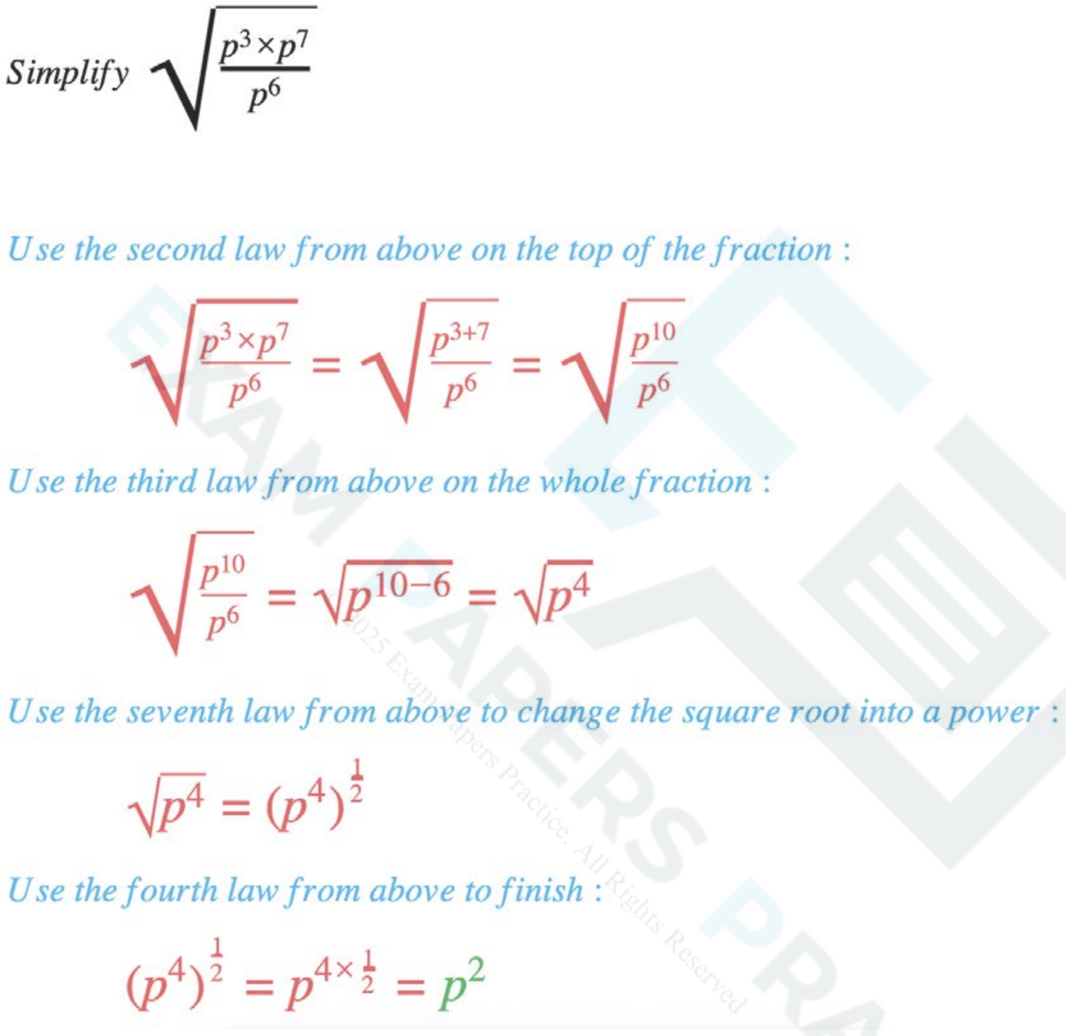
Laws	Explanations
$a^1 = a$	
$a^p \times a^q = a^{p+q}$	$a^3 \times a^2$ $= (a \times a \times a) \times (a \times a)$ $= a^5$
$a^p \div a^q = a^{p-q}$	$a^5 \div a^3$ $= \frac{a \times a \times a \times a \times a}{a \times a \times a}$ $= a^2$
$(a^p)^q = a^{p \times q}$	$(a^3)^2$ $= (a \times a \times a) \times (a \times a \times a)$ $= a^6$
$a^0 = 1$	$a^0 = a^{2-2} = a^2 \div a^2 = \frac{a^2}{a^2} = 1$
$a^{-p} = \frac{1}{a^p}$	$a^{-3} = a^{0-3} = a^0 \div a^3 = \frac{a^0}{a^3} = \frac{1}{a^3}$
$a^{\frac{1}{n}} = \sqrt[n]{a}$	$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1 = a = \sqrt{a} \times \sqrt{a}$
$a^{\frac{p}{q}} = (\sqrt[q]{a})^p = \sqrt[q]{(a)^p}$	$a^{\frac{3}{2}} = a^{\frac{1}{2} \times 3} = (a^{\frac{1}{2}})^3 = (\sqrt{a})^3$ $a^{\frac{3}{2}} = a^{3 \times \frac{1}{2}} = (a^3)^{\frac{1}{2}} = \sqrt{(a)^3}$

Exam Tip

Take it slowly and apply the laws one at a time.

1. Number

Worked Example



Simplify $\sqrt{\frac{p^3 \times p^7}{p^6}}$

Use the second law from above on the top of the fraction :

$$\sqrt{\frac{p^3 \times p^7}{p^6}} = \sqrt{\frac{p^{3+7}}{p^6}} = \sqrt{\frac{p^{10}}{p^6}}$$

Use the third law from above on the whole fraction :

$$\sqrt{\frac{p^{10}}{p^6}} = \sqrt{p^{10-6}} = \sqrt{p^4}$$

Use the seventh law from above to change the square root into a power :

$$\sqrt{p^4} = (p^4)^{\frac{1}{2}}$$

Use the fourth law from above to finish :

$$(p^4)^{\frac{1}{2}} = p^{4\times\frac{1}{2}} = p^2$$

1. Number

1.7.2 ROOTS & INDICES - HARDER

What are indices?

- An Index (plural = indices) is just a power that a number (called the base) is raised to:

Laws of indices – what you need to know

- There are lots of very important laws (or rules)
- It is important that you know and can apply these
- Understanding the explanations will help you remember them:

1. Number

Laws	Explanations
$a^1 = a$	
$a^p \times a^q = a^{p+q}$	$a^3 \times a^2$ $= (a \times a \times a) \times (a \times a)$ $= a^5$
$a^p \div a^q = a^{p-q}$	$a^5 \div a^3$ $= \frac{a \times a \times a \times a \times a}{a \times a \times a}$ $= a^2$
$(a^p)^q = a^{p \times q}$	$(a^3)^2$ $= (a \times a \times a) \times (a \times a \times a)$ $= a^6$
$a^0 = 1$	$a^0 = a^{2-2} = a^2 \div a^2 = \frac{a^2}{a^2} = 1$
$a^{-p} = \frac{1}{a^p}$	$a^{-3} = a^{0-3} = a^0 \div a^3 = \frac{a^0}{a^3} = \frac{1}{a^3}$
$a^{\frac{1}{n}} = \sqrt[n]{a}$	$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1 = a = \sqrt{a} \times \sqrt{a}$
$a^{\frac{p}{q}} = (\sqrt[q]{a})^p = \sqrt[q]{(a)^p}$	$a^{\frac{3}{2}} = a^{\frac{1}{2} \times 3} = (a^{\frac{1}{2}})^3 = (\sqrt{a})^3$ $a^{\frac{3}{2}} = a^{3 \times \frac{1}{2}} = (a^3)^{\frac{1}{2}} = \sqrt{(a)^3}$

Exam Tip

Write numbers in the question with the same BASE if possible.

1. Number

Worked Example

Express $\sqrt{\frac{8^2\times2^7}{4^3}}$ *as a single power of* 2.

Express 8 *as* 2^3 *and* 4 *as* 2^2 :

$$\sqrt{\frac{8^2\times2^7}{4^3}}=\sqrt{\frac{(2^3)^2\times2^7}{(2^2)^3}}$$

Use the fourth law from above to simplify :

$$\sqrt{\frac{(2^3)^2\times2^7}{(2^2)^3}}=\sqrt{\frac{2^{3\times2}\times2^7}{2^{2\times3}}}=\sqrt{\frac{2^6\times2^7}{2^6}}$$

Use the second law from above on the top of the fraction :

$$\sqrt{\frac{2^6\times2^7}{2^6}}=\sqrt{\frac{2^{6+7}}{2^6}}=\sqrt{\frac{2^{13}}{2^6}}$$

Use the third law from above on the whole fraction :

$$\sqrt{\frac{2^{13}}{2^6}}=\sqrt{2^{13-6}}=\sqrt{2^7}$$

Use the seventh law from above to change the square root into a power :

$$\sqrt{2^7}=\left(2^7\right)^{\frac{1}{2}}$$

Use the fourth law (again) from above to finish :

$$\left(2^7\right)^{\frac{1}{2}}=2^{7\times\frac{1}{2}}=2^{\frac{7}{2}}$$

1. Number

Exam Question: Easy

Simplify $(m^{-2})^5$

Exam Question: Medium

Simplify $(9x^8y^3)^{\frac{1}{2}}$

Exam Question: Hard

(a) Find the value of 2^{-3}

$5\sqrt{5}$ can be written in the form 5^k

(b) Find the value of k.

1. Number

1.8 ROUNDING & ESTIMATION

1.8.1 ROUNDING & ESTIMATION

Why use estimation?

- We **estimate** to find approximations for difficult sums
- Or to check our answers are about the right size (right order of magnitude)

How to estimate

- We **round** numbers to something sensible before **calculating**
- GENERAL RULE:
  Round numbers to 1 significant figure
    - 7.8 → 8
    - 18 → 20
    - 3.65 × 10-4 → 4 × 10-4
    - 1080 → 1000
- EXCEPTIONS:
  It can be more sensible (or easier) to round to something convenient
    - 16.2 → 15
    - 9.1 → 10
    - 1180 → 1200

  It wouldn't usually make sense to round a number to zero
- FRACTIONS get bigger when the top is bigger and/or the bottom is smaller and vice versa

1. Number

Worked Example

1. *Calculate an estimate for* $\frac{15.9\times3.87}{18.7}$.

3.87 → 4 — *1 − Round both to 1 significant figure*

18.7 → 20

15.9 → 15 — *2 − An exception as 15.9 is quite a long way from*

$\frac{15.9\times3.87}{18.7} \approx \frac{15\times4}{20}$ — *Now use the rounded figures to find an approximation*

$\approx \frac{60}{20}$

≈ 3

2. (*a*) *Use your calculator to work out Calculate an estimate for* 108.6 × 27.3.

(*b*) *Show an estimate to verify your answer to* (*a*) *is of the right order of magnitude.*

(*a*) 2964.78 — *Write down all the digits from your calculator display*

(*b*) 108.6 → 100 — *Round both numbers*

27.3 → 30

$108.6 \times 27.3 \approx 100 \times 30$ — *Use the rounded figures to find an approximation*

≈ 3000 — *Check both answers are the same order of magnitude*

Exam Question: Medium

Work out an estimate for $\frac{31\times9.87}{0.509}$

1. Number

Exam Question: Hard

Sanders has a water tank for storing rainwater.

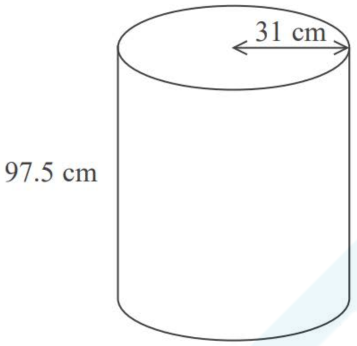


Diagram **NOT** accurately drawn

The tank is in the shape of a cylinder.
The radius of the cylinder is 31 cm.
The height of the cylinder is 97.5 cm.

The tank is full of water.

Work out an estimate for the volume of water in the tank.
Give your answer in litres.
You must show your working.

Use 1000 cm^3 = 1 litre.

1. Number

1.9 STANDARD FORM

1.9.1 STANDARD FORM - BASICS

What is standard form?

- Standard Form (sometimes called Standard Index Form) is a way of writing very big and very small numbers using powers of 10

Why do we use standard form?

- Writing big (and small) numbers in Standard Form allows us to:
    - write them more neatly
    - compare them more easily
    - and it makes things easier when doing calculations

How do we use standard form?

- Using Standard Form numbers are always written in the form:

  $\mathbf{a \times 10^n}$

- The rules:
    - $\mathbf{1 \le a < 10}$ so there is one non-zero digit before the decimal point
    - $\mathbf{n > 0}$ for LARGE numbers – how many times a is multiplied by 10
    - $\mathbf{n < 0}$ for SMALL numbers – how many times a is divided by 10
    - **Do calculations on a calculator** (if allowed)
      Otherwise follow normal rules (including indices) but adjust answer to fit Standard Form (move decimal point and change n

1. Number

Worked Example

1. *Without using a calculator, multiply* 5×10^{18} *by* 7×10^{-4}.

   Give your answer in standard form.

$5 \times 10^{18} \times 7 \times 10^{-4} = 5 \times 7 \times 10^{18} \times 10^{-4}$ *Separate into numbers and powers of 10*

$= 35 \times 10^{18+(-4)}$ *Use Laws of Indices on the powers of 10*

$= 35 \times 10^{14}$

$= 3.5 \times 10 \times 10^{14}$ *Write in standard form (this isn't as 35 > 10)*

$= 3.5 \times 10^{15}$

2. *Use your calculator to find* $\frac{1.275 \times 10^6}{3.4 \times 10^{-2}}$.

   Write your answer in the form $A \times 10^n$, *where* $1 \leq A < 10$ *and n is an integer.*

$\frac{1.275 \times 10^6}{3.4 \times 10^{-2}} = 37\ 500\ 000$

Your calculator will not necessarily give the answer in standard form. Copy the digits, especially those zeros, carefully!

$= 3.75 \times 10^7$

1. Number

1.9.2 STANDARD FORM - HARDER

Standard form – harder questions

- Make sure you are familiar with:
  Roots & Indices – Basics
  Standard Form – Basics
- Harder problems often combine algebra and laws of indices with numbers written in standard form
- Other areas of mathematics may be used and questions are often in context
- Units may be muddled up to make things trickier

1. Number

Worked Example

1. *Given that* $x = 25 \times 10^{4n}$ *write* $x^{\frac{3}{2}}$ *in standard form.*

$x^{\frac{3}{2}} = \left(25 \times 10^{4n}\right)^{\frac{3}{2}}$

$= 25^{\frac{3}{2}} \times \left(10^{4n}\right)^{\frac{3}{2}}$ *Apply the power to both factors in the bracket*

$= 25^{\frac{3}{2}} \times 10^{4n \times \frac{3}{2}}$ *Use the Laws of Indices on the power of 10*

$= 125 \times 10^{6n}$ *Simplify*

$= 1.25 \times 10^{2} \times 10^{6n}$

Write in Standard Form (this isn't as 125 > 10)

$= 1.25 \times 10^{6n+2}$

2. *The diameter of a hydrogen atom is* 1.06×10^{-10} *m.*

The nucleus of a hydrogen atom has diameter 2.40×10^{-13} *cm.*

The diameter of a hydrogen atom is k times the size of the nucleus of a hydrogen atom.

Find the value of k correct to three significant figures.

First spot that we have a mixture of m and cm!

$2.40 \times 10^{-13}\ cm = 2.40 \times 10^{-13} \div 100\ m$

Change one (doesn't matter which) so units match

$= 2.40 \times 10^{-15}\ m$

$k = 1.06 \times 10^{-10} \div 2.40 \times 10^{-15} = 44\ 166.6666\ ...$

To find k divide the (diameter of) the atom by the nucleus

$k = 44\ 200$ *Final answer for k rounded to 3 significant figures*

1. Number

Exam Question: Easy

(a) Write 7.8×10^{-4} as an ordinary number.

(b) Write 95 600 000 as a number in standard form.

Exam Question: Medium

Write 6.7×10^{-5} as an ordinary number.

Work out the value of $(3 \times 10^7) \times (9 \times 10^6)$
Give your answer in standard form.

1. Number

1.10 BOUNDS & ERROR INTERVALS

1.10.1 BOUNDS & ERROR INTERVALS - BASICS

What are bounds?

- Bounds are the smallest – the **Lower Bound (LB)** – and largest – the **Upper Bound (UB)** – numbers that a **rounded** number can lie between

How do we find bounds?

- The basic rule is "Half Up, Half Down"
- More formally:
  - UPPER BOUND – add on half the degree of accuracy
  - LOWER BOUND – take off half the degree of accuracy
  - ERROR INTERVAL: LB ≤ **x** < UB
- Note:
  It is very tempting to think that the Upper Bound should end in a 9, or 99, etc. but if you look at the Error Interval – LB ≤ x < UB – it does NOT INCLUDE the Upper Bound so all is well

Worked Example

1. *The length of a road,* l*, is given as* $l = 3.6$ *km, correct to 1 decimal place.*
   Find the Lower and Upper Bounds for l*.*

$UB \text{ for } l = 3.6 + 0.05$

$= 3.65$

1 – *The degree of accuracy is 1 decimal place (or 0.1) – half is 0.05.*

$LB \text{ for } l = 3.6 - 0.05 = 3.55$

$= 3.55$

2 – *As for the upper bound half the degree of accuracy is 0.05.*

3 – *This question doesn't require it but if the error interval had been asked for the final answer would be* $3.55 \le l < 3.65$*.*

1. Number

1.10.2 CALCULATIONS USING BOUNDS

What are bounds?

- Bounds are the smallest – the **Lower Bound (LB)** – and largest – the **Upper Bound (UB)** – numbers that a **rounded** number can lie between

How do we find bounds?

- The basic rule is "Half Up, Half Down"
- More formally:
    - UPPER BOUND – add on half the degree of accuracy
    - LOWER BOUND – take off half the degree of accuracy
    - ERROR INTERVAL: LB ≤ **x** < UB

Calculations using bounds

- Find bounds before calculating and then:
    - For FRACTIONS/DIVISION:
      UB = UB ÷ LB and
      LB = LB ÷ UB
    - Otherwise:
      UB = UB × UB etc.

Worked Example

1. Number

1. *A room measures* $4m$ *by* $7m$*, where each measurement is made to the nearest metre.*

   Find Upper and Lower Bounds for the area of the room.

$3.5 \leq 4 < 4.5$	*1, 2 − First find the bounds for each dimension*
$6.5 \leq 7 < 7.5$	*It is not essential to write these as an error intervals,*
	stating $LB = 3.5,\ UB = 4.5$*, etc is fine*
$Area\ LB = 3.5 \times 6.5$	*We do not have fractions so* $5 - LB = LB \times LB$
$Area\ LB = 22.75\ m^2$	
$Area\ UB = 4.5 \times 7.5$	$5 - UB = UB \times UB$
$Area\ UB = 33.75\ m^2$	

2. *David is trying to work out how many slabs he needs to buy in order to lay a garden path.*

   Slabs are $50\ cm$ *long, measured to the nearest* $10\ cm$*.*

   The length of the path is $6\ m$*, measured to the nearest* $10\ cm$*.*

   Find the maximum number of slabs David will need to buy.

$45 \leq 50 < 55$	*1, 2 − First find the bounds for each dimension*
$0.45 \leq 0.5 < 0.55$	*Change the units if necessary but do as a separate step*
$5.95 \leq 6 < 6.05$	
$6.05 \div 0.45 = 13.44...$	*The maximum number of slabs will be the Upper Bound*
	4 − This is a division calculation so use $UB = UB \div LB$
$Max\ no.\ of\ slabs\ =\ 14$	*The context of the question means it is sensible for the final*
	answer to be a whole number − but more than 13.44...

Exam Question: Easy

A number, n, is rounded to 2 decimal places.
The result is 4.76

Using inequalities, write down the error interval for n.

1. Number

Exam Question: Medium

A train travelled along a track in 110 minutes, correct to the nearest 5 minutes.

Jake finds out that the track is 270 km long.
He assumes that the track has been measured correct to the nearest 10 km.

(a) Could the average speed of the train have been greater than 160 km/h?
You must show how you get your answer.

Jake's assumption was wrong.
The track was measured correct to the nearest 5 km.

(b) Explain how this could affect your decision in part (a).

Exam Question: Hard

Dan does an experiment to find the value of π.
He measures the circumference and the diameter of a circle.

He measures the circumference, C, as 170 mm to the nearest millimetre.
He measures the diameter, d, as 54 mm to the nearest millimetre.

Dan uses $\pi = \frac{C}{d}$ to find the value of π.

Calculate the upper bound and the lower bound for Dan's value of π.

1. Number

Exam Question: V. Hard

$$m = \frac{\sqrt{s}}{t}$$

$s = 3.47$ correct to 2 decimal places
$t = 8.132$ correct to 3 decimal places

By considering bounds, work out the value of m to a suitable degree of accuracy.

You must show all your working and give a reason for your final answer.

1. Number

1.11 RECURRING DECIMALS

1.11.1 RECURRING DECIMALS

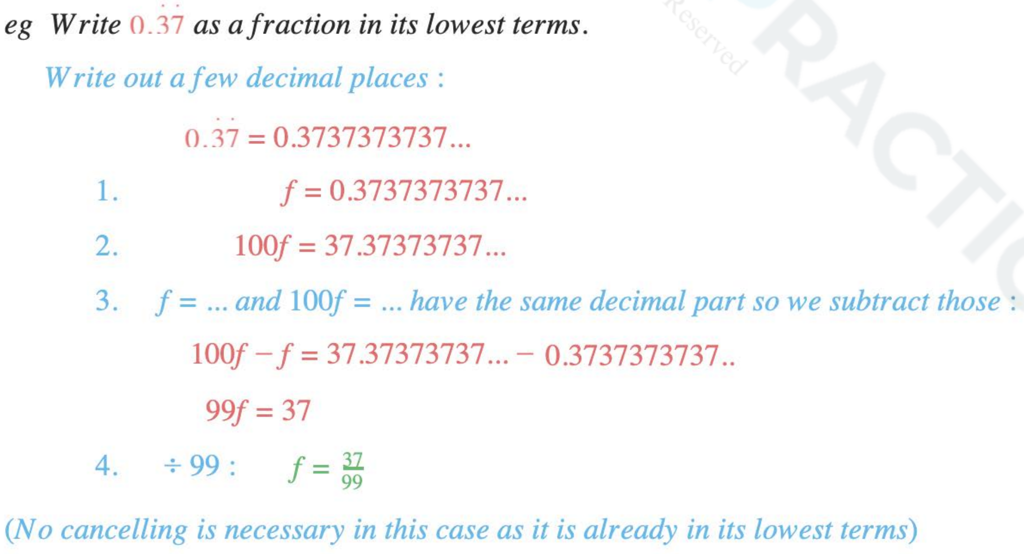
What are recurring decimals?

- A rational number is any number that can be written as an integer (whole number) divided by another integer
- When you write a **rational number as a decimal** you either get a decimal that stops (eg ¼ = 0.25) or one that recurs (eg ⅓ = 0.333333...)
- The recurring part can be written with a dot (or dots) over it instead as in the example below

What do we do with recurring decimals?

- Normally, you will be asked to write a recurring decimal as a fraction in its lowest terms
- To do this:
  Write out a few decimal places... ...and then:

1. Write as **f = ...**
2. **Multiply by 10** repeatedly until two lines have the same decimal part
3. **Subtract** those two lines
4. **DIVIDE to get f** = ... (and cancel if necessary to get fraction in lowest terms)



eg *Write* $0.\dot{3}\dot{7}$ *as a fraction in its lowest terms.*

Write out a few decimal places :

$$0.\dot{3}\dot{7} = 0.3737373737...$$

1. $f = 0.3737373737...$

2. $100f = 37.37373737...$

3. $f = ...$ *and* $100f = ...$ *have the same decimal part so we subtract those :*

$$100f - f = 37.37373737... - 0.3737373737..$$

$$99f = 37$$

4. $\div 99 :$ $f = \frac{37}{99}$

(No cancelling is necessary in this case as it is already in its lowest terms)

1. Number

Worked Example

Write $0.4\dot{2}\dot{7}$ *as a fraction in its lowest terms.*

Write out a few decimal places :

$0.4\dot{2}\dot{7} = 0.42727272727...$

1. $f == 0.42727272727...$

2. $10f = 4.2727272727...$

$100f = 42.727272727...$

$1000f = 427.27272727...$

3. *Here* $10f = ...$ *and* $1000f = ...$ *have the same decimal part so we subtract those :*

$1000f - 10f = 427.27272727... - 4.2727272727..$

$990f = 423$

4. $\div 990 : \quad f = \frac{423}{990}$

Both 423 *and* 990 *are multiples of* 9, *so cancel to get the fraction into its lowest terms :*

$$f = \frac{423}{990} = \frac{423 \div 9}{990 \div 9} = \frac{47}{110}$$

Exam Question: Easy

Prove algebraically that the recurring decimal $0.2\dot{5}$ has the value $\frac{23}{90}$

Exam Question: Medium

Write these numbers in order of size.
Start with the smallest number.

$0.2\dot{4}\dot{6}$ $\quad$ $0.24\dot{6}$ $\quad$ $0.\dot{2}4\dot{6}$ $\quad$ 0.246

1. Number

Exam Question: Hard

Express the recurring decimal $0.2\dot{8}\dot{1}$ as a fraction in its simplest form.

1. Number

1.12 SURDS

1.12.1 SURDS - BASICS

What is a surd?

- A surd is the square root of a non-square integer

What can we do with surds?

1. Multiplying surds – you can multiply numbers under square roots
eg. √3 × √5 = √3×5 = √15

2. Dividing surds – you can divide numbers under square roots
eg. √21 ÷ √7= √21 ÷ 7 = √3

3. Factorising surds – you can factorise numbers under square roots
eg. √35 = √5 × 7 = √5 × √7

4. Simplifying surds – separate out a square factor and square root it!
eg. √48 = √16 × 3 = √16 × √3 = 4 × √3 = 4√3

5. Adding or subtracting surds is very like adding or subtracting letters in algebra – you can only add or subtract multiples of "like" surds
eg. 3√5 + 8√5 = 11√5 or 7√3 – 4√3 = 3√3
Be very careful here! You can not add or subtract numbers under square roots. Think about √9 + √4= 3 + 2 = 5. It is not equal to √9 + 4= √13 = 3.60555...

6. All other algebraic rules apply – surds can be treated like letters (as in 5. above) and like numbers (as in 1. and 2. above)

1. Number

Worked Example

Write $\sqrt{54} - \sqrt{24}$ in the form $p\sqrt{q}$ where p and q are integers and q has no square factors.

4. $\sqrt{54} - \sqrt{24} = \sqrt{9 \times 6} - \sqrt{4 \times 6}$

$= \sqrt{9} \times \sqrt{6} - \sqrt{4} \times \sqrt{6}$

$= 3 \times \sqrt{6} - 2 \times \sqrt{6}$

5. $= 3\sqrt{6} - 2\sqrt{6}$

$= \sqrt{6}$

so $p = 1$ and $q = 6$

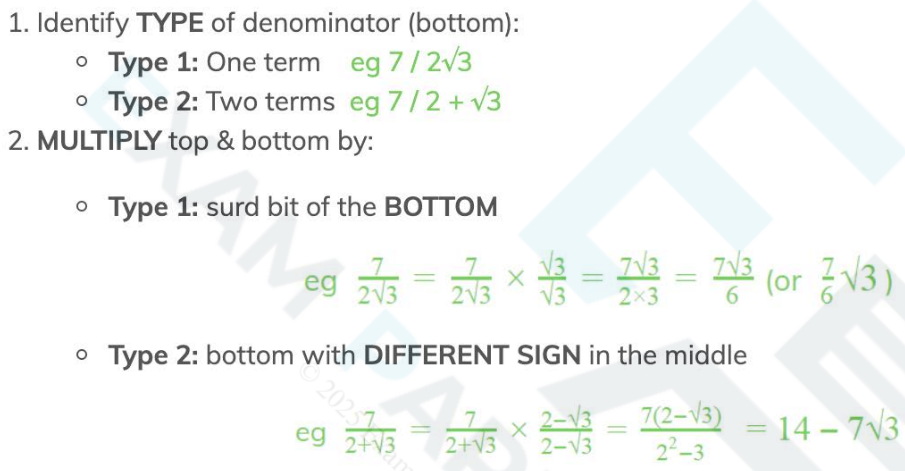
1. Number

1.12.2 SURDS - RATIONALISING DENOMINATORS

What is a surd?

- A surd is the square root of a non-square integer

How do we rationalise the denominator of a surd?



1. Identify **TYPE** of denominator (bottom):
    - **Type 1:** One term eg 7 / 2√3
    - **Type 2:** Two terms eg 7 / 2 + √3
2. **MULTIPLY** top & bottom by:
    - **Type 1:** surd bit of the **BOTTOM**

      eg $\frac{7}{2\sqrt{3}} = \frac{7}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{7\sqrt{3}}{2\times3} = \frac{7\sqrt{3}}{6}$ (or $\frac{7}{6}\sqrt{3}$)

    - **Type 2:** bottom with **DIFFERENT SIGN** in the middle

      eg $\frac{7}{2+\sqrt{3}} = \frac{7}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{7(2-\sqrt{3})}{2^2-3} = 14 - 7\sqrt{3}$

Worked Example

1. Number

Write $\frac{4}{\sqrt{6}-2}$ *in the form* $p + q\sqrt{r}$ *where* p, q *and* r *are integers and* r *has no square factors.*

1. *The bottom has two terms so this Type 2*
2. *Multiply top and bottom by* $\sqrt{6}+2$:

$$\frac{4}{\sqrt{6}-2} = \frac{4}{\sqrt{6}-2} \times \frac{\sqrt{6}+2}{\sqrt{6}+2}$$

Spot the Difference of Two Squares on the bottom :

$$= \frac{4(\sqrt{6}+2)}{6-2^2}$$

Multiply out the top and simplify :

$$= \frac{4\sqrt{6}+8}{2}$$

$$= 2\sqrt{6}+4$$

$$= 4+2\sqrt{6}$$

so $p = 4$, $q = 2$ *and* $r = 6$

Exam Question: Medium

(a) Rationalise the denominator of $\frac{12}{\sqrt{3}}$

(b) Work out the value of $(\sqrt{2} + \sqrt{8})^2$

1. Number

Exam Question: Hard

(a) Rationalise the denominator of $\frac{5}{\sqrt{2}}$

(b) Expand and simplify $(2 + \sqrt{3})^2 - (2 - \sqrt{3})^2$

1. Number

1.13 USING A CALCULATOR

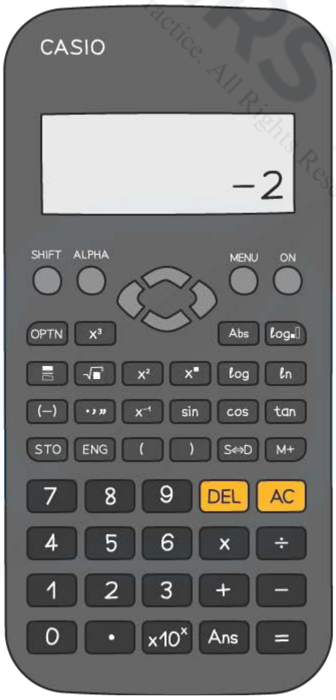
1.13.1 USING A CALCULATOR

Why the fuss about using a calculator?

- GCSE Mathematics goes beyond using the basic features of a calculator and explores many of the special functions of a scientific calculator
- It is important to get to know your calculator, the earlier you get one and learn about the scientific functions the better you will be at using them
- It's not just maths that uses these, some of the scientific functions can be used in science exams too

What do I need to know?

- The notes below apply to most if not all scientific calculators but the images are based on the Casio fx-83GTX
- The Casio fx-85GTX is the same model but also has solar power. Both are labelled "Classwiz" too but be careful here at there is a more advanced "Classwiz" calculator that is used at A level (fx-991EX)



1. Number

The Casio fx-83GTX Classwiz

- Be aware if you have an old or very basic scientific calculator that they may work backwards
- For example, if you wanted to find sin (57) you would type 57 then press the sin button
- Modern calculators tend to work in the order in which we write things

1. Mode/setup

- Make sure you know how to change the mode of your calculator, especially if someone else has used it
- The “Angle Unit” needs to be degrees – normally indicated by a “D” symbol across the top of the display
- Make sure you can switch between “exact” answers (fractions, surds, in terms of π, etc) and “approximate” answers (decimals)
- Most calculators default to “Math” mode with the word Math written across the top of the display or using a symbol
- When in “Math” mode you can switch whatever is on the answer line between exact and decimals by pressing the “S-D” button

2. Templates

- These are largely the shortcut buttons – the fraction button, the square, cube and power buttons, square roots

Calculator shortcut buttons

1. Number

3. Trigonometry (sin/cos/tan)

- Remember to use SHIFT (sometimes called 2nd or INV button) when finding angles
- When using these buttons you will find that before you type the angle the calculator automatically gives you an open bracket "(". You should get into the habit of making sure you use a closed bracket ")" after typing the angle in
- This is very important if there is something else to type in that comes after sin/cos/tan

4. Standard Form and π

- Find the $\times10^x$ button and know how to use it
- Modern calculators display standard form in the way it is written
- Older models may use a small capital letter "$_E$" in place of $\times10^x$ on the display line
- π is often near or under SHIFT with the standard form button

5. Memory

- The **ANS** (answer) button is very useful – especially when working with decimals in the middle of solutions that you should avoid rounding until your final answer
- ANS recalls the last answer the calculator calculated

6.Table

- If your calculator has a **table** function or mode, use it
- This can be extremely useful in those "complete the table of values and draw the graph" type questions

7. Brackets and negative numbers

- Use as you would in written mathematics
- Remember to use the (-) button for a negative number, not the subtract button

8. Judgement and special features

- The rule of thumb is to use your calculator to do one calculation at a time
- However, you can also make a judgement call on this as to how many marks are available in the question and whether a question asks you to "write down all the digits on your calculator display"
- You are better off writing too much down than not enough!

1. Number

9. Practise!

- This is a long list but we will finish by going back to the start – there is nothing better you can do than getting a calculator early and learning how to use it by practising the varying types of questions you are likely to come across

Exam Tip

Always put negative numbers in brackets. For a quick example, try using your calculator to work out -3^2 and then $(-3)^2$.

In working out always write down more digits than the final answer requires and don't round them (write something like 9.3564… using the three dots shows you haven't rounded). Use the ANS button when you next need that number on your calculator.

1. Number

Worked Example

3. Complete the table of values for $y = x^3 - 6x + 1$

x	-3	-2	-1	0	1	2	3
y		5					10

$(-3)^3 - 6 \times (-3) + 1 = -8$ Use brackets around negatives and (-) key

$(-1)^3 - 6 \times (-1) + 1 = 6$ Use arrow keys and change “3”s to “1”s

$0^3 - 6 \times 0 + 1 = 1$

$1^3 - 6 \times 1 + 1 = -4$ You can use the TABLE mode/feature

$2^3 - 6 \times 2 + 1 = -3$ of your calculator if it has one

x	-3	-2	-1	0	1	2	3
y	-8	5	6	1	-4	-3	10

4. Solve the quadratic equation $2x^2 + 6x + 3 = 0$, giving your answers in the form $\frac{a \pm b\sqrt{3}}{2}$

$“a" = 2, \ "b" = 6, \ "c" = 3$ a, b, (c) from quadratic formula have nothing to do with the a and b mentioned in question

$"b^2 - 4ac" = 6^2 4 \times 2 \times 3 = 12$ Find the discriminant first (the bit under square root)

$x = \frac{-6 \pm \sqrt{12}}{2 \times 2}$ Now put into full quadratic formula

$x = \frac{-3 \pm \sqrt{3}}{2}$ ... and your calculator will simplify for you

Note: You’ll have to choose “+” or “-“ when You type it into your calculator.

1. Number

1. Use your calculator to work out

$$\frac{\sqrt{4.69}}{0.34^3 + sin\,(45°)}$$

Give your answer as a decimal.

Write down all the figures on your calculator display.

$\sqrt{4.69} = 2.16564 \ldots$ To show your working write down the top

$0.34^3 + sin(45) = 0.746410 \ldots$ and bottom separately ...

2.901406085 ... but you can type it all in one go for the final answer using the fraction button, etc

2. $a^5 = \frac{p+q}{p^2q}$

Find the value of a when $p = 1.2 \times 10^{-4}$ and $q = 7.83 \times 10^5$

Give your answer to 3 decimal places.

$p + q = 783000.0001$ Show each stage as working

$p^2q = 0.0112752$ Use brackets when it gets long or awkward

$a^5 = 69444444.46$ Write down all digits at these stages

$\boldsymbol{a = 37.01071 \ldots = 37.011}$ Write more digits than you need, then round

1. Number

Exam Question: Easy

(a) Use your calculator to work out $\dfrac{38.5 \times 14.2}{18.4 - 5.9}$

Write down all the figures on your calculator display.
You must give your answer as a decimal.

(b) Write your answer to part (a) correct to 1 significant figure.

Exam Question: Medium

Calculate the value of $\sqrt{\dfrac{\tan 60° + 1}{\tan 60° - 1}}$

Write down all the figures on your calculator display.
You must give your answer as a decimal.

Exam Question: Hard

$$p^2 = \frac{x - y}{xy}$$

$x = 8.5 \times 10^9$
$y = 4 \times 10^8$

Find the value of p.
Give your answer in standard form correct to 2 significant figures.

1. Number

1.14 COUNTING

1.14.1 COUNTING

How to count combinations of things

- When you have a question like “How many ways...?”

1. **Systematic** listing can help in simple cases
2. Write down the thing you are counting with “AND”s and “OR”s
3. Remember that (like in Probability problems):
   “AND means ×”
   “OR means +”
4. Beware of **repetitions** and whether they are allowed or not
   (If they are not we reduce the number of options by one)

1. Number

Worked Example

1. *John is choosing an ice cream flavour and a topping.*

He has a choice of three flavours – vanilla (V), strawberry (S) and raspberry ripple (R).

He also has a choice of two toppings – nuts (N) and chocolate sauce (C).

How many different combinations of ice cream can John choose?

VN, VC — *1 – Be systematic – stick with one flavour at a time and*

SN, SC — *work through each of the toppings in turn*

RN, RC — *Abbreviations are fine – the question even suggests them!*

We could have used 2 and 3 – flavour AND topping – 3×2

4 – There are no repetitions to consider here

No of ice cream combinations is 6

2. *Jamil is out for lunch.*

The menu has 5 starters, 6 main courses and 4 puddings.

How many different two course meals (Starter & Main or Main & Pudding) could Jamil have?

1 – There would be far too many combinations to list

Starter AND Main OR Main AND Pudding

2 – Use "AND"s and "OR"s to state what Jamil could have

4 – There are no repetitions to consider here

No of meals $= 5 \times 6 + 6 \times 4$ — *3 – "AND means ×" and "OR means +"*

$= 30 + 24$

No of meals $= 54$

3. *Lauren is generating a 4 – digit passcode using the digits 0 – 9.*

She will pick each digit at random but does not want any digit to be repeated.

How many different passcodes will Lauren be able to generate?

1. Number

Exam Question: Medium

Jeff is choosing a shrub and a rose tree for his garden.
At the garden centre there are 17 different types of shrubs and some rose trees.

Jeff says,

> "There are 215 different ways to choose one shrub and one rose tree."

Could Jeff be correct?
You must show how you get your answer.

1. Number

1.15 BEST BUY

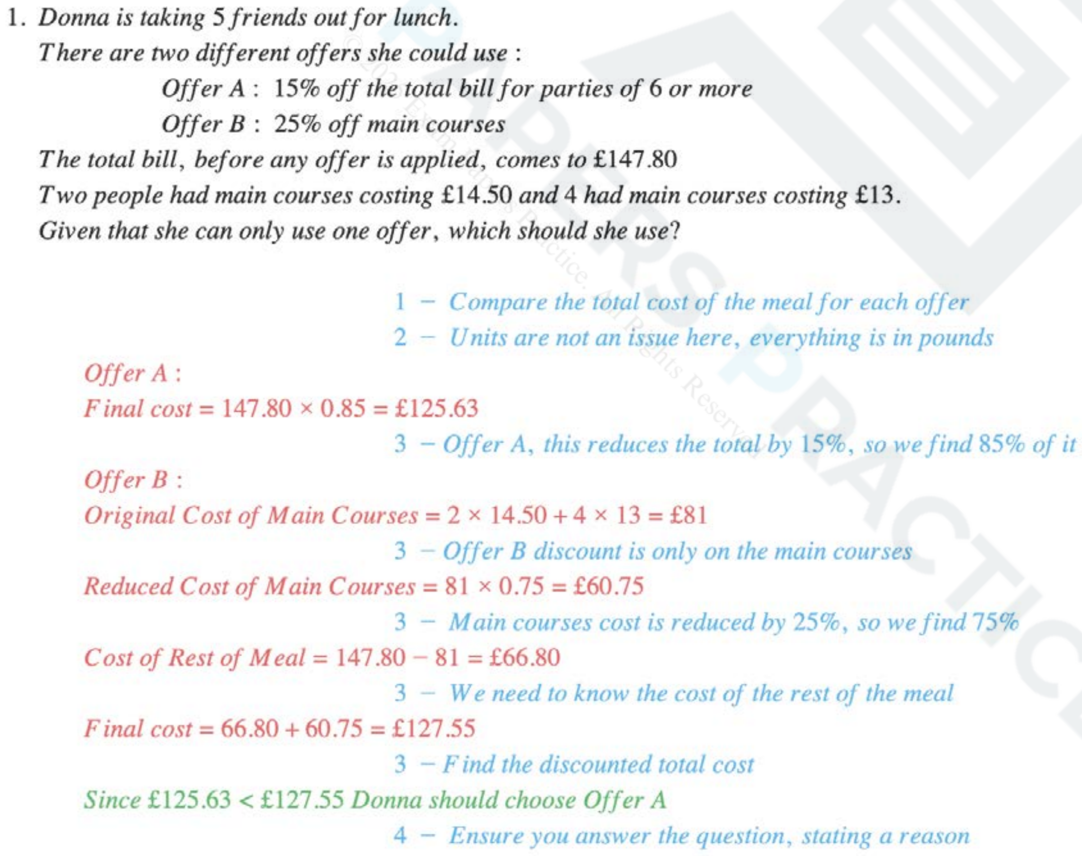
1.15.1 BEST BUY

How do you tackle “Best Buy” problems?

- You may need a range of skills...

1. **Decide on a suitable comparison** – eg total cost, cost/year, cost/m2 etc)
2. **Be consistent with units** – convert if necessary
3. **Do the calculations** – showing working carefully
4. Make sure you **answer the question**, with reasons

Worked Example



1. *Donna is taking 5 friends out for lunch.*
There are two different offers she could use :
Offer A : 15% off the total bill for parties of 6 or more
Offer B : 25% off main courses
The total bill, before any offer is applied, comes to £147.80
Two people had main courses costing £14.50 and 4 had main courses costing £13.
Given that she can only use one offer, which should she use?

1 – Compare the total cost of the meal for each offer
2 – Units are not an issue here, everything is in pounds

Offer A :
Final cost $= 147.80 \times 0.85 = £125.63$
3 – Offer A, this reduces the total by 15%, so we find 85% of it

Offer B :
Original Cost of Main Courses $= 2 \times 14.50 + 4 \times 13 = £81$
3 – Offer B discount is only on the main courses
Reduced Cost of Main Courses $= 81 \times 0.75 = £60.75$
3 – Main courses cost is reduced by 25%, so we find 75%
Cost of Rest of Meal $= 147.80 - 81 = £66.80$
3 – We need to know the cost of the rest of the meal
Final cost $= 66.80 + 60.75 = £127.55$
3 – Find the discounted total cost

Since $£125.63 < £127.55$ *Donna should choose Offer A*
4 – Ensure you answer the question, stating a reason

1. Number

Exam Question: Easy

Plants are sold in three different sizes of tray.

A small tray of 30 plants costs £6.50
A medium tray of 40 plants costs £8.95
A large tray of 50 plants costs £10.99

Kaz wants to buy the tray of plants that is the best value for money.

Which size tray of plants should she buy?
You must show all your working.

Exam Question: Medium

Henry is thinking about having a water meter.

These are the two ways he can pay for the water he uses.

Water Meter

A charge of £28.20 per year

plus

91.22p for every cubic metre of water used

1 cubic metre = 1000 litres

No Water Meter

A charge of £107 per year

Henry uses an average of 180 litres of water each day.

Henry wants to pay as little as possible for the water he uses.
Should Henry have a water meter?

1. Number

1.16 EXCHANGE RATES

1.16.1 EXCHANGE RATES

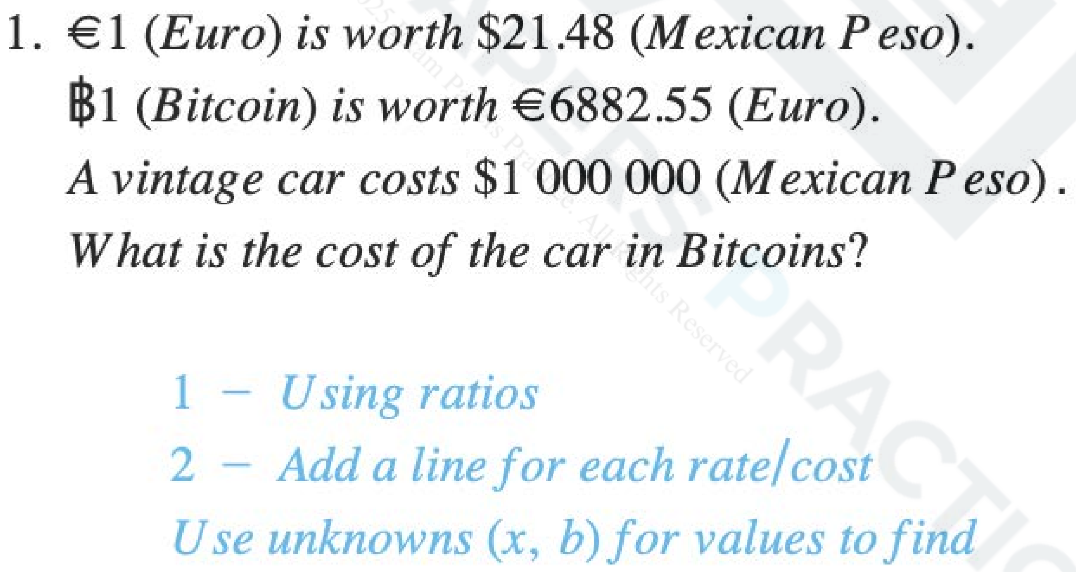
Simplifying exchange rate questions

Use ratios!

1. Put exchange rates in **ratio** form (use more than one line if necessary)
2. Add lines for prices/costs
3. Use **scale factors** to complete lines
4. Pick out the **answer!**

It can be that simple!

Worked Example



1. €1 (*Euro*) *is worth* \$21.48 (*Mexican Peso*).
฿1 (*Bitcoin*) *is worth* €6882.55 (*Euro*).
A vintage car costs \$1 000 000 (*Mexican Peso*).
What is the cost of the car in Bitcoins?

1 – *Using ratios*
2 – *Add a line for each rate/cost*
Use unknowns $(x,\ b)$ *for values to find*

1. Number

	€ EURO	:	$ MP	:	$ BITCOIN
RATE €/$	1	:	21.48	:	
RATE ฿/€	6882.50	:	x	:	1
RATE $/฿		:	1 000 000	:	b

$x = 21.48 \times 6882.55$ 3 − *Looking at the Euro column, the Scale Factor is* 6882.55

$x = \$147\ 837.174$

$\text{Scale Factor} = 1\ 000\ 000 \div 147837.174$

3 − *Looking at the MP column, we can find the Scale Factor to convert between the rate for ฿/€ and cost.*

$= 6.764$

$\text{Cost} = 1 \times 6.764 = ฿6.764$ (*Bitcoins*) 4 − *We can now pick out the answer in Bitcoins*

Exam Question: Medium

Linda is going on holiday to the Czech Republic.
She needs to change some money into koruna.

She can only change her money into 100 koruna notes.

Linda only wants to change up to £200 into koruna.
She wants as many 100 koruna notes as possible.

The exchange rate is £1 = 25.82 koruna.

How many 100 koruna notes should she get?

1. Number

Exam Question: Hard

In the UK, petrol cost £1.24 per litre.
In the USA, petrol cost 3.15 dollars per US gallon.

1 US gallon = 3.79 litres
£1 = 1.47 dollars

Was petrol cheaper in the UK or in the USA?

2. Algebra Basics

CONTENTS

2. Algebra Basics

2.1 EXPANDING ONE BRACKET

2.1.1 EXPANDING ONE BRACKET

What is expanding one bracket?

- When we expand brackets in algebra we ‘get rid of’ the brackets by multiplying out
- Expanding one bracket is done by multiplying everything inside the bracket by the factor that is outside the bracket

Beware of minus signs

Remember the basic rules:
“ $- \times - = +$ ”
“ $- \times + = -$ ”

How to multiply out

1. Identify the factor outside the bracket
2. Identify the terms inside the bracket
3. Link the outside factor to the inside terms
4. Follow the links to multiply the outside factor by each term inside the bracket

2. Algebra Basics

Worked Example

1. *Expand* $4x(2x - 3)$

   1. $Factor\ outside\ is\ "4x"$
   2. $Terms\ inside\ are\ "2x"\ and\ "-3"$
   3. $4x(2x - 3)$
   4. $= 4x \times 2x + 4x \times (-3)$

      $= 8x^2 - 12x$

1. *Expand* $-7x(4 - 5x)$

   1. $Factor\ outside\ is\ "-7x"$
   2. $Terms\ inside\ are\ "4"\ and\ "-5x"$
   3. $-7x(4 - 5x)$
   4. $= (-7x) \times 4 + (-7x) \times (-5x)$

      $= -28x + 35x^2$

Exam Question: Easy

Expand $3(2y - 5)$

2. Algebra Basics

2.2 SIMPLE FACTORISATION

2.2.1 SIMPLE FACTORISATION

What is simple factorisation?

Simple Factorisation is the opposite of Expanding One Bracket

Beware of minus signs

Remember the basic rules:

“ $- \times - = +$ ”

“ $- \times + = -$ ”

How to factorise

- Write each term in factorised form
- Identify the common factors
- Write those factors outside a bracket
- Write everything else inside the bracket
  You can check your answer by multiplying out (in your head!)

2. Algebra Basics

Worked Example

1. $Factorise\ 6x^2 - 15x$

1. $6x^2 - 15x = 2 \times 3 \times x \times x - 3 \times 5 \times x$
2. $Common\ Factors:\ 3 \times x$

3&4. $6x^2 - 15x = 3 \times x \times (2 \times x - 5)$

$$= 3x(2x - 5)$$

2. $Factorise\ 42x^2 - 18x$

1. $42x^2 - 18x = 7 \times 6 \times x \times x - 3 \times 6 \times x$
2. $Common\ Factors:\ 6 \times x$

3&4. $42x^2 - 18x = 6 \times x \times (7 \times x - 3)$

$$= 6x(7x - 3)$$

Exam Question: Easy

Factorise completely $8x^2 + 4xy$

Exam Question: Medium

Factorise fully $9x^2 - 6xy$

2. Algebra Basics

2.3 EXPANDING QUADRATICS

2.3.1 EXPANDING TWO BRACKETS - BASICS

What is expanding two brackets?

- When we expand brackets in algebra we ‘get rid of’ the brackets by multiplying out
- When we expand two brackets, each term in one bracket gets multiplied by each term in the other bracket

Beware of minus signs

“ $- \times - = +$ ”

“ $- \times + = -$ ”

How to multiply out two brackets

- Write FOIL above your working space
  F – First term in each bracket
  O – Outer pair of terms
  I – Inner pair of terms
  L – Last term in each bracket
- Write down the four multiplications
- Careful here – you have to be sure to include any minus signs!
- Simplify by collecting like terms (if there are any)

Worked Example

2. Algebra Basics

1. *Expand* $(2x-3)(x+4)$

1. $F \quad O \quad I \quad L$

2. $(2x-3)(x+4) = \overbrace{2x \times x} + \overbrace{2x \times 4} - \overbrace{3 \times x} - \overbrace{3 \times 4}$

$= 2x^2 + 8x - 3x - 12$

3. $= 2x^2 + 5x - 12$

2. *Expand* $(x-3)(3x-5)$

1. $F \quad O \quad I \quad L$

2. $(x-3)(3x-5) = \overbrace{x \times 3x} + \overbrace{x \times (-5)} - \overbrace{3 \times 3x} - \overbrace{3 \times (-5)}$

$= 3x^2 - 5x - 9x + 15$

3. $= 3x^2 - 14x + 15$

2. Algebra Basics

2.3.2 EXPANDING TWO BRACKETS - HARDER

Hidden FOIL (aka spot the hidden brackets)

- Always write $(a + b)^2$ as (a + b)(a + b)

Beware of minus signs

- Remember the basic rules:
  " − × − = + "
  " − × + = − "

How to multiply out two brackets

- Write FOIL above your working space
  F – First term in each bracket
  O – Outer pair of terms
  I – Inner pair of terms
  L – Last term in each bracket
- Write down the four MULTIPLICATIONS
  Careful here – you have to be sure to include any minus signs!
- Simplify by collecting LIKE TERMS (if there are any)

Worked Example

Expand $(2x+3)^2$

First write : $(2x+3)^2 = (2x+3)\ (2x+3)$

1. $F \qquad O \qquad I \qquad L$

2. $(2x+3)(2x+3) = \overbrace{2x \times 2x} + \overbrace{2x \times 3} + \overbrace{3 \times 2x} + \overbrace{3 \times 3}$

$$= 4x^2 + 6x + 6x + 9$$

3. $= 4x^2 + 12x + 9$

2. Algebra Basics

2.3.3 EXPANDING THREE BRACKETS

Expanding three brackets

- We’re still following the basic principle of expanding here – get rid of the brackets by multiplying out
- We just need a good method to make sure everything gets multiplied by everything else in the correct way

Beware of minus signs

- Remember the basic rules:
  “ $-\times-=+$ ”
  “ $-\times+=-$ ”

How to multiply out three brackets

- Multiply out one pair of brackets using FOIL (as for two brackets)
- If the new bracket has two terms use FOIL (again!)
- Otherwise link each term in the smaller bracket to each term in the larger bracket (that’s six links in total)
- Write down the six multiplications
- Careful here – you have to be sure to include any minus signs!
- Simplify by collecting like terms (if there are any)

2. Algebra Basics

Worked Example

1. *Expand* $(2x-3)(x+4)(3x-1)$

1. $(2x-3)(x+4) = \overbrace{2x \times x}^{F} + \overbrace{2x \times 4}^{O} - \overbrace{3 \times x}^{I} - \overbrace{3 \times 4}^{L}$

$= 2x^2 + 5x - 12$

2. $(2x-3)(x+4)(3x-1) = (2x^2 + 5x - 12)(3x-1)$

3. $(2x-3)(x+4)(3x-1) = 2x^2 \times 3x + 5x \times 3x - 12 \times 3x + 2x^2 \times (-1) + 5x \times (-1) - 12 \times (-1)$

$= 6x^3 + 15x^2 - 36x - 2x^2 - 5x + 12$

4. $= 6x^3 + 13x^2 - 41x + 12$

2. *Expand* $(x-3)(x+2)(2x-1)$

1. $(x-3)(x+2) = \overbrace{x \times x}^{F} + \overbrace{x \times 2}^{O} - \overbrace{3 \times x}^{I} - \overbrace{3 \times 2}^{L}$

$= x^2 - x - 6$

2. $(x-3)(x+2)(2x-1) = (x^2 - x - 6)(2x-1)$

3. $(x-3)(x+2)(2x-1) = x^2 \times 2x - x \times 2x - 6 \times 2x + x^2 \times (-1) - x \times (-1) - 6 \times (-1)$

$= 2x^3 - 2x^2 - 12x - x^2 + x + 6$

4. $= 2x^3 - 3x^2 - 11x + 6$

Exam Question: Easy

Expand and simplify $(p+9)(p-4)$

2. Algebra Basics

Exam Question: Medium

Expand and simplify $(2x + 1)(x - 4)$

Exam Question: Hard

Show that $(x + 1)(x + 2)(x + 3)$ can be written in the form $ax^3 + bx^2 + cx + d$ where a, b, c and d are positive integers.

2. Algebra Basics

2.4 FACTORISING QUADRATICS

2.4.1 FACTORISING QUADRATICS - BASICS

What is a quadratic expression?

- A quadratic expression looks like this:

  $ax^2 + bx + c$ (as long as $a \neq 0$)

- Note: If there are any higher powers of x (like x^3 say) then it is not a quadratic!

Factorising a 3-term quadratic expression (a = 1)

- Signs in quadratic determine signs in brackets:
  - if c is **positive** then both signs are the same as the sign of b
  - if c is **negative** then the signs are different and bigger number has the sign of b
- Using those signs find numbers p and q which
  - **multiply** to give c
  - **add** to give b
- Write down the brackets $(x + p)(x + q)$ – that's your answer!
  Don't forget – both p and q here can be negative!

Exam Tip

Make sure you know if you are being asked to:

• Solve an equation (look for the "=") or

• Factorise an expression (no "=")

Do not confuse the two things.

When the quadratic expression only has two terms check for:

• Simple factorisation (no number term, ie. when c = 0)

• Difference Of Two Squares (no x term, ie. when b = 0)

2. Algebra Basics

Worked Example

1. *Factorise* $x^2 - 2x - 8$

   1. $"-8"$ *means the signs will be different*
      $"-2"$ *means the bigger number will be negative*
   2. *The only numbers which multiply to give* -8 *and follow the sign rules in* 1. *are*

      -8×1 *and* -4×2

      But only the second pair add to give -2

      $-4 \times 2 =- 8$ *and* $-4 + 2 =- 2$

      So we have found $p =- 4$ *and* $q = 2$
   3. $x^2 - 2x - 8 = (x - 4)(x + 2)$

2. *Factorise* $x^2 - 5x + 6$

   1. $"+6"$ *means the signs will be the same*
      $"-5"$ *means that both signs will be negative*
   2. *The only numbers which multiply to give* $+6$ *and follow the sign rules in* 1. *are*

      -2×-3 *and* -1×-6

      But only the first pair add to give -5

      $-2 \times (-3) = 6$ *and* $-2 + (-3) =- 5$

      So we have found $p =- 2$ *and* $q =- 3$
   3. $x^2 - 5x + 6 = (x - 2)(x - 3)$

2. Algebra Basics

2.4.2 FACTORISING QUADRATICS - HARDER

What is a quadratic expression?

- A quadratic expression looks like this:

  $ax^2 + bx + c$ (as long as $a \neq 0$)

- Note: If there are any higher powers of x (like x^3 say) then it is not a quadratic!

Factorising a 3 term quadratic expression ($a \neq 1$)

- Signs in quadratic determine signs in brackets:
  - If c is **positive** then both signs are the same as the sign of b
  - If c is **negative** then the signs are different and bigger number has the sign of b
- Using those signs find numbers p and q which
  - **Multiply** to give a × c
  - **Add** to give b
  - Cheat by writing brackets as $(ax + p)(ax + q)$ – Yes, I know! This is **not** correct!
  - Uncheat by cancelling common factors in each bracket

Exam Tip

Check for a common factor – life will be a lot easier if you take it out before you try to factorise.

If a is negative take out out a factor of -1 – that will help, too.

2. Algebra Basics

Worked Example

1. $Factorise\ 6x^2 - 7x - 3$

1. $"-3"$ *means the signs will be different*
   $"-7"$ *means the bigger number will be negative*

2. $$a \times c = 6 \times (-3) =- 18$$

   The only numbers which multiply to give -18 *and follow the sign rules in* 1. *are*

   $$-18 \times 1 \ \ and \ \ -9 \times 2 \ \ and \ \ -6 \times 3$$

   But only the second pair add to give -7

   $$-9 \times 2 =- 18 \quad and \quad -9 + 2 =- 7$$

   So we have found $p =- 9$ *and* $q = 2$

3. *Cheat :* $(6x - 9)(6x + 2)$
   $= 3\,(2x - 3) \times 2\,(3x + 1)$ *This is not correct!*

4. *Uncheat :* $6x^2 - 7x - 3 = (2x - 3)\,(3x + 1)$

2. $Factorise\ \ 10x^2 + 9x - 7$

1. $"-7"$ *means the signs will be different*
   $"+9"$ *means the bigger number will be positive*

2. $$a \times c = 10 \times (-7) =- 70$$

   The only numbers which multiply to give -70 *and follow the sign rules in* 1. *are*

   $$-1 \times 70 \ \ and \ \ -2 \times 35 \ \ and \ \ -5 \times 14 \ and \ \ -7 \times 10$$

   But only the third pair add to give $+9$

   $$-5 \times 14 =- 70 \quad and \quad -5 + 14 = 9$$

   So we have found $p =- 5$ *and* $q = 14$

3. *Cheat :* $(10x - 5)(10x + 14)$
   $= 5\,(2x - 1) \times 2\,(5x + 7)$ *This is not correct!*

4. *Uncheat :* $10x^2 + 9x - 7 = (2x - 1)\,(5x + 7)$

2. Algebra Basics

2.4.3 DIFFERENCE OF TWO SQUARES

What is the difference of two squares?

- A “Difference Of Two Squares” is any expression in the form $p^2 - q^2$

Why is it so important?

- It comes up a lot – recognising it can save a lot of time

How does it factorise?

- $p^2 - q^2$ can always be factorised in the same way:

  $p^2 - q^2 = (p + q)(p - q)$
  (Try multiplying it out to see why…)

Worked Example

1. *Factorise* $9x^2 - 16$

$$9x^2 - 16 = (3x)^2 - 4^2 = (3x+4)(3x-4)$$

2. *Factorise* $4x^2 - 25$

$$4x^2 - 25 = (2x)^2 - 5^2 = (2x+5)(2x-5)$$

2. Algebra Basics

2.4.4 FACTORISING QUADRATICS - GENERAL

What is a quadratic expression?

- A quadratic expression looks like this:

  $ax^2 + bx + c$ (as long as $a \neq 0$)

- Note: If there are any higher powers of x (like x^3 say) then it is not a quadratic!

Does it factorise?

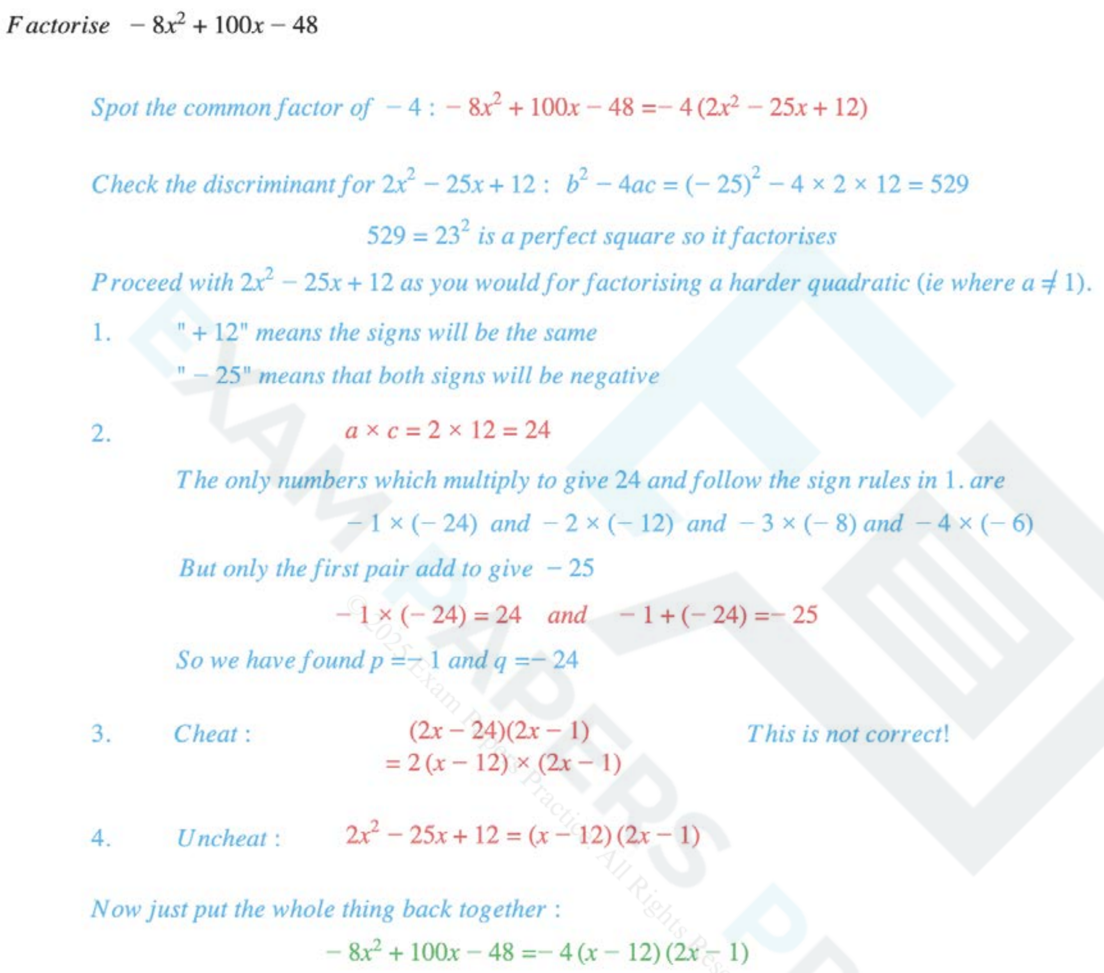
- It is sensible to check before trying to factorise a quadratic that it actually does factorise!
- A quadratic will factorise if (and only if) the discriminant ($b^2 - 4ac$) is a perfect square
- If it does factorise then you have to decide what to do:

Decisions, decisions…

- Always check for a Common (numerical) Factor before you factorise – spotting it early will make life a lot easier
- Take it out and just leave it in front of a big bracket
- When the quadratic expression only has two terms check for:
  - Simple factorisation (no number term, ie. when $c = 0$)
  - Difference Of Two Squares (no x term, ie. when $b = 0$)
- Once you have decided that it is a factorisable three term quadratic with no common numerical factors then you can follow the normal rules (see other Notes for those!)

2. Algebra Basics

Worked Example



Factorise $-8x^2 + 100x - 48$

Spot the common factor of -4 : $-8x^2 + 100x - 48 = -4(2x^2 - 25x + 12)$

Check the discriminant for $2x^2 - 25x + 12$: $b^2 - 4ac = (-25)^2 - 4 \times 2 \times 12 = 529$

$529 = 23^2$ *is a perfect square so it factorises*

Proceed with $2x^2 - 25x + 12$ *as you would for factorising a harder quadratic (ie where* $a \neq 1$).

1. "$+12$" *means the signs will be the same*
   "-25" *means that both signs will be negative*

2. $a \times c = 2 \times 12 = 24$

   The only numbers which multiply to give 24 and follow the sign rules in 1. are

   $-1 \times (-24)$ *and* $-2 \times (-12)$ *and* $-3 \times (-8)$ *and* $-4 \times (-6)$

   But only the first pair add to give -25

   $-1 \times (-24) = 24$ *and* $-1 + (-24) = -25$

   So we have found $p = -1$ *and* $q = -24$

3. *Cheat :* $(2x - 24)(2x - 1)$ *This is not correct!*

   $= 2(x - 12) \times (2x - 1)$

4. *Uncheat :* $2x^2 - 25x + 12 = (x - 12)(2x - 1)$

Now just put the whole thing back together :

$$-8x^2 + 100x - 48 = -4(x - 12)(2x - 1)$$

Exam Question: Easy

Factorise $x^2 + 3x - 10$

2. Algebra Basics

Exam Question: Medium

Factorise $x^2 - 49$

Exam Question: Hard

Solve $x^2 = 4(x - 3)^2$

2. Algebra Basics

2.5 REARRANGING FORMULAE

2.5.1 REARRANGING FORMULAE - BASICS

Choosing a method

- There are lots of different ways of remembering what to do when rearranging formulae – every teacher (and student) has their favourite, this is my recommendation....

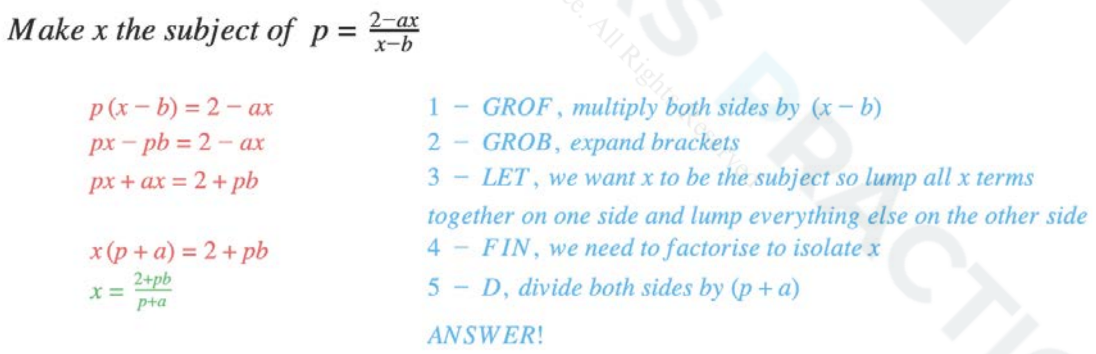
GROF GROBLET FIND ANSWER

- This is a mnemonic (way of remembering something) deals with everything except roots and powers (see the following set of notes for that) and can also be used for solving equations...

1. **GROF** – Get Rid Of Fractions
2. **GROB** – Get Rid Of Brackets
3. **LET** – Lump Everything Together (or Let's Examine Terms)
4. **FIN** – Factorise If Necessary
5. **D** – Divide

ANSWER!

Worked Example



Make x the subject of $p = \frac{2-ax}{x-b}$

$p(x-b) = 2 - ax$ 1 − *GROF*, *multiply both sides by* $(x-b)$

$px - pb = 2 - ax$ 2 − *GROB*, *expand brackets*

$px + ax = 2 + pb$ 3 − *LET*, *we want x to be the subject so lump all x terms together on one side and lump everything else on the other side*

$x(p+a) = 2 + pb$ 4 − *FIN*, *we need to factorise to isolate x*

$x = \frac{2+pb}{p+a}$ 5 − *D*, *divide both sides by* $(p+a)$

ANSWER!

2. Algebra Basics

2.5.2 REARRANGING FORMULAE - HARDER

SIR! GROF GROBLET FIND ANSWER!

- This is a mnemonic (way of remembering something) developed to deal with everything including roots and powers (for a simpler version see the previous set of notes) and can also be used for solving equations…

  SIR! – Squares, Indices, Roots

1. **SIR**! – Squares, Indices, Roots (can be used at any point)
2. **GROF** – Get Rid Of Fractions
3. **GROB** – Get Rid Of Brackets
4. **LET** – Lump Everything Together (or Let's Examine Terms)
5. **FIN** – Factorise if Necessary
6. **D** – Divide
   ANSWER!

Worked Example

Make x the subject of $p = \sqrt{\frac{2-ax^2}{x^2-b}}$

$p^2 = \frac{2-ax^2}{x^2-b}$	0 – *SIR! Square both sides*
$p^2(x^2-b) = 2-ax^2$	1 – *GROF, multiply both sides by* (x^2-b)
$p^2x^2 - p^2b = 2-ax^2$	2 – *GROB, expand brackets*
$p^2x^2 + ax^2 = 2+p^2b$	3 – *LET, x terms together, everything else on other side*
$x^2(p^2+a) = 2+p^2b$	4 – *FIN, factorise to isolate x term*
$x^2 = \frac{2+p^2b}{p^2+a}$	5 – *GROF, multiply both sides by* (x^2-b)
$x = \sqrt{\frac{2+p^2b}{p^2+a}}$	0 – *SIR! Square root both sides*
	ANSWER!

2. Algebra Basics

Exam Question: Easy

Make h the subject of the formula

$$t = \frac{gh}{10}$$

Exam Question: Medium

Make t the subject of the formula $2(d - t) = 4t + 7$

Exam Question: Hard

Make a the subject of the formula $p = \frac{3a + 5}{4 - a}$

Exam Question: V. Hard

Make t the subject of the formula

$$p = \frac{3 - 2t}{4 + t}$$

2. Algebra Basics

2.6 SOLVING LINEAR EQUATIONS

2.6.1 COLLECTING LIKE TERMS

Collecting like terms

- Expand brackets first!

1. TERMS are separated by + or –
2. “LIKE” terms must have exactly the same LETTER bit (the NUMBER bit can be different)
3. Add the COEFFICIENTS of like terms

Exam Tip

A “Coefficient” answers the question “how many?”

For example:

the coefficient of x in $2x^2 - 5x + 2$ is -5

and:

the coefficient of x in $ax^2 + bx + c$ is b

Worked Example

Simplify $x^2 - 3xy + 2x^2 + 4x - 2xy - x + 7$

1 – There are 7 terms to deal with

$x^2 + 2x^2 - 3xy - 2xy + 4x - x + 7$

2 – Grouping like terms – "x^2"s, "xy"s, "x"s and constants

3 – Add the coefficients

$= 3x^2 - 5xy + 3x + 7$

2. Algebra Basics

2.6.2 SOLVING LINEAR EQUATIONS

Solving equations…

… is just like rearranging formula so we'll use exactly the same method …

GROF GROBLET FIND ANSWER!

- This is a mnemonic (way of remembering something) developed to deal with everything except roots and powers (see Rearranging Formulae – Extra Bits notes for that) and can also be used for rearranging formulae…

1. **GROF** – Get Rid Of Fractions
2. **GROB** – Get Rid Of Brackets
3. **LET** – Lump Everything Together (or Let's Examine Terms)
4. **FIN** – Factorise If Necessary
5. **D** – Divide
   ANSWER!

Worked Example

Solve the equation $\frac{3x-2}{4-x} = -1$

$3x-2=-1(4-x)$	$1 - GROF$, *multiply both sides by* $(4-x)$, *careful with that* -1 !
$3x-2=-4+x$	$2 - GROB$, *expand brackets*
$2x=-2$	$3 - LET$, *x terms on one side, numbers on the other,* "$-x$" *and* "$+2$"
	$4 - FIN$ $-$ *you shouldn't need this step when solving equations*
$x=-1$	$5 - D$, *divide both sides by* 2
	ANSWER!

2. Algebra Basics

Exam Question: Easy

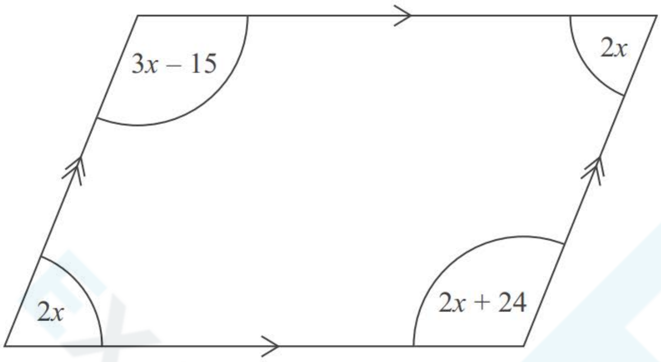


Diagram **NOT** accurately drawn

The diagram shows a parallelogram.
The sizes of the angles, in degrees, are

$2x$
$3x - 15$
$2x$
$2x + 24$

Work out the value of x.

2. Algebra Basics

Exam Question: Medium

ABC is a triangle.

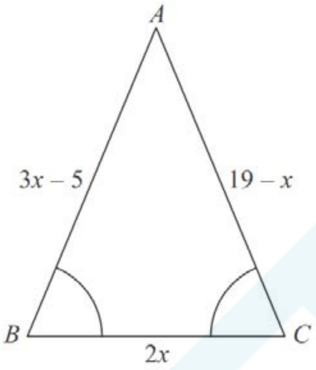


Diagram **NOT** accurately drawn

Angle *ABC* = angle *BCA*.

The length of side *AB* is $(3x - 5)$ cm.
The length of side *AC* is $(19 - x)$ cm.
The length of side *BC* is $2x$ cm.

Work out the perimeter of the triangle.
Give your answer as a number of centimetres.

Exam Question: Hard

Solve $\dfrac{4x-1}{5} + \dfrac{x+4}{2} = 3$

2. Algebra Basics

2.7 SUBSTITUTION

2.7.1 SUBSTITUTION

What is substitution?

- Substitution is where we replace letters in a formula with their values
- This allows you to find one other value that is in the formula

How do we substitute?

- Write down the FORMULA if not clearly stated in question
- SUBSTITUTE the numbers given – use ( ) around negative numbers
- SIMPLIFY if you can
- REARRANGE if necessary – it is usually easier to substitute first
- Do the CALCULATION – use a calculator if allowed

Worked Example

1. *Find the value of the expression* $2x(x+3y)$ *when* $x = 2$ *and* $y =- 4$

$2 \times 2 \times (2 + 3 \times (-4))$	1 – *Substitute the numbers given, use () around negatives*
$= 2 \times 2 \times (2 - 12)$	*You don't need all these lines of working but we've included*
$= 2 \times 2 \times (-10)$	*them here to remind you of order of operations*
$=- 40$	*Use a calculator if allowed*

2. *The formula* $P = 2l + 2w$ *is used to find the perimeter,* P *of a rectangle of length* l *and width* w
Given that a rectangle has a perimeter of 20 cm and a width of 4 cm, find its length.

$20 = 2 \times l + 2 \times 4$	1 – *Substitute the numbers given, no negatives*
$20 = 2l + 8$	2 – *Simplify*
$2l = 12$	3, 4 – *Rearrange and do the calculation*
$l = 6$	
Length is 6 cm	

2. Algebra Basics

Exam Question: Easy

You can work out the amount of medicine, c ml, to give to a child by using the formula

$$c = \frac{ma}{150}$$

m is the age of the child, in months.
a is an adult dose, in ml.

A child is 30 months old.
An adult's dose is 40 ml.

Work out the amount of medicine you can give to the child.

Exam Question: Medium

$A = 4bc$

$A = 100$
$b = 2$

Work out the value of c.

2. Algebra Basics

2.8 PROOF/REASONING

2.8.1 PROOF/REASONING - ALGEBRAIC

What is proof?

- Proof is the process of showing something is true in every case, often by using algebra

Given an algebraic expression…

1. Expand brackets
2. Simplify (work with left-hand side only if = sign involved)
3. Make sure you finish with the statement you are proving

Given just words…

- This is a bit trickier but turning it into algebra is usually the best thing to do

4. Give things names (use as few letters as possible):
   - **n** is "any integer" (or **m** or **k** or…)
   - **n + 1** is the integer after **n** ("consecutive")
   - **2n** is an even integer (**2n + 2** is the next one)
   - **2m** is a different even integer
   - **2n + 1** is an odd integer (and **2n + 3** is the next one)

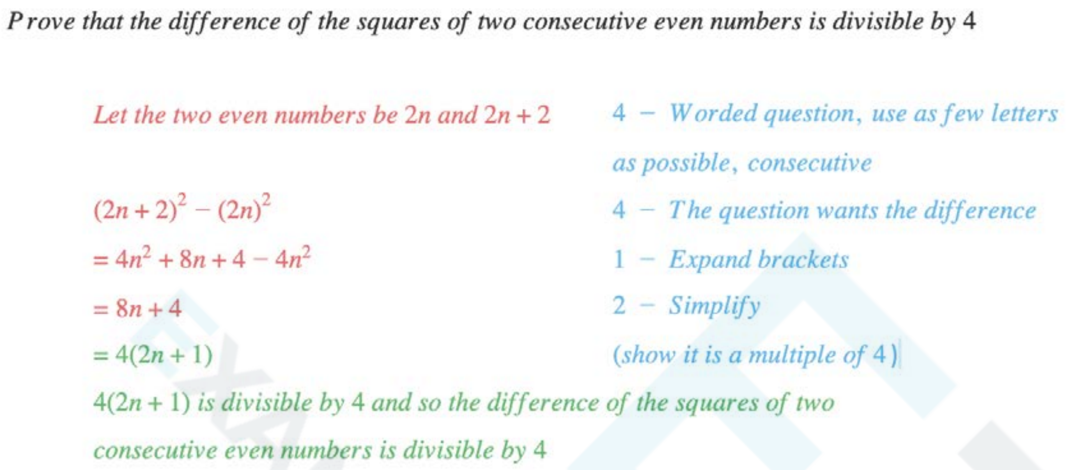
5. Then go on as above ("Given an Algebraic Expression")

Exam Tip

"A multiple of k" means it can be written as k(……) – ie. k × …

2. Algebra Basics

Worked Example



Prove that the difference of the squares of two consecutive even numbers is divisible by 4

Let the two even numbers be $2n$ *and* $2n+2$	4 – *Worded question, use as few letters as possible, consecutive*
$(2n+2)^2-(2n)^2$	4 – *The question wants the difference*
$=4n^2+8n+4-4n^2$	1 – *Expand brackets*
$=8n+4$	2 – *Simplify*
$=4(2n+1)$	(*show it is a multiple of* 4)

$4(2n+1)$ *is divisible by* 4 *and so the difference of the squares of two consecutive even numbers is divisible by* 4

Exam Question: Medium

Show that $(n+3)^2-(n-3)^2$ is an even number for all positive integer values of n.

Exam Question: Hard

Prove that

$$(2n+3)^2-(2n-3)^2 \text{ is a multiple of } 8$$

for all positive integer values of n.

2. Algebra Basics

Exam Question: V. Hard

Prove algebraically that the difference between the squares of any two consecutive integers is equal to the sum of these two integers.

2. Algebra Basics

2.9 FUNCTIONS

2.9.1 FUNCTIONS - BASICS

What is a function?

- A function is simply a mathematical "machine" that takes one set of numbers and changes them into another set of numbers according to a set rule
  eg. If the function (rule) is "double the number and add 1"
  - Putting 3 in to the function would give 2 × 3 + 1 = 7 out
  - Putting -4 in would give 2 × (-4) + 1 = -7 out
  - Putting a in would give 2a + 1 out
- The number being put into the function is often called the input
- The number coming out of the function is often called the output

What does a function look like?

- A function f can be written as:
  $f(x) = \ldots$ or $f : x \mapsto \ldots$
  which mean exactly the same thing
- eg. The function with the rule "triple the number and subtract 4" would be written:
  $f(x) = 3x - 4$ or $f{:}x \mapsto 3x - 4$
- In such cases, x would be the input and f would be the output
- Sometimes functions don't have names like f and can be written as y = …
  eg. $y = 3x - 4$

How does a function work?

1. A function has an INPUT (x) and OUTPUT (f or y)
2. Whatever goes in the bracket (instead of x)with f, replaces the x on the other side
   eg. For the function $f(x) = 2x + 1$
   - $f(3) = 2 \times 3 + 1 = 7$
   - $f(-4) = 2 \times (-4) + 1 = -7$
   - $f(a) = 2a + 1$

2. Algebra Basics

Worked Example

$A\ function\ is\ defined\ as\ f(x) = 3x^2 - 2x + 1$

$(a)\ Find\ f(7)$

$(b)\ Find\ f(x+3)$

$(a)\ f(7) = 3 \times 7^2 - 2 \times 7 + 1$ $2 -$ $Substitute\ x's\ for\ 7's\ on\ the\ right\ hand\ side$

$= 147 - 14 + 1$ $Use\ a\ calculator\ if\ allowed, if\ not\ take\ your$

$= 134$ $time\ and\ be\ accurate.$

$(b)\ f(x+3) = 3(x+3)^2 - 2(x+3) + 1$ $2 -\ Substitute\ x's\ for\ (x+3)'s$

$= 3(x^2 + 6x + 9) - 2x - 6 + 1$ $Expand\ and\ simplify$

$= 3x^2 + 18x - 2x + 27 - 5$

$= 3x^2 + 16x + 22$

2. Algebra Basics

2.9.2 COMPOUND FUNCTIONS

What is a compound function?

- A compound function is one function applied to the output of another function

What do compound functions look like?

- The notation you will see is:
  fg(x)
- it can be written as:
  f(g(x))
  and means "f applied to the output of g(x)" – ie. g(x) happens FIRST !

How does a compound function work?

- If you are putting a number into fg(x):

1. Put the number into g(x)
2. Put the output into f(x)
   eg. if f(x) = 2x + 1 and g(x) = 1/x
   then fg(2) = f(½) =2 × ½ + 1 = 2
   and gf(2) = g(2 × 2 + 1) = g(5) = 1/5

- If you are using algebra:

1. For fg(x) put g(x) wherever you see x in f(x)
2. Substitute g(x) with the right hand side of g(x)=…
3. SIMPLIFY if necessary
   eg. if f(x) = 2x + 1 and g(x) = 1 / x
   then fg(x) = f(1/x) = 2 × 1 / x + 1 = 2 / x +1
   and gf(x) = g(2 × x + 1) = g(2x + 1) = 1 / 2x+1

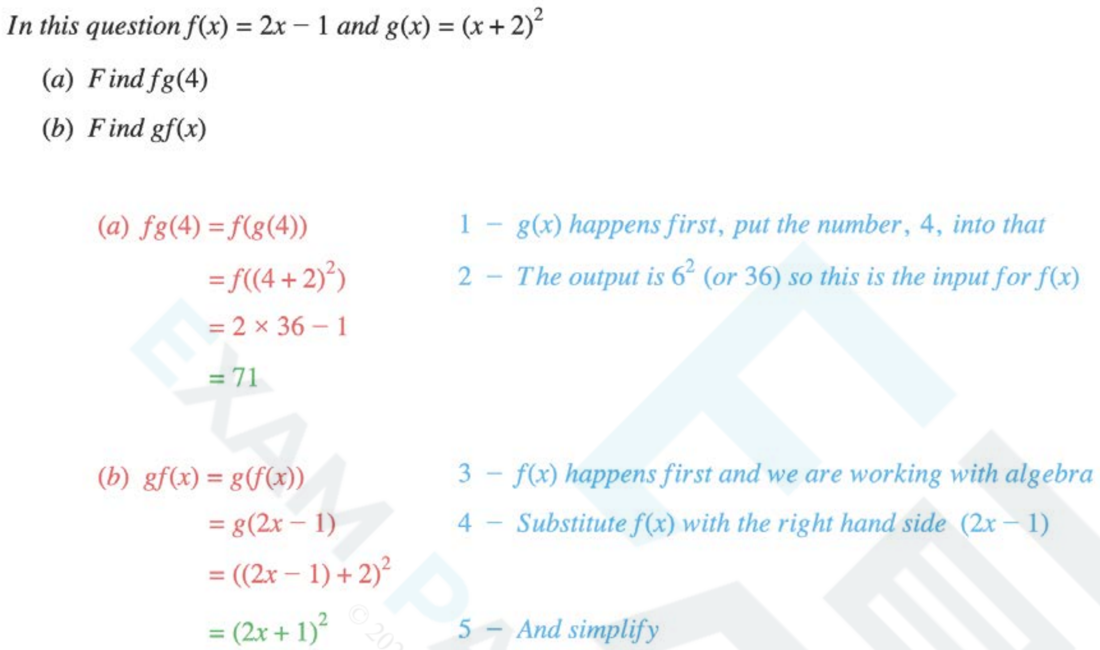
Exam Tip

Make sure you are applying the functions in the correct order.

The letter nearest the bracket is the function applied first.

2. Algebra Basics

Worked Example



In this question $f(x) = 2x - 1$ *and* $g(x) = (x+2)^2$

(*a*) *Find* $fg(4)$

(*b*) *Find* $gf(x)$

(*a*) $fg(4) = f(g(4))$ — 1 – *$g(x)$ happens first, put the number, 4, into that*

$= f((4+2)^2)$ — 2 – *The output is 6^2 (or 36) so this is the input for $f(x)$*

$= 2 \times 36 - 1$

$= 71$

(*b*) $gf(x) = g(f(x))$ — 3 – *$f(x)$ happens first and we are working with algebra*

$= g(2x - 1)$ — 4 – *Substitute $f(x)$ with the right hand side $(2x-1)$*

$= ((2x-1)+2)^2$

$= (2x+1)^2$ — 5 – *And simplify*

2. Algebra Basics

2.9.3 INVERSE FUNCTIONS

What is an inverse function?

- An inverse function does the exact opposite of the function it came from
- Eg. if the function “doubles the number and adds 1”
  then its inverse will “subtract 1 and halve the result”
- It is the **INVERSE** operations in the **reverse** order

What do inverse functions look like?

- An inverse function f-1 can be written as:

  $f^{-1}(x)$ = … or f^{-1} : x ↦ …
- Eg. if f(x) = 2x + 1 its inverse can be written as:

  $f^{-1}(x)$ = x – 1 / 2 or f^{-1}: x ↦ x – 1 / 2

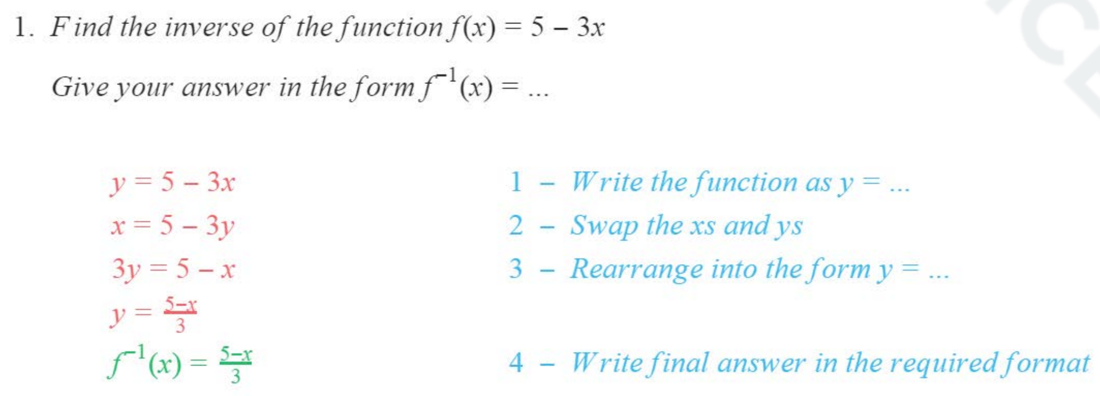
What do you find an inverse function?

1. Write the function in the form y=…
2. SWAP the xs and ys to get x=…
3. REARRANGE to give y=… again
4. Write as $f^{-1}(x)$ = … (or f^{-1} : x ↦ …)

- Eg. if f(x)=2x+1 its inverse can be found as follows …
  y= 2x + 1 1 – 1. Write the function as y=…
  x = 2y + 1 2 – 2. Swap the xs and ys
  x – 1= 2y 3 – 3. Rearrange into the form y=… (or …=y)
  x – 1 / 2 = y

  $f^{-1}(x)$ = x – 1 / 2 – 4. Write as $f^{-1}(x)$=…

Worked Example



1. *Find the inverse of the function* $f(x) = 5 - 3x$

   Give your answer in the form $f^{-1}(x) = \ldots$

$y = 5 - 3x$	1 – *Write the function as* $y = \ldots$
$x = 5 - 3y$	2 – *Swap the xs and ys*
$3y = 5 - x$	3 – *Rearrange into the form* $y = \ldots$
$y = \frac{5-x}{3}$	
$f^{-1}(x) = \frac{5-x}{3}$	4 – *Write final answer in the required format*

2. Algebra Basics

Exam Question: Medium

f and g are functions such that

$$f(x) = \frac{2}{x^2} \qquad \text{and} \qquad g(x) = 4x^3$$

(a) Find f(–5)

(b) Find fg(1)

Exam Question: Hard

The function f is such that

$$f(x) = 4x - 1$$

(a) Find $f^{-1}(x)$

The function g is such that

$g(x) = kx^2$ where k is a constant.

Given that fg(2) = 12

(b) work out the value of k

2. Algebra Basics

2.10 ALGEBRAIC FRACTIONS

2.10.1 ALGEBRAIC FRACTIONS - ADDING & SUBTRACTING

What is an algebraic fraction?

- An algebraic fraction is simply a fraction with an algebraic expression on the top (numerator) and/or the bottom (denominator)

Adding & subtracting

- Rules same as numeric fractions:

1. Find the lowest common bottom (denominator)
2. Write fractions with new bottoms
3. Multiply tops by the same as bottoms
4. Write as a single fraction (take care if subtracting)
5. Simplify the top

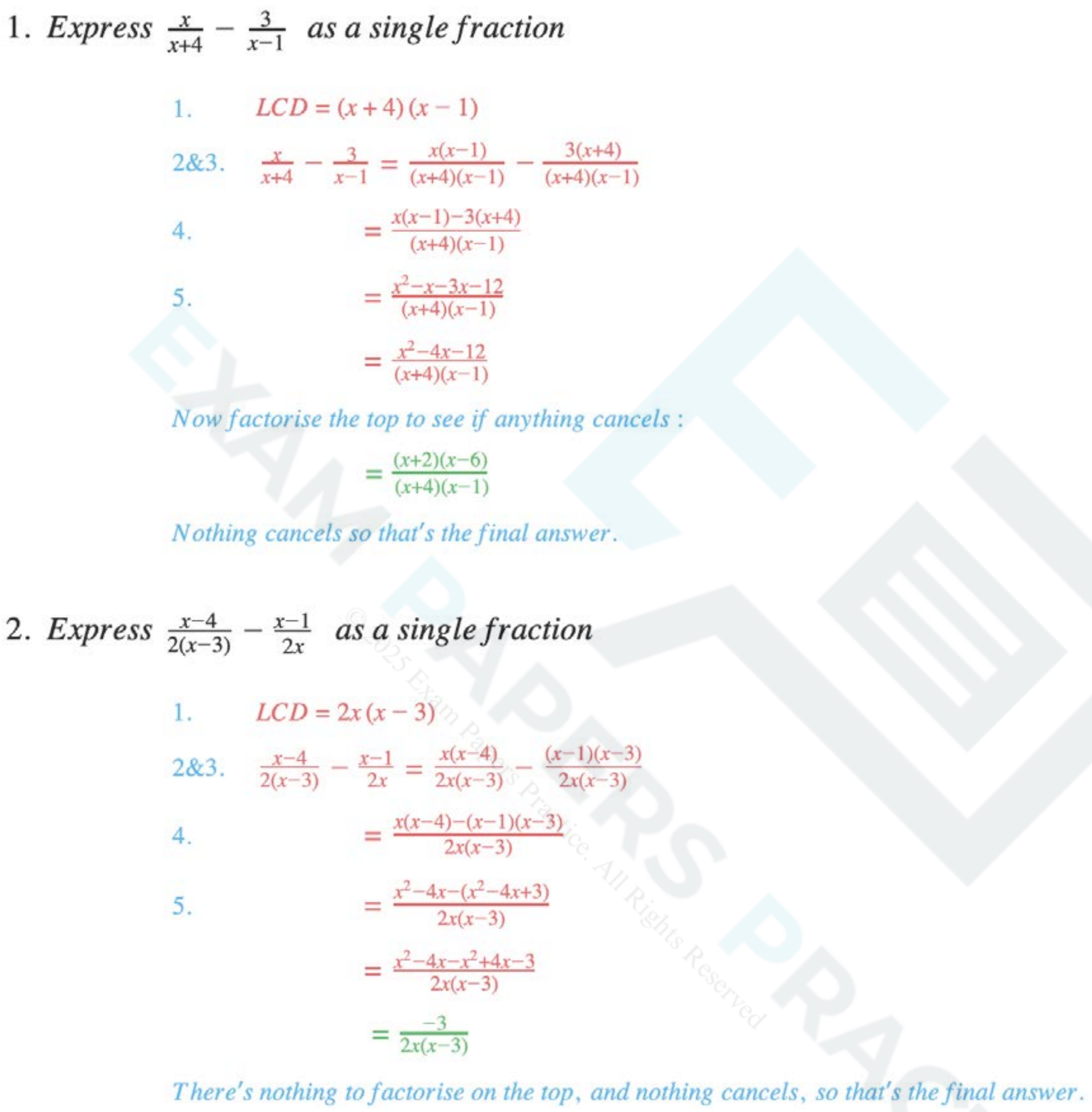
- Always leave your answer in as simple a form as possible (see Algebraic Fractions – Simplifying)

Exam Tip

Leaving the top and bottom of the fraction in factorised form will help you see if anything cancels.

2. Algebra Basics

Worked Example



1. *Express* $\frac{x}{x+4} - \frac{3}{x-1}$ *as a single fraction*

1. $LCD = (x+4)(x-1)$

2&3. $\frac{x}{x+4} - \frac{3}{x-1} = \frac{x(x-1)}{(x+4)(x-1)} - \frac{3(x+4)}{(x+4)(x-1)}$

4. $= \frac{x(x-1)-3(x+4)}{(x+4)(x-1)}$

5. $= \frac{x^2-x-3x-12}{(x+4)(x-1)}$

$= \frac{x^2-4x-12}{(x+4)(x-1)}$

Now factorise the top to see if anything cancels :

$= \frac{(x+2)(x-6)}{(x+4)(x-1)}$

Nothing cancels so that's the final answer.

2. *Express* $\frac{x-4}{2(x-3)} - \frac{x-1}{2x}$ *as a single fraction*

1. $LCD = 2x(x-3)$

2&3. $\frac{x-4}{2(x-3)} - \frac{x-1}{2x} = \frac{x(x-4)}{2x(x-3)} - \frac{(x-1)(x-3)}{2x(x-3)}$

4. $= \frac{x(x-4)-(x-1)(x-3)}{2x(x-3)}$

5. $= \frac{x^2-4x-(x^2-4x+3)}{2x(x-3)}$

$= \frac{x^2-4x-x^2+4x-3}{2x(x-3)}$

$= \frac{-3}{2x(x-3)}$

There's nothing to factorise on the top, and nothing cancels, so that's the final answer.

2. Algebra Basics

2.10.2 ALGEBRAIC FRACTIONS - SIMPLIFYING FRACTIONS

What is an algebraic fraction?

- An algebraic fraction is simply a fraction with an algebraic expression on the top (numerator) and/or the bottom (denominator)

How do you simplify an algebraic fraction?

- When you have a single Algebraic Fraction (or two multiplied together) you may be able to simplify things by cancelling common factors
- Factorise top and bottom
- Cancel common factors
  That's it!

Exam Tip

If you are asked to simplify an algebraic fraction and have to factorise the top or bottom it is very likely that one of the factors will be the same on the top and the bottom – you can use this to help you factorise difficult quadratics!

Worked Example

$$Simplify\ \frac{4x+6}{2x^2-7x-15}$$

1. $$Top : 4x + 6 = 2\,(2x + 3)$$

$Use\ the\ fact\ that\ there\ is\ a\ factor\ of\ \ (2x + 3)\ to\ help\ factorise\ the\ bottom :$

$$Bottom : 2x^2 - 7x - 15 = (2x + 3)\,(x - 5)$$

$$so \qquad \frac{4x+6}{2x^2-7x-15} = \frac{2(2x+3)}{(2x+3)(x-5)}$$

2. $$= \frac{2}{(x-5)}$$

2. Algebra Basics

2.10.3 ALGEBRAIC FRACTIONS - MULTIPLYING & DIVIDING

What is an algebraic fraction?

- An algebraic fraction is simply a fraction with an algebraic expression on the top (numerator) and/or the bottom (denominator)

Dividing algebraic fractions

- **Never** try to divide fractions
- **Instead** “flip’n’times”
- So “÷a/b” becomes “×b/a”
  and then follow the rules for multiplying…

Multiplying algebraic fractions

1. **Simplify** by factorising and cancelling (ignore the × between the fractions)
2. Multiply the **tops** (numerators)
3. Multiply the **bottoms** (denominators)
4. **Simplify** by factorising and cancelling (if you missed something earlier).

Worked Example

Divide $\frac{x+3}{x-4}$ *by* $\frac{2x+6}{x^2-16}$, *giving your answer as a simplified fraction.*

First "flip'n'times" :

$$\frac{x+3}{x-4} \div \frac{2x+6}{x^2-16} = \frac{x+3}{x-4} \times \frac{x^2-16}{2x+6}$$

Now follow rules for multiplying :

1.

$$\frac{x+3}{x-4} \times \frac{x^2-16}{2x+6} = \frac{x+3}{x-4} \times \frac{(x-4)(x+4)}{2(x+3)}$$

Cancel the $(x+3)$ *s and the* $(x-4)$ *s :*

2&3.

$$= \frac{1}{1} \times \frac{1\times(x+4)}{2\times 1}$$

$$= \frac{x+4}{2}$$

4. *No need to factorise and cancel again as that was done in* 1.

2. Algebra Basics

Exam Question: Medium

(a) Simplify fully $\dfrac{x^2 + 3x - 4}{2x^2 - 5x + 3}$

(b) Write $\dfrac{4}{x+2} + \dfrac{3}{x-2}$ as a single fraction in its simplest form.

Exam Question: Hard

Write as a single fraction in its simplest form

$$\frac{2}{y+3} - \frac{1}{y-6}$$

Exam Question: V. Hard

Simplify $\dfrac{x^2 - 9}{2x^2 + 5x - 3}$

3. Solving Equations & Inequalities

CONTENTS

3.1 QUADRATIC FORMULA

3.1.1 QUADRATIC FORMULA

3. Solving Equations & Inequalities

What is a quadratic equation?

- A quadratic equation looks like (or can/should be made to look like) this:
  $ax^2 + bx + c = 0$ (as long as $a \neq 0$)

What is the quadratic formula?

- The Quadratic Formula is a way of finding the roots (also called solutions) of a quadratic equation:

$$If\ ax^2 + bx + c = 0\ then\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The part of the formula under the square root (b2 – 4ac) is called the discriminant and it tells you a lot about the roots:
  - If $b^2 - 4ac > 0$ then there are two distinct (different) real roots (*)
  - If $b^2 - 4ac = 0$ then there is one real root (two repeated roots)
  - If $b^2 - 4ac < 0$ then there are no real roots (and the equation cannot be solved)
- Also (and not a lot of people know this!):
- If $b^2 - 4ac$ is a perfect square (1, 4, 9, 16, ...) then the quadratic can be factorised
  (*) We have to call them "real" roots for a very mathsy reason – don't worry about it too much – all the numbers you come across at GCSE are real!

A quick note about calculators

If you have an "advanced" scientific calculator (like ones from the Casio Classwiz range) you may find it will solve quadratic equations for you. However, at GCSE, you should avoid using this feature as marks are awarded for method. At best you can use them to check your final answers but be aware that such calculators often show solutions even when the roots are not real.

When do you use the quadratic formula?

You can use the Quadratic Formula whenever you need to solve a quadratic equation UNLESS you are specifically asked to do it by Factorising or Completing the Square. (You could still use the formula to check your answer!)

If the question asks you to give your answers to a certain degree of accuracy (eg 3 significant figures or 2 decimal places etc) then you will not be able to factorise and you MUST use the formula.

3. Solving Equations & Inequalities

How do you use the quadratic formula?

- The safest way to avoid errors is to break the formula down into stages:

1. WRITE DOWN the values of a, b and c – ie a = … , b = … and c = …
2. Work out the DISCRIMINANT, $d = b^2 - 4ac$
3. Substitute values into the QUADRATIC FORMULA including d
4. Work out the TWO ANSWERS on your calculator
   (Type in the formula twice, once with the "+" and once with the "-")

Exam Tip

Write down more digits than you need from your calculator display.

Then do the rounding as a separate stage.

This will avoid errors and could save you a mark.

3. Solving Equations & Inequalities

Worked Example

Find the roots of the equation $3x^2 - 2x - 4 = 0$, *giving your answers to 2 decimal places.*

	Exam Tip 2 – we have to use the formula
$a = 3,\ b = -2,\ c = -4$	*1 – Write down the values of a, b and c*
$d = b^2 - 4ac$	*2 – Work out the value of d, the discriminant*
$d = (-2)^2 - 4 \times 3 \times (-4)$	*Always be careful with negatives, use () around them*
$d = 52$	
$x = \frac{2 \pm \sqrt{52}}{2 \times 3}$	*3 – Substitute values into the quadratic formula*
$x = 1.53518375...$ *or* $-0.86851709...$	*4 – Using the " + " and using the " – "*
	Exam Tip 3 – write down more digits than required
$x = 1.54$ *or* -0.87 *to 2 dp*	*Round to 2 decimal places for final answer*

Exam Question: Medium

Solve $3x^2 - 4x - 2 = 0$
Give your solutions correct to 3 significant figures.

3. Solving Equations & Inequalities

Exam Question: Hard

The diagram shows a trapezium.

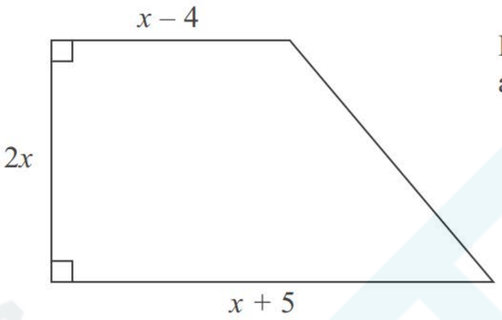


Diagram **NOT** accurately drawn

All the measurements are in centimetres.

The area of the trapezium is 351 cm^2.

(a) Show that $2x^2 + x - 351 = 0$

(b) Work out the value of x.

3. Solving Equations & Inequalities

3.2 COMPLETING THE SQUARE

3.2.1 COMPLETING THE SQUARE

What is completing the square?

- “Completing the square” is something that can be done to a quadratic expression (to make it easier to work with or more useful in some way)
- It involves writing the quadratic expression $x^2 + px + q$ in the form $(x + a)^2 + b$

Exam Tip

A question on this topic (see the question below) may use the **Identity Symbol** “≡” instead of an **Equals Sign** “=”.

This tells you that what is on the left is exactly the same as what is on the right (no matter what value any letters take).

You should never try to solve an identity like you might try to solve an equation.

Worked Example

3. Solving Equations & Inequalities

1. *Find integers a and b such that* $x^2 - 2x - 4 \equiv (x+a)^2 + b$.

$a = \frac{1}{2} \times (-2) = -1$	1 − *Halve the coefficient of the x term :* $a = \frac{1}{2}p$
$b = -4 - (-1)^2 = -5$	2 − *Subtract the square of a from q :* $b = q - a^2$
$x^2 - 2x - 4 \equiv (x-1)^2 - 5$	3 − *Write down your answer using a and b*

2. *Solve the equation* $4x^2 - 16x + 15 = 0$ *by completing the square*.

$4x^2 - 16x + 15 = 4(x^2 - 4x + \frac{15}{4})$	*Special case :* $m \neq 1$, *divide by* 4
$= 4((x-2)^2 + \frac{15}{4} - (-2)^2)$	1, 2 − *Halve* −4 *and subtract* $(-2)^2$ *from q*
$= 4((x-2)^2 - \frac{1}{4})$	3 − *Finish completing the square*
$= 4(x-2)^2 - 1$	*Special case : Multiply by m*
$4(x-2)^2 - 1 = 0$	*Now we can solve by rearranging*
$4(x-2)^2 = 1$	
$(x-2)^2 = \frac{1}{4}$	
$x - 2 = \pm \frac{1}{2}$	$\sqrt{\frac{1}{4}} = \pm \frac{1}{2}$
$x = 2 \pm \frac{1}{2}$	
$x = \frac{5}{2}$ *or* $x = \frac{3}{2}$	*Decimals or fractions fine for final answers*
	unless a specific format is asked for

3. Solving Equations & Inequalities

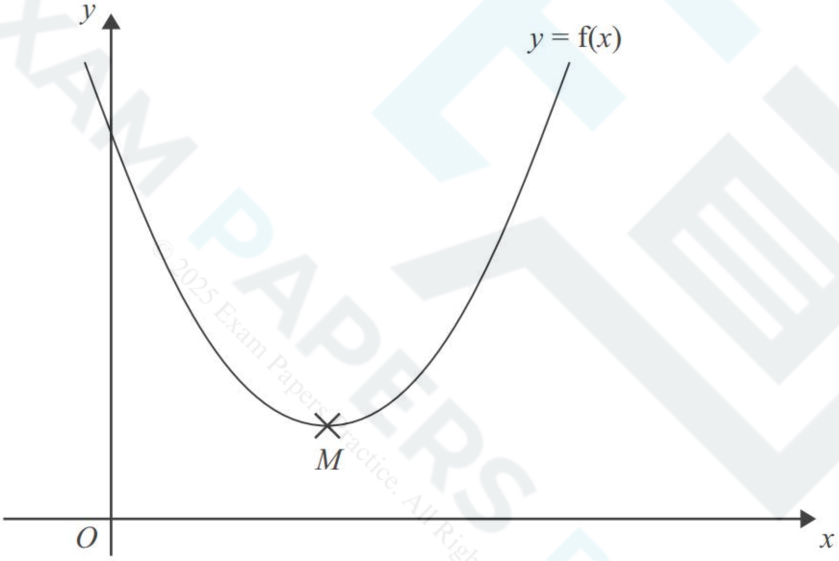
Exam Question: Medium

The expression $x^2 - 8x + 21$ can be written in the form $(x - a)^2 + b$ for all values of x.

(a) Find the value of a and the value of b.

The equation of a curve is $y = f(x)$ where $f(x) = x^2 - 8x + 21$

The diagram shows part of a sketch of the graph of $y = f(x)$.



The minimum point of the curve is M.

(b) Write down the coordinates of M.

Exam Question: Hard

(a) Write $2x^2 + 16x + 35$ in the form $a(x + b)^2 + c$ where a, b, and c are integers.

(b) Hence, or otherwise, write down the coordinates of the turning point of the graph of $y = 2x^2 + 16x + 35$

3. Solving Equations & Inequalities

> CHECK YOUR ANSWERS AT SAVEMYEXAMS.CO.UK

3. Solving Equations & Inequalities

3.3 LINEAR SIMULTANEOUS EQUATIONS

3.3.1 LINEAR SIMULTANEOUS EQUATIONS

What are linear simultaneous equations?

- When there are two unknowns (say x **and** y) in a problem, we need two equations to be able to find them both: these are called **Simultaneous Equations**

- If they just have x and y in them (no x^2 or y^2) then they are **Linear** Simultaneous Equations (They can be represented by two straight lines on a graph – the two answers are the x and y coordinates of the point of intersection of the lines)

- You may have to use the information in the question to write down the equations
- Look for an "is" (or equivalent) in the question and make sure you know what the letters you are using stand for
    - For example, if the question says:
      "The cost of 5 apples and 3 bananas is $1.35"
      then you can write down the equation: 5a + 3b = 135
      where a is the cost of an apple in cents (and b is the cost of a banana in cents)

How do you solve linear simultaneous equations?

- The method described here is called the Elimination (or Balance) Method:**Label** the equations **A** and **B**

1. **Multiply A** and/or **B** by numbers to make the coefficients of x or y the same size
2. **Add** or **subtract** appropriate equations to eliminate that variable
   (Note: if the coefficients you made the same size in 1. have different signs you will be adding. If they have the same signs you will be subtracting.)
3. **Solve** the resulting equation
4. **Substitute** back into **A** or **B** and solve to find the other variable
5. **Check** your answer by substituting into the equation you didn't use in 4

Exam Tip

If one of the equations is written in the form **y = …** or **x = …** then you can use the method of **Substitution** (see Simultaneous Equations – Quadratic).

3. Solving Equations & Inequalities

Worked Example

Solve the simultaneous equations :

$3x - y = 13$ $\quad A$

$2x + 3y = 5$ $\quad B$

1. $A \times 3$: $\quad 9x - 3y = 39$

The coefficients of y in the last two equations are the same size but have different signs so we can add these equations together to eliminate y :

2. $B + A \times 3$: $\quad 11x = 44$
3. *Solving gives :* $\quad x = 4$
4. *Sub x into A :* $\quad 3 \times 4 - y = 13$

   and solve : $\quad - y = 1$

   $y =- 1$
5. *Check in B :* $\quad 2 \times 4 + 3 \times (- 1) = 5$ ✓ *It works!*

So the answer is

$x = 4, y =- 1$

Exam Question: Easy

Solve the simultaneous equations

$4x + y = 25$

$x - 3y = 16$

3. Solving Equations & Inequalities

Exam Question: Medium

Solve the simultaneous equations

$$5x + 2y = 11$$
$$4x - 3y = 18$$

Exam Question: Hard

A cinema sells adult tickets and child tickets.

The total cost of 3 adult tickets and 1 child ticket is £30
The total cost of 1 adult ticket and 3 child tickets is £22

Work out the cost of an adult ticket and the cost of a child ticket.

3. Solving Equations & Inequalities

3.4 QUADRATIC SIMULTANEOUS EQUATIONS

3.4.1 QUADRATIC SIMULTANEOUS EQUATIONS

What are quadratic simultaneous equations?

- When there are two unknowns (say x and y) in a problem, we need two equations to be able to find them both: these are called **Simultaneous Equations**
- If there is an x^2 or y^2 in one of the equations then they are **Quadratic** (or **Non-Linear**) Simultaneous Equations
  (They can be represented by a straight line and a curve on a graph – the two pairs of answers are the points of intersection of the line and the curve)

How do you solve quadratic simultaneous equations?

- This is called the **Substitution Method**:**Label** the equations **A** and **B**

1. Rearrange the **linear** equation to **y = …** (or **x = …**)
2. **Substitute** for **y** (or **x**) in the **quadratic** equation
3. **Multiply out** brackets
4. **Rearrange** to “quadratic = 0”
5. **Solve** using appropriate method (Factorisation or Formula)
6. **Substitute** back into the linear equation to find the other variable
7. **Check** your answer by substituting into the equation you didn’t use in 6

Exam Tip

If the resulting quadratic has a **repeated root** then the line is a **tangent** to the curve.

If the resulting quadratic has **no roots** then the line does not intersect with the curve – or you have made a mistake!

When giving your final answer, make sure you indicate which x and y values go together. If you don’t make this clear you can lose marks for an otherwise correct answer.

3. Solving Equations & Inequalities

Worked Example

Solve the simultaneous equations :

$$3x - y = 13 \quad A$$

$$x^2 + y^2 = 17 \quad B$$

1. *Rearrange A :* $y = 3x - 13$

2. *Substitute into B :* $x^2 + (3x - 13)^2 = 17$

3. *Multiply out :* $x^2 + 9x^2 - 78x + 169 = 17$

4. *Rearrange :* $10x^2 - 78x + 152 = 0$

5. *Solve (÷ 2 first) :* $5x^2 - 39x + 76 = 0$

Using the Formula or Factorising gives :

$$x = 4 \quad or \quad x = \frac{19}{5}$$

6. *Substitute into A :* $y = -1 \quad or \quad y = -\frac{8}{5}$

7. *Now check in B :* $4^2 + (-1)^2 = 17$ ✓ *That one works!*

$$\left(\frac{19}{5}\right)^2 + \left(-\frac{8}{5}\right)^2 = 17$$ ✓ *So does that one!*

So the answers are

$$x = 4, \ y = -1$$

$$or \ \ x = \frac{19}{5}, \ y = -\frac{8}{5}$$

3. Solving Equations & Inequalities

Exam Question: Medium

Solve the equations

$$x^2 + y^2 = 36$$

$$x = 2y + 6$$

Exam Question: Hard

Solve the simultaneous equations
$$\begin{aligned} x^2 + y^2 &= 9 \\ x + y &= 2 \end{aligned}$$

Give your answers correct to 2 decimal places.

3. Solving Equations & Inequalities

3.5 INEQUALITIES

3.5.1 SOLVING INEQUALITIES - LINEAR

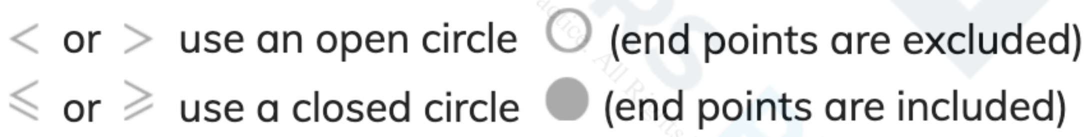
What is a linear inequality?

- An **Inequality** tells you that one expression is greater than (“>”) or less than (“<”) another
    - “⩾” means “greater than or equal to”
    - “⩽” means “less than or equal to”
- A **Linear Inequality** just has an **x** (and/or a **y**) etc in it and no **x^2** or similar
- For example, 3x + 4 ⩾ 7 would be read “3x + 4 is greater than or equal to 7”

Solving linear inequalities

- Solving linear inequalities is just like **Solving Linear Equations** (so review these notes first)
- You also need to know how to use **Number Lines** and deal with **“Double” Inequalities**

1. Same rules as solving equations: **GROF GROBLET FIND ANSWER!**
But do **NOT** multiply (or divide) by negative numbers
2. When drawing **NUMBER LINES:**



< or > use an open circle ○ (end points are excluded)
⩽ or ⩾ use a closed circle ● (end points are included)

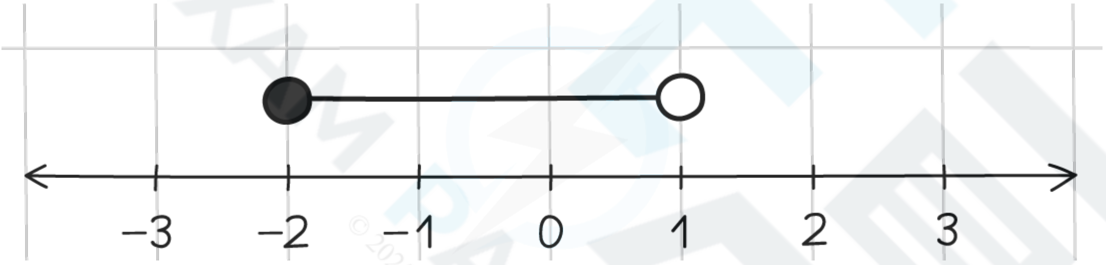
3. For “double” inequalities just do the same thing to all three parts

3. Solving Equations & Inequalities

Worked Example

1. *Solve the inequality* $-7 \le 3x - 1 < 2$ *illustrating your answer on a number line.*

$-7 \le 3x - 1 < 2$	3 – *First thing to notice is that this is a double inequality*
$-6 \le 3x < 3$	1 – *We are trying to isolate x in the middle so first step is to add 1 to all three parts*
$-2 \le x < 1$	1 – *Divide all three parts by 3*



2 – *Illustrate your final answer on a number line, using open and closed circles as appropriate*

3. Solving Equations & Inequalities

3.5.2 SOLVING INEQUALITIES - QUADRATIC

What is a quadratic inequality?

- An **Inequality** tells you that one expression is greater than (“>”) or less than (“<”) another
  - “⩾” means “greater than or equal to”
  - “⩽” means “less than or equal to”
- A **Quadratic Inequality** has an **x^2** (or a **y^2** etc) in it
- For example, $3x^2 + 4x \geqslant 7$ would be read “$3x^2 + 4x$ is **greater than or equal to** 7”

Solving quadratic inequalities

- Solving quadratic inequalities requires you to be able to factorise quadratics so you might want to have a look at those notes first
- You also need to know how to **SKETCH** graphs and use **Number Lines**

1. **REARRANGE** so that your “quadratic >0” (or <0)
2. **FACTORISE** the quadratic
3. **SKETCH** a graph of the quadratic
4. Use x-axis as a **NUMBER LINE** to show where the graph is above (if ≥ or >) or below (if ≤ or <) the x-axis
5. Turn the number line into a **PAIR** of inequalities or a **DOUBLE** inequality

Exam Tip

• When dealing with quadratic inequalities always make sure that you end up with a positive number in front of the x^2

• For example, to solve the inequality $4 – 2x^2 > 5x$ start by adding the $2x^2$ to (and subtracting 4 from both sides to get $0 > 2x^2 + 5x -4$) rather than subtracting the 5x (and getting $4 – 5x -2x^2 > 0$)

• Why? Because it’s much easier to factorise a quadratic when the coefficient of x^2 is positive

3. Solving Equations & Inequalities

Worked Example

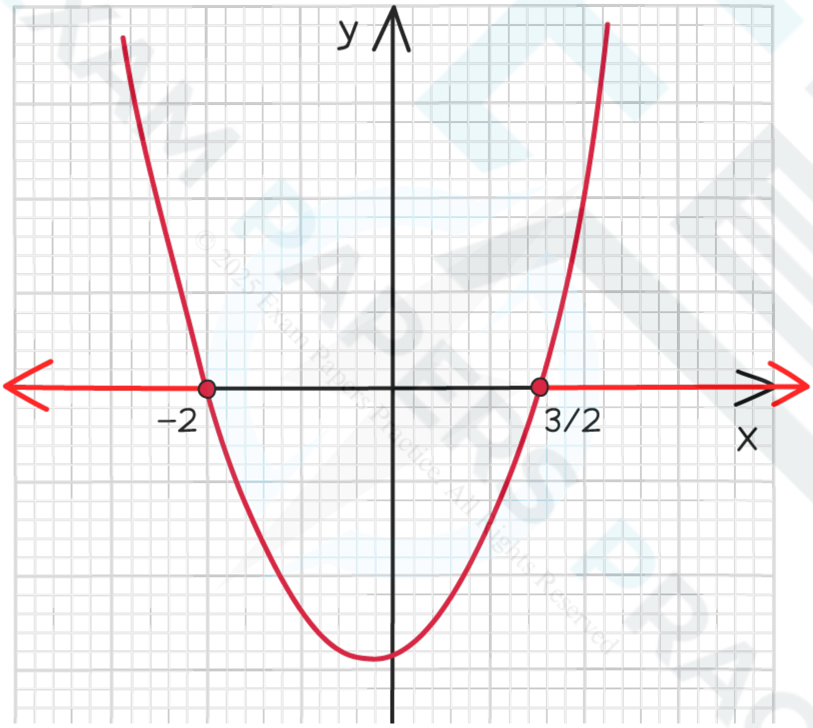
1. *Solve the inequality* $6 - 2x^2 \le x$ *illustrating your answer on a number line*

$0 \le 2x^2 + x - 6$ *Exam Tip, 1 – Add* $2x^2$ *to and subtract 6 from both sides*

If you prefer you can rewrite this as $2x^2 + x - 6 \ge 0$

$0 \le (2x-3)(x+2)$ *2 – Factorise the quadratic*

3 – Sketch the graph of $y = (2x-3)(x+2)$



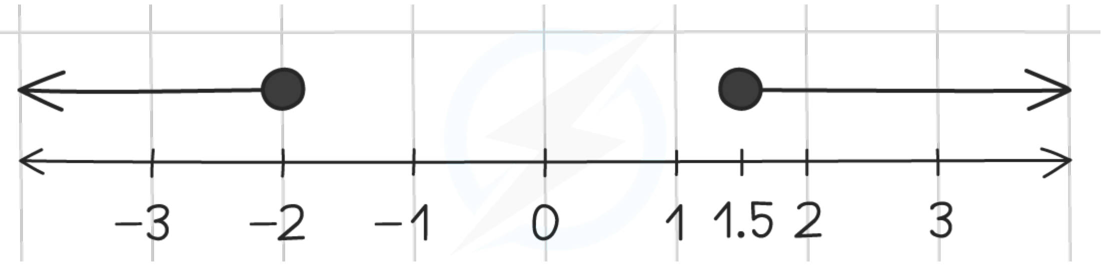
Note that this is not your final answer

number line diagram

4 – The quadratic $0 \le 2x^2 + x - 6$ *is true when the graph is above the x axis (x values marked in red)*

$x \le -2$ *or* $x \ge 3/2$ *5 – The solution is a pair of inequalities*

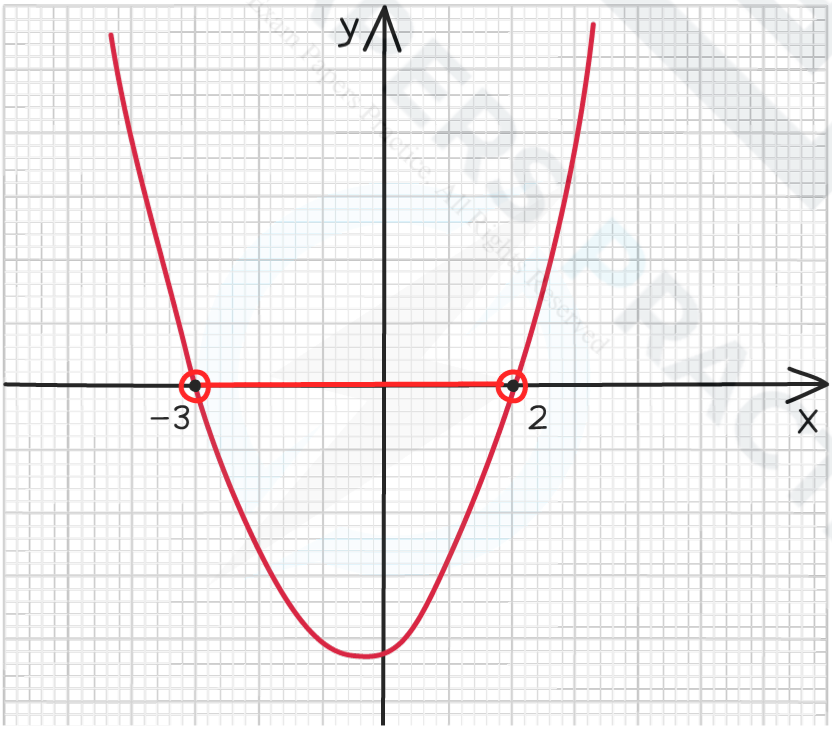
And illustrate on a number line, using closed circles



2. *Solve the inequality* $2 - x + 6x^2 \le 5x^2 - 2x + 8$.

$x^2 + x - 6 \le 0$	1 – *Take* $5x^2$ *from both sides to leave a positive* x^2 *term*
$(x+3)(x-2) \le 0$	2 – *Factorise the quadratic*

3 – *Sketch the graph of* $y = (x+3)(x-2)$



3. Solving Equations & Inequalities

$-3 < x < 2$

4 – The quadratic $x^2 + x - 6 < 0$ is true when the graph is below the x axis (x values marked in red)

5 – The solution is a double inequality

This question does not require the answer on a number line

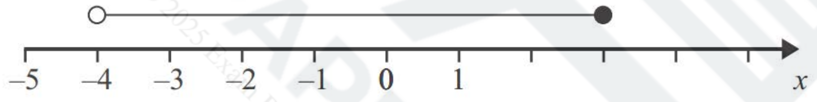
Exam Question: Easy

(a) n is an integer.

$$-1 \leqslant n < 4$$

List the possible values of n.

(b)



Write down the inequality shown in the diagram.

(c) Solve $y - 2 > 5$

Exam Question: Medium

m is an integer such that $-2 < m \leqslant 3$

(a) Write down all the possible values of m.

(b) Solve $7x - 9 < 3x + 4$

3. Solving Equations & Inequalities

Exam Question: Hard

Solve $2x^2 + 3x - 2 > 0$

3. Solving Equations & Inequalities

3.6 INEQUALITIES ON A GRAPH

3.6.1 INEQUALITIES ON GRAPHS - DRAWING

How do we draw inequalities on a graph?

- First, see Straight Line Graphs and Linear Inequalities

Once you do…

1. **DRAW** the line (as if using "=") for each inequality
   Use a **solid** line for ≤ or ≥ (to indicate the line is included)
   Use **dotted** line for < or > (to indicate the line is not included)
2. **DECIDE** which side of line is wanted:
   Below line if ≤ or <
   Above line if ≥ or >
   (Use the point (0, 0) as a test if unsure)
3. Shade **UNWANTED** side of each line (unless the question says otherwise)
   This is because it is easier, with pen/pencil/paper at least, to see which region has not been shaded at all than it is to look for a region that has been shaded 2-3 times or more
   (Graphing software often shades the area that is required but this is easily overcome by reversing the inequality sign)

Worked Example

1. *On the axes given show the region that satisfies the three inequalities*

$$3x + 2y \geq 12 \qquad y < 2x \qquad x < 3$$

Label the region R.

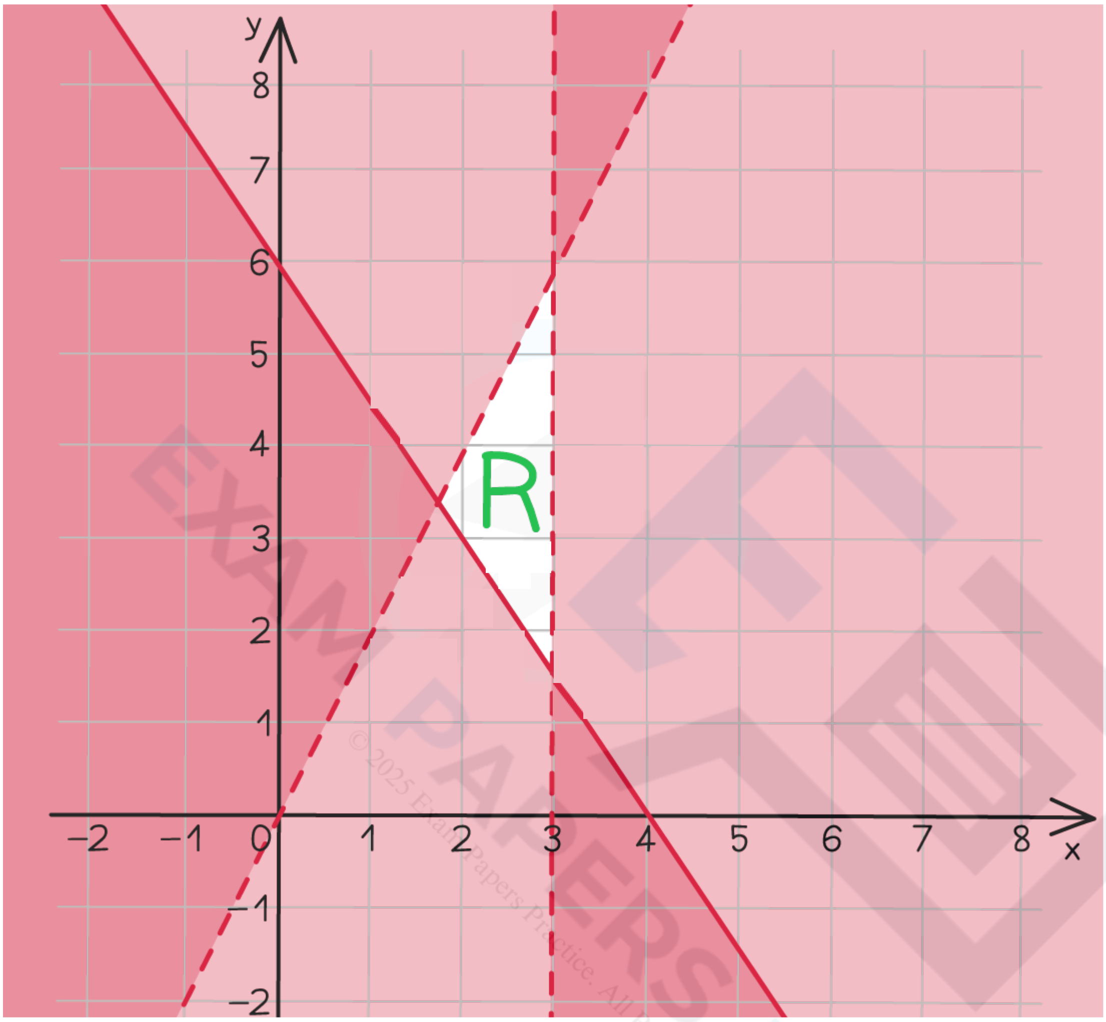
1, 2, 3 − *Draw* $3x + 2y = 12$ *with a solid line and shade below it*

(The point $(0, 0)$ *is below this line)*

1, 2, 3 − *Draw* $y = 2x$ *with a dotted line and shade above it*

1, 2, 3 − *Draw* $x = 3$ *with a dotted line and shade to the right of it*

3. Solving Equations & Inequalities



3. Solving Equations & Inequalities

3.6.2 INEQUALITIES ON GRAPHS - INTERPRETING

How do we interpret inequalities on a graph?

- First, see Inequalities and Straight Line Graphs

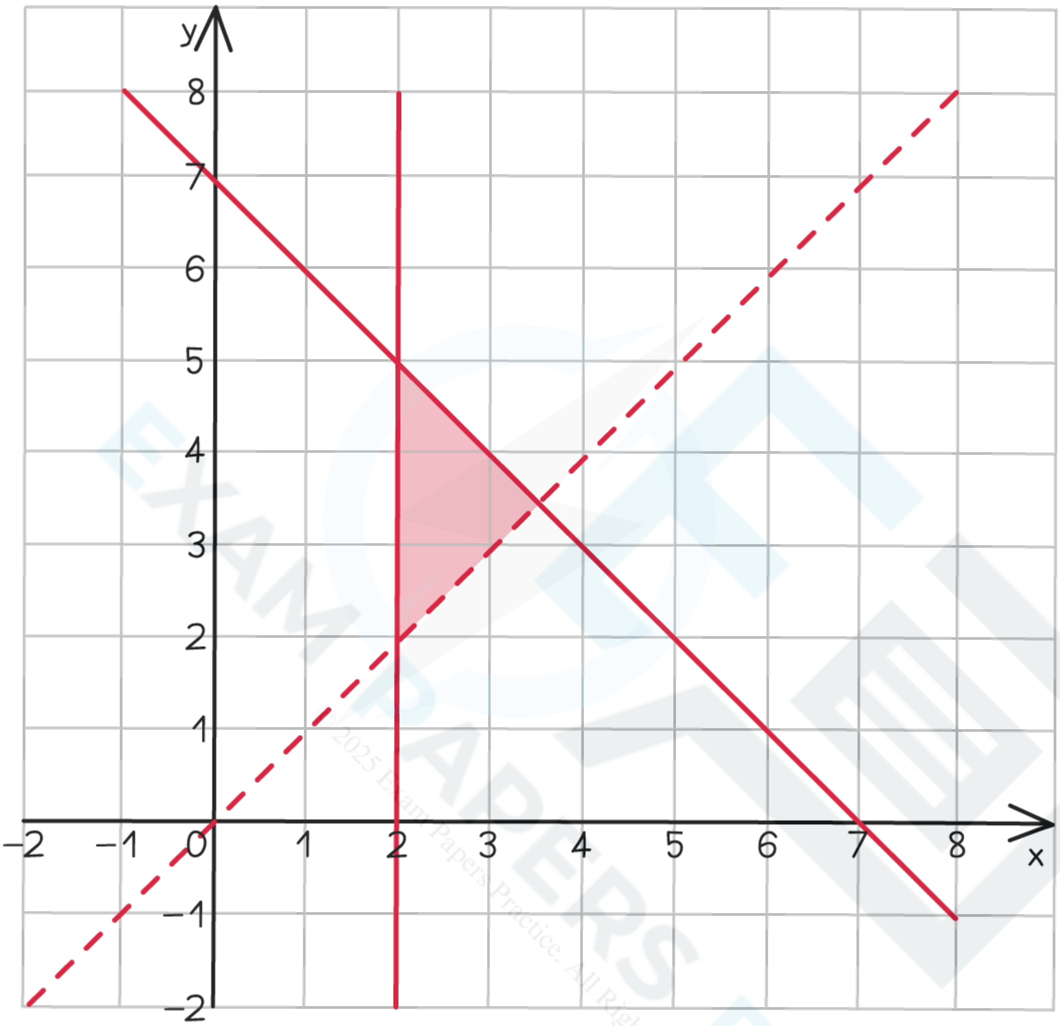
Once you do...

1. Write down the **EQUATION** of each line on the graph
2. **REMEMBER** that lines are drawn with:
   A **solid** line for ≤ or ≥ (to indicate line included in region)
   A **dotted** line for < or > (to indicate line not included)
3. **REPLACE** = sign with:
   ≤ or < if shading **below** line ≥ or
   > if shading **above** line
   (Use a point to test if not sure)

Worked Example

1. *Write down the three inequalities which define the shaded region on the axes below.*

3. Solving Equations & Inequalities

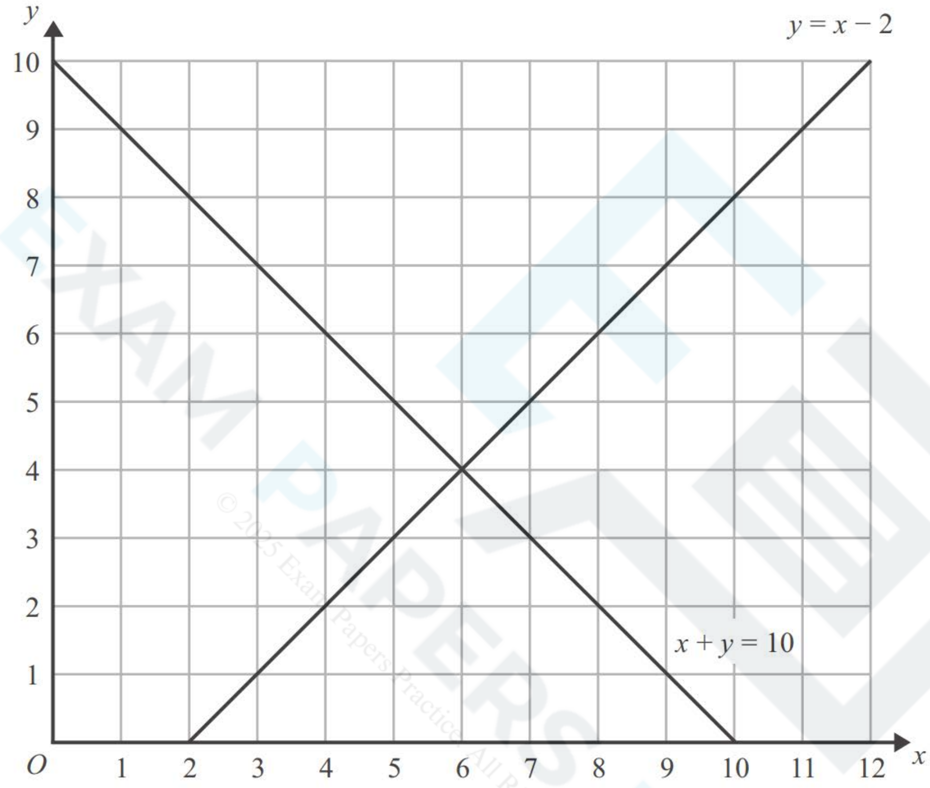


$y > x$	1 − *The dotted line has equation* $y = x$ 2, 3 − *Dotted line and region is above it so* $>$ (*If we tested the point* $(3,\ 4)$ *from the shaded region then* $4 > 3$ *and so* $y > x$)
$x \geq 2$	1 − *The solid vertical line is* $x = 2$ 2, 3 − *Solid line and region is to the right of it so* $\geq$
$x + y \leq 7$	1 − *The other solid line has equation* $x + y = 7$ 2, 3 − *Solid line and region is below it so* $\leq$

3. Solving Equations & Inequalities

Exam Question: Medium

The lines $y = x - 2$ and $x + y = 10$ are drawn on the grid.



On the grid, mark with a cross (×) each of the points with integer coordinates that are in the region defined by

$y > x - 2$
$x + y < 10$
$x > 3$

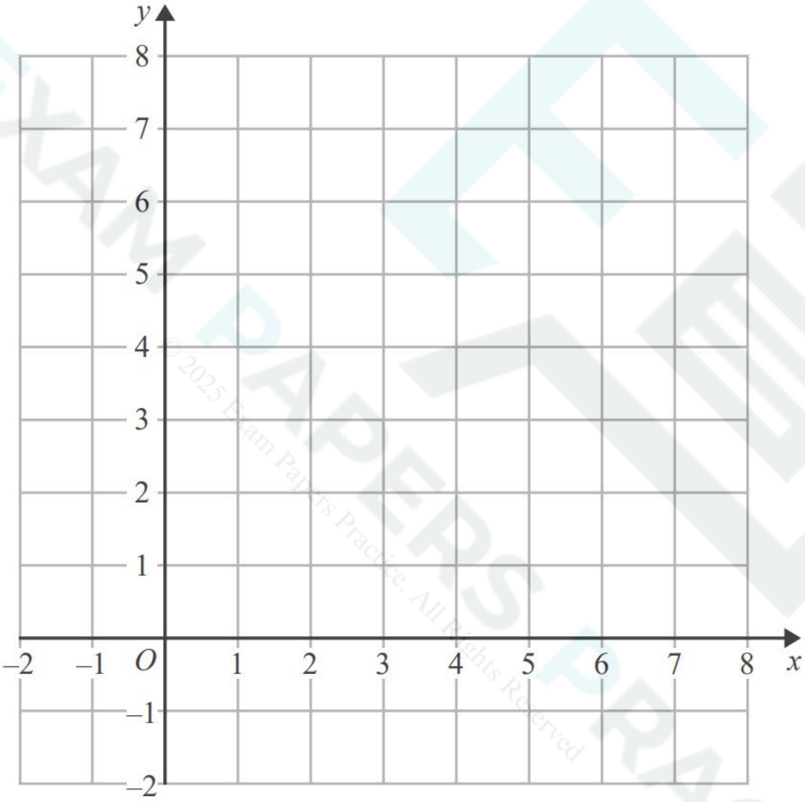
3. Solving Equations & Inequalities

Exam Question: Hard

On the grid show, by shading, the region that satisfies all three of the inequalities

$x + y < 7$ $y < 2x$ $y > 3$

Label the region **R**.



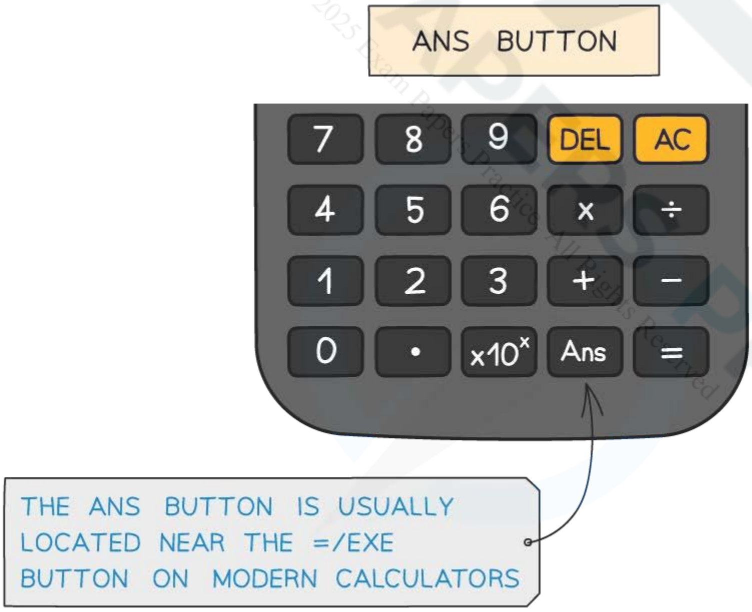
3. Solving Equations & Inequalities

3.7 ITERATION

3.7.1 ITERATION - USING A CALCULATOR

What is iteration?

- Some equations do not have “nice” solutions.
    - They are not integers, fractions or simple decimals.
    - Some equations have irrational
- **Iteration** is a repeated process, in this case used to solve such equations
    - the process starts with an **initial value** (starting value)
    - after each stage of the process (after each iteration) a solution is produced
    - the solutions get more accurate as more iterations are performed
- Scientific calculators allow us to perform iterations very quickly using the ANS button



3. Solving Equations & Inequalities

So how do I do iteration?

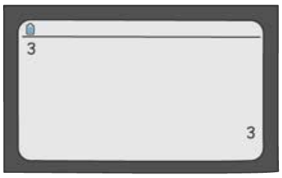
- If not already done, then the equation will need rewriting into the form **$x = f(x)$**
- You may have to deduce a suitable **initial value (x_0)** (starting value)
- Both of these are covered in the Revision Notes on Iteration – Applications

- Once **x_0** and the **iterative formula** are known the work is largely done by calculator

- STEP 1 Enter the initial value **x_0** and press **=/EXE**
- STEP 2 Enter (the right-hand side of) the **iterative formula** using **ANS** instead of **x_n**
- STEP 3 Press **=/EXE** to obtain **x_1** (be careful not to double press **=/EXE** !)
- STEP 4 Repeat STEP 3 as many times as required

e.g. USE THE ITERATIVE FORMULA $x_{n+1} = 5 - 2\sqrt[3]{x}$
WITH $x_0 = 3$ TO FIND x_1, x_2, x_3 AND x_4.
GIVE YOUR ANSWERS TO 3 DECIMAL PLACES.

STEP 1: ENTER INITIAL VALUE x_0 AND PRESS =/EXE


3

3

STEP 2: ENTER (THE RIGHT HAND SIDE OF) THE ITERATIVE FORMULA USING ANS INSTEAD OF x_n.


$5 - 2\sqrt[3]{ANS}$

STEP 3: PRESS =/EXE TO OBTAIN x_1


$5 - 2\sqrt[3]{ANS}$

2.115500859

$x_1 = 2.116$

STEP 4: REPEAT STEP 3 AS NECESSARY


$5 - 2\sqrt[3]{ANS}$

2.4325555

$x_2 = 2.433$


$5 - 2\sqrt[3]{ANS}$

2.310214912

$x_3 = 2.310$


$5 - 2\sqrt[3]{ANS}$

2.356085196

$x_4 = 2.356$

3. Solving Equations & Inequalities

- Do not be put off by the awkward-looking iterative formula
    - **n** and **n+1** are just counters – **n+1** is simply one more than **n**
    - **n** starts at **0** so the process starts with $\mathbf{x_0}$ – the initial value
    - $\mathbf{x_0}$ allows us to find $\mathbf{x_1}$, $\mathbf{x_1}$ allows us to find $\mathbf{x_2}$, etc.
      (See Iteration – Applications)

Exam Tip

Be careful to not press =/EXE more than once at a time. If you do the best thing to do is to restart from the beginning.
Look carefully at the rounding level required – more digits are often needed in iteration questions than elsewhere on the exam. If you prefer you could write down all digits from your calculator display for $\mathbf{x_1}$, $\mathbf{x_2}$, etc. and round them after.

Worked Example

?

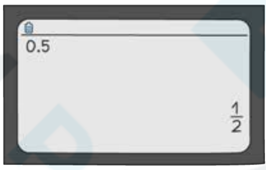
Use the iterative formula

$$x_{n+1} = \frac{\sqrt{x_n^{\,5} - x_n + 2}}{3}$$

with $x_0 = 0.5$ to find values for x_1, x_2, and x_3.

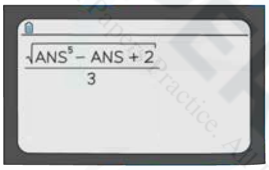
Give your answers to 5 decimal places.

STEP 1: ENTER INITIAL VALUE INTO CALCULATOR AND PRESS =

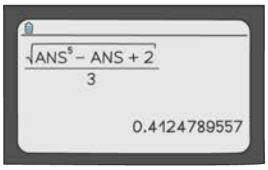


YOU MAY HAVE TO PRESS S⇔D BUTTON TO SEE AS A DECIMAL

STEP 2: ENTER ITERATIVE FORMULA USING ANS INSTEAD OF x_n

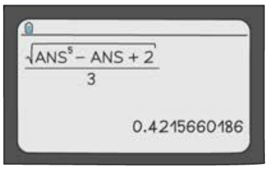


STEP 3: PRESS =/EXE TO CALCULATE x_1

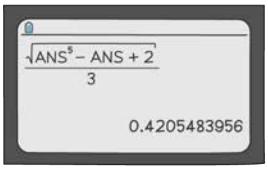


$x_1 = 0.41248$

STEP 4: REPEAT STEP 3 AS MANY TIMES AS NECESSARY



$x_2 = 0.42157$



$x_3 = 0.42055$

3. Solving Equations & Inequalities

3.7.2 ITERATION - APPLICATIONS

How do I solve problems with iteration?

- To solve equations using iteration two things are required:
- The equation **f(x) = 0** needs to be rewritten into the form **x = g(x)**
    - Some equations may need rearranging to get to the **f(x) = 0** stage too!
    - There may be more than one way to rearrange to get the form **x = g(x)** but a question will state which one to show
- An **initial value** (**x_0**) (or starting value)
    - This is the first **estimate** of a **solution** to the equation **f(x) = 0**

e.g. $\sqrt[3]{x} - x = 4$

NOT IN THE FORM f(x) = 0

$\sqrt[3]{x} - x - 4 = 0$

REARRANGE SO IT IS IN THE FORM f(x) = 0

A REARRANGEMENT IN THE FORM x = a(x) IS ALSO REQUIRED

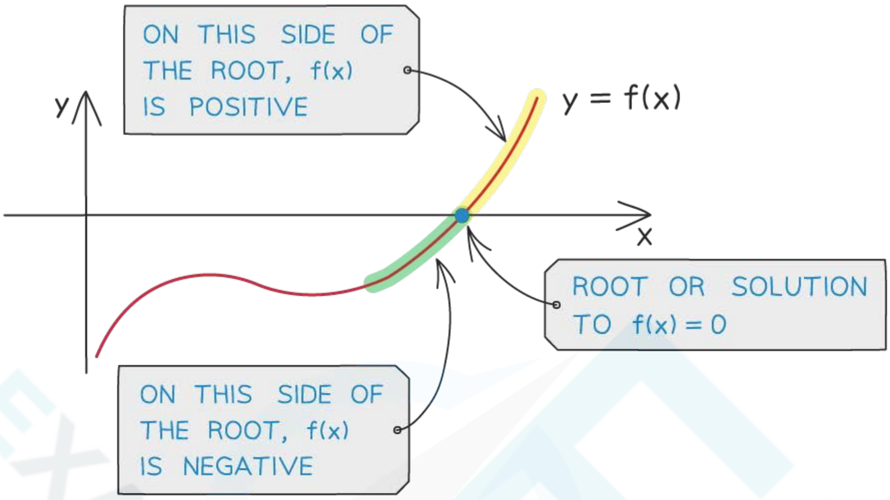
$x = \sqrt[3]{x} - 4$

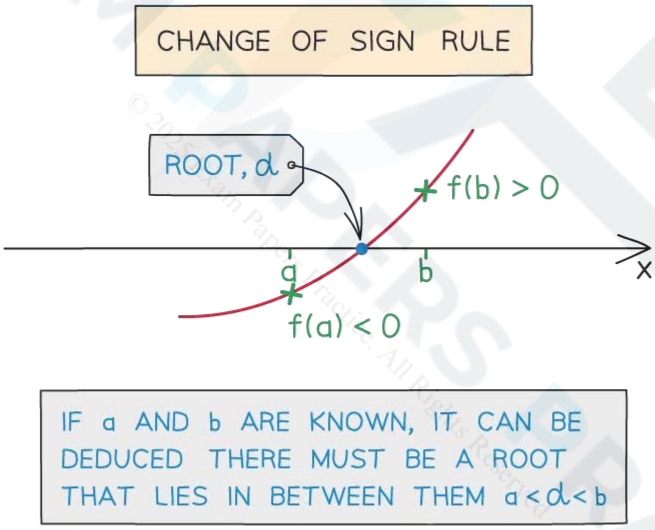
TAKE x TO THE OTHER SIDE OF THE EQUATION

NOW f(x) = 0 AND x = g(x) ARE KNOWN, IT IS POSSIBLE TO DEDUCE AN **INITIAL VALUE** (x_0) AND THEN PERFORM THE ITERATION PROCESS

3. Solving Equations & Inequalities

- A solution to the equation **f(x) = 0** is also called a **root**
    - On a graph, a root is where the curve crosses the **x**-axis
    - On one side of the root the value of **f(x)** will be positive
    - On the other side of the root the value of **f(x)** will be negative
- This is called the change of sign rule
    - Two values, **a** and **b**, chosen appropriately, lead to **f(a)** and **f(b)** having different signs
    - It does not matter which one is positive and which is negative, they just need to have different signs
    - In exam questions **a** and **b** are usually given but can be hidden
      Look for phrases such as "... solution between ...", "... root in the interval ..."
- This information can then be used to find an initial estimate (x_0) of a solution





- Once **x = g(x)** and **x_0** are known, the iterative process can begin
- See Revision Notes **Iteration – Using a Calculator**

Exam Tip

When writing down an iterative formula, write it down without the **n** and **n+1**

Go back and add these in afterwards.

3. Solving Equations & Inequalities

Worked Example

(a) Show that the equation $x^3 - 2x = 3 - 2x^2$ has a solution between 1 and 2.

(b) Show that the equation $x^3 - 2x = 3 - 2x^2$ can be rearranged into the form

$$x = \sqrt[3]{3 + 2x - 2x^2}$$

(c) The iterative formula

$$x_{n+1} = \sqrt[3]{3 + 2x_n - 2x_n^{\,2}}$$

is to be used to find an estimate to a solution of the equation $x^3 + 2x^2 - 2x - 3 = 0$

Using your answer to part (a), state an appropriate initial value, x_0, and use this value to find the values of x_1, x_2 and x_3.

Give your answers for x_1, x_2 and x_3 to 4 decimal places.

a) $x^3 - 2x = 3 - 2x^2$

$x^3 + 2x^2 - 2x - 3 = 0$

$f(x) = 0$

STEP 1: REARRANGE TO f(x) = 0

$f(1) = 1^3 + 2 \times 1^2 - 2 \times 1 - 3 = -2 \quad < 0$

$f(2) = 2^3 + 2 \times 2^2 - 2 \times 2 - 3 = 9 \quad > 0$

STEP 2: f(a) & f(b)

STEP 3: STATE f(a) & f(b) HAVE DIFFERENT SIGNS

f(1) AND f(2) HAVE DIFFERENT SIGNS
SO $x^3 - 2x = 3 - 2x^2$ HAS A SOLUTION
BETWEEN 1 AND 2

b) $x^3 - 2x = 3 - 2x^2$

$x^3 = 3 - 2x^2 + 2x$

$x = \sqrt[3]{3 + 2x - 2x^2}$

CLUE IS $\sqrt[3]{}$ SO REARRANGE SUCH THAT $x^3 = \dots$

c) $x_0 = 1.5$

IN PART a), A ROOT WAS FOUND BETWEEN 1 AND 2

$x_1 = 1.1447$
$x_2 = 1.3871$
$x_3 = 1.2442$
(ALL TO 4 dp)

USE ANS BUTTON.
SEE REVISION NOTES:
ITERATION – USING A CALCULATOR

3. Solving Equations & Inequalities

Exam Question: Medium

Using $x_{n+1} = -2 - \frac{4}{x_n^2}$

with $x_0 = -2.5$

(a) find the values of x_1, x_2 and x_3

Explain the relationship between the values of x_1, x_2 and x_3 and the equation $x^3 + 2x^2 + 4 = 0$

Exam Question: Hard

(a) Show that the equation $x^3 + 7x - 5 = 0$ has a solution between $x = 0$ and $x = 1$

(b) Show that the equation $x^3 + 7x - 5 = 0$ can be arranged to give $x = \frac{5}{x^2 + 7}$

(c) Starting with $x_0 = 1$, use the iteration formula $x_{n+1} = \frac{5}{x_n^{\ 2} + 7}$ three times to find an estimate for the solution of $x^3 + 7x - 5 = 0$

(d) By substituting your answer to part (c) into $x^3 + 7x - 5$, comment on the accuracy of your estimate for the solution to $x^3 + 7x - 5 = 0$

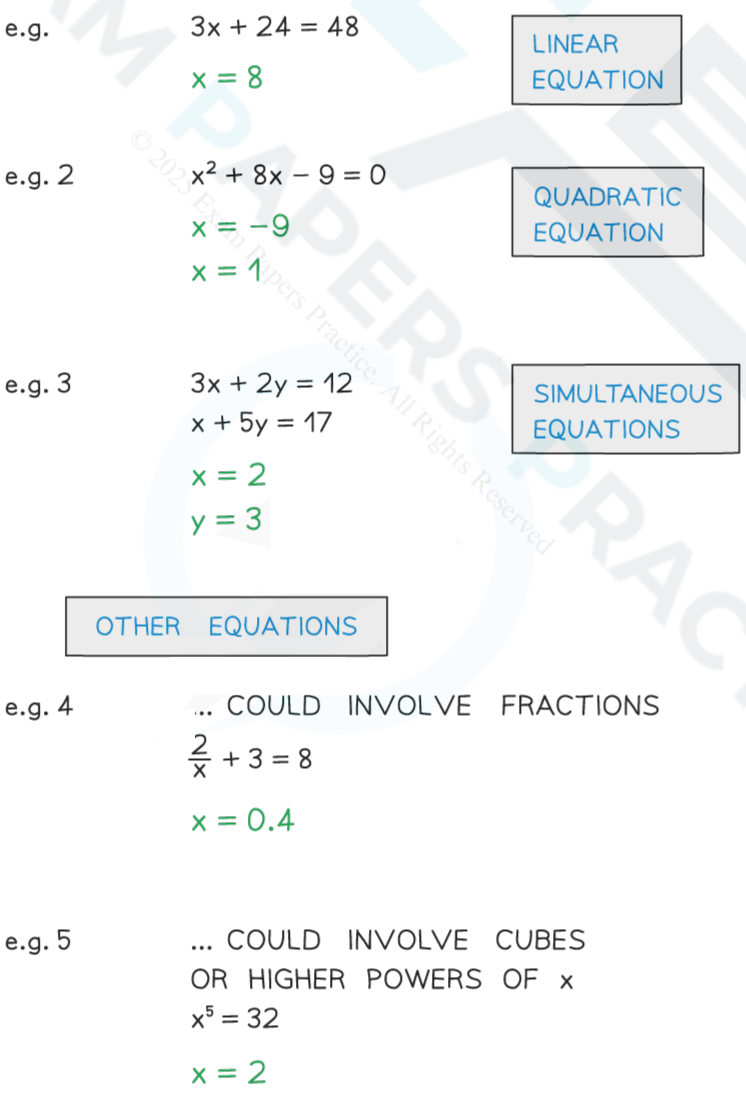
3. Solving Equations & Inequalities

3.8 EQUATIONS & PROBLEM SOLVING

3.8.1 EQUATIONS & PROBLEM SOLVING

What is problem solving?

- Problem solving in mathematics involves using several stages, across a variety of topics, to answer a question
- In this set of notes all the problems will involve **equations**
  - These could be **linear** equations, **quadratic** equations or **simultaneous** equations and **other**, relatively straightforward equations



e.g. $3x + 24 = 48$

$x = 8$

LINEAR EQUATION

e.g. 2 $x^2 + 8x - 9 = 0$

$x = -9$

$x = 1$

QUADRATIC EQUATION

e.g. 3 $3x + 2y = 12$

$x + 5y = 17$

$x = 2$

$y = 3$

SIMULTANEOUS EQUATIONS

OTHER EQUATIONS

e.g. 4 ... COULD INVOLVE FRACTIONS

$\frac{2}{x} + 3 = 8$

$x = 0.4$

e.g. 5 ... COULD INVOLVE CUBES OR HIGHER POWERS OF x

$x^5 = 32$

$x = 2$

3. Solving Equations & Inequalities

- You may notice there are not many subheadings in these notes
- That is deliberate so the examples are not labelled or dealt with in an order
    - This is the nature of problem solving questions!
    - You never know exactly what's coming … !

e.g. AT A FIREWORKS STALL A CUSTOMER PAYS £9 FOR SIX BANGERS AND TWELVE SPARKLERS. ANOTHER CUSTOMER BUYS NINE BANGERS AND TEN SPARKLERS FOR £12.30. FIND THE COST OF 5 BANGERS AND 15 SPARKLERS.

TWO "THINGS" – B's AND S's
TWO UNKNOWNS – COST OF B's AND S's
TWO PIECES OF INFO – TWO EQUATIONS
SIMULTANEOUS EQUATIONS

NO "NICE" LINK FROM 6 OR 9 TO 5,
NOR FROM 12 OR 10 TO 15, SO FIND
COST OF 1B AND 1S

NOTICE HOW MUCH WORK AND THINKING SHOULD HAPPEN IN YOUR HEAD BEFORE ATTEMPTING A SOLUTION!

LET B BE THE PRICE OF BANGERS, IN POUNDS.
LET S BE THE PRICE OF SPARKLERS, IN POUNDS.

3. Solving Equations & Inequalities

$6B + 12S = 9$ ①

$9B + 10S = 12.3$ ②

① × 3: $18B + 36S = 27$ ③

② × 2: $18B + 20S = 24.6$ ④

③ − ④: $16S = 2.4$

$$S = \frac{2.4}{16}$$

$S = 0.15$

SUB. IN ③: $18B + 36 \times 0.15 = 27$

$18B = 21.6$

$$B = \frac{21.6}{18}$$

$B = 1.2$

SO, $5B + 15S = 5 \times 1.2 + 15 \times 0.15$

$= 8.25$

FIVE BANGERS AND FIFTEEN SPARKLERS WILL COST £8.25

A GOOD ALTERNATIVE QUESTION HERE TO TEST INTERPRETATION OF YOUR ANSWER WOULD BE TO FIND THE CHANGE FROM £10. (£1.75 !)

- In an ordinary mathematics question you would be given an equation to solve
- In a problem solving question you would have to **generate** the equation …
  - … using information from the question
  - … using your knowledge of standard mathematical results

3. Solving Equations & Inequalities

e.g. AN IRONMAN COMPETITION CONSISTS OF A SWIM, A BICYCLE RIDE AND A MARATHON WITH THE DISTANCES OF EACH SPLIT IN THE RATIO $\frac{x}{10} : 2(2x+5) : x$

COMPETITOR'S COVER A TOTAL DISTANCE OF 137.5 MILES.
FIND THE DISTANCE OF EACH COMPONENT OF THE COMPETITION.

RATIO – TOTAL GIVEN

$$\frac{x}{10} + 2(2x+5) + x = 137.5$$

$$x + 20(2x+5) + 10x = 1375$$

x10 TO "GET RID OF FRACTIONS" (GROF)

$$x + 40x + 100 + 10x = 1375$$

EXPAND TO "GET RID OF BRACKETS" (GROB)

$$51x + 100 = 1375$$
$$51x = 1275$$
$$x = 25$$

SO $\frac{x}{10} : 2(2x+5) : x = \frac{25}{10} : 2(2 \times 25 + 5) : 25$

$$= 2.5 : 110 : 25$$

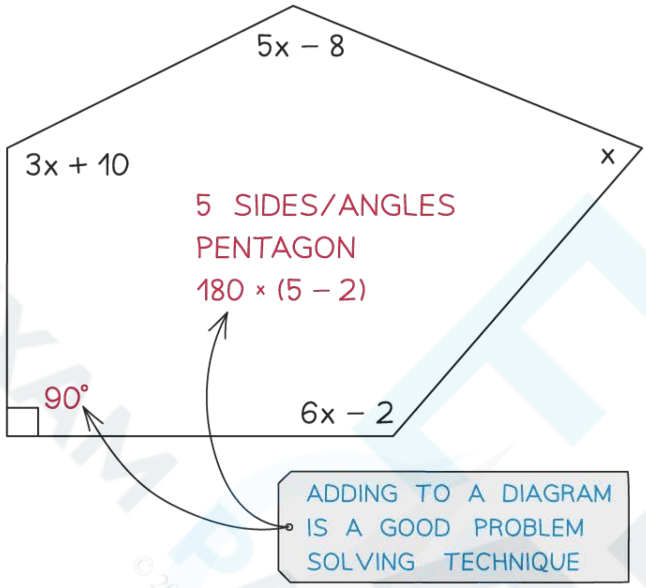
SWIM IS 2.5 MILES
BICYCLE RIDE IS 110 MILES
MARATHON IS 25 MILES

AN ALTERNATIVE HERE WOULD BE TO ASK HOW MANY TIMES FURTHER IS THE BICYCLE RIDE THAN THE SWIM (44)

3. Solving Equations & Inequalities

- A key feature of problem solving questions is to **interpret** the answer in **context**
- An answer on a calculator may be **2**
    - If the question was about money then your final answer should be **£1.20**
- A **quadratic** equation can have **two** solutions
    - Only **one** may be valid if **only** positive values are relevant (eg distance)

e.g. THE DIAGRAM BELOW SHOWS AN IRREGULAR POLYGON.
ALL ANGLES ARE IN DEGREES.
FIND THE VALUE OF x.



ADDING TO A DIAGRAM IS A GOOD PROBLEM SOLVING TECHNIQUE

NOT TO SCALE – CAN'T MEASURE IT!
IRREGULAR POLYGON, BUT, 5 SIDES PENTAGON
INTERIOR ANGLES GIVEN
∟ IS A RIGHT-ANGLE, 90°
SUM OF INTERIOR ANGLES IN **ANY** POLYGON

$$\text{SUM OF ANGLES} = 180(n-2)$$
$$= 180(5-2)$$
$$= 180 \times 3$$
$$= 540$$

$$3x + 10 + 5x - 8 + x + 6x - 2 + 90 = 540$$

$$15x + 90 = 540$$

$$15x = 450$$

$$x = 30°$$

A GOOD ALTERNATIVE QUESTION HERE WOULD BE TO FIND THE SMALLEST OR LARGEST ANGLE (30°, 178°!)

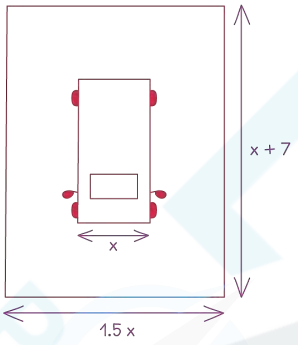
3. Solving Equations & Inequalities

- In problem solving questions you are typically given less information about the type of maths involved
- It is impossible to list every type of problem solving question you could see
    - There are endless contexts questions can be set in
    - There is no one-fits-all step-by-step method to solving problems
- **Practice**, **experience** and **familiarity** are the keys to solving problems successfully

e.g. A TYPICAL DISABLED CAR PARKING SPACE HAS TO BE ONE ANS A HALF TIMES THE WIDTH OF AN AVERAGE CAR WHEN MEASURED IN METRES.
THE LENGTH OF THE PARKING SPACE HAS TO BE 7 m LONGER THAN THE WIDTH OF AVERAGE CAR.
GIVEN THAT THE AREA OF A TYPICAL DISABLED CAR PARKING SPACE IS 25 m² FIND THE WIDTH OF AN AVERAGE CAR.

WIDTHS, LENGTHS, AREAS... DRAW A DIAGRAM !!

LET x m BE THE WIDTH OF AN AVERAGE CAR.



AREA = LENGTH × WIDTH

$1.5x(x+7) = 25$

$1.5x^2 + 10.5x = 25$ QUADRATIC

$1.5x^2 + 10.5x - 25 = 0$ MAKE = 0

$"b^2 - 4ac" = (10.5)^2 - 4 \times 1.5 \times -25$

$= 260.25$

NUMBERS AWKWARD SO USE FORMULA

$$x = \frac{-10.5 \pm \sqrt{260.25}}{2 \times 1.5}$$

$x = 1.877421...$ $x = -8.877421...$

WRITE DOWN MORE DIGITS THAN NEEDED AND INTERPRET AND ROUND AFTERWARDS

x IS A DISTANCE SO CAN'T BE NEGATIVE.
REJECT $x = -8.877421...$ AS A SOLUTION

$x = 1.88$ m (2 dp)

METRES SO 2 dp IS CENTIMETRES AND SENSIBLE

THE WIDTH OF AN AVERAGE CAR IS 1.88m

A GOOD ALTERNATIE HERE WOULD BE TO ASK ABOUT THE LENGTH OF THE DISABLED PARKING SPACE (8.88m !)

3. Solving Equations & Inequalities

- Do **not** necessarily expect whole number (**integer**) or “nice” solutions
    - Especially where a calculator is allowed
- **Rounding appropriately** may be one of the skills being tested
    - eg Rounding a value in **cm** only needs to be to one decimal place; so it indicates **mm**

Exam Tip

Do **not** start by focusing on what the question has asked you to find, but on what maths you **can** do.

If your attempt turns out to be unhelpful, that’s fine, you may still pick up some marks.

If your attempt is relevant it could nudge you towards the full solution – and full marks!

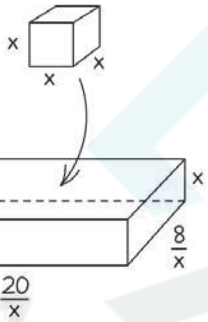
Add information to a **diagram** as you work through a problem. If there is no diagram, try **sketching** one.

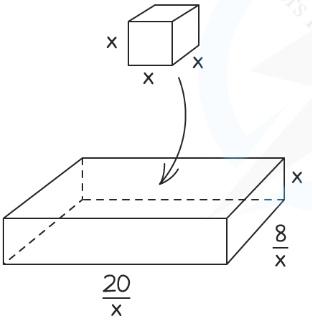
Worked Example

Cubes of side length x cm are placed into a cuboid measuring $\frac{20}{x}$ cm by $\frac{8}{x}$ cm by x cm.

The cuboid holds exactly 10 of the cubes.

Find the volume of one of the cubes.





$V_1 = x^3$

ADD TO THE DIAGRAM

$V_2 = \frac{20}{x} \times \frac{8}{x} \times x$

$V_2 = \frac{160x}{x^2}$

$V_2 = \frac{160}{x}$

$V_2 = 10V_1$ ← CUBOID HOLDS 10 CUBES

$\frac{160}{x} = 10x^3$

$16 = x^4$ ← MULTIPLY (BOTH SIDES) BY x DIVIDE (BOTH SIDES) BY 10

$x = 2$

$x = -2$ TOO BUT x IS A DISTANCE SO $x > 0$

VOLUME OF ONE CUBE $= 2 \times 2 \times 2$

$= 8\,\text{cm}^3$

3. Solving Equations & Inequalities

Exam Question: Medium

The diagram shows a trapezium.

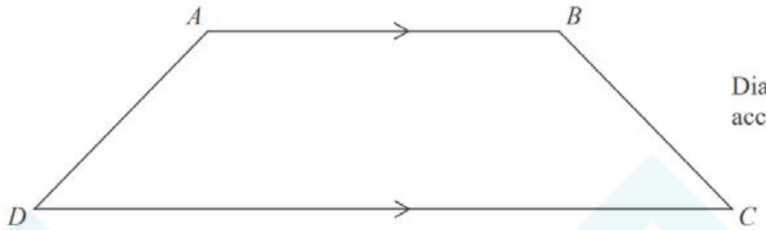


Diagram **NOT** accurately drawn

$AD = x$ cm.
BC is the same length as AD.
AB is twice the length of AD.
DC is 4 cm longer than AB.

The perimeter of the trapezium is 38 cm.

Work out the length of AD.

Exam Question: Hard

Julie and Liam write down the same number.

Julie multiplies the number by 5 and then adds 4 to the result.
She writes down her answer.

Liam subtracts the number from 10
He writes down his answer.

Julie's answer is two thirds of Liam's answer.

Work out the number that Julie and Liam started with.
You must show your working.

4. Sequences

CONTENTS

4.1 SEQUENCES - LINEAR

4.1.1 SEQUENCES - LINEAR

What is a sequence?

- A sequence is simply a set of numbers (or objects) in an order

What is a linear sequence?

- A linear sequence is one where the numbers go up (or down) by the same amount each time
    - eg 1, 4, 7, 10, 13, ... (add 3 to get the next term)
    - 15, 10, 5, 0, -5, ... (subtract 5 to get the next term)
- If we look at the differences between the terms, we see that they are **constant**

What can we do with linear sequences?

- You should be able to recognise and continue a linear sequence
- You should also be able to find a formula for the **nth term** of a linear sequence in terms of n
- This formula will be in the form:

  nth term = dn + b
  where
  d is the common difference, b is a constant that makes the first term "work"

How to find the nth term formula for a linear sequence

- Find the common **difference** between the terms – this is d
- Put the first term and n=1 into the formula, then solve to find b

4. Sequences

Worked Example

For the sequence $5, 7, 9, 11, 13, \ldots,$ *find the next three terms and a formula for the* n^{th} *term.*

Looking at the differences between the terms, we see that they are all 2,

so this is a Linear Sequence with common difference 2.

The next three terms are :

$$13 + 2 = 15, \quad 15 + 2 = 17, \quad 17 + 2 = 19$$

To find the n^{th} *term formula :*

1. *The common difference between the terms is* 2, *so* $d = 2$. *This means*

$$n^{th}\ term\ =\ 2n + b$$

2. *The first term is* 5, *so we put this and* $n = 1$ *into the formula :*

$$5\ =\ 2 \times 1 + b$$

$$b = 5 - 2 = 3$$

$$n^{th}\ term\ =\ 2n + 3$$

Exam Question: Medium

Here are the first 5 terms of an arithmetic sequence.

$$3 \qquad 9 \qquad 15 \qquad 21 \qquad 27$$

(a) Find an expression, in terms of n, for the nth term of this sequence.

Ben says that 150 is in the sequence.

(b) Is Ben right?

You must explain your answer.

4. Sequences

4.2 SEQUENCES - QUADRATIC

4.2.1 SEQUENCES - QUADRATIC

What is a sequence?

- A sequence is simply a set of numbers (or objects) in an **order**

What is a quadratic sequence?

- Unlike in a linear sequence, in a quadratic sequence the differences between the terms (the **first differences**) are not constant
- However, the differences between the differences (the **second differences**) are constant
- Another way to think about this is that in a quadratic sequence, the sequence of differences is a linear sequence
  eg Sequence 2, 3, 6, 11, 18, ...
  1st Differences 1 3 5 7 (a Linear Sequence)
  2nd Differences 2 2 2 (Constant)
- Because the second differences there are constant, we know that the example is a 1quadratic sequence

What can we do with quadratic sequences?

- You should be able to recognise and continue a quadratic sequence
- You should also be able to find a formula for the nth term of a quadratic sequence in terms of n
- This formula will be in the form:

  nth term = an2 + bn + c
  (The process for finding a, b, and c is given below)

How to find the nth term formula for a quadratic sequence

1. Work out the sequences of first and second differences
   Note: check that the first differences are not constant and the second differences are constant, to make sure you have a quadratic sequence!
2. Use the first and second differences to find a, b, and c in the nth term formula
   Follow these steps in order:
   2a is the 1st second difference (or any second difference, as in a quadratic sequence they are all the same!)
   3a + b is the 1st first difference
   a + b + c is the 1st term

4. Sequences

Exam Tip

Before doing the very formal process to find the nth term, try comparing the sequence to the square numbers 1, 4, 9, 16, 25, ... and see if you can spot the formula.

For example:

Sequence 4, 7, 12, 19, 28, ...

Square Numbers 1 4 9 16 25

We can see that each term of the sequence is 3 more than the equivalent square number so the formula is:

nth term = n2 + 3

This could save you a lot of time!

4. Sequences

Worked Example

For the sequence 5, 7, 11, 17, 25, ..., *find the next three terms and a formula for the* n^{th} *term.*

1. *Sequence* 5, 7, 11, 17, 25, …

1^{st} *Differences* 2 4 6 8 *(a Linear Sequence)*

2^{nd} *Differences* 2 2 2 *(Constant)*

So we know this is a Quadratic Sequence.

The first differences increase by 2 each time, so the next three terms are :

$25 + 10 = 35,\ 35 + 12 = 47,\ 47 + 14 = 61$

2. $n^{th}\ term = an^2 + bn + c$

Looking at the 1^{st} *second difference :*

$2a = 2 \ so\ a = 1$

Looking at the 1^{st} *first difference :*

$3a + b = 2\ so\ 3 \times 1 + b = 2\ so\ b =- 1$

Looking at the 1^{st} *term :*

$a + b + c = 5\ so\ 1 - 1 + c = 5\ so\ c = 5$

Which gives :

$n^{th}\ term = n^2 - n + 5$

Exam Question: Medium

The nth term of a number sequence is $n^2 + 1$

Write down the first three terms of the sequence.

4. Sequences

Exam Question: Hard

Here are the first five terms of a sequence.

4 11 22 37 56

Find an expression, in terms of n, for the nth term of this sequence.

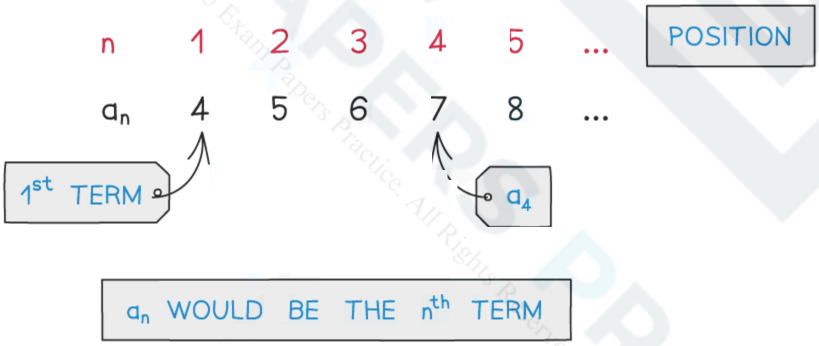
4. Sequences

4.3 FIBONACCI & GEOMETRIC

4.3.1 SEQUENCES - BASICS

What are sequences?

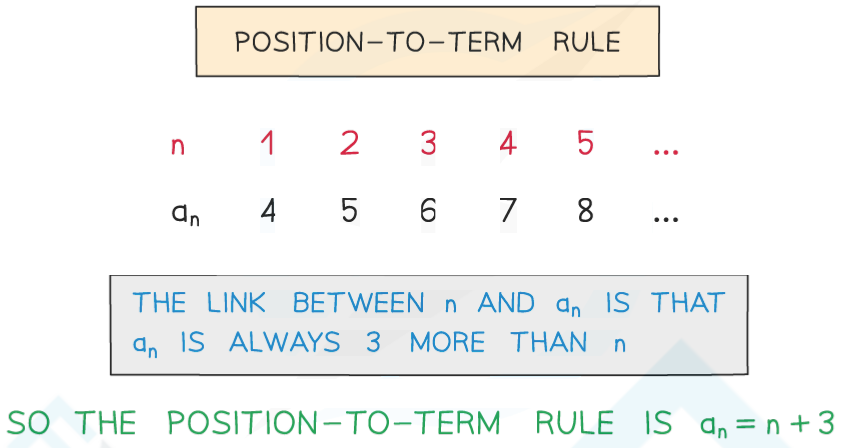
- A sequence is an order set of (usually) numbers
- Each number in a sequence is called a **term**
- The **location** of a **term** within a **sequence** is called its **position**
    - The letter **n** is often used for (an unknown) **position**
- Subscript notation is used to talk about a particular term
    - $\mathbf{a_1}$ would be the **first** term in a sequence
    - $\mathbf{a_7}$ would be the **seventh** term
    - $\mathbf{a_n}$ would be the **n**th term



What is a position-to-term rule?

- A **position-to-term** rule gives the **n**th term of a sequence in terms of **n**
    - This is a very powerful piece of mathematics
    - With a position-to-term rule the **100**th term of a sequence can be found without having to know or work out the first 99 terms!

4. Sequences



POSITION–TO–TERM RULE

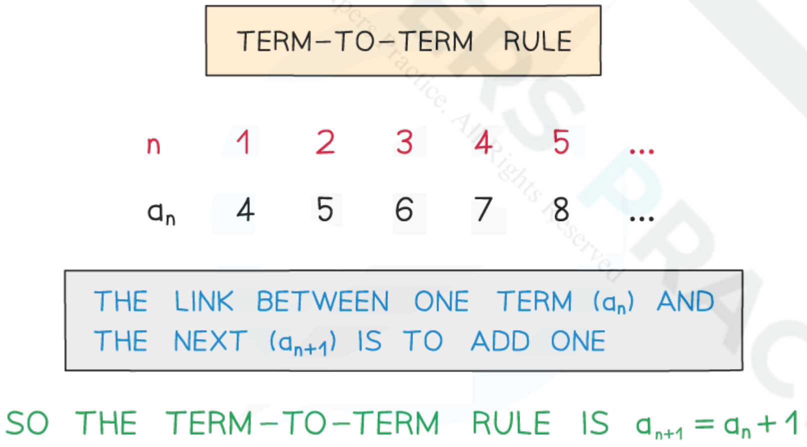
n	1	2	3	4	5	...
a_n	4	5	6	7	8	...

THE LINK BETWEEN n AND a_n IS THAT a_n IS ALWAYS 3 MORE THAN n

SO THE POSITION–TO–TERM RULE IS $a_n = n + 3$

What is a term-to-term rule?

- A **term-to-term** rule gives the **(n+1)**th term in terms of the **n**th term
    - ie $\mathbf{a_{n+1}}$ is given in terms of $\mathbf{a_n}$
    - If a term is known, the next one can be worked out



TERM–TO–TERM RULE

n	1	2	3	4	5	...
a_n	4	5	6	7	8	...

THE LINK BETWEEN ONE TERM (a_n) AND THE NEXT (a_{n+1}) IS TO ADD ONE

SO THE TERM–TO–TERM RULE IS $a_{n+1} = a_n + 1$

How do I use the position-to-term and term-to-term rules?

- These can be used to generate a sequence
- From a given sequence the rules can be deduced
- Recognising and being aware of the types of sequences helps
    - Linear and quadratic sequences
    - Geometric sequences
    - Fibonacci sequences
    - Other sequences

4. Sequences

e.g. THE POSITION-TO-TERM RULE FOR A SEQUENCE IS $a_n = 2n - 3$

a) FIND THE FIRST FIVE TERMS OF THE SEQUENCE

$n = 1,\quad a_1 = 2 \times 1 - 3 = -1$

$n = 2,\quad a_2 = 2 \times 2 - 3 = 1$

$n = 3,\quad a_3 = 2 \times 3 - 3 = 3$

$n = 4,\quad a_4 = 2 \times 4 - 3 = 5$

$n = 5,\quad a_5 = 2 \times 5 - 3 = 7$

FIRST FIVE TERMS ARE:
$-1, 1, 3, 5, 7$

b) WRITE DOWN THE TERM-TO-TERM RULE FOR THE SEQUENCE

$$-1, \underset{+2}{\curvearrowright} 1, \underset{+2}{\curvearrowright} 3, \underset{+2}{\curvearrowright} 5, 7$$

$a_{n+1} = a_n + 2$

Exam Tip

Write the position numbers above (or below) each term in a sequence.

This will make it much easier to recognise and spot common types of sequence.

4. Sequences

(a) Find the first 5 terms of the following sequences

(i) n^{th} term $= 20 - 2n$(ii) n^{th} term $= 2n^2 - 3$

(iii) n^{th} term $= \dfrac{1}{2n}$

(b) Find the 2nd, 3rd and 4th terms in each of these sequences

(i) $a_{n+1} = a_n + 6$ with $a_1 = 3$

(ii) $a_{n+1} = 3{a_n}^2 - 1$ with $a_1 = 2$

(iii) $a_{n+2} = a_{n+1} + a_n$ with $a_1 = 4$ and $a_2 = 6$

a)

"n^{th} TERM = ..." MEANS WE HAVE THE POSITION-TO-TERM RULE

i)

n	1	2	3	4	5
a_n	18	16	14	12	10

$20 - 2 \times 1$

THE CALCULATIONS ARE USUALLY EASY SO YOU CAN DO THEM MENTALLY

YOU MAY SPOT THIS IS A LINEAR SEQUENCE

ii)

n	1	2	3	4	5
a_n	−1	5	15	29	47

$2 \times 3^2 - 3$

YOU MAY SPOT THIS IS A QUADRATIC SEQUENCE

iii)

n	1	2	3	4	5
a_n	1/2	1/4	1/6	1/8	1/10

$\dfrac{1}{2 \times 4}$

4. Sequences

b)

THIS TIME WE HAVE BEEN GIVEN THE TERM–TO–TERM RULE

i)

n	1	2	3	4
a_n	3	9	15	21

$a_2 = a_1 + 6$

$a_2 = 3 + 6$

NOTE: THIS IS A LINEAR SEQUENCE

ii)

n	1	2	3	4
a_n	2	11	362	393 131

$a_4 = 3a_3^2 - 1$

$a_4 = 3 \times 362^2 - 1$

SOMETIMES A CALCULATOR IS HANDY!

iii)

n	1	2	3	4
a_n	4	6	10	16

$a_4 = a_3 + a_2$

$a_4 = 10 + 6$

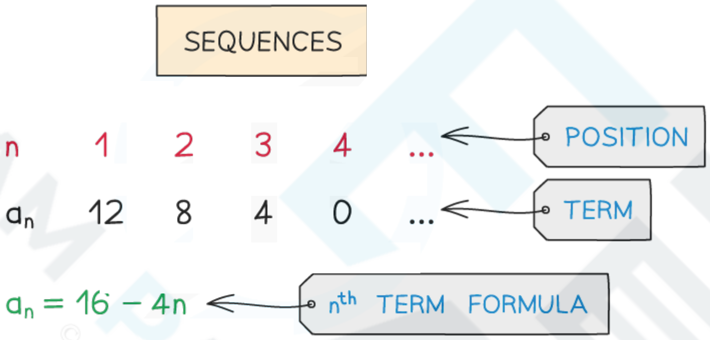
THIS IS A FIBONACCI SEQUENCE BUT NOT THE FIBONACCI SEQUENCE

4. Sequences

4.3.2 SEQUENCES - IDENTIFYING

What are sequences?

- A sequence is an order set of (usually) numbers
- Make sure you are familiar with the **basics** and **notation** used with sequences



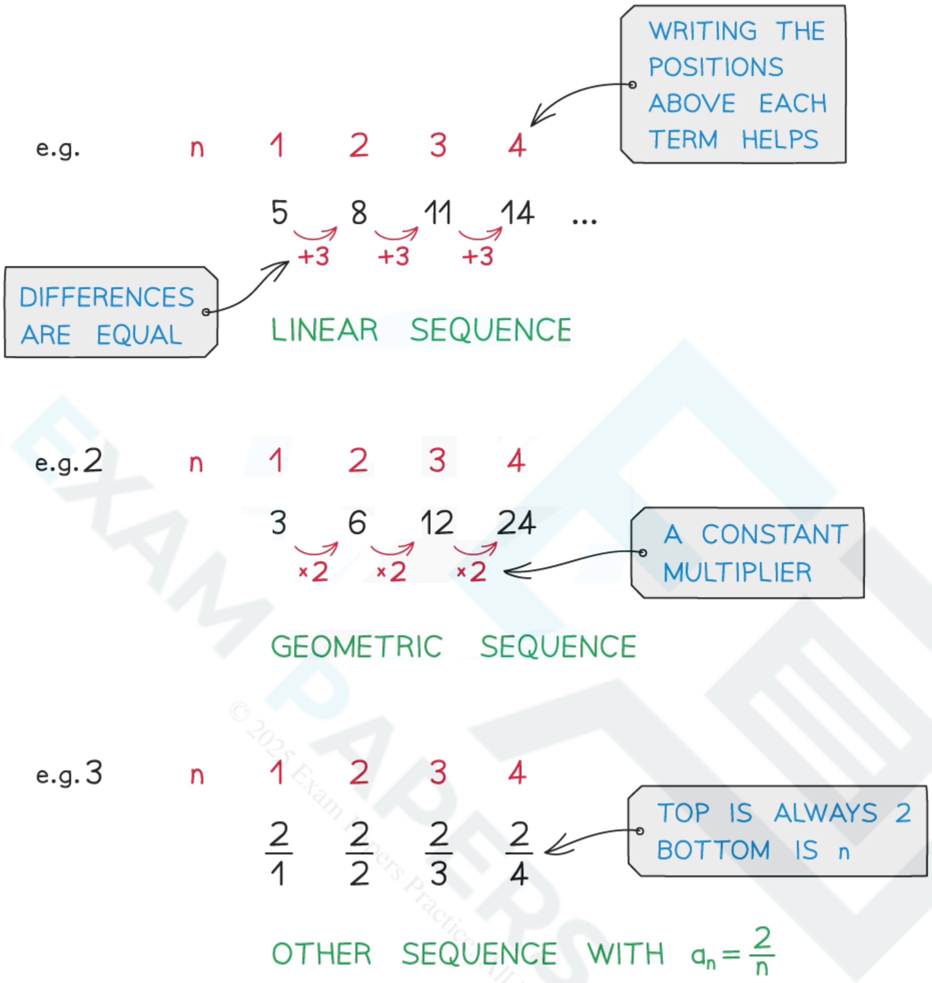
How do I identify a sequence?

- Is it obvious?
- Does it tell you in the question?
- Is there is a number that you multiply to get from one term to the next?
  - If so then it is a **geometric** sequence
- Next, look at the **differences** between the terms

  If **1st differences** are constant – it is a **linear** sequence

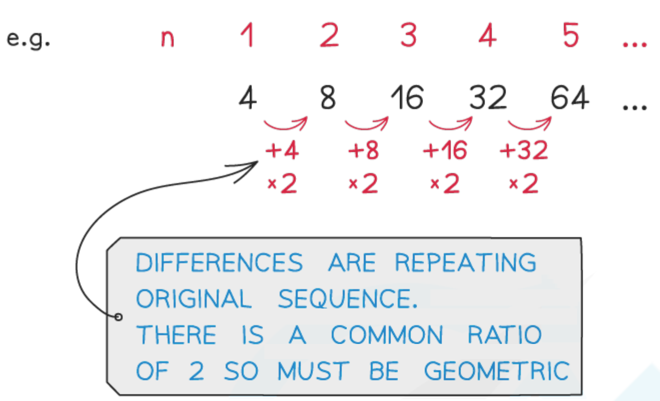
  If **2nd differences** are constant – it is a **quadratic** sequence

4. Sequences



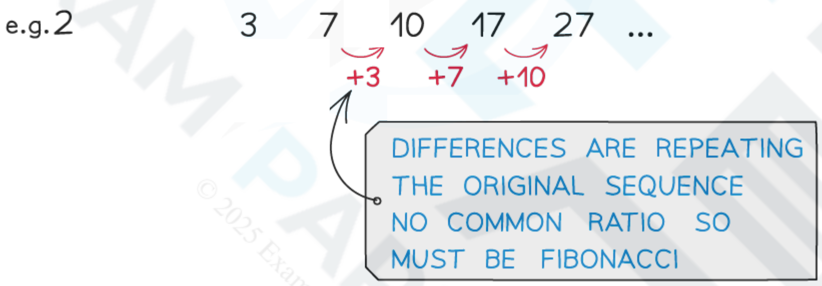
- Special cases to be aware of:
- If the differences **repeat** the original sequence
    - It is a **geometric** sequence with **common ratio** 2
- **Fibonacci** sequences also have **differences** that **repeat** the original sequence
    - However questions usually indicate if a Fibonacci sequence is involved

4. Sequences



DIFFERENCES ARE REPEATING ORIGINAL SEQUENCE. THERE IS A COMMON RATIO OF 2 SO MUST BE GEOMETRIC

GEOMETRIC SEQUENCE



DIFFERENCES ARE REPEATING THE ORIGINAL SEQUENCE NO COMMON RATIO SO MUST BE FIBONACCI

FIBONACCI SEQUENCE

4. Sequences

Worked Example

?

(a) Identify the types of sequence below

(i) 4, 5, 9, 14, 23, 37, 60, ...

(ii) 6, 10, 16, 24, 34

(iii) 12, 7, 2, -3, ...

(b) Write down a formula for the n^{th} term (in terms of n) for each of the following sequences

(i) $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6},$

(ii) 2, 6, 12, 20, 30, 42, …

a) i) 4 5 9 14 23 37 60 ...

+4 +5 +9 +14 +23

DIFFERENCES ARE REPEATING THE ORIGINAL SEQUENCE
GEOMETRIC – NO, AS NO COMMON RATIO.
FIBONACCI ✓

FIBONACCI SEQUENCE

ii) 6 10 16 24 34

+4 +6 +8 +10

+2 +2 +2

1st DIFFERENCES ARE **NOT** EQUAL

2nd DIFFERENCES ARE EQUAL

QUADRATIC SEQUENCE

iii) 12 7 2 −3

−5 −5 −5

1st DIFFERENCES ARE EQUAL

SEQUENCE IS GOING DOWN SO DIFFERENCES ARE NEGATIVE

LINEAR SEQUENCE

b) i)

n	1	2	3	4
a_n	$\frac{1}{3}$	$\frac{2}{4}$	$\frac{3}{5}$	$\frac{4}{6}$

IT HELPS TO WRITE THE POSITIONS AT

TOP IS SAME AS n...
BOTTOM IS 2 MORE THAN n...

n^{th} TERM, $a_n = \frac{n}{n+2}$

ii)

n	1	2	3	4	5
a_n	2	6	12	20	30

NOTHING OBVIOUS, NO COMMON RATIO.
LOOK AT DIFFERENCES

$$\begin{array}{ccccccccc} 2 & & 6 & & 12 & & 20 & & 30 \\ & +4 & & +6 & & +8 & & +10 & \\ & & +2 & & +2 & & +2 & & \end{array}$$

2nd DIFFERENCES ARE EQUAL – QUADRATIC
COMPARE TO THE SQUARE NUMBERS

a_n	2	6	12	20	30
n^2	1	4	9	16	25
n	1	2	3	4	5

NOW I CAN SEE...

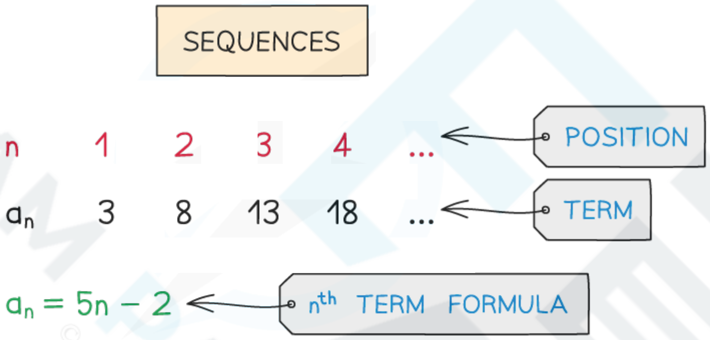
$a_n = n^2 + n$

4. Sequences

4.3.3 SEQUENCES - OTHERS

What are sequences?

- A sequence is an order set of (usually) numbers
- Make sure you are familiar with the **basics** and **notation** used with sequences



What are “other” sequences?

- **Linear** and **quadratic** sequences are particular types of sequence covered in previous notes
- “Other” sequences include geometric and Fibonacci sequences, both briefly mentioned here Sequences – Identifying
- Geometric and Fibonacci are looked at in more detail below
- But other sequences can include fractions, decimals
    - Anything that makes the position-to-term and/or the term-to-term rule easy to spot

4. Sequences

TYPES OF SEQUENCES

LINEAR

e.g. 2 6 10 14 18 ...

+4 +4 +4 +4

FIRST DIFFERENCES CONSTANT

QUADRATIC

e.g. 5 9 15 23 33 ...

+4 +6 +8 +10

+2 +2 +2

SECOND DIFFERENCES CONSTANT

GEOMETRIC

e.g. 4 12 36 108 324 ...

×3 ×3 ×3 ×3

CONSTANT MULTIPLIER (COMMON RATIO)

FIBONACCI

e.g. 2 4 6 10 16 26 42 ...

ADD THE PREVIOUS TWO TERMS

OTHER

e.g.

n	1	2	3	4
	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$ ...

n^{th} TERM, $a_n = \frac{1}{n}$

SUCH SEQUENCES DON'T FALL INTO ANY CATEGORY BUT THE LINK BETWEEN n AND a_n IS FAIRLY EASY TO SPOT

4. Sequences

What is a geometric sequence?

- In a geometric sequence, the term-to-term rule would be to multiply by a constant
    - $a_{n+1} = ra_n$
- **r** is called the **common ratio**
- In the sequence 4, 8, 16, 32, 64, ... the common ratio would be 2

GEOMETRIC SEQUENCES

IN A GEOMETRIC SEQUENCE, A TERM IS FOUND BY MULTIPLYING THE PREVIOUS TERM BY A CONSTANT

i.e. THE TERM–TO–TERM RULE IS $a_{n+1} = ra_n$

r IS THE CONSTANT AND IS CALLED THE COMMON RATIO

e.g. FIND THE FIRST FOUR TERMS IN THE GEOMETRIC SEQUENCE WITH FIRST TERM 2 AND COMMMON RATIO 4.

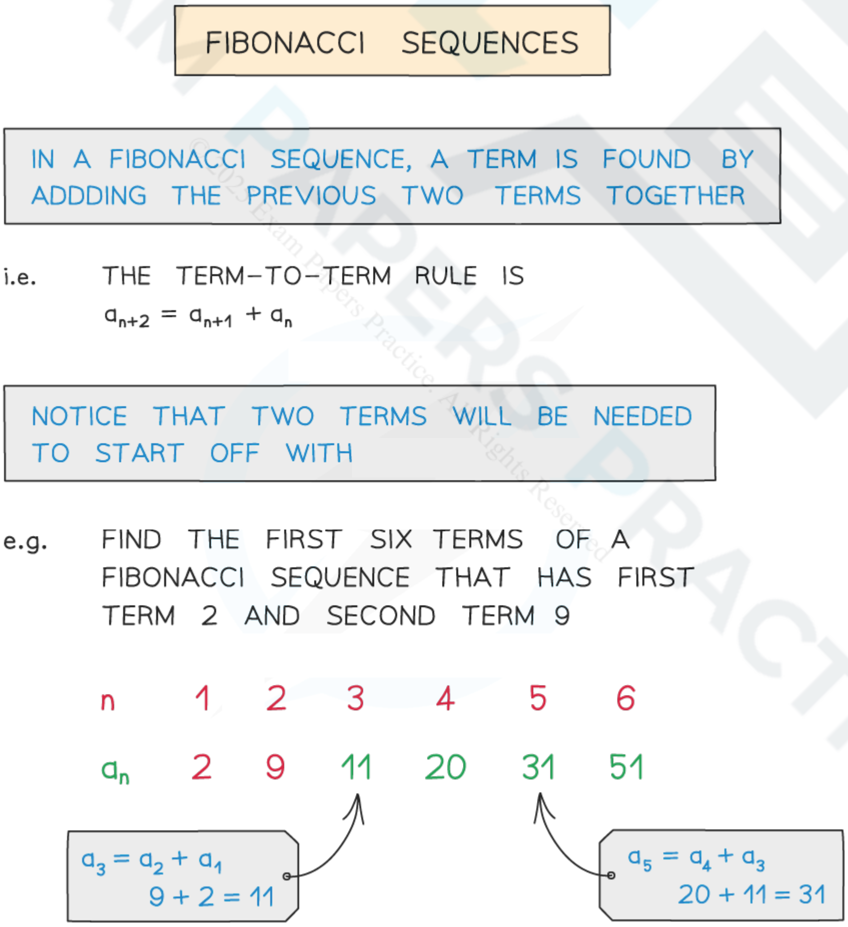
n	1	2	3	4
a_n	2	8	32	128

$$2 \xrightarrow{\times 4} 8 \xrightarrow{\times 4} 32 \xrightarrow{\times 4} 128$$

4. Sequences

What is a Fibonacci sequence?

- **THE** Fibonacci sequence is **1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...**
- The sequence starts with the **first two terms** as **1**
- Each subsequent term is the **sum** of the **previous two**
  - ie The term-to-term rule is $\mathbf{a_{n+2} = a_{n+1} + a_n}$
  - Notice that two terms are needed to start a Fibonacci sequence
- Any sequence that has the term-to-term rule of adding the previous two terms is called a **Fibonacci** sequence but the first two terms will not both be 1
- Fibonacci sequences occur a lot in nature such as the number of petals of flowers



FIBONACCI SEQUENCES

IN A FIBONACCI SEQUENCE, A TERM IS FOUND BY ADDDING THE PREVIOUS TWO TERMS TOGETHER

i.e. THE TERM–TO–TERM RULE IS $a_{n+2} = a_{n+1} + a_n$

NOTICE THAT TWO TERMS WILL BE NEEDED TO START OFF WITH

e.g. FIND THE FIRST SIX TERMS OF A FIBONACCI SEQUENCE THAT HAS FIRST TERM 2 AND SECOND TERM 9

n	1	2	3	4	5	6
a_n	2	9	11	20	31	51

$a_3 = a_2 + a_1$
$9 + 2 = 11$

$a_5 = a_4 + a_3$
$20 + 11 = 31$

Problem solving and sequences

- When the type of sequence is known it is possible to find unknown terms within the sequence
- This can lead to problems involving setting up and solving equations

  - Possibly simultaneous equations
- Other problems may involve sequences that are related to common number sequences such as square numbers, cube numbers and triangular numbers

e.g. IN A FIBONACCI SEQUENCE THE 4th TERM IS 2a, AND THE 5th TERM IS b + 1

a) WRITE DOWN EXPESSIONS FOR THE 6th AND 7th TERMS

6th TERM = (b + 1) + 2a

$a_6 = a_5 + a_4$

6th TERM = 2a + b + 1

7th TERM = (2a + b + 1) + (b + 1)

$a_7 = a_6 + a_5$

7th TERM = 2a + 2b + 2

b) GIVEN $a_6 = 20$ AND $a_7 = 32$
FIND THE VALUES OF a AND b

$2a + b + 1 = 20$
$2a + 2b + 2 = 32$

SOLVE AS SIMULTANEOUS EQUATIONS

$2a + b = 19$
$2a + 2b = 30$
$b = 11$

SUBTRACT

$2a + 11 = 19$
$2a = 8$
$a = 4$

SUBSTITUTE VALUE OF b INTO ANY EQUATION USED

$a = 4$ AND $b = 11$

4. Sequences

Worked Example

? (a) Three consecutive terms in a geometric sequence are

$5x + 9$, $2x$ and $x - 3$

Find the two possible values of x.

(b) The 3rd and 6th terms in a Fibonacci sequence are 7 and 31 respectively.

Find the 1st and 2nd terms of the sequence.

(c) Write down a formula for a_n for the following sequences

(i) 6, 10, 15, 21, 28, ...

(ii) 0, 7, 26, 63, 124, ...

(iii) 5, 15, 45, 135, 405, ...

a) $r(5x + 9) = 2x$

$r(2x) = x - 3$

$r = \frac{2x}{5x+9}$ $\quad$ $r = \frac{x-3}{2x}$

SO, $\frac{2x}{5x+9} = \frac{x-3}{2x}$

ONCE YOU ARE FAMILIAR WITH GEOMETRIC SEQUENCES YOU CAN START WITH THE LINE ABOVE. DO TRY TO UNDERSTAND WHERE IT HAS COME FROM

$(2x)(2x) = (x - 3)(5x + 9)$

MULTIPLY TO GET RID OF FRACTIONS (GROF)

$4x^2 = 5x^2 - 6x - 27$

EXPAND (USING FOIL MAYBE?) TO GET RID OF BRACKETS (GROB)

$x^2 - 6x - 27 = 0$ QUADRATIC EQUATION

$(x-9)(x + 3) = 0$

$x = 9$, $\quad x = -3$

IN THIS QUESTION, BOTH ANSWERS ARE VALID. ON ANOTHER QUESTION, YOU MAY BE TOLD THAT $x > 0$, IN WHICH CASE ONLY $x = 9$ WOULD BE VALID

4. Sequences

b)

WRITE AT WHAT YOU DO KNOW ABOUT THE SEQUENCE

n	1	2	3	4	5	6	7
a_n		x	7	x + 7	x + 14	31	

$a_{n+2} = a_{n+1} + a_n$

FIBONACCI – TOLD IN QUESTION

$a_3 = a_2 + a_1$ $\qquad$ $a_2 + a_1 = /$

$a_6 = a_5 + a_4$ $\qquad$ $a_5 + a_4 = 31$

LET $a_2 = x$, THEN, $a_4 = x + 7$

AND $a_5 = (x + 7) + 7$

$= x + 14$

SO $(x + 14) + (x + 7) = 31$

$a_6 = a_5 + a_4 = 31$

$2x + 21 = 31$

$2x = 10$

$x = 5$ $\qquad$ $a_2 = 5$

$a_3 = a_2 + a_1$

$7 = 5 + a_1$ $\qquad$ $a_1 = 2$

4. Sequences

c) i)

n	1	2	3	4	5
a_n	6	10	15	21	28

CAREFUL! IF YOU IMMEDIATELY CONSIDER DIFFERENCES THIS WILL LEAD TO A QUADRATIC SEQUENCE. WHICH IS CORRECT BUT ITS STLL TRICKY TO GET THE n^{th} TERM FORMULA AS IT IS NOT "OBVIOUS"

YOU SHOULD SPOT THAT 6, 10, 15, ... ARE TRIANGULAR NUMBERS, SO COMPARE THE SEQUENCE TO THOSE

n	1	2	3	4	5
$\frac{1}{2}n(n+1)$	1	3	6	10	15
a_n	6	10	15	21	28

$a_n = \frac{1}{2}(n+2)(n+2+1)$

THE TRIANGULAR NUMBERS ARE "2 AHEAD" SO REPLACE "n" WITH "n + 2" IN THE n^{th} TERM FORMULA

$a_n = \frac{1}{2}(n+2)(n+3)$

ii)

n	1	2	3	4	5
a_n	0	7	26	63	124

NUMBERS GROW QUICKLY SO NOT LIKELY LINEAR OR QUADRATIC.
NOT FIBONACCI, NOTHING LIKE TRIANGULAR.
TRY CUBE NUMBERS

n	1	2	3	4	5
n^3	1	8	27	64	125
a_n	0	7	26	63	124

a – ha!

$a_n = n^3 - 1$

iii)

n	1	2	3	4	5
a_n	5	15	45	135	405

$$5 \xrightarrow{\times 3} 15 \xrightarrow{\times 3} 45 \xrightarrow{\times 3} 135 \xrightarrow{\times 3} 405$$

GEOMETRIC SEQUENCE

TERM–TO–TERM IS $a_{n+1} = 3a_n$

POSITION–TO–TERM IS NOT THAT OBVIOUS – UNTIL YOU'VE SEEN IT ONCE! HERE GOES...

$a_2 = 3a_1$

$a_3 = 3a_2 = 3(3a_1)$

$a_4 = 3a_3 = 3(3(3a_1))$

$a_5 = 3a_4 = 3(3(3(a_1)))$ etc

i.e. $a_2 = 3a_1$

$a_3 = 3^2a_1$

$a_4 = 3^3a_1$

$a_5 = 3^4a_1$

WE KNOW $a_1 = 5$, SO...

$a_n = 5 \times 3^{n-1}$

4. Sequences

Exam Question: Hard

S is a geometric sequence.

(a) Given that $(\sqrt{x} - 1)$, 1 and $(\sqrt{x} + 1)$ are the first three terms of S, find the value of x.
You must show all your working.

(b) Show that the 5th term of S is $7 + 5\sqrt{2}$

5. Graphs

CONTENTS

5.1 COORDINATES

5.1.1 COORDINATES

5. Graphs

What are coordinates?

- When we want to plot a point on a graph we need to know where to put it
- If the horizontal axis is labelled **x** and the vertical axis is labelled **y**, then the **x** and **y coordinates** are how far we go along the **x** and **y** axes to plot the point

What can we do with coordinates?

- If we have two points with coordinates (x_1, y_1) and (x_2, y_2) then we should be able to find
    - the **gradient** of the line through them
    - the **midpoint** of the two points
    - the **distance** between the two points
- Don't get fazed by the horrid notation (x_1, y_1) – this is just "point 1" and the other is "point 2" so we put the little numbers (subscripts) in so that we know which coordinate we are referring to. In questions there'll be lots of nice numbers
  Here's how we do each of those:

1. **GRADIENT** $= \frac{RISE}{RUN} = \frac{y_2 - y_1}{x_2 - x_1}$

2. **MIDPOINT** is the "Average Point": $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

3. **DISTANCE** is found using Pythagoras Theorem:

$$Distance = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- You should also know that three points A, B and C lie on the same STRAIGHT LINE if AB and AC (or BC) have the same gradient

Exam Tip

If in doubt, SKETCH IT!

A quick, reasonably accurate sketch can make things a lot clearer.

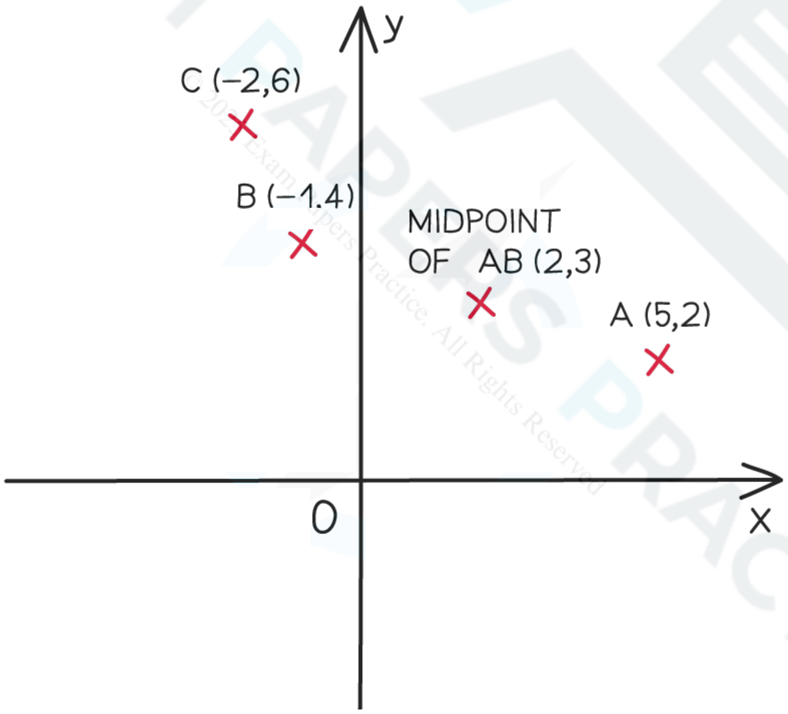
All of the above can also be applied to 3D coordinates in the form (x, y, z), on a 3D axis.

5. Graphs

Worked Example

1. *Three points A, B and C have coordinates* (5, 2), (− 1, 4) *and* (− 2, 6) *respectively.*
   - (a) *Find the midpoint of AB,*
   - (b) *Find the distance between the midpoint of AB and point C*
   - (c) *Show that AB and BC are not part of the same straight line.*

Exam Tip − Sketch the points, from which you should be able see roughly where the midpoint should go and so is a quick check to see if your answer to (a) is about right.
Add to your diagram as you work through the question



5. Graphs

(a) $Midpoint\ is\ \left(\frac{5+(-1)}{2}, \frac{2+4}{2}\right)$

$= (2,\ 3)$

2 − *Find the midpoint of AB*

(b) $Distance = \sqrt{(6-3)^2 + ((-2)-2)^2}$

$= \sqrt{25}$

$= 5$

3 − *Distance between* $(2,\ 3)$ *and* $(-2,\ 6)$

(Positive square root only as it is a distance)

(c) $Gradient\ of\ AB\ = \frac{4-2}{(-1)-5} = \frac{2}{-6} = -\frac{1}{3}$

1 − *Showing that gradients are unequal will be enough to show AB and BC are two different lines.*

$Gradient\ of\ BC\ = \frac{6-4}{(-2)-(-1)} = \frac{2}{-1} = -2$

Gradients of AB and BC are not equal and so AB and BC are not part of the same straight line.

Exam Question: Medium

5. Graphs

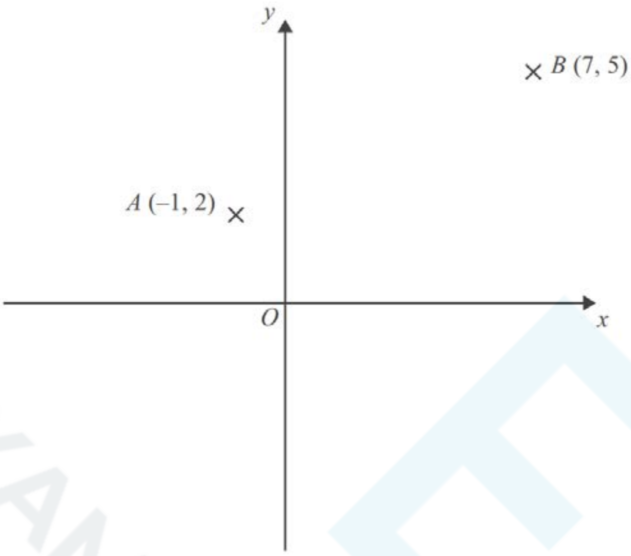


Diagram **NOT** accurately drawn

A is the point (–1, 2)
B is the point (7, 5)

(a) Find the coordinates of the midpoint of *AB*.

P is the point (–4, 4)
Q is the point (1, –5)

(b) Find the gradient of *PQ*.

5. Graphs

Exam Question: Hard

Here is a cuboid drawn on a 3-D grid.

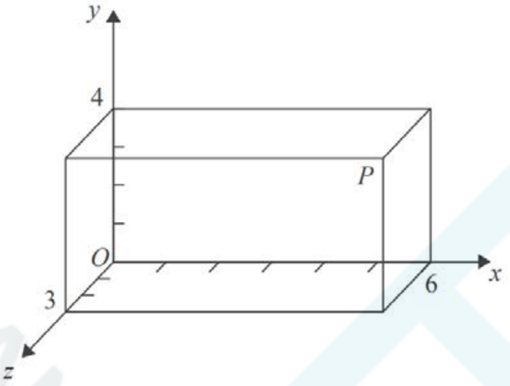


Diagram **NOT** accurately drawn

P is a vertex of the cuboid.

T divides the line *OP* in the ratio 1 : 2

Find the coordinates of *T*.

5. Graphs

5.2 DRAWING GRAPHS

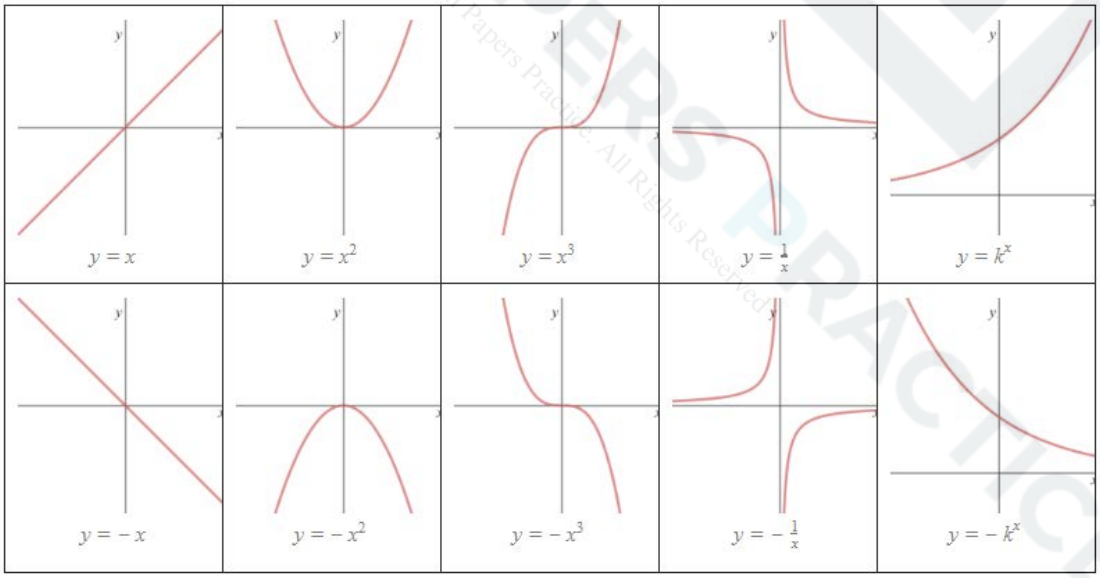
5.2.1 DRAWING GRAPHS - SHAPES

Why do we need to know what graphs look like?

- Graphs are used in various aspects of mathematics – but in the real world they can take on specific meanings
- For example a **linear (straight line)** graph could be the path a ship needs to sail along to get from one port to another
- An **exponential** graph ($y=k^x$) can be used to model population growth – for instance to monitor wildlife conservation projects

Drawing graphs – shapes

- Recalling facts alone won’t do much for boosting your GCSE Mathematics grade!
- But being familiar with the general shapes of graphs will help you quickly recognise the sort of maths you are dealing with and features of the graph a question may refer to

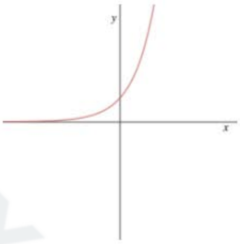


5. Graphs

Worked Example

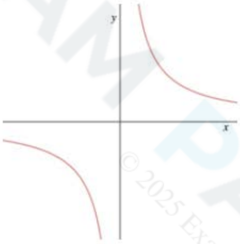
1. *Match the graphs to the equations.*

(A)



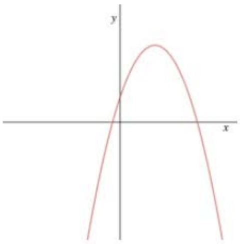
(1) $y = 0.6x + 2$

(B)



(2) $y = 3^x$

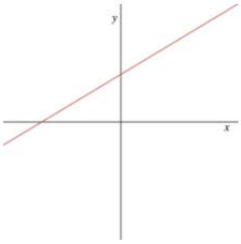
(C)



(3) $y =- 0.7x^3$

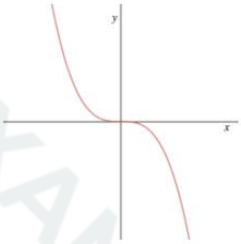
5. Graphs

(D)



(4) $y = \frac{4}{x}$

(E)



(5) $y = -x^2 + 3x + 2$

Graph A $\rightarrow$ Equation 2
Graph B $\rightarrow$ Equation 4
Graph C $\rightarrow$ Equation 5
Graph D $\rightarrow$ Equation 1
Graph E $\rightarrow$ Equation 3

5. Graphs

5.2.2 DRAWING GRAPHS - USING A TABLE

How do we draw a graph using a table of values?

- Use your calculator
- THINK what the graph might look like – see the previous notes on being familiar with shapes of graphs
- Find the TABLE function on your CALCULATOR
- Enter the FUNCTION – f(x)
  (use ALPHA button and x or X, depending on make/model)
  (Press = when finished)
  (If you are asked for another function, g(x), just press enter again)
- Enter **Start**, **End** and **Step** (gap between **x** values)
- Press = and scroll up and down to see **y** values
- PLOT POINTS and join with a SMOOTH CURVE

- If your calculator does not have a TABLE function then you will have to work out each y value separately using the normal mode on your calculator
- To avoid errors always put negative numbers in brackets and use the (-) key rather than the subtraction key

Worked Example

1. (a) *Complete the table of values for the function* $f(x) = x^3 - 5x + 2$

   (b) *On the graph paper provided draw the graph of* $y = x^3 - 5x + 2$

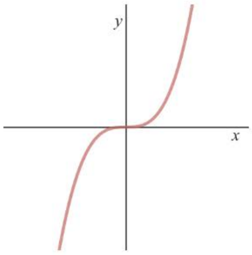
(a)

x	-3	-2	-1	0	1	2	3
$f(x)$	**-10**	4	**6**	**2**	**-2**	**0**	14

2, 3, 4, 5 – *Complete the table using you calculator*

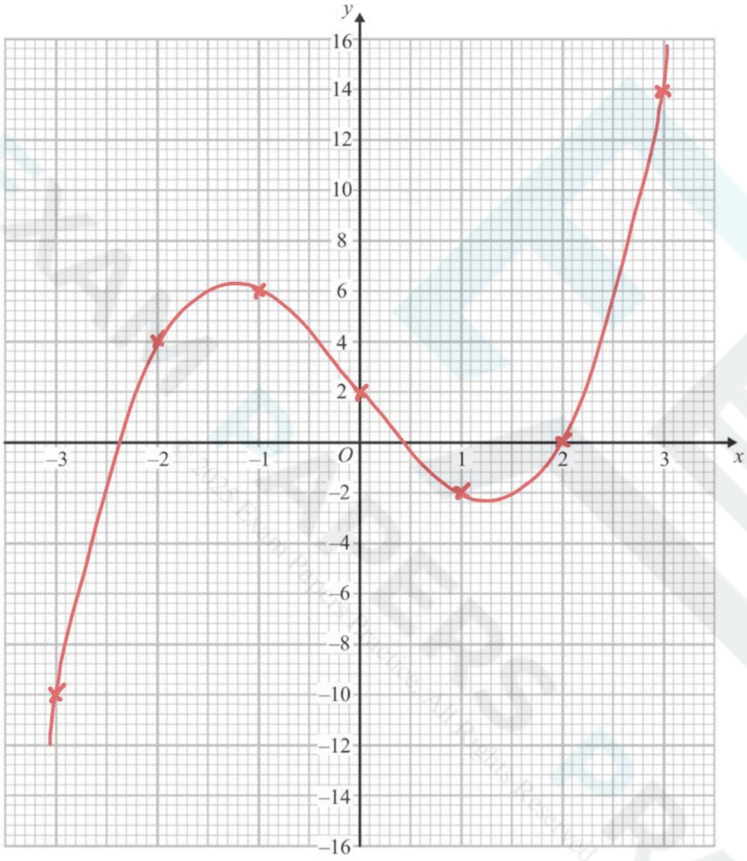
(b)

5. Graphs



1 – It's a positive cubic graph so you know it should look a bit like this

It is not essential to sketch the graph but you should've at least thought about it



6 – Use the table from part (a) to plot the points carefully and join them up wit a smooth curve
If you don't/can't get a smooth curve flowing nicely through the points you may have made a mistake so go back and double check your values and plotting

5. Graphs

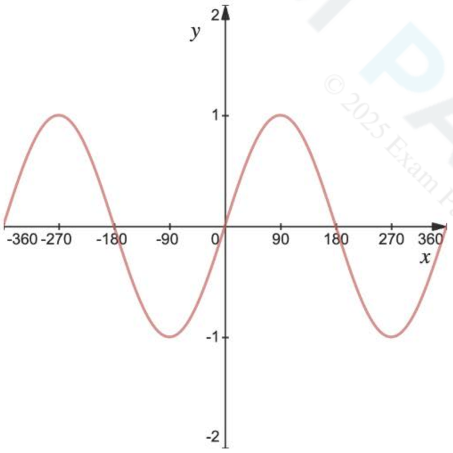
5.2.3 DRAWING GRAPHS - TRIG GRAPHS

Why do we need to know what trigonometric graphs look like?

- Trigonometric Graphs are used in various applications of mathematics – for example, the oscillating nature can be used to model how a pendulum swings

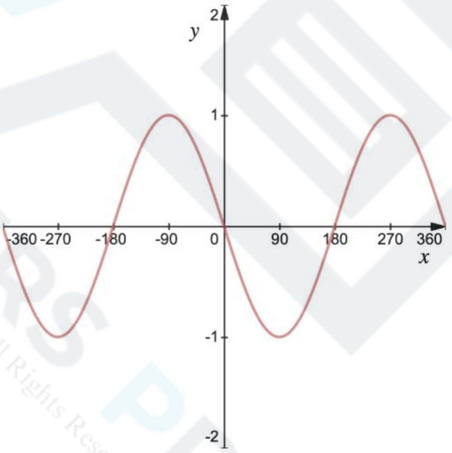
Drawing graphs – trig graphs

- As with other graphs, being familiar with the general style of trigonometric graphs will help you sketch them quickly and you can then use them to find values or angles alongside your calculator
- All trigonometric graphs follow a pattern – a “starting point” and then “something happens every 90°”.



$y = sin\ x$

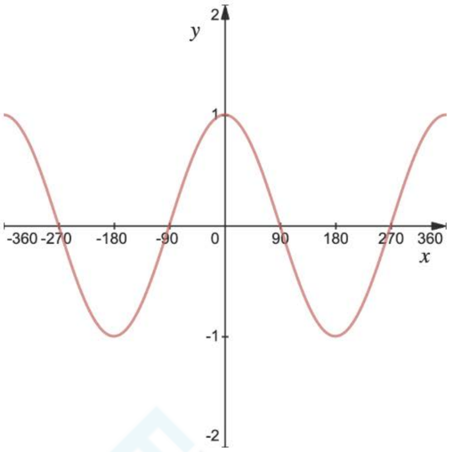
Starts at $(0,\ 0)$
Every $90\,°$ *it cycles through* $1,\ 0,\ -1,\ 0,\ \ldots$



$y = -\ sin\ x$

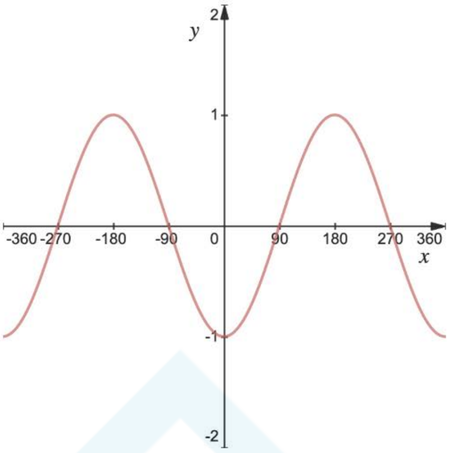
Learn how this relates to $y = sin\ x$

5. Graphs



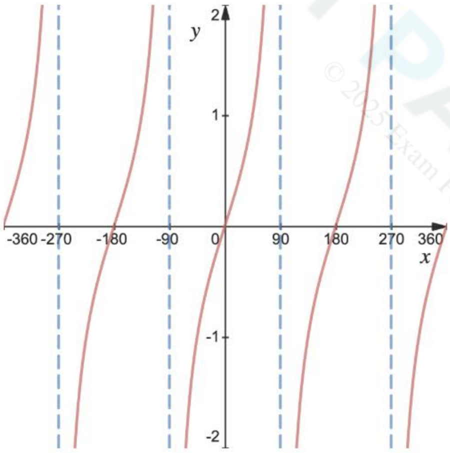
$y = \cos x$

Starts at $(0, 1)$
Every 90° it cycles through $0, -1, 0, 1, \ldots$



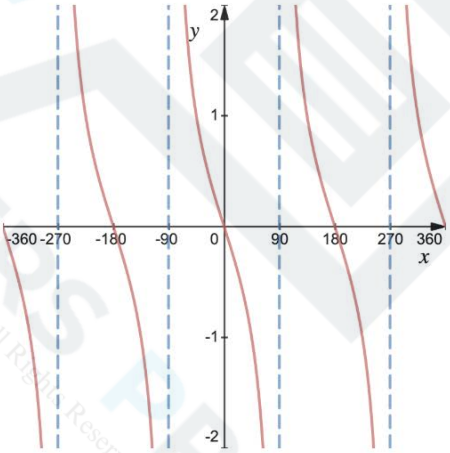
$y = -\cos x$

Learn how this relates to $y = \cos x$



$y = \tan x$

Starts at $(0, 0)$
Every 90° it is either 0 or an asymptote $(*)$



$y = -\tan x$

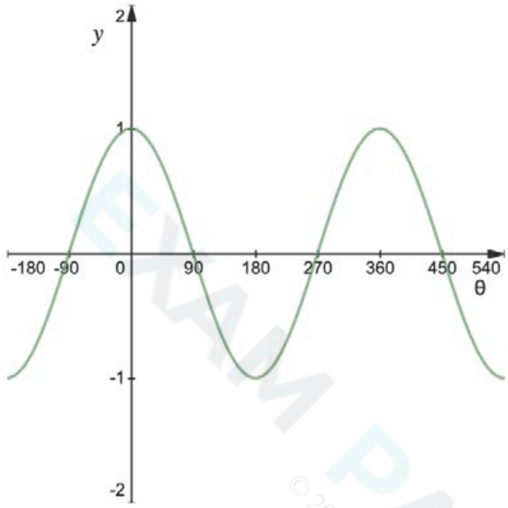
Learn how this relates to $y = \tan x$

- (*) An asymptote is a line that a graph gets ever closer to without ever crossing or touching it

5. Graphs

1. *Sketch the graph of* $y = \cos\theta$ *for* $-180° \le \theta \le 540°$.

Do not be put off by the use of θ (theta), it is a Greek letter often used in mathematics to denote angles
Okay we have the positive cos graph, so we start at (0, 1) and cycle round 0, −1, 0, 1, 0, −1, 0, etc



Note that it is easier to deal with the positive angles first, then work backwards from (0, 1) to deal with the negative angles

It is easy to see that −180° is linked to "every 90°" but not so obvious that 540° is − just count up in 90° steps until you reach this value or past it.

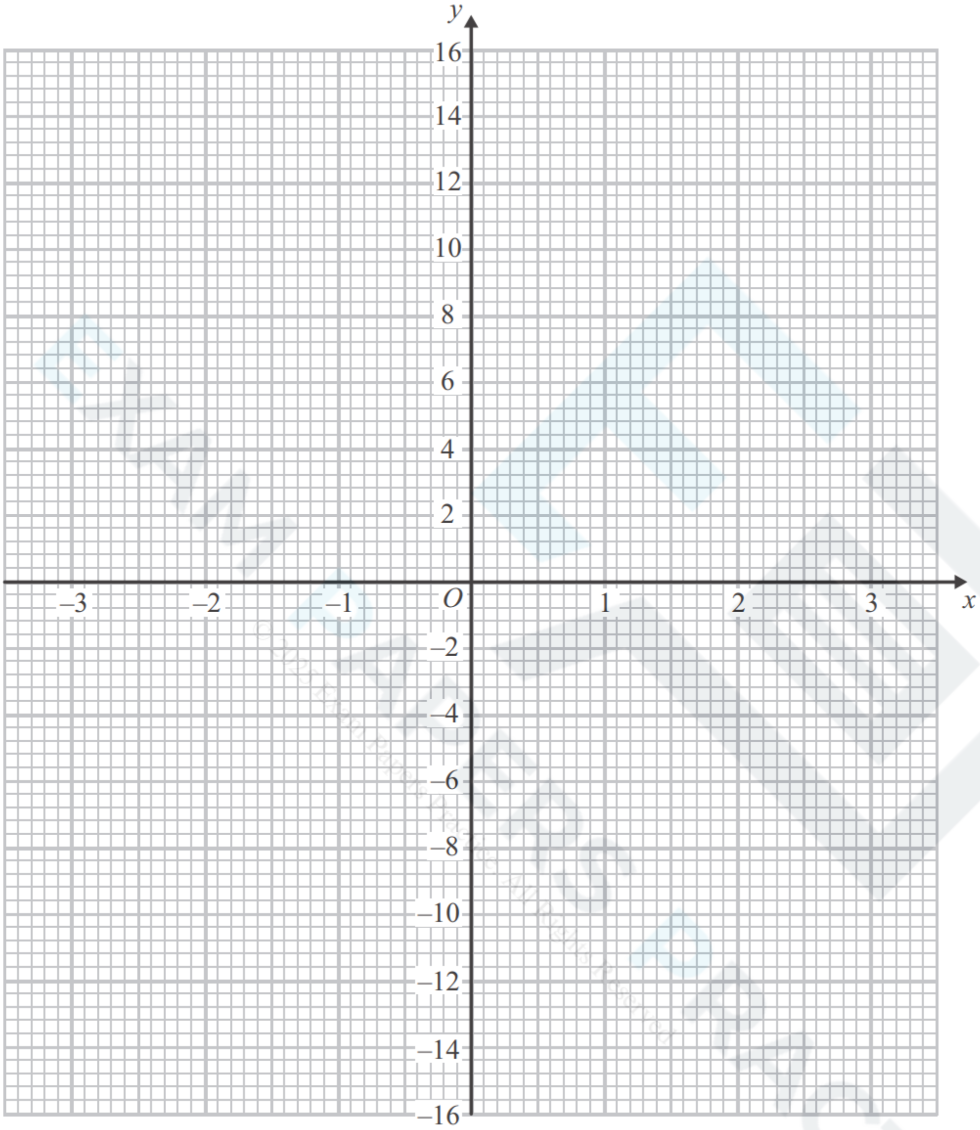
Exam Question: Medium

(a) Complete the table of values for $y = x^3 - 4x$

x	−3	−2	−1	0	1	2	3
y			3	0			15

5. Graphs

(b) On the grid, draw the graph of $y = x^3 - 4x$ from $x = -3$ to $x = 3$



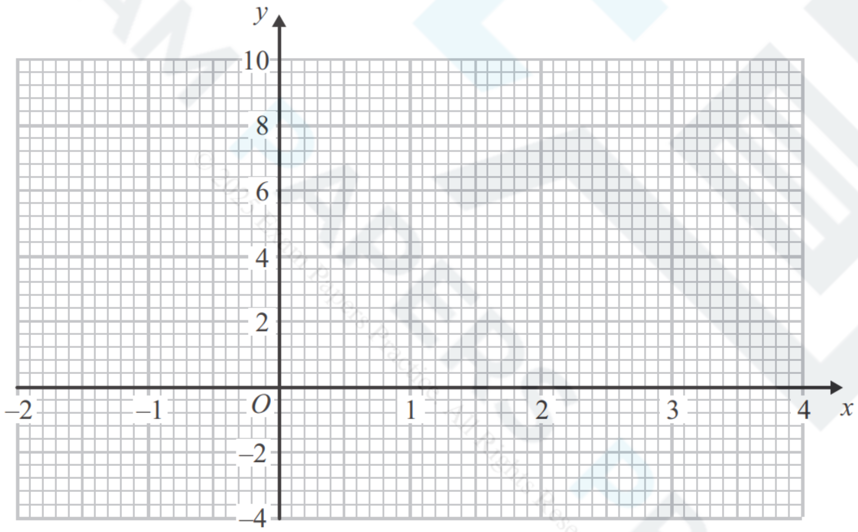
5. Graphs

Exam Question: Hard

(a) Complete the table of values for $y = x^2 - 2x$

x	–2	–1	0	1	2	3	4
y		3	0			3	

(b) On the grid, draw the graph of $y = x^2 - 2x$ for values of x from –2 to 4



(c) Solve $x^2 - 2x - 2 = 1$

5. Graphs

5.3 EQUATION OF A LINE / Y = MX + C

5.3.1 STRAIGHT LINES - FINDING EQUATIONS

Why do we want to know about straight lines and their equations?

- Straight Line Graphs (Linear Graphs) have lots of uses in mathematics – one use is in navigation
- We may want to know the equation of a straight line so we can program it into a computer that will plot the line on a screen, along with several others, to make shapes and graphics

How do we find the equation of a straight line?

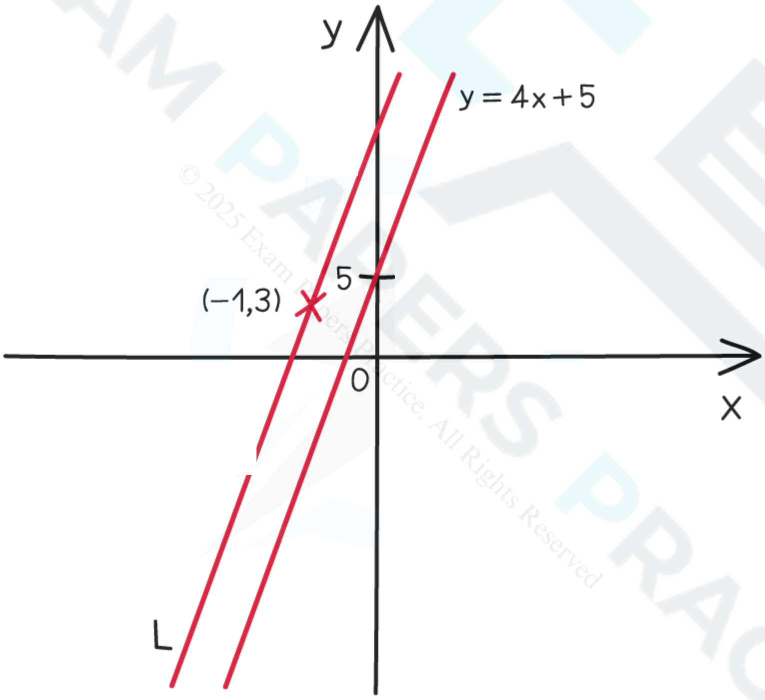
- The general EQUATION of a straight line is **y = mx + c**
    - where **m** is the gradient
    - **c** is the **y**-axis intercept
- To find the EQUATION of a straight line you need TWO things:
    - the gradient, m
    - any point on the line
- You might find these things from a graph, another equation or two points
- You may be asked to give the equation in the form **ax + by + c = 0**
  (especially if **m** is a fraction)
  If in doubt, SKETCH IT!

5. Graphs

Worked Example

1. *The line P has equation* $y = 4x + 5$.
Find the equation of the line L parallel to P which passes through the point $(-1, 3)$.
Give your answer in the form $ax + by + c = 0$

Sketch not essential but if it doubt ...
Sketching the given line and roughly where the new lineL should be will give you a good idea if your final answer is realistic or not



5. Graphs

$y = mx + c$ $m = 4$ $y = 4x + c$	$1 -$ *Using the general equation of a straight line* $2 -$ *Since the line L is parallel to* $y = 4x + 5$ *We now know that the equation of L looks like this*
$3 = 4 \times (-1) + c$ $c = 7$	$2 -$ *The second thing we need is a point* *Use the point* $(-1,\ 3)$ $-$ *ie* $x =- 1,\ y = 3$ *to find c*
$y = 4x + 7$ $0 = 4x - y + 7$ $4x - y + 7 = 0$	$3 -$ *This is the equation of L but it's not in the correct format so rearrange, in stages if need be, until it is.*

5. Graphs

5.3.2 STRAIGHT LINES - DRAWING GRAPHS

How do we draw the graph of a straight line from an equation?

- Before you start trying to draw a straight line, make sure you understand how to find the equation of a straight line – that will help you understand this
- How we draw a straight line depends on what form the equation is given in
- There are two main forms you might see:
  y = mx + c and **ax + by = c**
- Different ways of drawing the graph of a straight line:

1. From the form **y = mx + c**
   (you might be able to rearrange to this form easily)
   plot **c** on the **y**-axis
   go 1 across, **m** up (and repeat until you can draw the line)
2. From **ax + by = c**
   put **x = 0** to find **y**-axis intercept
   put **y = 0** to find **x**-axis intercept
   (You may prefer to rearrange to **y = mx + c** and use above method)

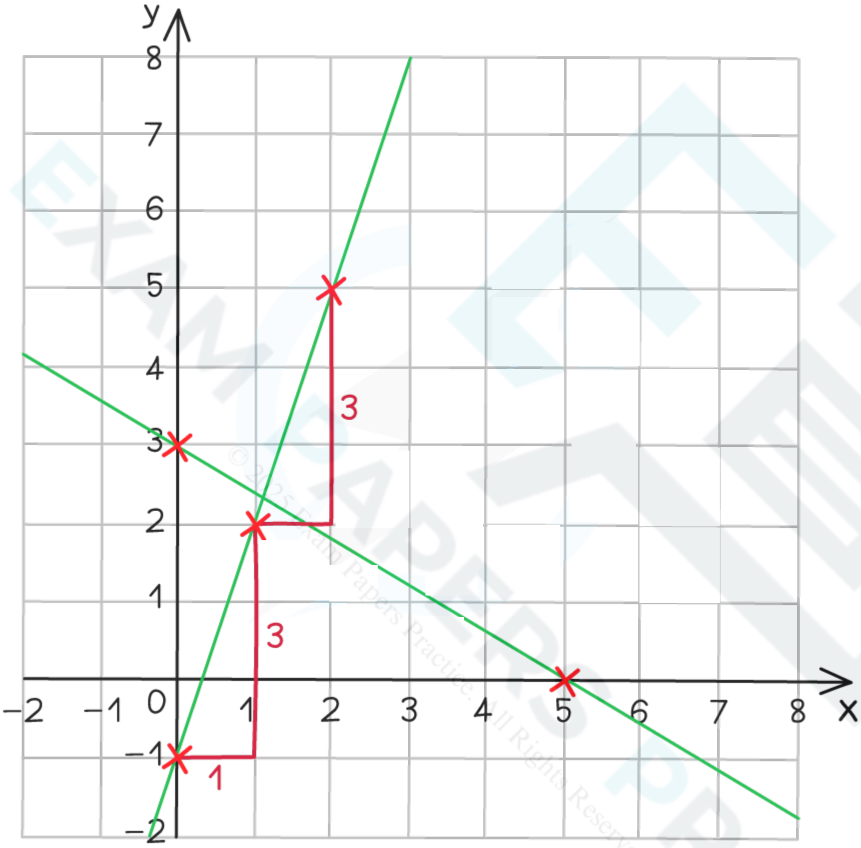
Exam Tip

It might be easier just to plot ANY two points on the line (a third one as a check is not a bad idea either) or use the TABLE function on your calculator.

5. Graphs

Worked Example

1. *On the axes below draw the lines* $y = 3x - 1$ *and* $3x + 5y = 15$



1 − $y = 3x - 1$ *First plot c, which is* -1 *on the* y − *axis*
Then go across 1, *m, which is* 3, *up* − *this has been done twice on the diagram*
Doing the second part twice gives you three points that are on the line and so you can be confident you have the right answer

2 − $3x + 5y = 15$ *When* $x = 0$, $5y = 15$ *and so* $y = 3$, *therefore* (0, 3) *is on the line*
When $y = 0$, $3x = 15$ *and so* $x = 5$, *therefore* (5, 0) *is on the line*
Plot the two points (0, 3) *and* (5, 0) *and join them up to get your line*

5. Graphs

5.4 PERPENDICULAR LINES

5.4.1 PERPENDICULAR LINES

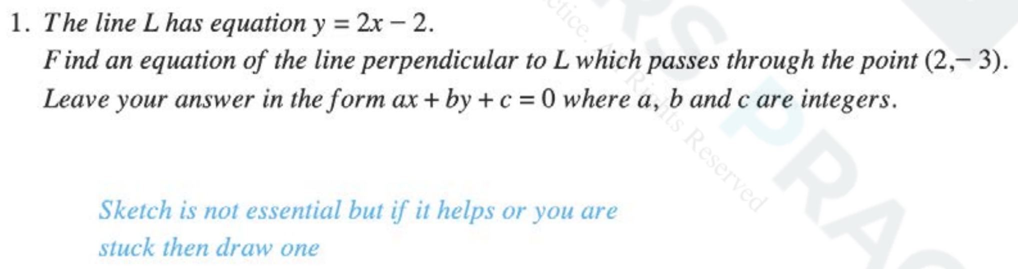
What are perpendicular lines?

- You should already know that PARALLEL lines have **equal** gradients
- PERPENDICULAR LINES do meet each other and where they do the two lines form a right angle – ie they meet at 90°

What's the deal with perpendicular gradients (and lines)?

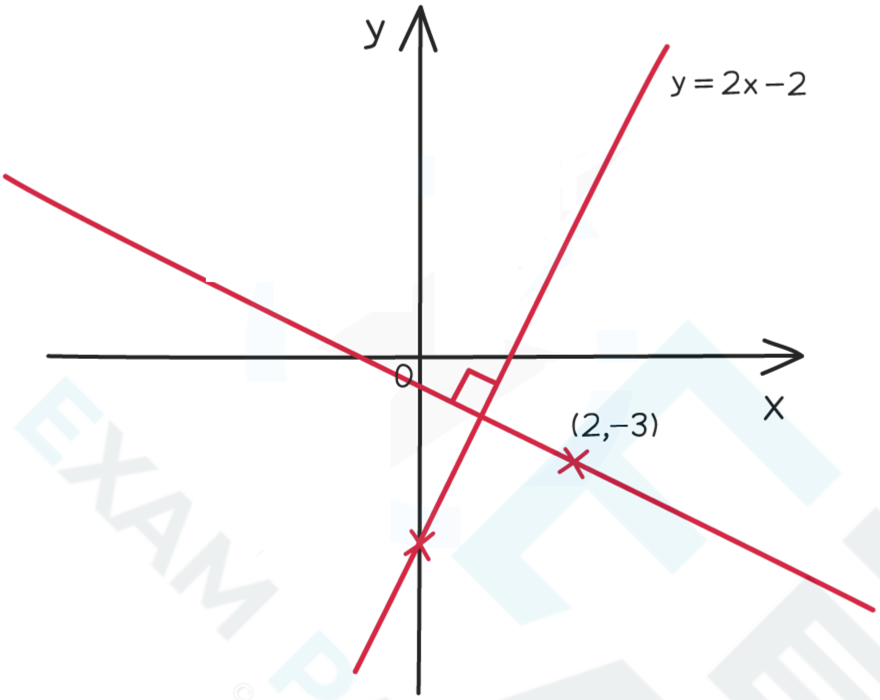
- Before you start trying to work with perpendicular gradients and lines, make sure you understand how to find the equation of a straight line – that will help you do the sorts of questions you will meet
- Gradients $\mathbf{m_1}$ and $\mathbf{m_2}$ are PERPENDICULAR if $\mathbf{m_1 \times m_2 = -1}$
- We can use $\mathbf{m_2 = -1 \div m_1}$ to find a perpendicular gradient
  (This is called the NEGATIVE RECIPROCAL)
  If in doubt, SKETCH IT!

Worked Example



1. *The line L has equation* $y = 2x - 2$.
Find an equation of the line perpendicular to L which passes through the point $(2, -3)$.
Leave your answer in the form $ax + by + c = 0$ *where* a, b *and* c *are integers*.

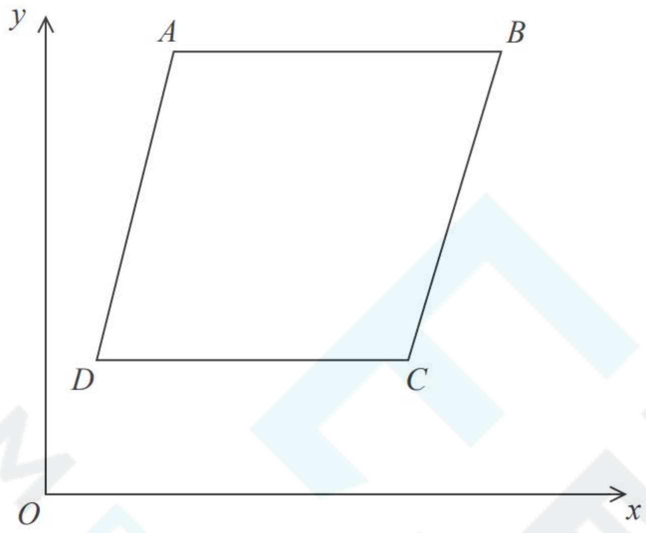
Sketch is not essential but if it helps or you are stuck then draw one



$m_1 = 2$	*L is in the form $y = mx + c$ so we can see that its gradient is 2*
$m_2 = -1 \div 2$ $m_2 = -\frac{1}{2}$	1, 2 − *The gradient of the line perpendicular to L will be the negative reciprocal of 2*
	See *Finding Equations of Straight Lines*
$y = -\frac{1}{2}x + c$ $-3 = -\frac{1}{2} \times 2 + c$ $c = -2$	*We know the gradient of the line we're after but we still need to find c* *Find c by substituting the point (2,− 3) in :*
$y = -\frac{1}{2}x - 2$ $2y = -x - 4$ $x + 2y + 4 = 0$	*This is the line we want but it is not in the correct form* *Multiply by 2 to get rid of fractions (GROF) and rearrange until we do have it in the required format*

5. Graphs

Exam Question: Medium



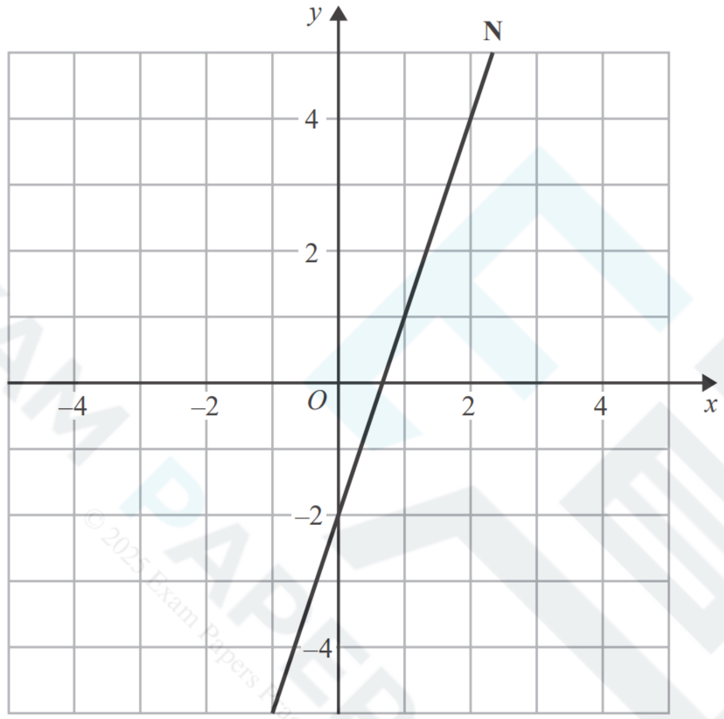
ABCD is a rhombus.
The coordinates of *A* are (5, 11)
The equation of the diagonal *DB* is $y = \frac{1}{2}x + 6$

Find an equation of the diagonal *AC*.

5. Graphs

Exam Question: Hard

The line **N** is drawn below.



Find an equation of the line perpendicular to line **N** that passes through the point (0, 1).

Exam Question: V. Hard

A is the point with coordinates (1, 3)

B is the point with coordinates (4, –1)
The straight line *L* goes through both *A* and *B*.

Is the line with equation $2y = 3x - 4$ perpendicular to line *L*?
You must show how you got your answer.

5. Graphs

5.5 TRANSFORMATIONS OF GRAPHS

5.5.1 TRANSFORMATIONS OF GRAPHS

What are transformations of graphs?

- A transformation is simply a change of some sort
- When it comes to graphs you need to know about two different sorts of transformation:
  - **Reflections** (using either the **x**-axis or **y**-axis as a mirror line)
  - **Translations** (moving the whole curve in the **x** and/or **y** direction)
- You should be able to recognise these two different sorts of transformation and apply them to a given graph.

How do we recognise and apply them?

- If you are given the graph of a function, say **y = f(x)** then:

1. **REFLECTIONS** look like this:
   In **x**-axis: **y = -f(x)** (minus **outside** the function)
   In **y**-axis: **y = f(-x)** (minus **inside** the function)
2. **TRANSLATIONS**:
   + a units in the **x** direction: **y = f(x – a)**
   + b units in the **y** direction: **y = f(x) + b**
3. **BEWARE** the highlighted change of sign!

Exam Tip

Translation by the vector (-3 2) means a translation of -3 in the x direction and +2 in the y direction.

Applied to y = f(x) this would give y = f(x + 3) + 2

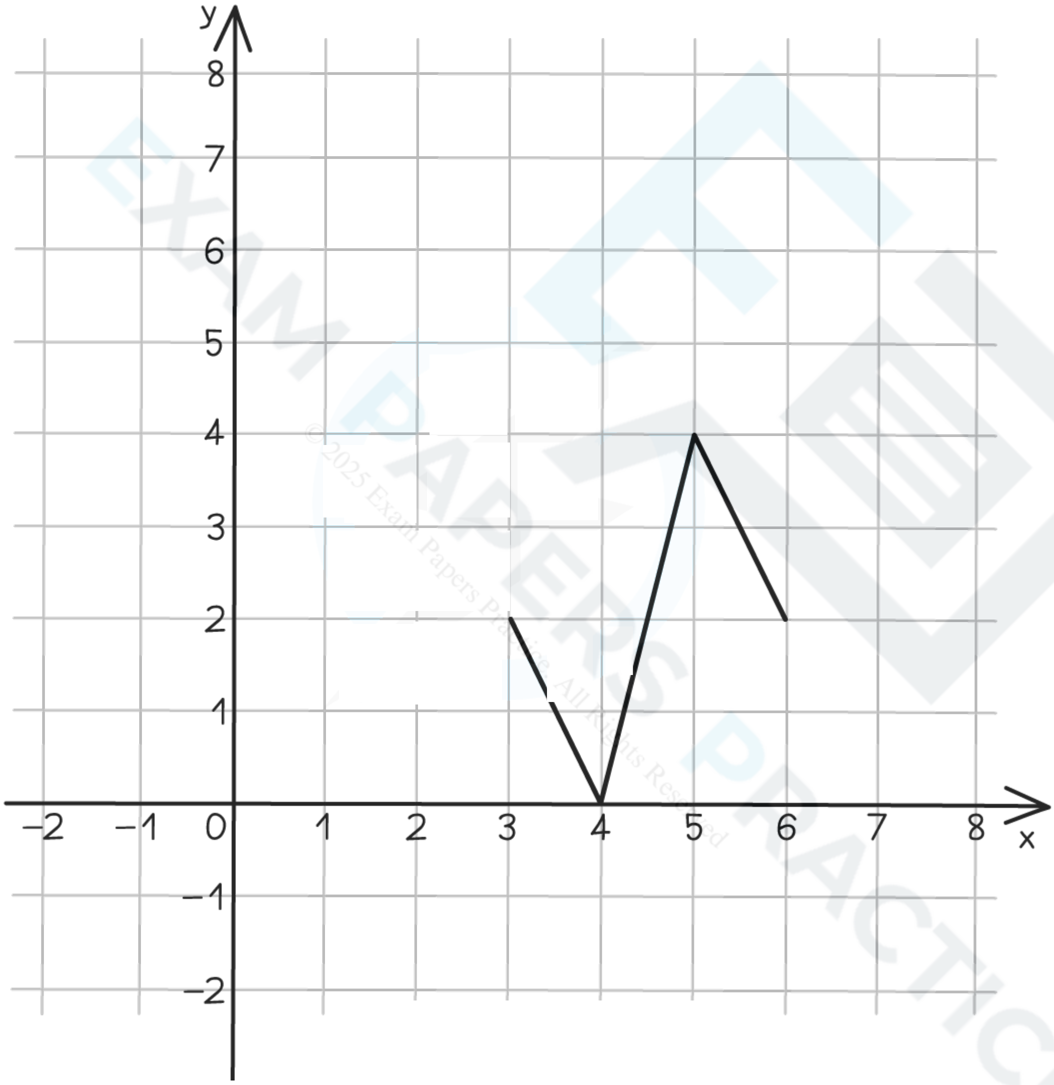
(Note the change of sign in the x direction as pointed out in 2. above)

5. Graphs

Worked Example

The graph of $y = f(x)$ is shown on the graph below.

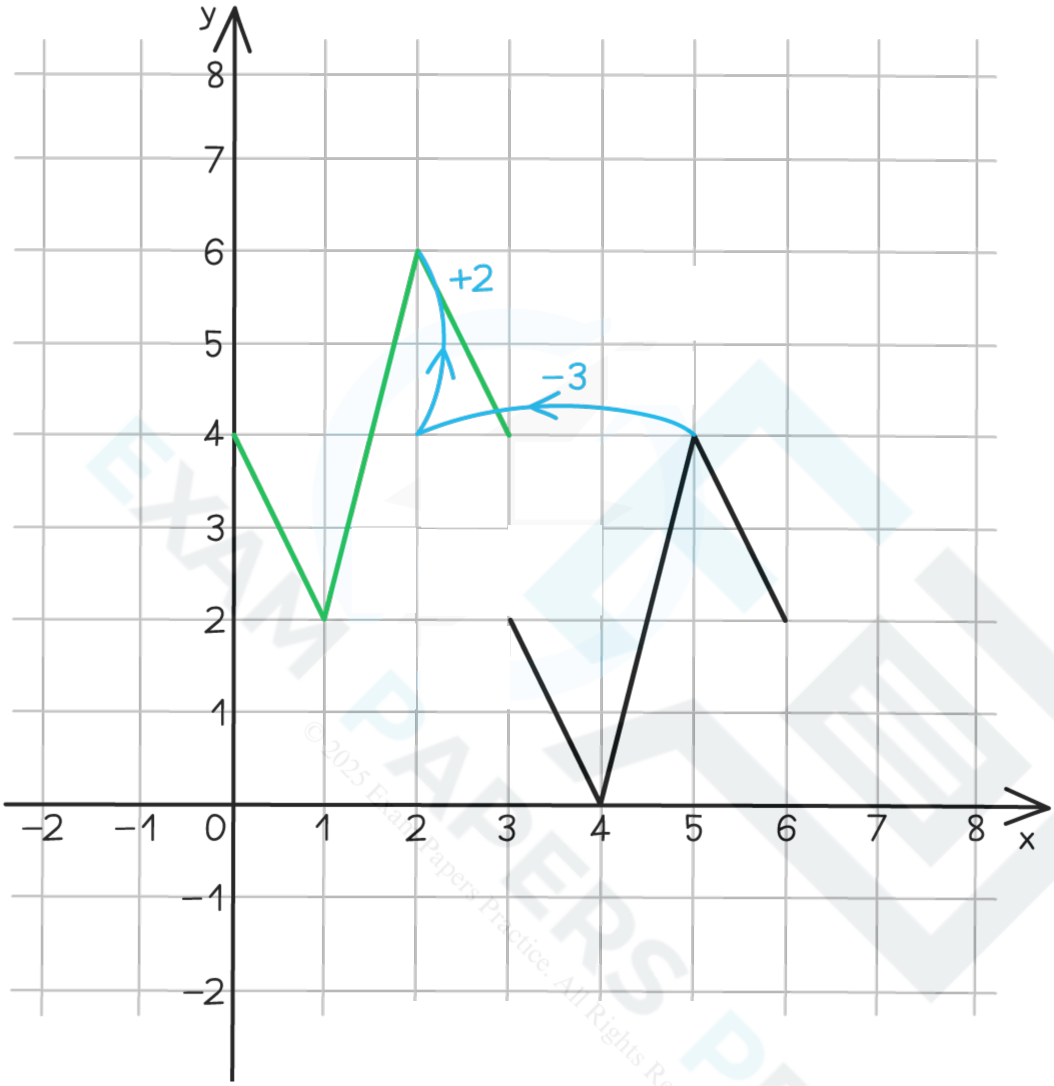
On the same graph sketch $y = f(x+3)+2$



This is a translation by -3 in the x direction (ie 3 to the left, note the change of sign again) and $+2$ in the y direction (ie 2 up).

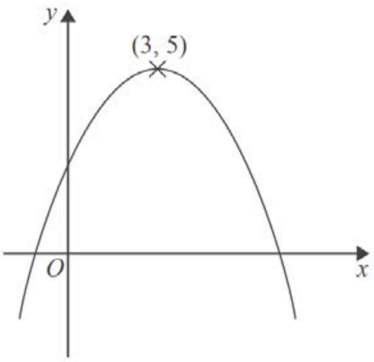
So we copy the given (slightly unusual) graph in its new position :

5. Graphs



Exam Question: Easy

5. Graphs



The diagram shows part of the curve with equation $y = f(x)$.
The coordinates of the maximum point of the curve are (3, 5).

(a) Write down the coordinates of the maximum point of the curve with equation

(i) $y = f(x + 3)$

(.................... , ...)

(ii) $y = -f(x)$

(.................... , ...)

(iii) $y = f(-x)$

(.................... , ...)

The curve with equation $y = f(x)$ is transformed to give the curve with equation $y = f(x) - 4$

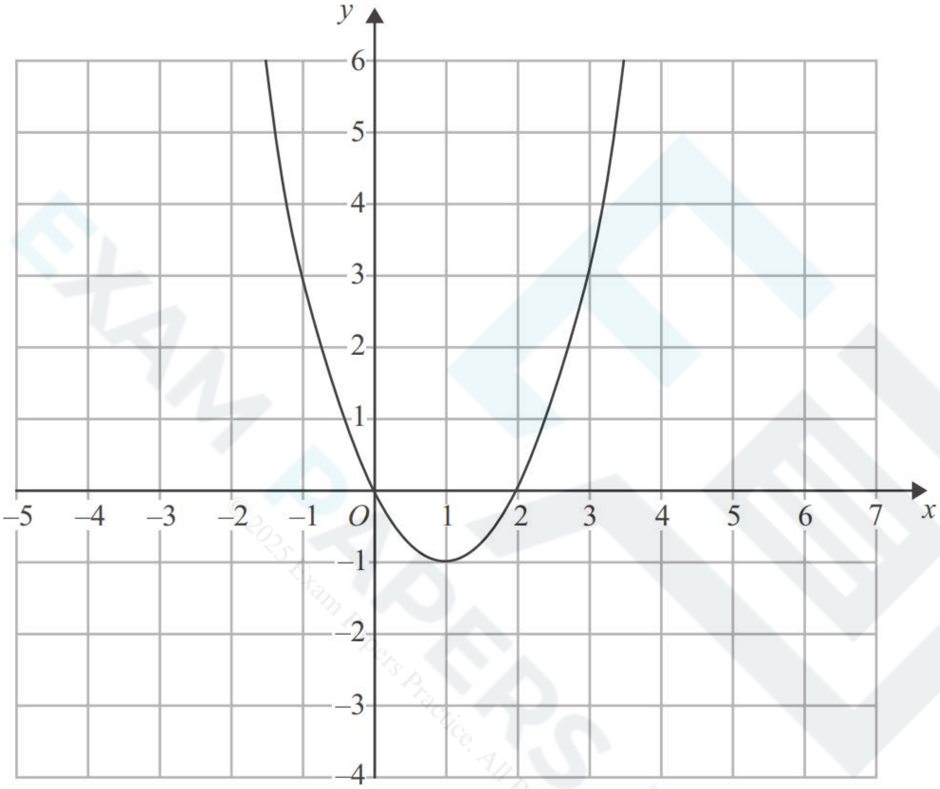
(b) Describe the transformation.

Exam Question: Medium

5. Graphs

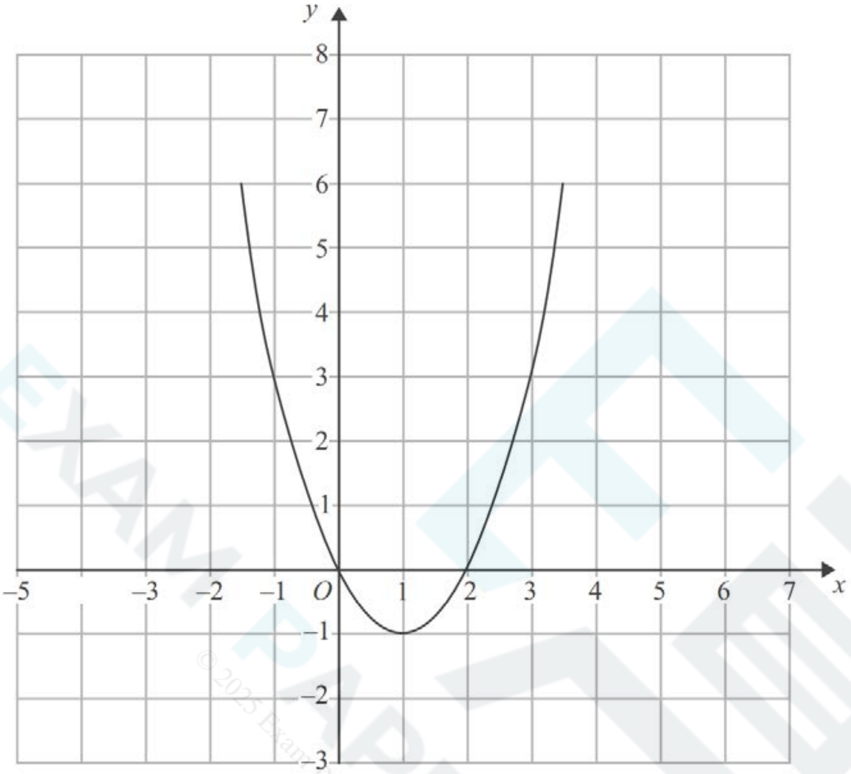
The graph of $y = f(x)$ is shown on each of the grids.

(a) On this grid, sketch the graph of $y = f(x - 3)$



5. Graphs

(b) On this grid, sketch the graph of $y =$ f(-x) + 2



5. Graphs

5.6 D-T / V-T GRAPHS

5.6.1 DISTANCE-TIME GRAPHS

What is a distance-time graph?

- **Distance-time** graphs show distance from a fixed point at different times
    - Distance is on the vertical axis, and time is on the horizontal axis.

- The gradient of the graph is the speed:
  Speed =RISE / RUN=DISTANCE / TIME
- **Straight** line = Constant speed
- **Horizontal** line = Stationary (not moving!)

Exam Tip

It is easy to get confused between different types of graph.

Look at the label on the vertical axis to make sure you are looking at a DISTANCE-time graph (not speed-time).

5. Graphs

Worked Example

One afternoon Mary cycled to her grandparents's house, 8 km from her own home.

Part of the travel graph for her journey is shown below.

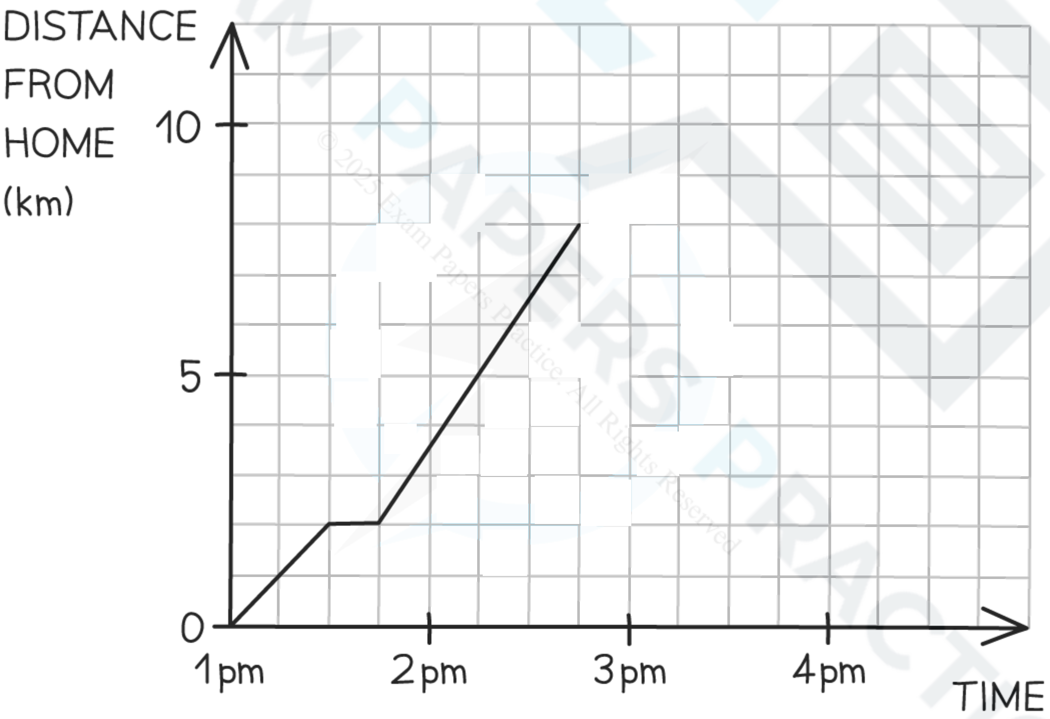
Mary stayed at her grandparent's house for half an hour.

She then cycled home at a steady speed, without stopping, arriving home at 4pm.

(a) *Complete the travel graph for Mary's journey.*

(b) *For how long did Mary stop on the way to her grandparent's house?*

(c) *What is Mary's speed between 1.45pm and 2.45pm?*

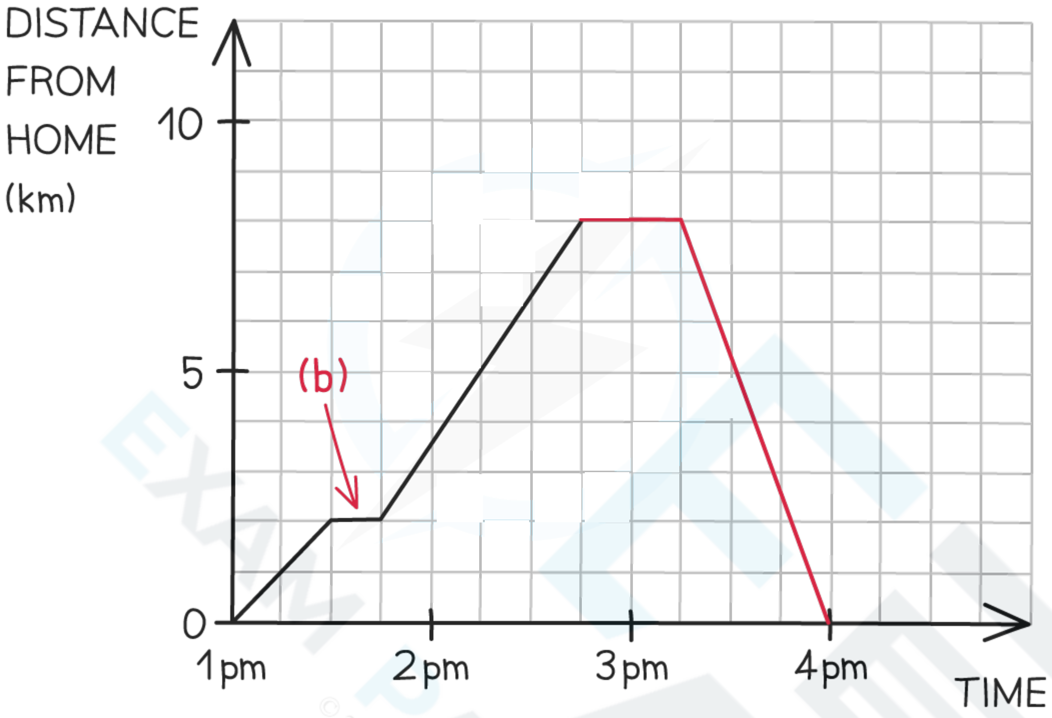


Note the scale on the time axis – one square is 15 minutes.

(a) *She is not moving for half an hour so we draw a horizontal line for 2 squares.*

Her cycle home is represented by a straight line (steady speed) drawn from the end of her stay to 4pm on the time axis (where the distance from home is zero).

5. Graphs



(b) *Mary's stop on the way is the short horizontal line indicated.*

It is 1 square long so Time Stopped = 15 minutes

(c) $Speed = \frac{RISE}{RUN} = \frac{DISTANCE}{TIME}$

$Speed = \frac{6\ km}{1\ hour} = 6\ km/h$

5. Graphs

5.6.2 SPEED-TIME GRAPHS

What is a speed-time graph?

- **Speed-time** graphs show speed at different times
    - Speed is on the vertical axis, and time is on the horizontal axis.

- The **gradient** of the graph is the **acceleration**:
  Acceleration = RISE / RUN = SPEED / TIME
  Area under graph = Distance covered
  Horizontal line = Constant speed (so zero acceleration)

Exam Tip

It is easy to get confused between different types of graph.

Look at the label on the vertical axis to make sure you are looking at a SPEED-time graph (not distance-time).

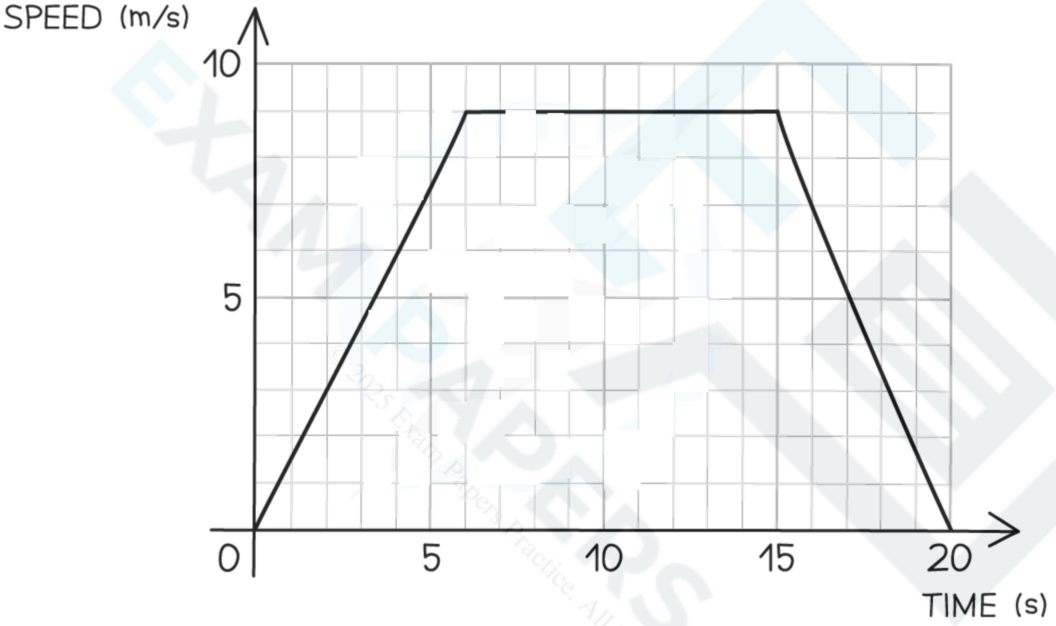
5. Graphs

Worked Example

The speed – time graph for a car travelling between two sets of traffic lights is shown below.

(a) Calculate the acceleration in the first 6 seconds.

(b) Work out the distance covered by the car.



(a) $Acceleration = \frac{\underline{RISE}}{RUN} = \frac{\underline{SPEED}}{TIME}$

$Acceleration = \frac{9\ m/s}{6\ s} = 1.5\ ms^{-2}$

(b) $Distance = Area\ under\ the\ graph$

$In\ this\ case\ it's\ a\ trapezium\ so\ A = \frac{1}{2}(a+b)h$

$Distance = \frac{1}{2} \times (9+20) \times 9$

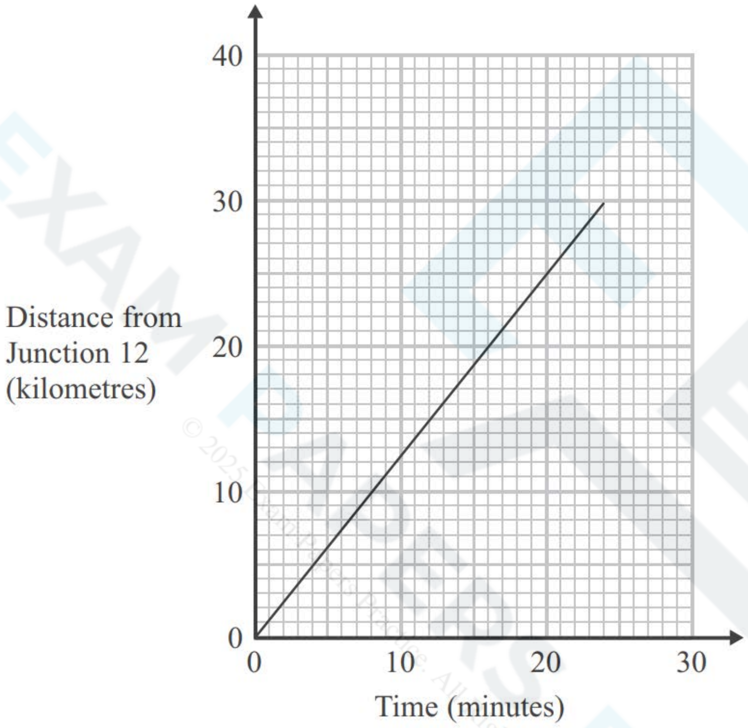
$Distance = 130.5\ m$

5. Graphs

Exam Question: Easy

Debbie drove from Junction 12 to Junction 13 on a motorway.

The travel graph shows Debbie's journey.



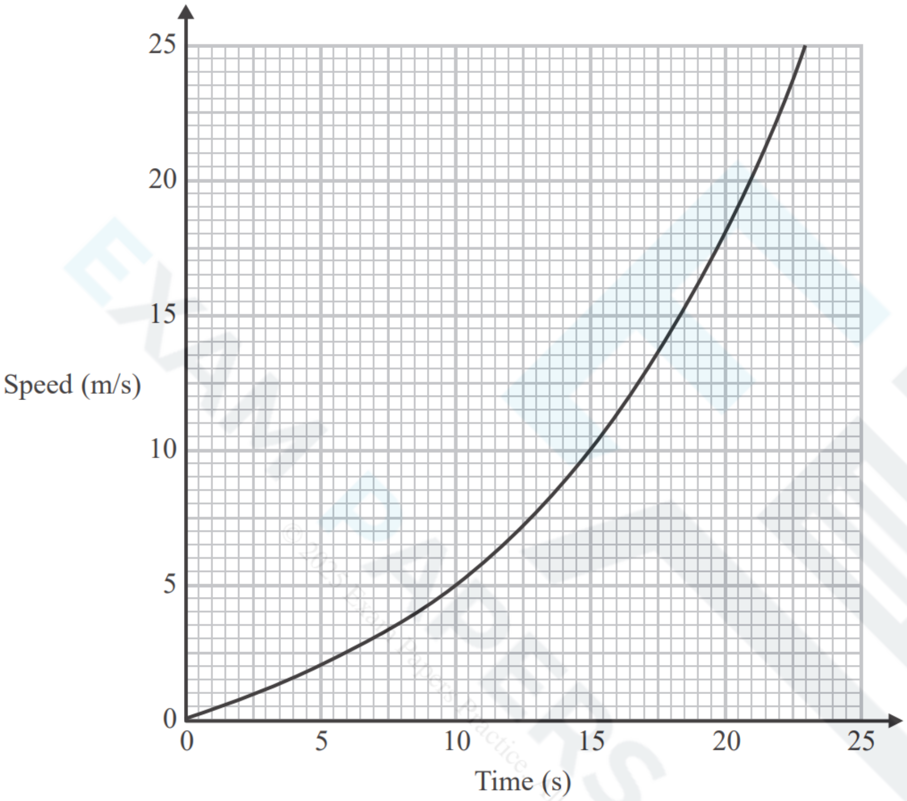
Ian also drove from Junction 12 to Junction 13 on the same motorway.
He drove at an average speed of 66 km/hour.

Who had the faster average speed, Debbie or Ian?
You must explain your answer.

5. Graphs

Exam Question: Medium

Here is a speed-time graph for a train.



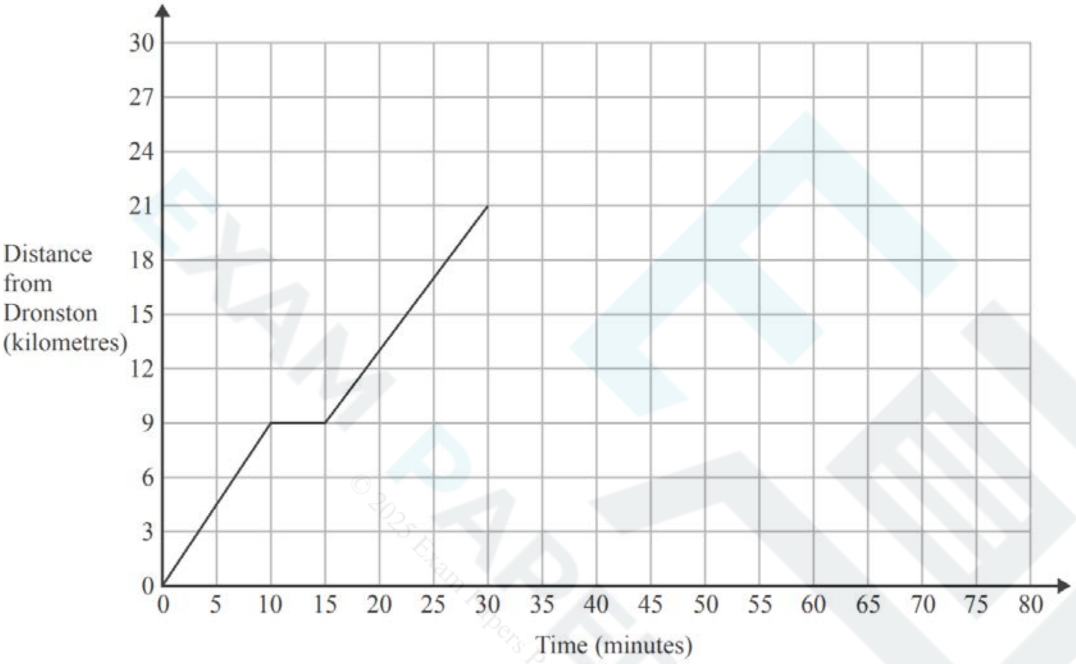
(a) Work out an estimate for the distance the train travelled in the first 20 seconds.
Use 4 strips of equal width.

(b) Is your answer to (a) an underestimate or an overestimate of the actual distance the train travelled?
Give a reason for your answer.

5. Graphs

Exam Question: Hard

A coach travels from Dronston to Luscoe.
The travel graph for this journey is shown below.



(a) Work out the average speed of the coach, in kilometres per hour, for the first 10 minutes of the journey.

The coach stops in Luscoe for 15 minutes.
The coach then returns to Dronston at a constant speed of 42 km/h.

(b) Show this information on the travel graph.

5. Graphs

5.7 SOLVING EQUATIONS USING GRAPHS

5.7.1 SOLVING EQUATIONS USING GRAPHS

How do we use graphs to solve equations?

- Solutions are always read off the **x-axis**
- Solutions of **f(x) = 0** are where the graph of **y = f(x**) cuts the **x**-axis
- If given g(x)=0 instead (Q: “by drawing a suitable straight line”) then:
    - Rearrange into **f(x) = mx + c** and draw the line **y = mx + c**
    - Solutions are the **x**-coordinates of where the **line** crosses the **curve**
      Note that **solutions** may also be called **roots**

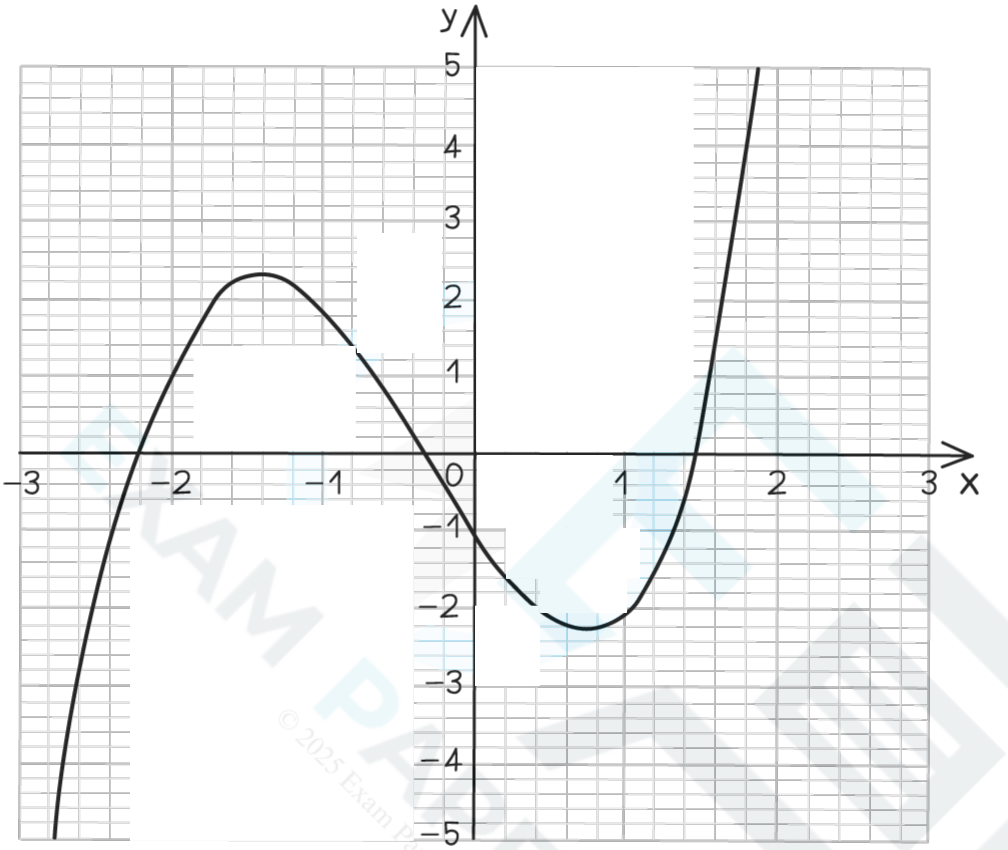
Worked Example

The graph of $y = x^3 + x^2 - 3x - 1$ *is shown below.*

Use the graph to estimate the solutions (*to* 1 *d.p.*) *of the equation*

$$x^3 + x^2 - 4x = 0.$$

5. Graphs



3. *We are given a different equation to the one plotted so we must rearrange it to* $f(x) = mx + c$.

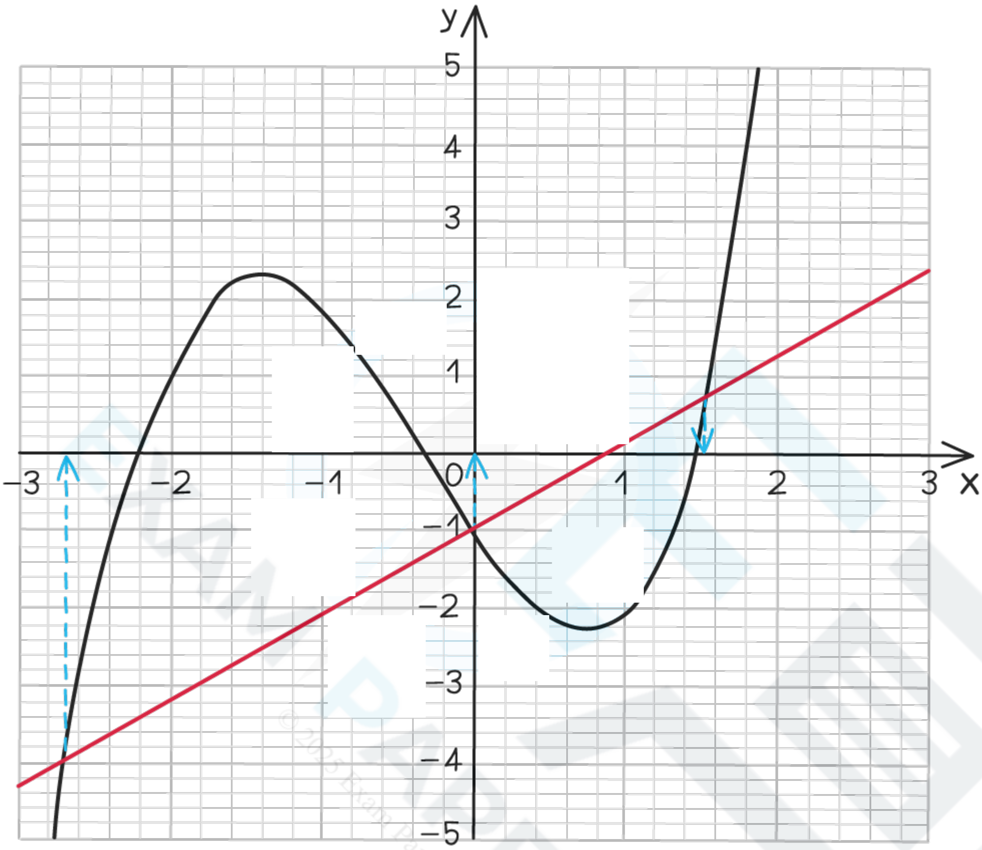
$$x^3 + x^2 - 4x = 0$$

Add $x - 1$ *to both sides :*

$$x^3 + x^2 - 3x - 1 = x - 1$$

Now plot $y = mx + c$ *(which is* $y = x - 1$ *here) on the graph − this is the solid red line on the graph below.*

5. Graphs



The solutions are the x coordinates of the crossing points so :

$x = -2.6,\ x = 0,\ x = 1.6$

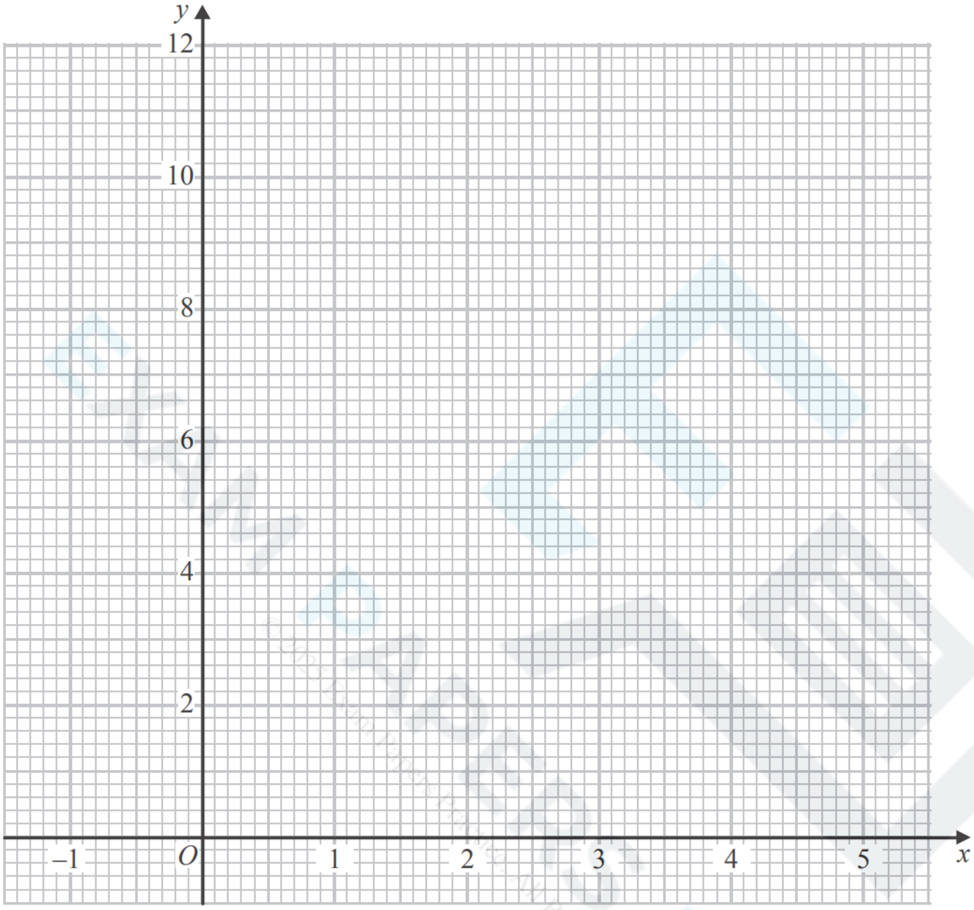
Exam Question: Hard

(a) Complete the table of values for $y = x^2 - 3x + 2$

x	−1	0	1	2	3	4	5
y	6				2		12

5. Graphs

(b) On the grid, draw the graph of $y = x^2 - 3x + 2$ for values of x from –1 to 5



(c) Find estimates for the solutions of the equation $x^2 - 3x + 2 = 4$

5. Graphs

5.8 EQUATION OF A CIRCLE

5.8.1 CIRCLES - EQUATION & GRAPHS

What is the equation of a circle?

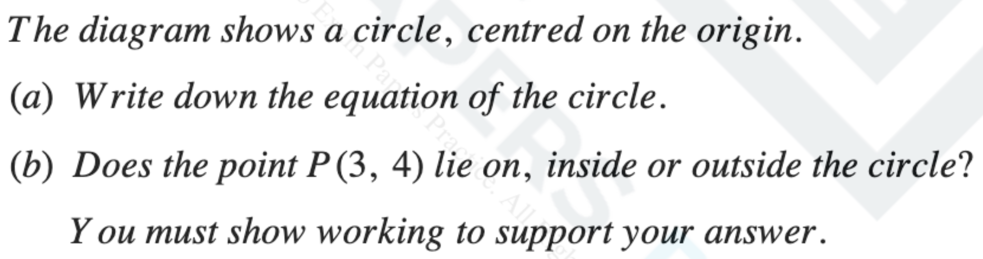
- A circle centred on the origin with radius **r** has equation $\mathbf{x^2 + y^2 = r^2}$

1. **(a, b)** lies on the circle if $\mathbf{a^2 + b^2 = r^2}$

   Note: If $\mathbf{a^2 + b^2 < r^2}$ then **(a, b)** lies inside the circle

   If $\mathbf{a^2 + b^2 > r^2}$ then **(a, b)** lies outside the circle
2. The circle cuts the **x**– and **y**-axes at **±r** (“plus or minus r”)
3. Don’t forget: Diameter = 2**r**
   Circumference = 2**πr**

Worked Example



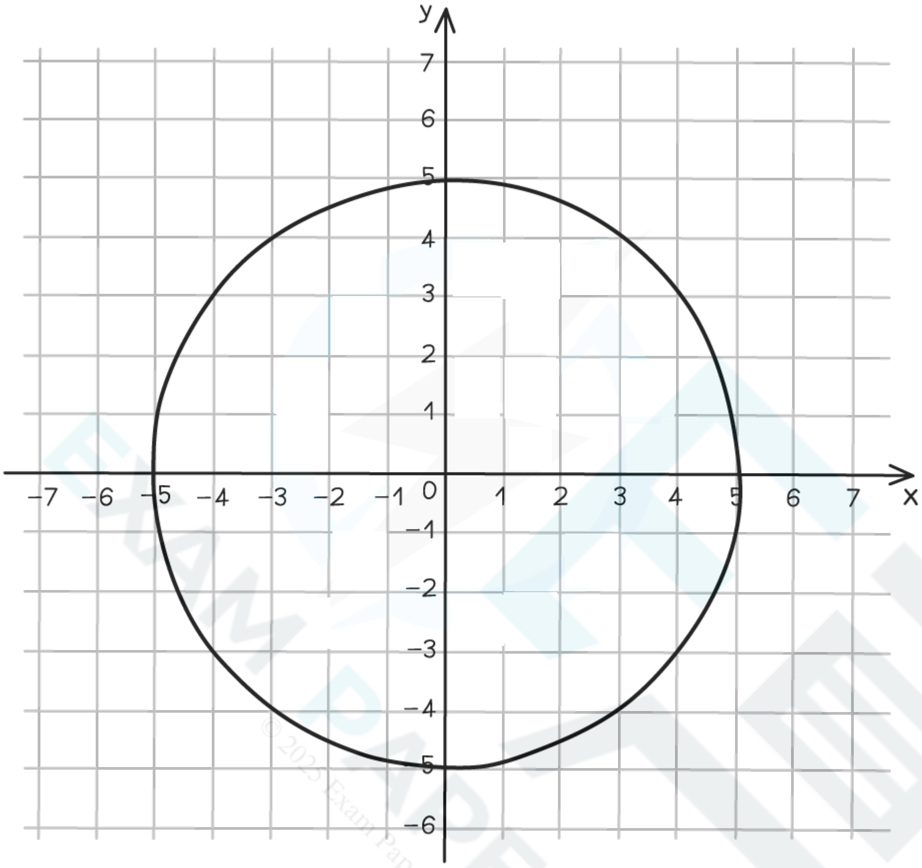
The diagram shows a circle, centred on the origin.

(a) *Write down the equation of the circle.*

(b) *Does the point* $P(3, 4)$ *lie on, inside or outside the circle?*

You must show working to support your answer.

5. Graphs



(a) $r = 5$ so the equation is simply :

$$x^2 + y^2 = 5^2$$

$$x^2 + y^2 = 25$$

1. (b) Substitute $x = 3,\ y = 4$ into $x^2 + y^2$:

$$3^2 + 4^2 = 9 + 16 = 25$$

Since this equals 25 the point (3, 4) lies on the circle.

5. Graphs

5.8.2 CIRCLES - FINDING TANGENTS

What is the equation of a circle?

- A circle centred on the origin with radius **r** has equation $x^2 + y^2 = r^2$

1. **(a, b)** lies on the circle if $a^2 + b^2 = r^2$

   Note: If $a^2 + b^2 < r^2$ then **(a, b)** lies inside the circle

   If $a^2 + b^2 > r^2$ then **(a, b)** lies outside the circle
2. The circle cuts the **x**– and **y**-axes at **±r** ("plus or minus r")
3. Don't forget:
   Diameter = 2**r**
   Circumference = 2**πr**

How do we find the equation of a tangent to a circle?

- First, make sure you are familiar with **Equations of Straight Lines** and **Perpendicular Gradients**

1. A **TANGENT** just touches a circle (but does not cross it)
2. The gradient of the tangent at point A is **PERPENDICULAR** to the gradient of the **radius** OA (remember, perpendicular gradients multiply to **-1**)
3. Use **y = mx + c** to find the equation of the tangent

Exam Tip

Solving **simultaneous equations** of circle and tangent only gives **ONE SOLUTION.**

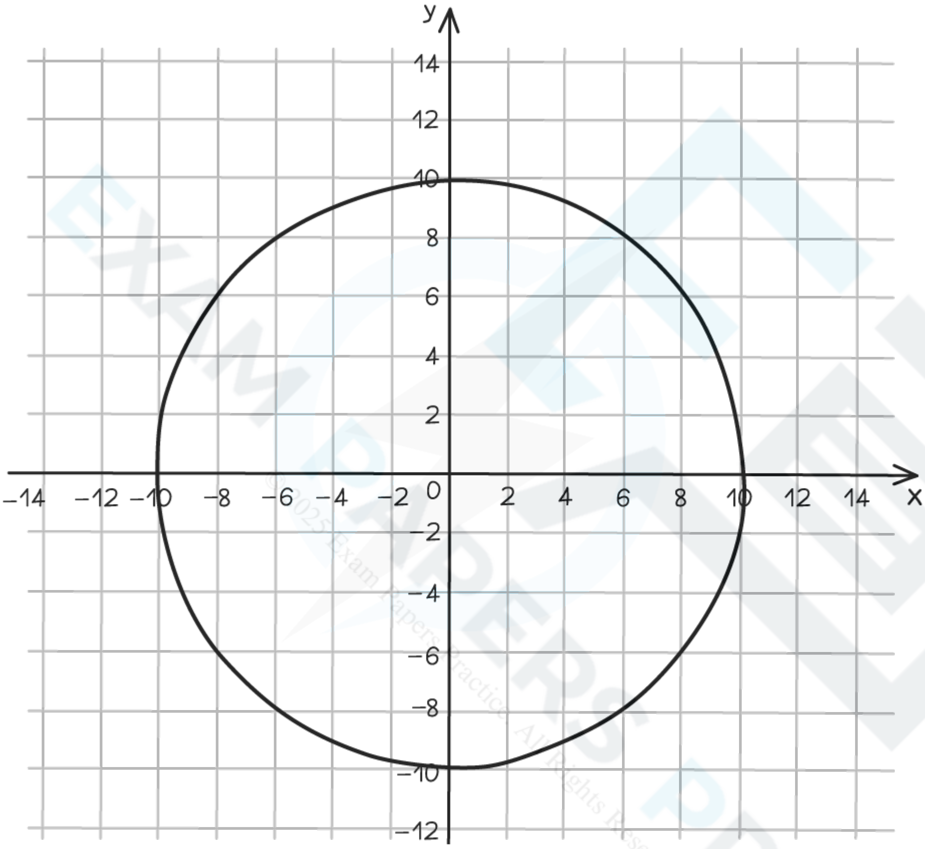
So if you are asked to show a line is a tangent, solve the simultaneous equations and show there is only one solution.

ALWAYS draw a diagram to help!

5. Graphs

Worked Example

The graph shows the circle with equation $x^2 + y^2 = 100$.



Find an equation of the tangent to the circle at the point $P(8,\ 6)$.

5. Graphs

$First\ draw\ the\ tangent\ on\ the\ diagram.$

$Find\ the\ gradient\ of\ the\ radius\ OP:$

$$Gradient\ OP = \frac{RISE}{RUN} = \frac{6}{8} = \frac{3}{4}$$

$The\ gradient,\ m,\ of\ the\ tangent\ is\ perpendicular\ to\ this\ so:$

$$For\ the\ tangent,\ m = -\frac{4}{3}\ \ (since\ \frac{3}{4} \times -\frac{4}{3} = -1)$$

$Substitute\ m = -\frac{4}{3},\ x = 8\ and\ y = 6\ into\ y = mx + c\ to\ find\ c:$

$$6 = -\frac{4}{3} \times 8 + c$$

$$c = 16\ \frac{2}{3}$$

$So\ an\ equation\ of\ the\ tangent\ is:$

$$y = -\frac{4}{3}x + 16\ \frac{2}{3}$$

Exam Question: Hard

L is the circle with equation $x^2 + y^2 = 4$

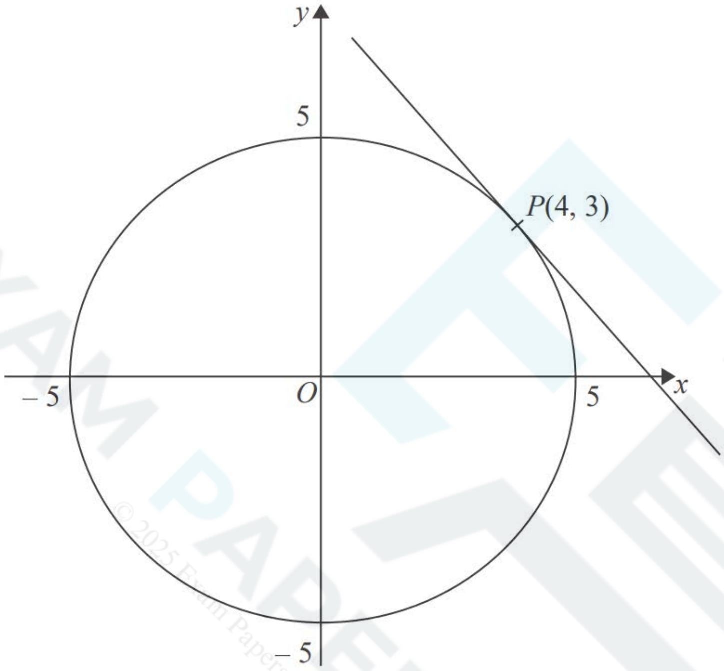
$P\left(\frac{3}{2}, \frac{\sqrt{7}}{2}\right)$ is a point on **L**.

Find an equation of the tangent to **L** at the point P.

5. Graphs

Exam Question: V. Hard

Here is a circle, centre *O*, and the tangent to the circle at the point *P*(4, 3) on the circle.



Find an equation of the tangent at the point *P*.

5. Graphs

5.9 ESTIMATING AREAS & GRADIENTS OF GRAPHS

5.9.1 ESTIMATING AREAS & GRADIENTS OF GRAPHS

What is the gradient of a graph?

- The gradient of a graph at any point is equal to the gradient of the tangent to the curve at that point
- Remember that a tangent is a line that just touches a curve (and doesn't cross it)

How to estimate the gradient of & the area under a graph

- To find an estimate for the gradient:
  - Draw a tangent to the curve
  - Find the gradient of the tangent using **Gradient =RISE ÷ RUN**
- To find an estimate for the **area**:
  - Split area into vertical **strips**
  - Draw **straight lines** at top of strips
  - Find area of strips (trapeziums) using **Area = ½(a + b)h**

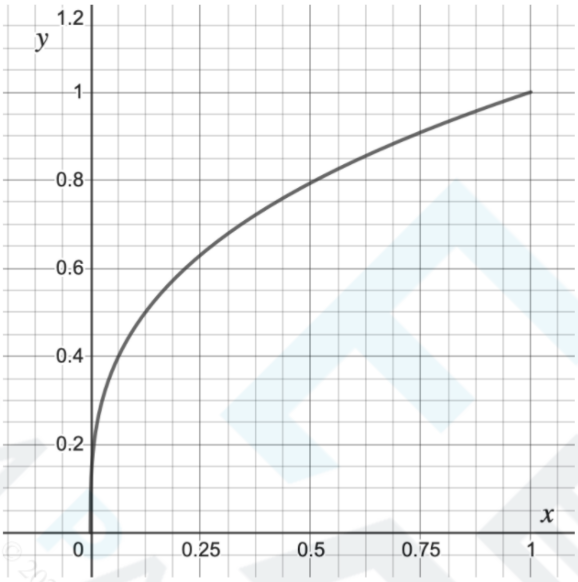
Exam Tip

This is particularly useful when working with Speed-Time and Distance-Time graphs if they are curves and not straight lines.

5. Graphs

Worked Example

The graph below shows $y = \sqrt[3]{x}$ *for* $0 \le x \le 1$.

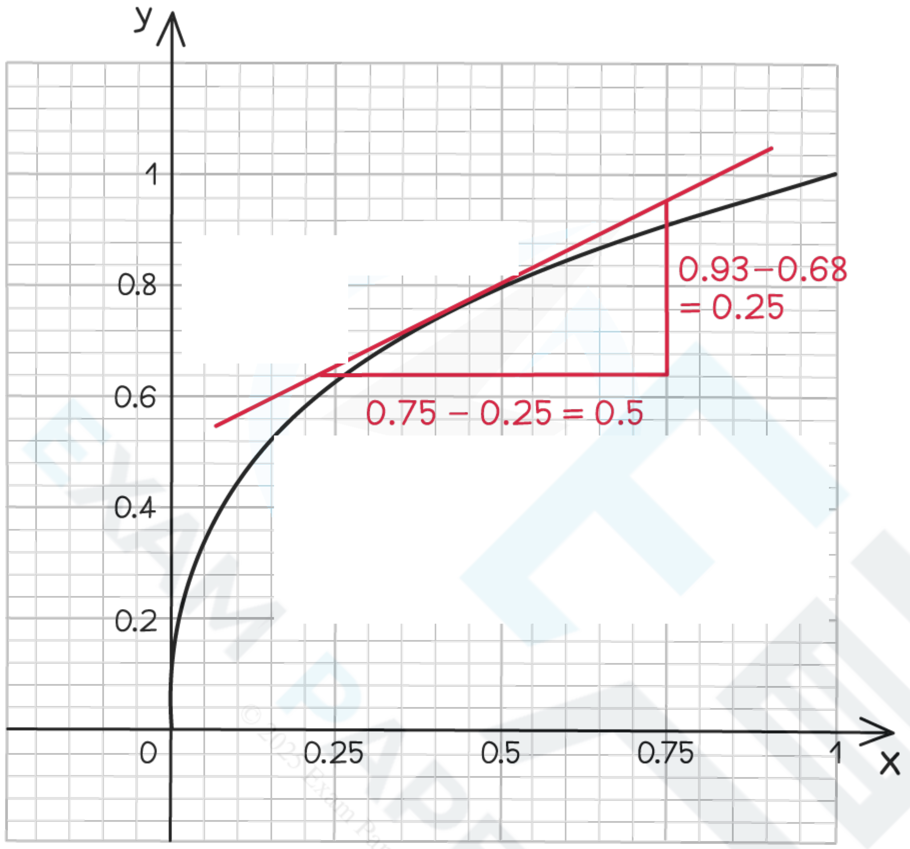


(a) *Find an estimate of the gradient of the curve at the point where* $x = 0.5$.

(b) *Find an estimate for the area between the curve and the x axis.*
Use four strips of equal width.

1. (a) *Tangent drawn on curve at* $x = 0.5$ *and RISE and RUN found*

5. Graphs



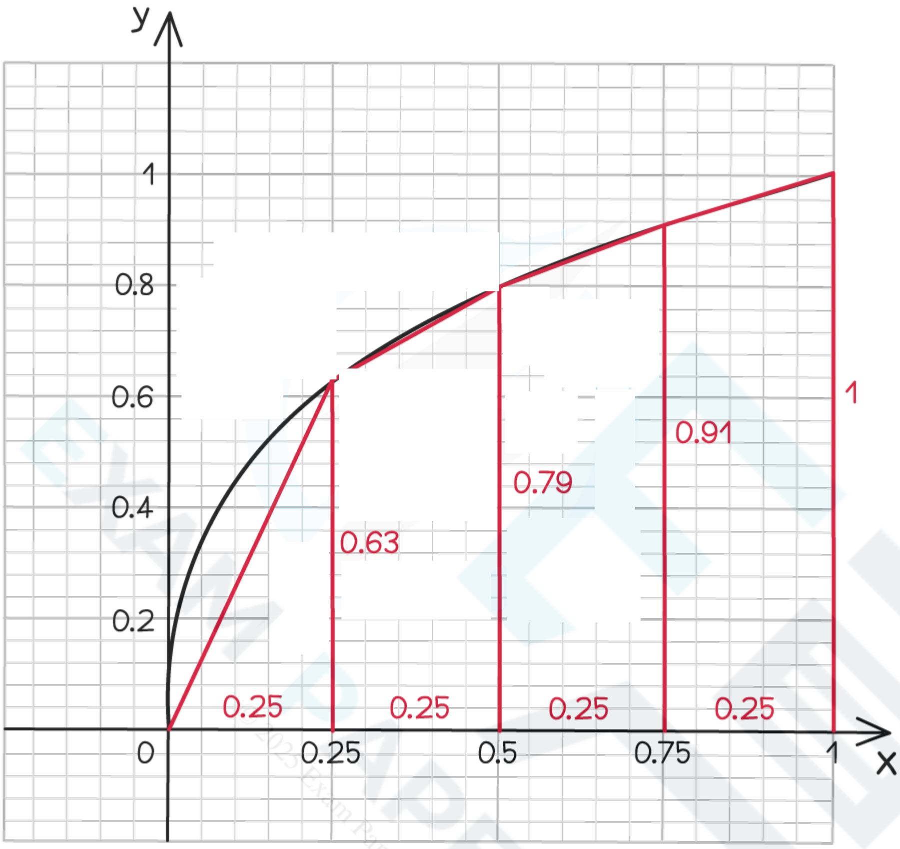
$Gradient = \frac{RISE}{RUN} = \frac{0.25}{0.5} = 0.5$

2. (b) Area split up using straight lines into one triangle and two trapezia

Now use $Area = \frac{1}{2}(a + b)h$ for each trapezium ($h = 0.25$ for each one)

(and $Area = \frac{1}{2}bh$ for the triangle on the left)

5. Graphs



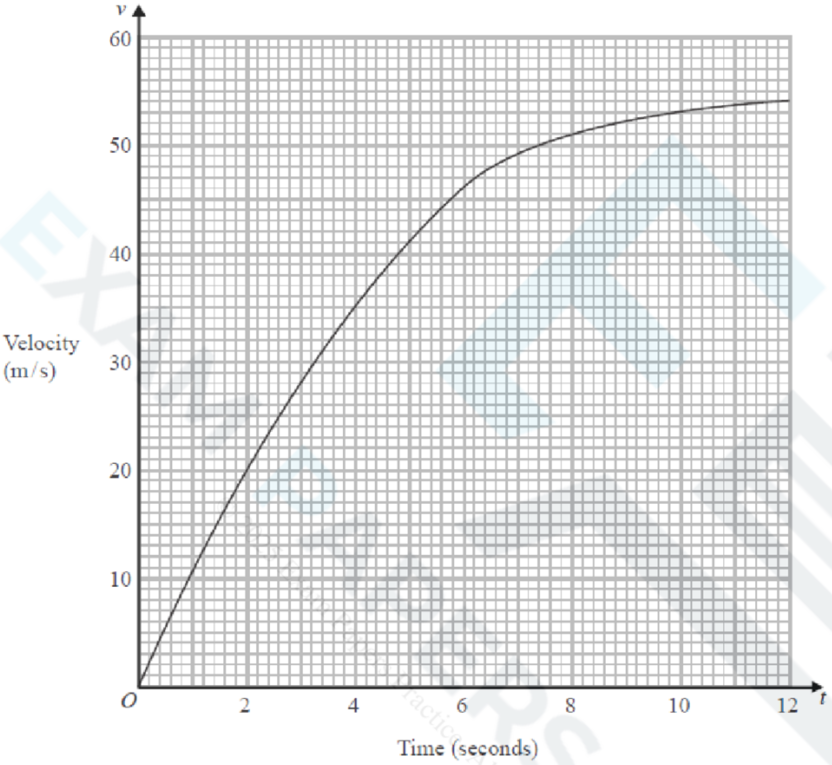
$$Total\ Area \simeq \tfrac{1}{2} \times 0.25 \times 0.63 + \tfrac{1}{2} \times (0.63 + 0.79) \times 0.25 + \tfrac{1}{2} \times (0.79 + 0.91) \times 0.25 + \tfrac{1}{2} \times (0.91 + 1) \times 0.25$$

$$Total\ Area \simeq 0.71$$

5. Graphs

Exam Question: Medium

The graph shows information about the velocity of a parachutist after jumping from a plane.



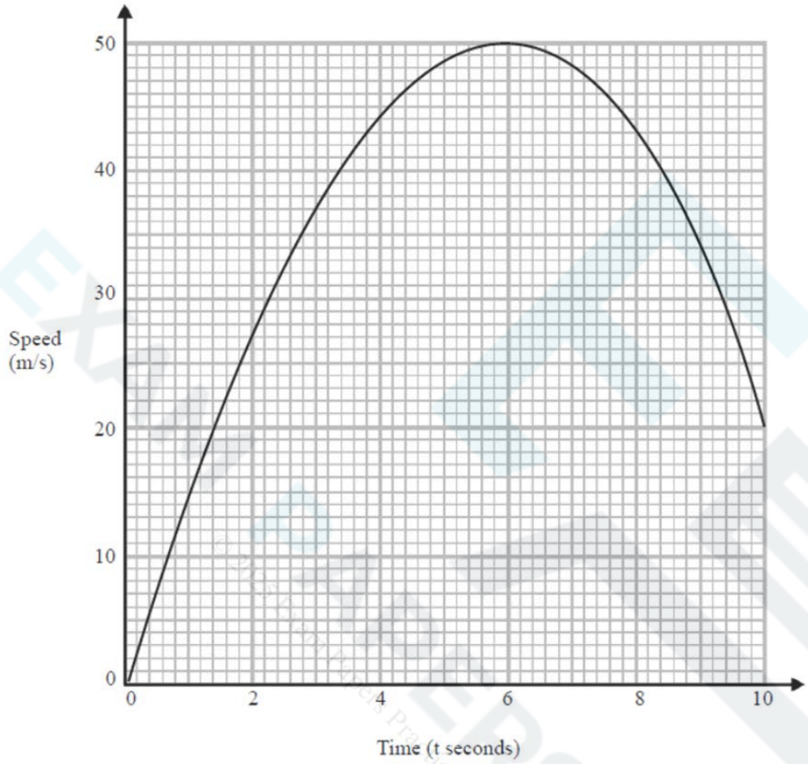
(a) By drawing a suitable tangent, find an estimate of the gradient of the curve after 3 seconds.

(b) Interpret the value of the gradient.

5. Graphs

Exam Question: Hard

Here is a speed-time graph.



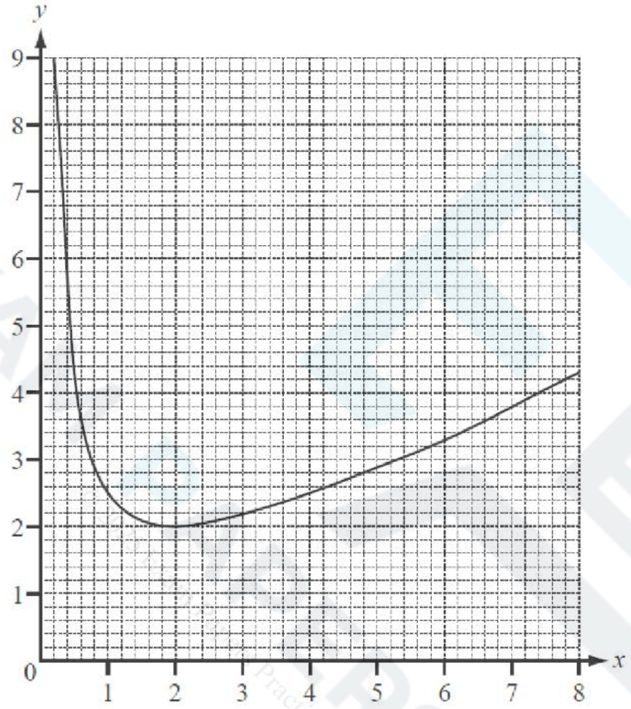
(a) By drawing a suitable tangent, find an estimate of the gradient of the curve at $t = 8$. Give the units of your answer.

(b) Explain why your answer to (a) can only be an estimate.

5. Graphs

Exam Question: V. Hard

The diagram shows the graph of $y = \frac{x}{2} + \frac{2}{x}$ for $0 < x \leq 8$.



(a) Use the graph to solve the equation $\frac{x}{2} + \frac{2}{x} = 3$.

(b) By drawing a suitable tangent, find an estimate of the gradient of the graph when $x = 1$.

6. Ratios, Proportion & Rate of Change

CONTENTS

6.1 RATIOS

6.1.1 RATIOS

What is a ratio?

- A **ratio** is a way of comparing one part of a whole to another
- A **ratio** can also be expressed as a **fraction** (of the whole)
- We often use a ratio (instead of a fraction) when we are trying to show how things are shared out or in any situation where we might use **scale factors**

How to work with ratios

1. Put what you know in **RATIO** form (use more than one line if necessary)
2. Add "extra bits" (eg Total, Difference, Sum) if you think they might be useful
3. Use **SCALE FACTORS** to complete lines
4. Pick out the ANSWER!

Exam Tip

One less obvious place to use Ratios is when dealing with Currency Conversion problems – see Exchange Rates.

Worked Example

1. *Jemima is 5 years old, Karl is 8 and Lonnie is 11. They are sharing some money in the ratio of their ages. If Karl gets £5.60, what is the total amount of money they are sharing?*

$Total\ of\ ages = 5 + 8 + 11 = 24$

	J	:	K	:	L	:	$Total$
Age	5	:	8	:	11	:	24
Money	j	:	5.60	:	l	:	t

1, 2 – *Put the given information into ratio form – and in this case it may be useful to look at a total column (since a total amount is what we've been asked to find)*

Note the use of (useful) letters to mark unknown values in the table – we may use none, some or all of them later

$Scale\ Factor = \frac{5.6}{8} = 0.7$

3 – *To find t (and j and l too!) we need to compare the age row with the money row.*

The same scale factor applies to the Total column

$\frac{t}{24} = 0.7$

$t = 0.7 \times 24$

$t = £16.80$

Remember your final answer is money in this case

6. Ratios, Proportion & Rate of Change

Exam Question: Easy

Here are the ingredients needed to make 12 shortcakes.

Shortcakes

Makes **12** shortcakes

50 g	of sugar
200 g	of butter
200 g	of flour
10 m*l*	of milk

Liz makes some shortcakes.
She uses 25 m*l* of milk.

(a) How many shortcakes does Liz make?

Robert has	500 g of sugar 1000 g of butter 1000 g of flour 500 m*l* of milk

(b) Work out the greatest number of shortcakes Robert can make.

6. Ratios, Proportion & Rate of Change

Exam Question: Medium

Here are the ingredients needed to make 16 gingerbread men.

Ingredients
to make **16** gingerbread men

180 g	flour
40 g	ginger
110 g	butter
30 g	sugar

Hamish wants to make 24 gingerbread men.

Work out how much of each of the ingredients he needs.

Exam Question: Hard

Talil is going to make some concrete mix.
He needs to mix cement, sand and gravel in the ratio 1 : 3 : 5 by weight.

Talil wants to make 180 kg of concrete mix.

Talil has

15 kg of cement
85 kg of sand
100 kg of gravel

Does Talil have enough cement, sand and gravel to make the concrete mix?

6. Ratios, Proportion & Rate of Change

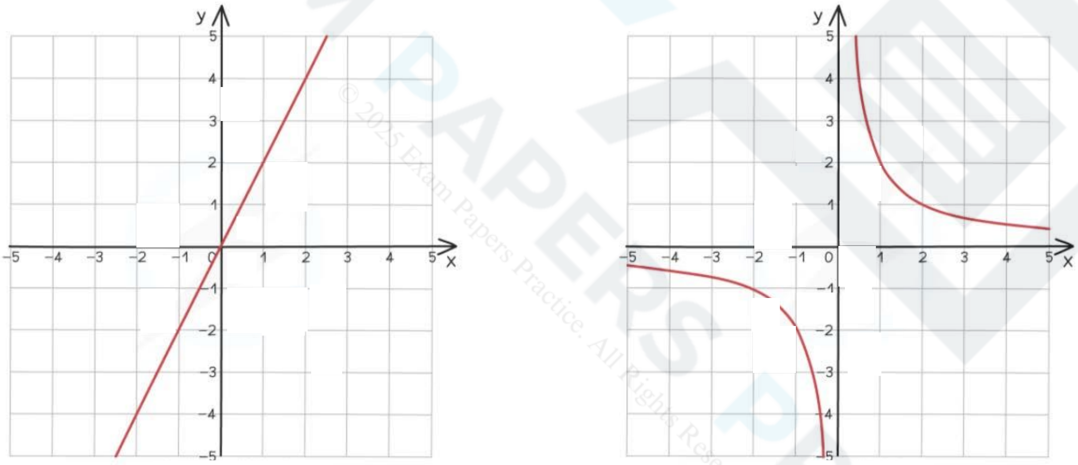
6.2 DIRECT & INVERSE PROPORTION

6.2.1 DIRECT & INVERSE PROPORTION

What is proportion?

- **Proportion** is a way of talking about how two **variables** are related to each other
- **Direct** proportion: as one variable goes **up** the other goes up by the same **factor**
  eg. If one variable is multiplied by 3, so is the other
- **Inverse** proportion: as one variable goes up the other goes **down** by the same **factor**
  eg. If one variable is multiplied by 3, the other is **divided** by 3

You should know what the graphs look like:



How do we deal with proportion questions?

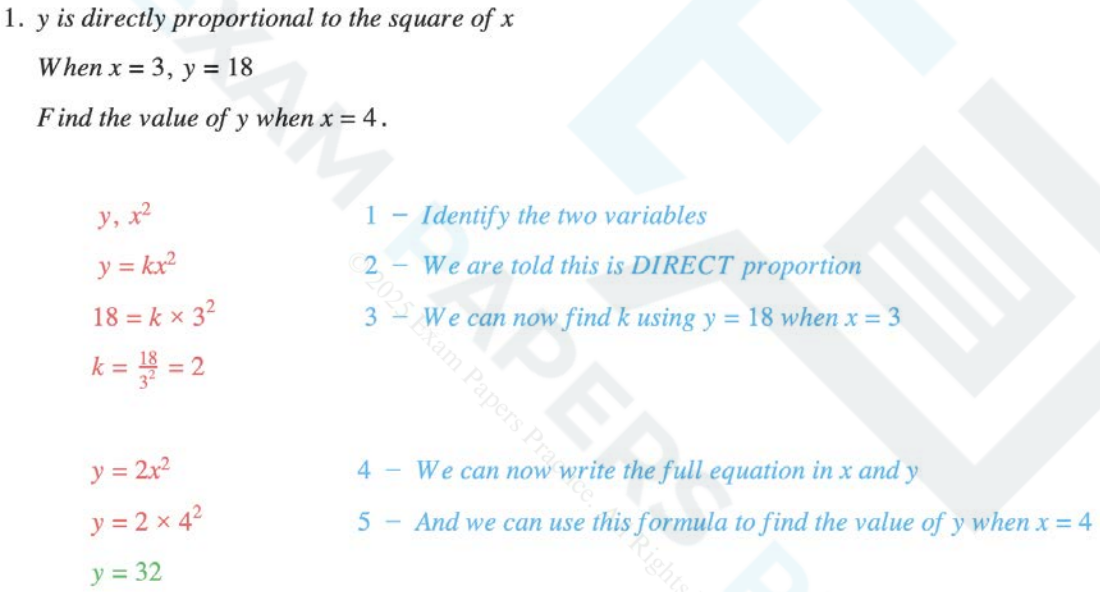
1. Identify the two **VARIABLES** (called **A** and **B** below)
2. Choose **TYPE** of proportion:
   A is **DIRECTLY** proportional to **B** use formula **A = kB**
   A is **INVERSELY** proportional to **B** use formula **A = k÷B**
3. **FIND k** using the values given in the question
4. Write **FORMULA** for **A** in terms of **B** (using your value of **k**)
5. **USE** formula to find the required quantity

6. Ratios, Proportion & Rate of Change

Exam Tip

Even if the question doesn't ask for a formula (Step 4. above) it is always worth working one out and using it in all but the simplest cases.

Worked Example



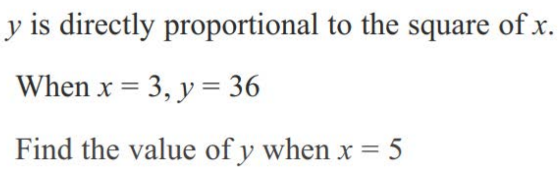
1. y is directly proportional to the square of x

When $x = 3$, $y = 18$

Find the value of y when $x = 4$.

$y,\ x^2$	1 – Identify the two variables
$y = kx^2$	2 – We are told this is DIRECT proportion
$18 = k \times 3^2$	3 – We can now find k using $y = 18$ when $x = 3$
$k = \frac{18}{3^2} = 2$	
$y = 2x^2$	4 – We can now write the full equation in x and y
$y = 2 \times 4^2$	5 – And we can use this formula to find the value of y when $x = 4$
$y = 32$	

Exam Question: Medium



y is directly proportional to the square of x.

When $x = 3$, $y = 36$

Find the value of y when $x = 5$

6. Ratios, Proportion & Rate of Change

Exam Question: Hard

h is inversely proportional to the square of r.

When $r = 5$, $h = 3.4$

Find the value of h when $r = 8$

6. Ratios, Proportion & Rate of Change

6.3 SPEED, DENSITY & PRESSURE

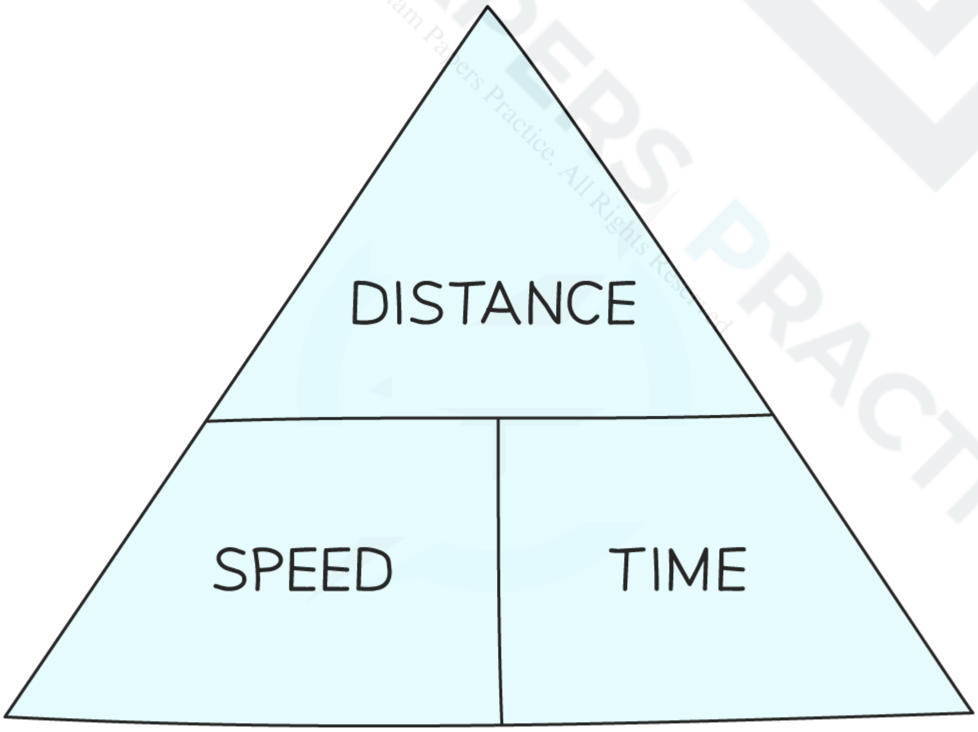
6.3.1 SPEED, DENSITY & PRESSURE

What connects speed, density & pressure?

- Speed, density and pressure are all examples of variables which are calculated by dividing one thing by another:
  Speed = Distance ÷ Time
  Density = Mass ÷ Volume
  Pressure = Force ÷ Area
- In that respect they can all be treated in the same way

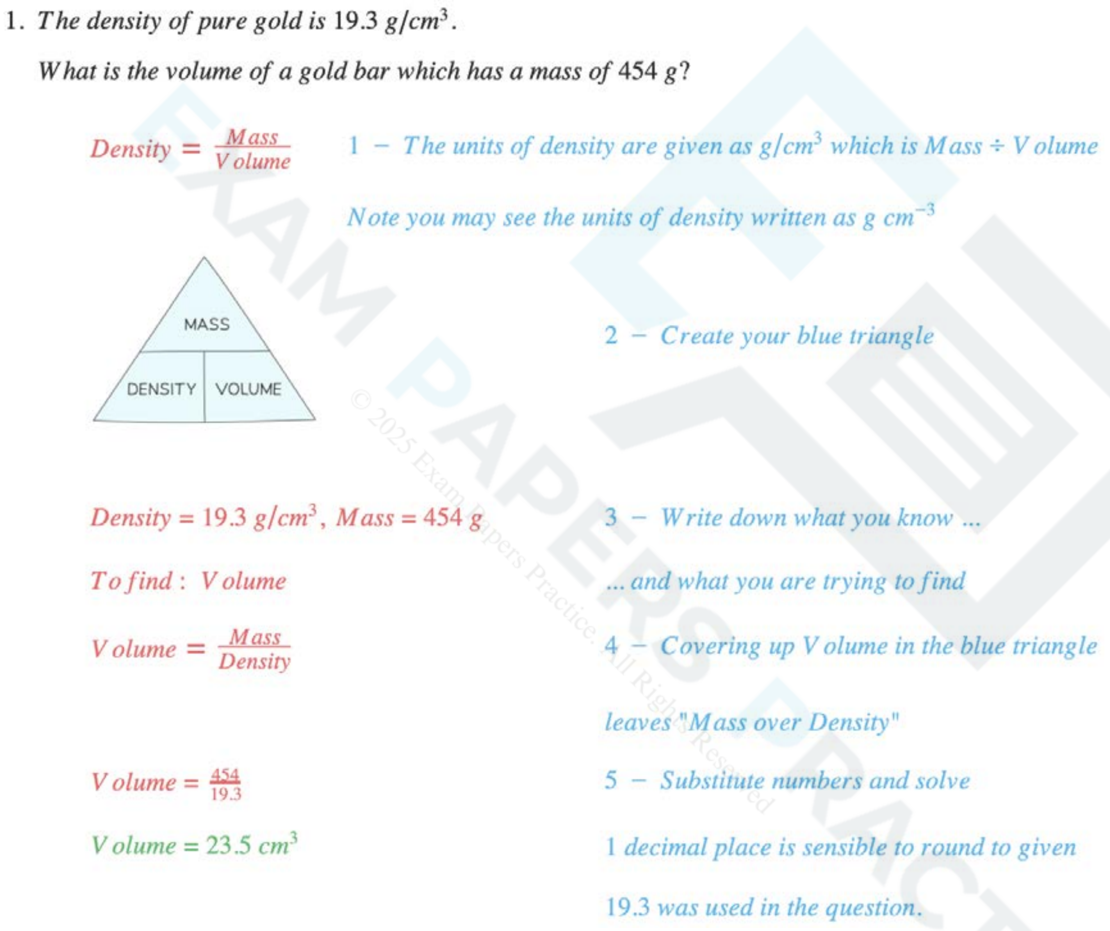
Doing speed, density & pressure questions

1. Use **UNITS** in Q (or other info) to write down a formula
2. Create BLUE TRIANGLE:
   e.g. for Speed, Distance and Time



3. For each part of the Q write down
what you know
what you want to know
4. Use Blue Triangle to **REARRANGE** formula (if necessary)
5. **SUBSTITUTE** numbers and SOLVE

Worked Example



1. The density of pure gold is $19.3\ g/cm^3$.

What is the volume of a gold bar which has a mass of $454\ g$?

$Density = \frac{Mass}{Volume}$

1 – The units of density are given as g/cm^3 which is $Mass \div Volume$

Note you may see the units of density written as $g\ cm^{-3}$

2 – Create your blue triangle

$Density = 19.3\ g/cm^3$, $Mass = 454\ g$

3 – Write down what you know ...

To find : Volume

... and what you are trying to find

$Volume = \frac{Mass}{Density}$

4 – Covering up Volume in the blue triangle

leaves "Mass over Density"

$Volume = \frac{454}{19.3}$

5 – Substitute numbers and solve

$Volume = 23.5\ cm^3$

1 decimal place is sensible to round to given

19.3 was used in the question.

6. Ratios, Proportion & Rate of Change

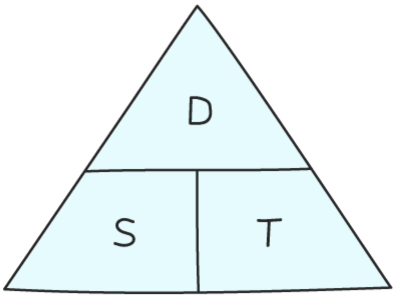
6.3.2 SPEED, DENSITY & PRESSURE - HARDER

What are speed, density and pressure?

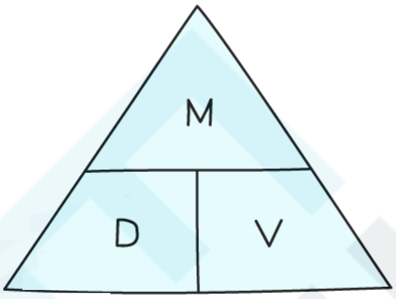
- **Speed**, **density** and **pressure** are compound measures – they are made from other measures
    - **Speed** is related to the measures **distance** and **time**
    - **Density** is related to **mass** and **volume**
    - **Pressure** is related to **force** and **area**
- The relationship between each of these sets of measures follows the same pattern – what we refer to as “blue triangles”

6. Ratios, Proportion & Rate of Change

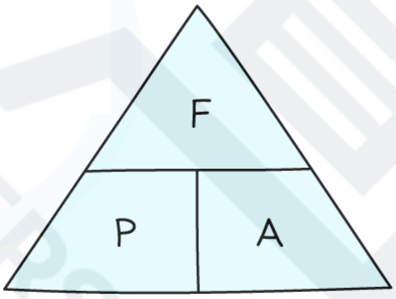
SPEED, DISTANCE, TIME



DENSITY, MASS, VOLUME

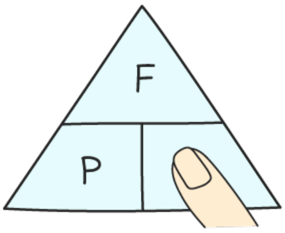


PRESSURE, FORCE, AREA



BLUE TRIANGLES WORK BY COVERING UP THE MEASURE YOU ARE TRYING TO FIND

e.g. FOR FINDING AREA...



$$\text{AREA} = \frac{\text{FORCE}}{\text{PRESSURE}}$$

- If you do not remember the blue triangles – do not worry, these can often be deduced from information given in the question – see the examples below
- It is important you understand and can do the basic questions with speed, density and pressure

6. Ratios, Proportion & Rate of Change

Speed, distance and time harder problems

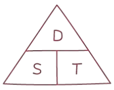
- **Speed** is commonly measured in **metres per second** (**m/s**) or **miles per hour** (**mph**)
    - There are other possibilities such as **kilometres per hour** (**kmph**)
    - The units indicate **speed** is **distance per time** ie **speed = distance ÷ time**
- "**Speed**" (in this formula) means "**average speed**"
- In harder problems there are often two journeys – or two parts to one longer journey

e.g. A CYCLIST'S JOURNEY IS SPLIT INTO TWO PARTS.
IN THE FIRST PART OF THE JOURNEY THE CYCLIST TRAVELS AT AN AVERAGE SPEED OF 13 mph FOR 45 MINUTES.
IN THE SECOND PART OF THE JOURNEY THE CYCLIST TRAVELS 10 MILES AT AN AVERAGE SPEED OF 16 mph.
FIND THE AVVERAGE SPEED FOR THE WHOLE JOURNEY.

STEP 1: USE UNITS FROM THE QUESTION TO DEDUCE FORMULA

$S = \frac{D}{T}$ "MILES PER HOUR"

STEP 2: CREATE A "BLUE" TRIANGLE



THIS MAY FEEL LIKE A REPEAT OF STEP 1 BUT COULD BE USEFUL LATER

STEP 3: FOR EACH PART OF THE QUESTION WRITE DOWN WHAT YOU KNOW, AND WHAT YOU DON'T KNOW

1st $S = 13$ $D = ?$ $T = 0.75$

NEEDS TO BE IN HOURS
45 mins = 0.75 h

2nd $S = 16$ $D = 10$ $T = ?$

WHOLE JOURNEY → W $S = ?$ $D = ?$ $T = ?$

USE "BLUE" TRIANGLE TO FIND 1st D,
SAME FOR 2nd T.
CAN THEN FIND D AND T FOR WHOLE JOURNEY, AND SO WORKOUT S

STEP 4: USE BLUE TRIANGLE TO REARRANGE IF NECESSARY

1st $D = ST$

2nd $T = \frac{D}{S}$

STEP 5: SUBSTITUTE AND SOLVE

1st $D = 13 \times 0.75 = 9.75$

2nd $T = \frac{10}{16} = 0.625$

W $D = 9.75 + 10 = 19.75$
$T = 0.75 + 0.625 = 1.375$

STEP 4: USE BLUE TRIANGLE TO REARRANGE IF NECESSARY

W $S = \frac{D}{T}$

STEP 5: SUBSTITUTE AND SOLVE

W $S = \frac{19.75}{1.375} = 14.363636...$

$S = 14.4$ mph ← ROUND TO A SENSIBLE DEGREE OF ACCURACY

6. Ratios, Proportion & Rate of Change

Density, mass and volume harder problems

- Density is usually measured in **grams per cubic centimetre** (**g/cm^3**) or **kilograms per cubic metre** (**kg/m^3**)
    - The units indicate that **density** is **mass per volume** ie **density = mass ÷ volume**
- In harder problems there are often two metals (alloys), liquids or gases that have been combined rather than working with a single substance

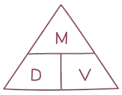
6. Ratios, Proportion & Rate of Change

e.g. A COIN IS TO BE MADE FROM AN ALLOY OF COPPER AND NICKEL. THE DENSITY OF NICKEL IS 8.91 g/cm^3 AND THE DENSITY OF COPPER IS 8.96 g/cm^3. THE COIN IS TO BE 75% COPPER AND 25% NICKEL BY MASS.
FIND THE DENSITY OF 0.5 kg OF THE ALLOY.

STEP 1: USE UNITS FROM THE QUESTION TO DEDUCE FORMULA

$D = \frac{M}{V}$ "g PER cm^3"

STEP 2: CREATE A "BLUE" TRIANGLE



COULD BE USEFUL LATER

STEP 3: FOR EACH PART OF THE QUESTION WRITE DOWN WHAT YOU KNOW, AND WHAT YOU DON'T KNOW

NICKEL → Ⓝ $D = 8.91$ $M = 25\%$ OF 500g $= 125g$ $V = ?$

MASS IS GIVEN INDIRECTLY!

0.5kg = 500g

COPPER → Ⓒ $D = 8.96$ $M = 75\%$ OF 500g $= 375g$ $V = ?$

ALLOY → Ⓐ $D = ?$ $M = 500g$ $V = ?$

USE "BLUE" TRIANGLE TO FIND V FOR Ⓝ SAME FOR V FOR Ⓒ CAN THEN FIND V FOR Ⓐ AND SO FIND D OF Ⓐ

STEP 4: USE BLUE TRIANGLE TO REARRANGE IF NECESSARY

Ⓝ, Ⓒ $V = \frac{M}{D}$

STEP 5: SUBSTITUTE AND SOLVE

Ⓝ $V = \frac{125}{8.91} = 14.0291...$

Ⓒ $V = \frac{375}{8.96} = 41.8526...$

Ⓐ $V = 14.0291... + 41.8526... = 55.881...$

STEP 4: USE BLUE TRIANGLE TO REARRANGE IF NECESSARY

Ⓐ $D = \frac{M}{V}$

STEP 5: SUBSTITUTE AND SOLVE

Ⓐ $D = \frac{500}{55.881} = 8.94744...$

$D = 8.95\ g/cm^3$ ← ROUND TO A SENSIBLE DEGREE OF ACCURACY

6. Ratios, Proportion & Rate of Change

Pressure, force and area harder problems

- Pressure is usually measured in Newtons per square metre (**N/m^2**)

  The units of pressure are often called **Pascals** (**Pa**) rather than **N/m^2**

  - The units indicate that **pressure** is **force per area**
    ie **pressure = force ÷ area**

- Remember that **weight** is a **force;** it is different to **mass**

Exam Tip

Do look out for a mixture of units:

• **Time** can be given as **minutes** but common phrases like “half an **hour**”

(ie 30 minutes) could also be used in the same question.

• Any mixed units should be those in common use and easy to convert

g to **kg** (and vice versa)

m to **km** (and vice versa)

6. Ratios, Proportion & Rate of Change

Worked Example

?

Nichrome is a metal alloy, made from nickel and chromium, one use of which is for the heating element in toasters.

The density of nickel is 8.91 g/cm^3 and the density of chromium is 7.19 g/cm^3.

To create nichrome, nickel and chromium are mixed in the ratio 4:1.

Find the density of 2 kg of nichrome.

STEP 1: USE UNITS FROM THE QUESTION TO DEDUCE FORMULA

$D = \frac{M}{V}$ g/cm³ IS MASS PER VOLUME

STEP 2: CREATE A "BLUE" TRIANGLE

COULD BE USEFUL LATER

STEP 3: FOR EACH PART OF THE QUESTION WRITE DOWN WHAT YOU KNOW, AND WHAT YOU DON'T KNOW

NICKEL Ⓝ : D = 8.91 M = ? V = ?

CHROMIUM Ⓒ : D = 7.19 M = ? V = ?

NICHROME (Ni) : D = ? M = 2000 V = ?

UNITS! 2kg = 2000g

FIND THE MASS OF Ⓝ & Ⓒ BY USING THE GIVEN RATIO

Ⓝ : Ⓒ (Ni)

4 : 1 5

1600 : 400 2000

STEP 4: USE BLUE TRIANGLE TO REARRANGE IF NECESSARY

$V = \frac{M}{D}$

STEP 5: SUBSTITUTE AND SOLVE

Ⓝ : $V = \frac{1600}{8.91} = 179.5735...$

Ⓒ : $V = \frac{400}{7.19} = 55.6328...$

DON'T ROUND TOO EARLY – USE CALCULATOR MEMORY

(Ni) : $V = 179.5735... + 55.6328...$

$V = 235.2063...$

STEP 4: $D = \frac{M}{V}$

STEP 5: $D = \frac{2000}{235.2063...} = 8.5031...$

DENSITY OF 2kg OF NICHROME IS 8.50 g/cm³ TO 2 DECIMAL PLACES

QUESTION GIVES OTHER DENSITIES TO 2dp SO IT MAKES SENSE TO DO THE SAME

ALTERNATIVE SOLUTION

THE BLUE TRIANGLE FORMULA WORKS ON TWO ASSUMPTIONS:

- NO MASS OR VOLUME IS LOST WHEN THE TWO METALS ARE COMBINED
- DENSITY IS CONSTANT, NO MATTER HOW MUCH OF THE SUBSTANCE THERE IS

THE SECOND BULLET POINT MEANS THE 2kg IS NOT NEEDED – JUST USE THE RATIO AS MASSES

STEP 3:

Ⓝ : $D = 8.91$ $M = 4$ $V = ?$

Ⓒ : $D = 7.19$ $M = 1$ $V = ?$

(Ni) : $D = ?$ $M = 5$ $V = ?$

STEP 5:

Ⓝ : $V = \frac{4}{8.91} = 0.44893...$

Ⓒ : $V = \frac{1}{7.19} = 0.13908...$

(Ni) : $V = 0.44893... + 0.13908...$

$V = 0.58801...$

STEP 4: $D = \frac{M}{V}$

STEP 5: $D = \frac{5}{0.58801...}$

$D = 8.50$ g/cm³ (2 dp)

IF YOU UNDERSTOOD THIS ALTERNATIVE SOLUTION, SEE IF YOU CAN APPLY SIMILAR SKILLS TO THE DENSITY PROBLEM IN THE REVISION NOTES ABOVE

6. Ratios, Proportion & Rate of Change

Exam Question: Easy

Peter goes for a walk.
He walks 15 miles in 6 hours.

(a) Work out Peter's average speed.
Give your answer in miles per hour.

5 miles = 8 km.
Sunita says that Peter walked more than 20 km.

*(b) Is Sunita right?
You must show all your working.

Exam Question: Medium

The diagram shows a solid triangular prism.

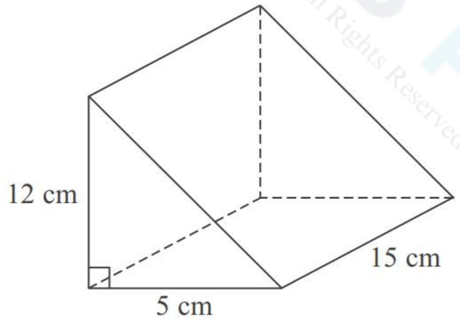


Diagram **NOT** accurately drawn

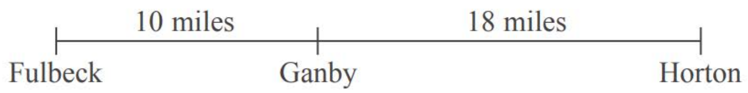
The prism is made from metal.
The density of the metal is 6.6 grams per cm^3.

Calculate the mass of the prism.

6. Ratios, Proportion & Rate of Change

Exam Question: Hard

The distance from Fulbeck to Ganby is 10 miles.
The distance from Ganby to Horton is 18 miles.



Raksha is going to drive from Fulbeck to Ganby.
Then she will drive from Ganby to Horton.

Raksha leaves Fulbeck at 10 00
She drives from Fulbeck to Ganby at an average speed of 40 mph.

Raksha wants to get to Horton at 10 35

Work out the average speed Raksha must drive at from Ganby to Horton.

8. Probability

CONTENTS

8.1 BASIC PROBABILITY

8.1.1 BASIC PROBABILITY

What do we mean by basic probability?

- There are both simple concepts and very hard topics in probability
- Probability is used in many areas and industries, eg. insurance costs and the rate of interest on loans

8. Probability

What do I need to know?

- Be aware of how we write and use **probability notation:**

1. Event

- When we have a **trial** or **experiment** there may be several outcomes
- We use capital letters to denote an event – this is something that might happen in our trial
- For example, A could be the event of getting an even number when rolling a dice
- Note that it is more common to see reference to "outcomes" rather than "events" but they broadly mean the same thing.

2. "A-dash"

- If A is an event, then A' (spoken: "A-dash") is the event "not A"
- Notice we're not necessarily interested in what has happened, just that A hasn't

3. Probability of …

- **P(A)** means the probability that event A happens (eg. rolling a six on a dice)
- **P(A')** means the probability that event A does not happen (eg. getting any other number when rolling a dice)
- The letter **n** is often used to talk about the number of times A happens (or might happen) if the trial is **repeated** several times

4. Total probability

- All the different events for a trial have a total probability of 1 (**certainty**)
- This should make sense in that something will happen from a trial (eg. when rolling a (normal) dice it is certain you will get a number between 1 and 6)

5. Experimental probability

- Also known as theoretical probability
- This is where the probability of an event can be determined by considering all the possible outcomes (eg. on a dice we would say the probability of getting each individual number is 1/6) without performing any trials
- However sometimes we don't know this (maybe because we know we have a **biased** dice but don't know **how** it is biased) so we can only talk about probabilities once we have done some trials:

$$P\ (A) = \frac{Number\ of\ times\ A\ happens}{Total\ number\ of\ trials\ (n)}$$

- The more trials that are done, the more the experimental probability will reflect the true probability of the event

6. Number of outcomes

- If we want to **estimate** the number of times an event will happen out of a total of n trails we calculate:

No. of times A happens=n ×P(A)

- You will sometimes see/hear this being called the “**expected** number of times A happens”

7. Mutually exclusive (OR means **+)**

- Mutually exclusive events cannot happen at the same time, rolling a 2 on a dice and rolling a 4 on a dice
- This leads to the result that

$$P\ (2)\ \ OR\ P\ (4) = P\ (2) + P\ (4) = \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{2}{6}$$

8. Independent events (**AND** means)

- This is the first situation where we might combine different events from different trials
- For example:
  A is the event “Rolling a 3” and B is the event “Flipping heads”
  These events are **independent** because one does not affect (the probability of) the other
  So to find the probability of both events happening we multiply their individual probabilities together:

$$P\ (A)\ \ AND\ P\ (B) = \tfrac{1}{6} \times \tfrac{1}{2} = \tfrac{1}{12}$$

- Note that this is **NOT** the opposite of mutually exclusive – because both often crop up together it is easy to think they must be linked

8. Probability

Exam Tip

It is unusual in probability questions that you will be asked to simplify fractions – so don't, in case you mess it up! You could use your calculator to do it automatically but this topic can appear on all papers.

In probability questions, it is usually easiest to use whatever number format the question does. Probabilities can be fractions, decimals or percentages (nothing else!). If no format is indicated in the question then fractions are normally best.

Worked Example

1. Emilia is using a spinner that has outcomes and probabilities as shown in the table below.

Outcome	Blue	Yellow	Green	Red	Purple
Probability		0.2	0.1		0.4

The probability of spinning a Blue is twice the probability of spinning a Red

(a) Complete the probability table.

(b) Emilia spins the spinner twice. Work out the probability she gets a Yellow on the first spin and a Green on the second spin.

(c) Emilia spins the spinner 150 times. Find an estimate for the number of times the spinning lands on Yellow or Purple.

8. Probability

(a)

If we say $P\ (Blue) = x$ and $P\ (Red) = y$ then

$x = 2y$

Also

$x + 0.2 + 0.1 + y + 0.4 = 1$

$x + y = 1 - 0.7$

$x + y = 0.3$

Then

$3y = 0.3$

$y = 0.1$

So $x = 0.2$

4 – This is a very formal way of solving this problem and you don't need to do it like this

However many mistakes are made by not taking the time to ensure things like "blue is double red"

The key here is all probabilities add up to 1

Outcome	Blue	Yellow	Green	Red	Purple
Probability	0.2	0.2	0.1	0.1	0.4

(b)

$P\ (Yellow)\ AND\ P\ (Green) = 0.2 \times 0.1$

$= 0.02$

8 – "**AND** means ×"

Getting yellow on the first spin does not affect the probability of getting green on the second spin

(c)

$P\ (Yellow)\ OR\ P\ (Purple) = 0.2 + 0.4 = 0.6$

$150 \times 0.6 = 90$

7 – "**OR** means +"

6 – Expected number of outcomes

2. Jake is throwing a biased coin. He throws it 200 times and it lands on heads 145 times.
   (a) Estimate the probability of getting a tails with this coin.
   (b) Comment on the reliability of this estimate.

8. Probability

(a)

$P\,(Tails) \approx \frac{200-145}{200}$

$= \frac{55}{200}$

5 – **Experimental Probability**

Notice they ask for tails!

No need to simplify but if on calculator it would do it for you ($\frac{11}{40}$)

(b)

As Jake has thrown the coin a large number of times, 200, this is a reliable estimate for the probability of throwing tails with it.

Exam Question: Easy

The probability that a biased dice will land on a five is 0.3

Megan is going to roll the dice 400 times.

Work out an estimate for the number of times the dice will land on a five.

Exam Question: Medium

Bill has some counters in a bag.

3 of the counters are red.
7 of the counters are blue.
The rest of the counters are yellow.

Bill takes at random a counter from the bag.

The probability that he takes a yellow counter is $\frac{2}{7}$

How many yellow counters are in the bag before Bill takes a counter?

8. Probability

8.2 VENN DIAGRAMS & TWO WAY TABLES

8.2.1 PROBABILITY - VENN DIAGRAMS

What is a venn diagram?

- **Venn diagrams** allow us to show **two** (or more) characteristics of a situation where there is overlap between the characteristics
- For example, students in a VI Form can study Biology or Chemistry but there may be students who study both

What do I need to know?

- You can be asked to draw a Venn diagram and/or interpret a Venn diagram
- Strictly speaking the rectangle (box) is always **essential** on a Venn diagram as it represents everything that can happen in the situation
- You may see the letters ε or ζ written inside or just outside the box – this means "the set of all possible outcomes" – basically it just means "everything"!
- The words **AND** and **OR** become very important in both drawing and interpreting Venn diagrams
- You will need to be familiar with the symbols ∩ and ∪ – **intersection** and **union**, loosely speaking these mean **AND** and **OR** (respectively)

1. Drawing a Venn Diagram

- You'll need a "box" and overlapping "bubbles" depending on how many characteristics you are dealing with
- You will not normally be given all the values for every section in your diagram
- You will be expected to work out missing information in order to complete your Venn diagram
- Remember to consider **AND** and **OR**

2. Interpretation

- This is where "not A" (A') and similar probability that event A does not happen can get confusing
- The symbols ∩ and ∪ are often used here too

8. Probability

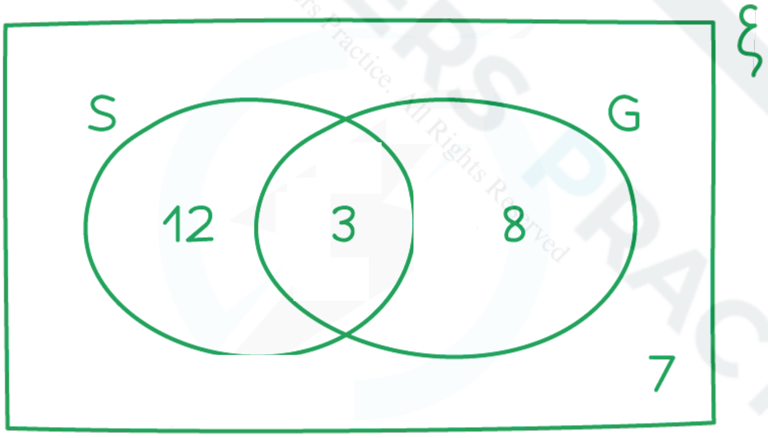
Exam Tip

Lightly highlighting the part of the Venn diagram you need can help but make sure you can still read the whole diagram for later parts of the question if you do this.

Worked Example

1. In a class of 30 students, 15 study Spanish, 3 of whom also study German.
   7 students study neither Spanish nor German.
   (a) Draw a Venn diagram to show this information
   (b) Use your Venn diagram to find the probability that a student, selected at random from the class, studies Spanish but not German.

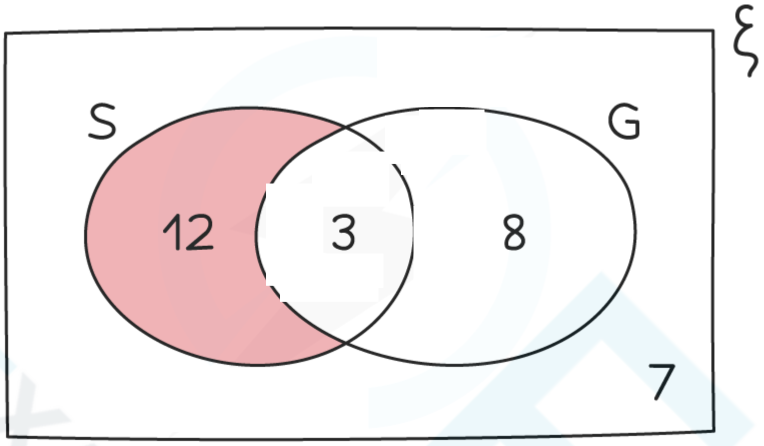
(a)



1 – You should start with the 3 in the overlap ("middle") then deduce that the "Spanish only" bubble will have 12 in it. 7 needs to be outside both bubbles but within the box. With a total of 30 required you can now work out how many study "German only" and complete the diagram.

8. Probability

(b)



2 – Highlight the part of the Venn diagram you need ("Spanish only")

Students studying Spanish only = 12

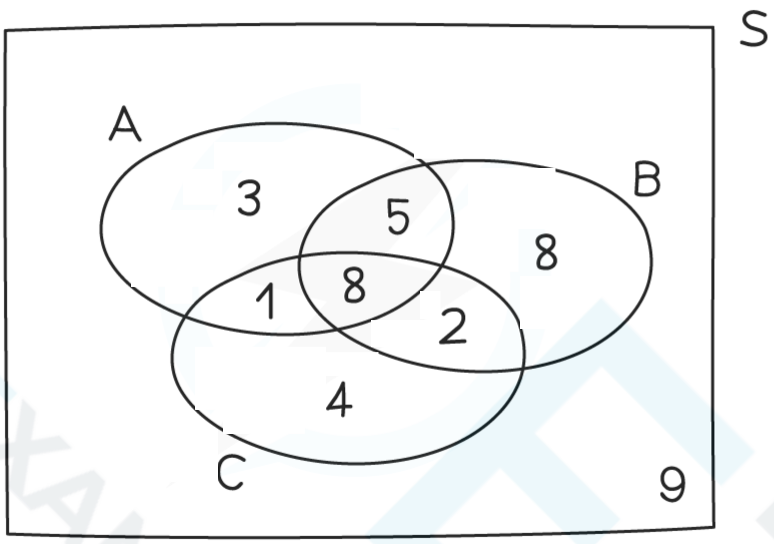
Total number of students = 30

Pick out the numbers you need carefully

P (random student studies S only) = $\frac{12}{30}$

2. Given the Venn diagram below answer the following questions:

8. Probability



(a) $P(A)$

(b) $P(A \cap B \cap C)$

(c) $P(B' \cap C)$

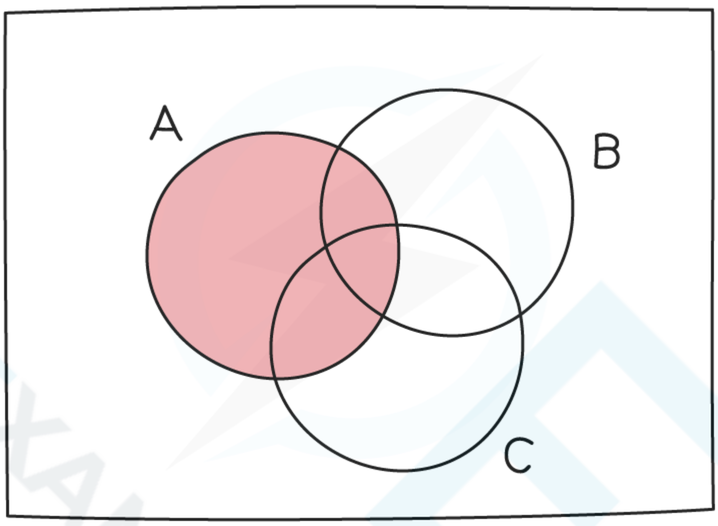
(d) $P(A \cup B)$

(e) $P(A \cup B \cup C)$

(f) $P(A' \cup B')$

8. Probability

(a)



Draw a quick sketch of the diagram without the details when multiple parts to a question

See these types of questions as "ways to win" – here if you are in "bubble A" you "win", B and C don't come into it at all

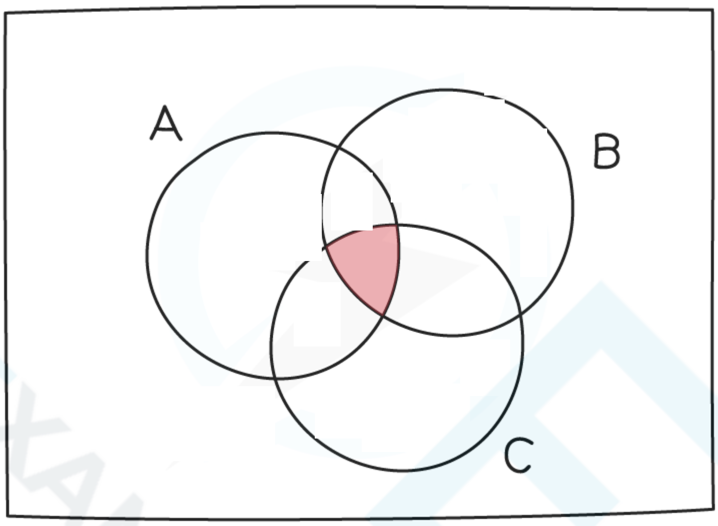
$Total\ in\ A = 3 + 5 + 1 + 8 = 17$

$Total = 3 + 5 + 1 + 8 + 2 + 4 + 8 + 9 = 40$

$P(A) = \frac{17}{40}$

8. Probability

(b)



∩ - intersection – AND – you "win" if "in A" AND "in B" AND "in C"

$Total\ in\ A, B\ and\ C = 8$

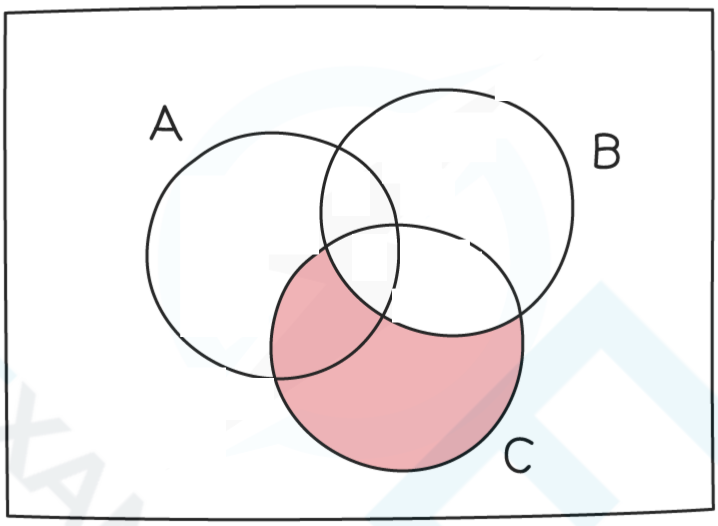
$Total = 40$

Worked out in part (a)

$P(A \cap B \cap C) = \frac{8}{40}$

8. Probability

(c)



∩ - intersection – AND – you "win" if "not in B" AND "in C"

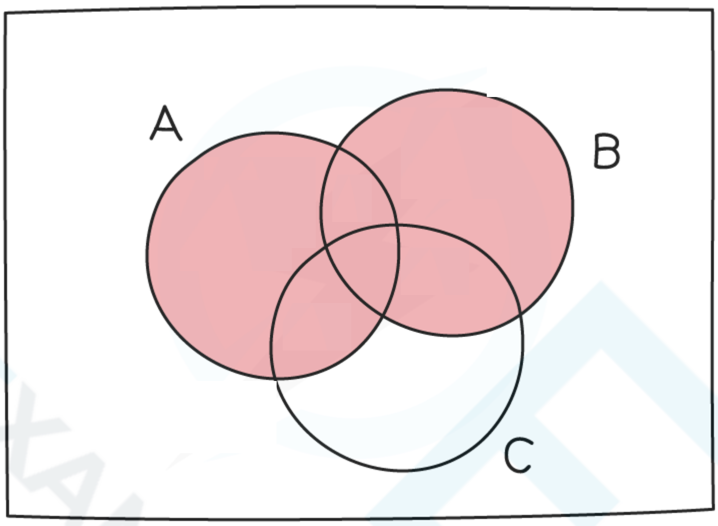
$Total\ in\ not\ B\ and\ C\ = 1 + 4 = 5$

$Total = 40$

$P(B' \cap C) = \frac{5}{40}$

8. Probability

(d)



∪ - union – OR – you “win” if “in A” OR “in B”

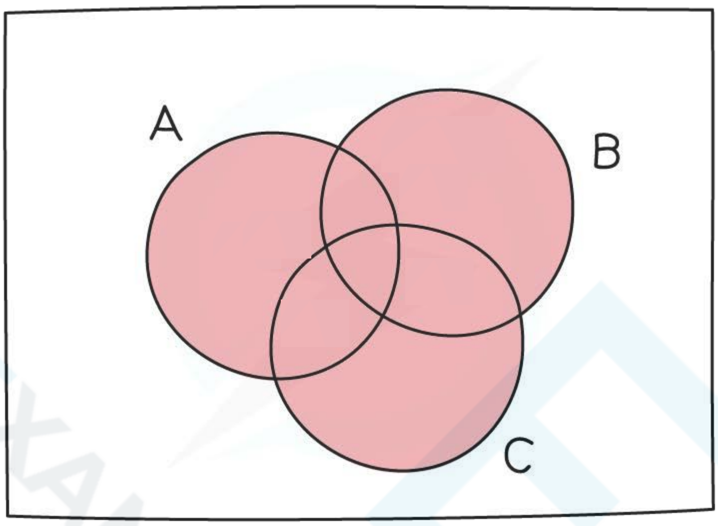
$Total\ in\ A\ or\ B = 3 + 1 + 8 + 5 + 2 + 8 = 27$

$Total = 40$

$P(A \cup B) = \frac{27}{40}$

8. Probability

(e)



∪ - union – OR – you “win” if “in A” OR “in B” OR “in C”

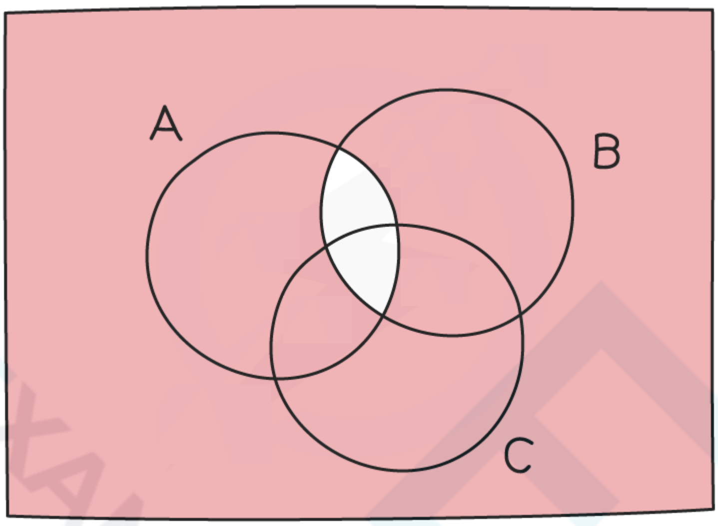
$Total\ in\ A, B\ or\ C = 40 - 9 = 31$ Easy to subtract from whole in this case

$Total = 40$

$P(A \cup B \cup C) = \frac{31}{40}$

8. Probability

(f)



∪ - union – OR – you "win" if "not in A" OR "not in B"

This one is particular difficult to see without a friend!

$Total\ in\ A'\ OR\ B' = 40 - 8 - 5 = 27$ Easier to subtract again

$Total = 40$

$P(A' \cup B') = \frac{27}{40}$

8. Probability

8.2.2 PROBABILITY - TWO WAY TABLES

What are two-way tables?

- While **Venn diagrams** are great at showing **overlap** they can only show one feature (**characteristic**) of a situation at one time
- In the notes on Venn diagrams we had an example talking about students studying either Spanish or German, or both
  However we may **also** be interested in how many **boys** and **girls** were studying Spanish and/or German as well
- This is where we need a two-way table – one of the **characteristics** will be the columns and the other will be presented in rows
- Once we have our table we can use the numbers within to determine probabilities

What do I need to know?

- You'll need to be able to construct a two-way table from information given in words and then use a table to calculate probabilities
- So you'll need to be familiar with the basics and notation around probability

1. Total row/column

- It may not be obvious from the wording but a total row and column can be really helpful in two-way table questions
- If they're not mentioned, or included when given a table, add them in

2. Completing a table

- It may not be possible to add numbers to the table from every sentence, one at a time
- You will usually have to combine one piece of information with another in order to fully complete a table

3. Conditional probability

- Two-way tables in particular give rise to using **conditional probability**
- This can get complicated but with two-way tables, it is usually straightforward to see which parts of the table the question is referring to

8. Probability

Worked Example

At an art group children are allowed to choose between four activities:

'Colouring', 'Painting', 'Clay Modelling' and 'Sketching'.

There is a total of 60 children attending the art group.

12 of the boys chose the activity 'Colouring'.

A total of 20 children chose 'Painting' and a total of 15 chose 'Clay Modelling'.

13 girls chose to do 'Clay Modelling'.

8 of the 30 boys chose 'Sketching', and did 4 of the girls.

(a) Construct a two-way table to show this information.

(b) Find the probability that:

- (i) a randomly selected child chose 'Colouring',
- (ii) a randomly selected child is a boy who chose 'Sketching',
- (iii) a randomly selected child is a boy, given that they chose 'Painting',
- (iv) a randomly selected child chose 'Clay Modelling', given that they're a girl

(a)

	Colouring	Painting	Clay Modelling	Sketching	Total
Boys	12	8	2	8	30
Girls	1	12	13	4	30
Total	13	20	15	12	60

1, 2 - Construct the table carefully, including total row and column

The values highlighted you should've been able to complete from the information given in the question

Work your way round the table, not necessarily in order – if you get to a stage where you can't complete a value then you would've missed something from the question so go back and have another look

8. Probability

(b)

(i) $P(Colouring) = \frac{13}{60}$

Total colouring ÷ Total Children

(ii) $P(Boy\ AND\ Sketching) = \frac{8}{60}$

Boy & Sketching ÷ Total Children

(iii) $P(Boy\ GIVEN\ THAT\ Painting) = \frac{8}{20}$

3 – Conditional probability

Boy & Painting ÷ Total Painting

(iv) $P(Clay\ Modelling\ GIVEN\ THAT\ Girl) = \frac{13}{30}$

3 – Conditional probability

Girl & Clay Modelling ÷ Total Girls

8. Probability

8.2.3 SET NOTATION & VENN DIAGRAMS

What do I need to know?

- You'll be drawing **Venn diagrams** so make sure you are familiar with those first
- **Notation**

  ξ is the **universal** set (the set of **everything**)a ∈ B means a is an **element** of B (a is in the set B)

  A ∩ B means the **intersection** of A and B (the **overlap** of A **and** B)

  A ∪ B means the **union** of A and B (**everything** in A **or** B **or** both)

  A' is "**not** A" (everything **outside** A)

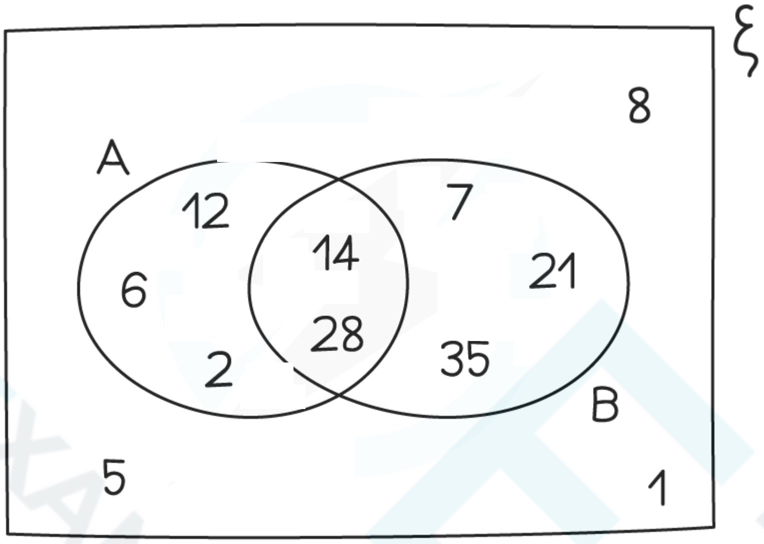
  Sets can be written as a list of **elements** (**members**) or described in words – either way curly brackets are used:

  - A={3, 6, 9, 12, 15, 18}
  - A={Multiples of 3 less than 20}

Worked Example

1. Use the Venn diagram below to answer the following questions.

8. Probability



(a) Write down all the members of the set A,

(b) Describe in words the members of the set B,

(c) Write down all the members of the set $A' \cap B$,

(d) Suggest, with justification, where the element 49 could go within the Venn diagram,

(e) If one of the numbers in the Venn diagram is chosen at random, find the probability that the number is in the set $A \cup B$.

8. Probability

(a)

2, 6, 12, 14, 28

They don't have to be in order but make sure you get them all!

(b)

Multiples of 7 less than 40

(c)

$A' \cap B = \{7, 21, 35\}$

Draw a quick sketch if and shade the area(s) required if unsure

(d)

49 could go in set B as these are multiples of 7.

There are alternative answers – especially if there is a limit to how high the multiples of 7 in set B can be (see answer (b)!)

(e)

Number of elements in the set $A \cup B$ is 8

Number of elements in total 11

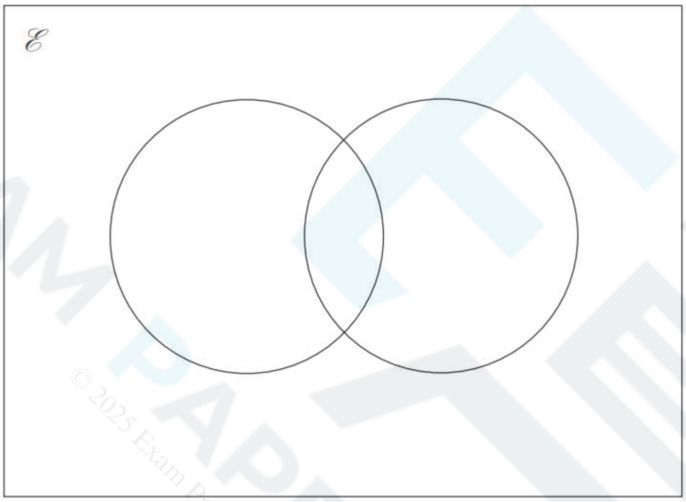
P (randomly chosen number is in set $A \cup B$) = $\frac{8}{11}$

8. Probability

Exam Question: Easy

$\mathscr{E}$ = {odd numbers less than 30}
A = {3, 9, 15, 21, 27}
B = {5, 15, 25}

(a) Complete the Venn diagram to represent this information.



A number is chosen at random from the universal set, $\mathscr{E}$.

(b) What is the probability that the number is in the set $A \cup B$?

8. Probability

Exam Question: Hard

50 people each did one activity at a sports centre.

Some of the people went swimming.
Some of the people played squash.
The rest of the people used the gym.

21 of the people were female.
6 of the 8 people who played squash were male.
18 of the people used the gym.
9 males went swimming.

Work out the number of females who used the gym.

8. Probability

8.3 COMBINED PROBABILITY

8.3.1 COMBINED PROBABILITY - BASICS

What do we mean by combined probabilities?

- This can mean lots of things as you'll see over these notes and the next set
- In general it means we have more than one 'thing' (trial/event) to bear in mind and these things may be **independent**, **mutually exclusive** or may involve an event that follows on from a previous event drawing a second counter from a bag

1. Tree diagrams

- Especially useful when we have more than one trial but are only concerned with two outcomes from each
- Even more useful when probabilities change for the second experiment

2. Replacement

- Are items being selected at random replaced or not?
- If not then numbers will decrease as the situation progresses and so probabilities change – this is often called conditional probability

3. AND's and **OR**'s

- **AND** means for **independent** events
- **OR** means for **mutually exclusive** events

4. Sum of all probabilities is 1

- This is a very basic fact that gets lost along the way in more complicated probability questions – but it is one of the best 'tricks' you can use!
- A good example of its use is when you want the probability of something being "non zero": P(x ≥ 1) = 1 – P(x = 0)

8. Probability

Worked Example

1. A box contains 3 blue counters and 8 red counters.

A counter is taken at random and **not** replaced.

Work out the probability:

- (a) Both counters are red,
- (b) There is one of each colour,
- (c) Both are the same colour

Do note, as far as the maths is concerned, taking a counter out at random, and not replacing it, is the same as taking two counters out at the same time.

(a)

$P(1^{st}\ is\ red) = \frac{8}{11}$ There's 8 reds to start with, 8 + 3 = 11 in total

$P(2^{nd}\ is\ red) = \frac{7}{10}$ 2 – not replaced so **reds and total** decrease by 1

$P(both\ red) = \frac{8}{11} \times \frac{7}{10}$ 3 – a hidden AND – red AND red

$= \frac{56}{110}$ When values get big it may be wise to simplify, particularly if you're likely to use them later

(b)

$P(red/blue\ OR\ blue/red) = P(red\ AND\ blue)\ OR\ P(blue\ AND\ red)$

3 – a hidden AND **and** OR question!

$= \frac{8}{11} \times \frac{3}{10} + \frac{3}{11} \times \frac{8}{10}$

1 – you may prefer a tree diagram here

$= \frac{24}{110} + \frac{24}{110}$

$= \frac{48}{110}$

8. Probability

(c)

$P(both\ blue\ OR\ both\ red) = P(blue\ AND\ blue)\ OR\ P(red\ AND\ red)$

3 – a hidden AND **and** OR question!

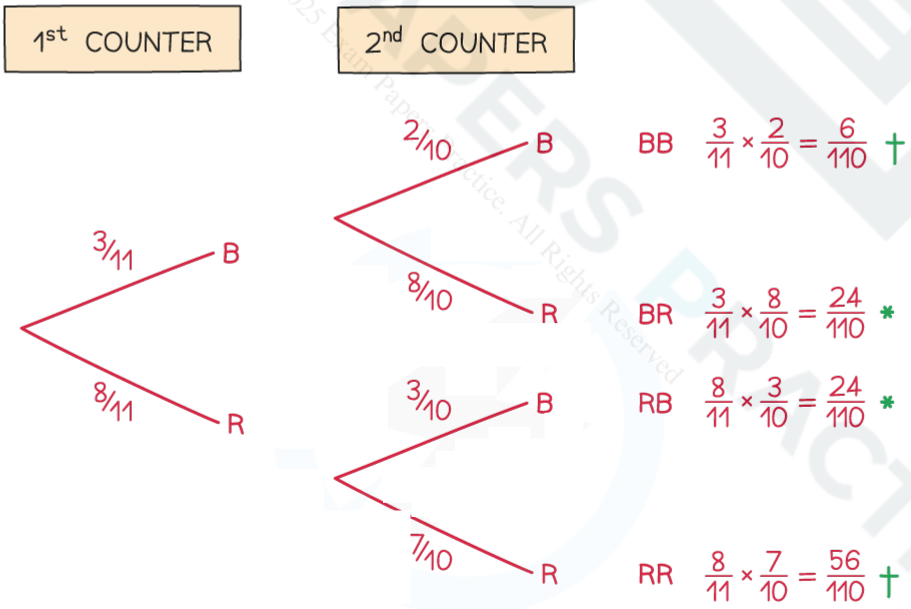
$$= \frac{3}{11} \times \frac{2}{10} + \frac{8}{11} \times \frac{7}{10}$$

1 – you may prefer a tree diagram here

$$= \frac{6}{110} + \frac{56}{110}$$

$$= \frac{62}{110}$$

Note that you could draw a tree diagram to help with this question – particularly parts (b) and (c). The diagram below is what we would describe as complete, but you don't need to go that far in the exam – draw as much or as little of the tree diagram as you need to help you understand and see your way through the problem



* (b) ONE OF EACH COLOUR: $\frac{24}{110} + \frac{24}{110} = \frac{48}{110}$

† (c) BOTH SAME COLOUR: $\frac{6}{110} + \frac{56}{110} = \frac{62}{110}$

8. Probability

2. The probability of winning a fairground game is known to be 26%.

If the game is played 4 times what is the probability that there is **at least one** win.

Write down one assumption you have made.

$P(at\ least\ one\ win) = 1 - P(zero\ wins)$ 4 – Use $P(lose) = 1 - P(win)$

$= 1 - (0.74)^4$ $1 - 0.26 = 0.74$

3 – 'Loose' AND 'Lose' AND 'Lose' AND 'Lose'

$= 0.7001$ **(4 decimal places)**

I have made the assumption that each attempt at the fairground game is independent – that the outcome of one game does not affect the outcome of the next.

8. Probability

8.3.2 COMBINED PROBABILITY - HARDER

What do I need to know?

- Be aware of the basics, ie. tree diagrams, replacement, AND and OR, the sum of all probabilities is 1 (see Combined Probability – Basics)
- Note that there may be some problem-solving and algebra involved (see question 2 of the Worked Example)

Worked Example

1. José passes two sets of traffic lights on his commute to work.

The probability of him being stopped at the first set of lights is 0.3.

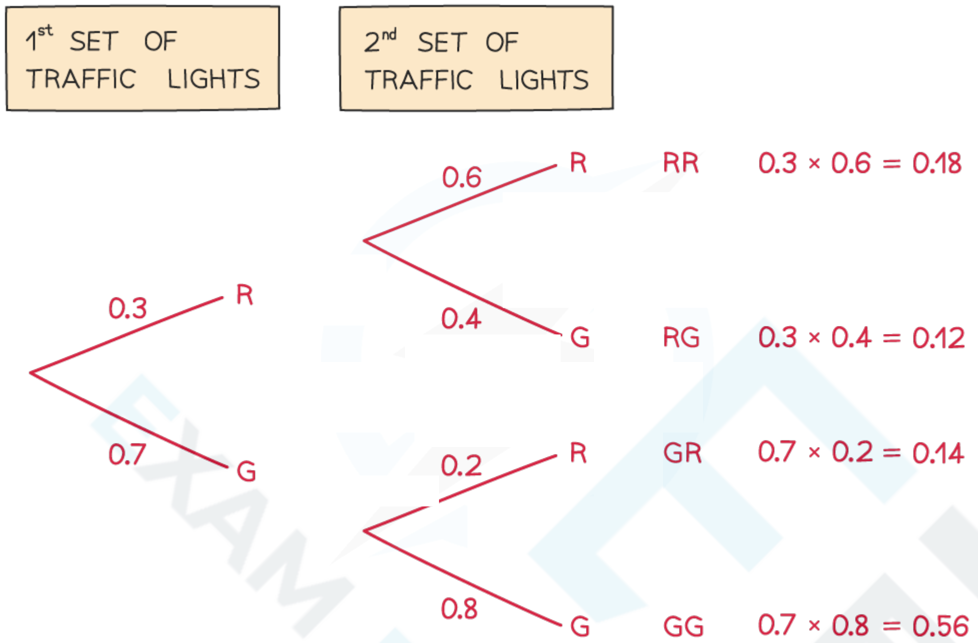
If he **is stopped** at the first set of lights the probability of him being stopped at the second set is 0.6.

If he **is not stopped** at the first set, the probability of José being stopped at the second set is 0.2.

Find the probability that José is stopped at **exactly** one set of traffic lights.

A complicated question with lots of words – so use a tree diagram!

8. Probability



We've used R and G for red and green on traffic lights (we can ignore amber for the purposes of this question!) but you can use any system you like but it's usually easiest to stick to letters (An alternative could be S for stopped, N for not stopped.)

$P(1^{st}\ set\ is\ green) = 1 - 0.3 = 0.7$ 4 – The good old "1 –" trick!

$P(2^{nd} set\ is\ green\ given\ 1^{st}\ is\ red) = 1 - 0.6 = 0.4$

$P(2^{nd} set\ is\ green\ given\ 1^{st}\ is\ green) = 1 - 0.2 = 0.8$

Now we have all the probabilities to complete the diagram

$P(stopped\ at\ exactly\ one\ set\ of\ lights) = P(R\ AND\ G)\ OR\ P(G\ AND\ R)$

Make sure those ANDs and ORs are the right way round by using words if need be

$= 0.3 \times 0.4\ +\ 0.7 \times 0.2$

If you've written these on your diagram (like above) there is no need to write them again

$= 0.12\ +\ 0.14$

$= 0.26$

8. Probability

2. A bag contains 7 red counters and b blue counters.

Two counters are taken at random from the bag.

The probability they are both red is $\frac{7}{40}$.

(a) Show that $b^2 + 13b - 198 = 0$

(b) Find b and hence find the probability that the two counters are of a different colour.

(a)

$P(R\ AND\ R) = P(R) \times P(R) = \frac{7}{40}$

You can use a combination of words and letters to help you understand the problem

So, $\frac{7}{b+7} \times \frac{6}{b+6} = \frac{7}{40}$

There are b+7 counters to start with,

$\frac{42}{(b+7)(b+6)} = \frac{7}{40}$

this decreases by one once the first counter

$\frac{42\times40}{7} = (b+7)(b+6)$

is selected so there are b+6 counters when

$240 = b^2 + 13b + 42$

the second one is selected

$b^2 + 13b - 198 = 0$

Lots of algebra here but should be straightforward once you get started

(b)

$(b+22)(b-9) = 0$

Solve the quadratic, use the formula if you cannot do the factorising

~~$b = -22$~~ or $b = 9$

We reject b = -22 as we cannot have a negative number of counters

$b = 9$

$P(one\ of\ each\ colour) = P(R\ AND\ B)\ \ OR\ \ P(B\ AND\ R)$

As you do more of these you will get quicker at them – you could always use a tree diagram here

$= \frac{7}{16} \times \frac{9}{15} + \frac{9}{16} \times \frac{7}{15}$

$= \frac{126}{240}$

If you do this on your calculator you will automatically get the rounded answer of $\frac{21}{40}$

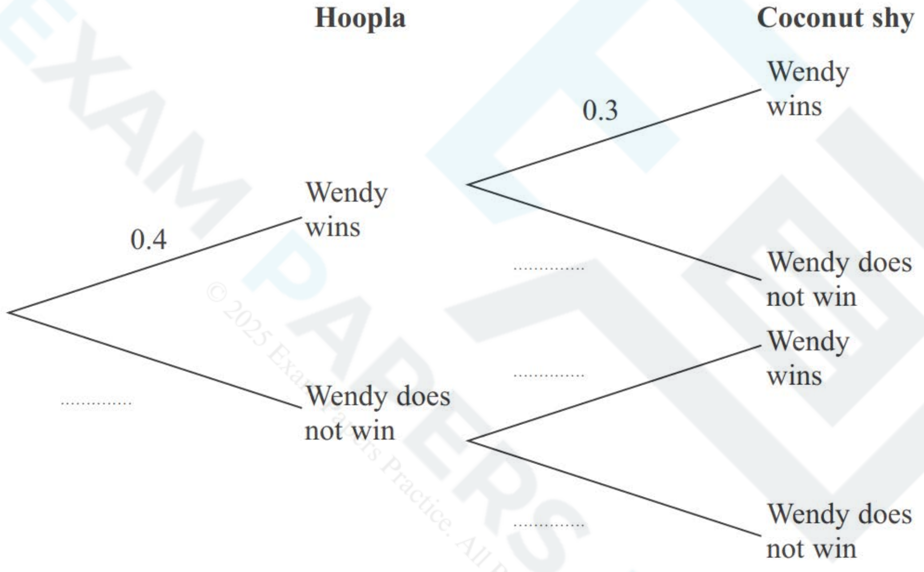
8. Probability

Exam Question: Easy

Wendy goes to a fun fair.
She has one go at Hoopla.
She has one go on the Coconut shy.

The probability that she wins at Hoopla is 0.4
The probability that she wins on the Coconut shy is 0.3

(a) Complete the probability tree diagram.



(b) Work out the probability that Wendy wins at Hoopla and also wins on the Coconut shy.

8. Probability

Exam Question: Medium

Here are seven tiles.

1 1 2 2 2 3 3

Jim takes at random a tile.
He does **not** replace the tile.

Jim then takes at random a second tile.

(a) Calculate the probability that both the tiles Jim takes have the number 1 on them.

(b) Calculate the probability that the number on the second tile Jim takes is greater than the number on the first tile he takes.

Exam Question: Hard

Carolyn has 20 biscuits in a tin.

She has

12 plain biscuits
5 chocolate biscuits
3 ginger biscuits

Carolyn takes at random two biscuits from the tin.

Work out the probability that the two biscuits were **not** the same type.

8. Probability

Exam Question: V. Hard

Paul has 8 cards.

There is a number on each card.

2 3 3 4 5 5 5 5

Paul takes at random 3 of the cards.
He adds together the 3 numbers on the cards to get a total *T*.

Work out the probability that *T* is an odd number.

9. Statistics

CONTENTS

9.1 MEAN, MEDIAN & MODE

9.1.1 MEAN, MEDIAN & MODE

Why do we have different types of average?

- You'll hear the phrase "on average" used a lot, from politicians talking about the economy to sports analysts to shops talking about their "average customer"
- However not all data is numerical (eg the party people voted for in the last election) and even when it is numerical, some of the data may lead to misleading results
- This is why we have **3 types of average**

9. Statistics

What do I need to know?

1. Mean

- This is what is usually meant by “average” – it’s like an ideal world where everybody has the same, everything is shared out equally
- It is the **TOTAL** of all the values **DIVIDED** by the **NUMBER OF VALUES**
  Find the mean of 4, 6, 7, 9

  4 + 6 + 7 + 9 = 26

  26 ÷ 4 = 6.5

  Mean = 6.5
- Problems with the mean occur when there are one or two unusually high (or low) values in the data (**outliers**) which can make the mean too high (or too low) to reflect any patterns in the data

9. Statistics

2. Median

- This is similar to the word medium, which can mean in the middle
- So the median is the middle value – but beware, the data has to be arranged into numerical order first
  Find the median of 20, 43, 56, 78, 92, 56, 48

  In order: 20, 43, 48, 56, 56, 78, 92

  To find the median cross out numbers from either end until you meet in the middle (cross them out lightly so you can still read them)
  This may not be necessary with small lists but is more important when working with lots of data

  ~~20~~, 43, 48, 56, 56, 78, ~~92~~

  ~~20~~, ~~43~~, 48, 56, 56, ~~78~~, ~~92~~

  ~~20~~, ~~43~~, ~~48~~, 56, ~~56~~, ~~78~~, ~~92~~

  Median = 56

- We would use the median instead of the mean if we did not want extreme values (outliers) affecting our data
- If we have an even number of values we would get two values in the middle
- In these cases we take the half-way point between these two values. This is usually obvious but, if not, we **add the two middle values and divide by 2** (this is the same as finding the mean of the middle two values)
  20, 43, 46, 48, 56, 56, 78, 92 (as above with an extra 46 in there!)

  When crossed out we get

  ~~20~~, ~~43~~, ~~46~~, 48, 56, ~~56~~, ~~78~~, ~~92~~

  So the two middle values are 48 and 56.

  Halfway is 52 but if you can’t spot that you can work it out …

  48 + 56 = 104

  104 ÷ 2 =52

  Median = 52

9. Statistics

3. Mode

- Not all data is numerical and that is where we use mode
- **MO**de means the **M**ost **O**ften
- So it is often used for things like "favourite …" or "… sold the most" or "… were the most popular"
- Mode is sometimes referred to as **modal** – so you may see phrases like "**modal value**" – but they still mean the mode

  12 people were asked about their favourite crisp flavour. The responses are below:

  Salt and vinegar, Prawn cocktail, Ready salted,

  Ready salted, Salt and vinegar, Salt and vinegar,

  Smokey bacon, Ready salted, Salt and vinegar

  Salt and vinegar, Cheese and onion, Ready salted

  With only a few pieces of data it is quite quick and easy to see here that Salt and vinegar is chosen the most

  With more data it may be wise to create a tally chart or similar to help count the number of each flavour

  Mode is Salt and vinegar
- Be aware that the mode can apply to numerical data as well (from the data used in the example for the median the mode would have been 56)
- Sometimes if no value/data occurs more often than others we say there is no mode
- If two values occur the most we may say there are two modes (**bi-modal**) – whether it is appropriate to do this will depend on what the data is about

Worked Example

1. (a) Briefly explain why the mean is not a suitable average to use in order to analyse the way people voted in the last general election.

   (b) Suggest a better measure of average that can be used.

(a)

Political parties have names and so the data is not numerical

(b)

The mode average can be used for non-numerical data

9. Statistics

2. 15 students were timed how long it took them to solve a maths problem. Their times, in seconds, are given below.

12	10	15	(37)
14	17	11	(42)
12	13	9	(34)
21	14	20	(55)
19	16	23	(58)

(a) Find the mean and median times.

(b) What can you say about the mode of the data?

(a)

Mean:

12 + 10 + 15 + 14 + 17 + 11 + 12 + 13 + 9 + 21 + 14 + 20 + 19 + 16 + 23 = 226

1 - You could do the adding up in bits by adding the rows (or columns) as above in brackets

226 ÷ 15 = 15.066...

Do the mean in two stages to avoid confusion around using brackets on your calculator and to show all stages of working

Mean = 15.1 (to one decimal place)

Round final answer to something sensible if not asked to

Median:

~~9~~ ~~10~~ ~~11~~ ~~12~~ ~~12~~ ~~13~~ ~~14~~ 14 ~~15~~ ~~16~~ ~~17~~ ~~19~~ ~~20~~ ~~21~~ ~~23~~

Median = 14

2 - Make sure you write them in order and do not miss any out, it's a good idea to lightly cross them off the original list

(b)

Two values occur more than any others, 12 and 14

3 - Notice how this is easy to see once the data is in order from finding the median in part (a)

So we can say there are two modes – 12 and 14, or, we could say that there is no mode

9. Statistics

9.1.2 AVERAGES FROM TABLES & CHARTS

How do we find averages if there are lots of values?

- In reality there will be far more data to work with than just a few numbers
- In these cases the data is usually organised in such a way to make it easier to follow and understand – for example in a **table** or **chart**
- We can still find the **mean**, **median** and **mode** but have to ensure we understand what the table or chart is telling us

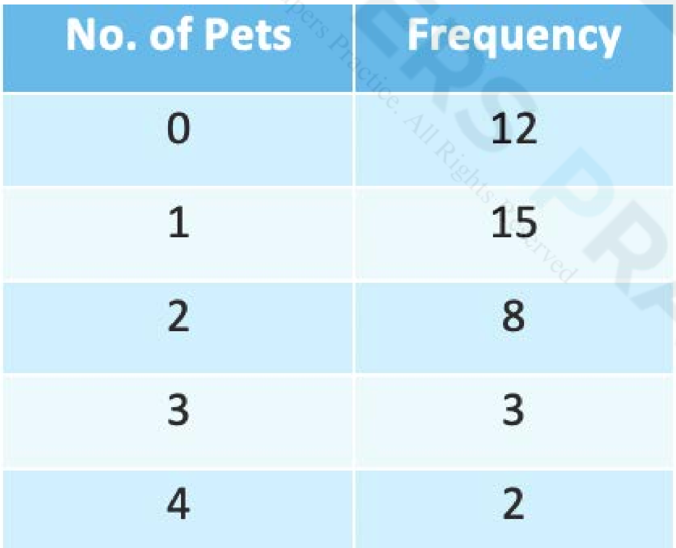
What do I need to know?

- Finding the median and mode from tables/charts is fairly straightforward once you understand what the table/chart is telling you so these notes focus mainly on finding the mean

1. Finding the mean from (discrete) data presented in tables

- Tables allow data to be summarised neatly – and quite importantly it puts it into order

Eg. the number of pets owned by 40 pupils in year 11 are summarised in the table below:



No. of Pets	Frequency
0	12
1	15
2	8
3	3
4	2

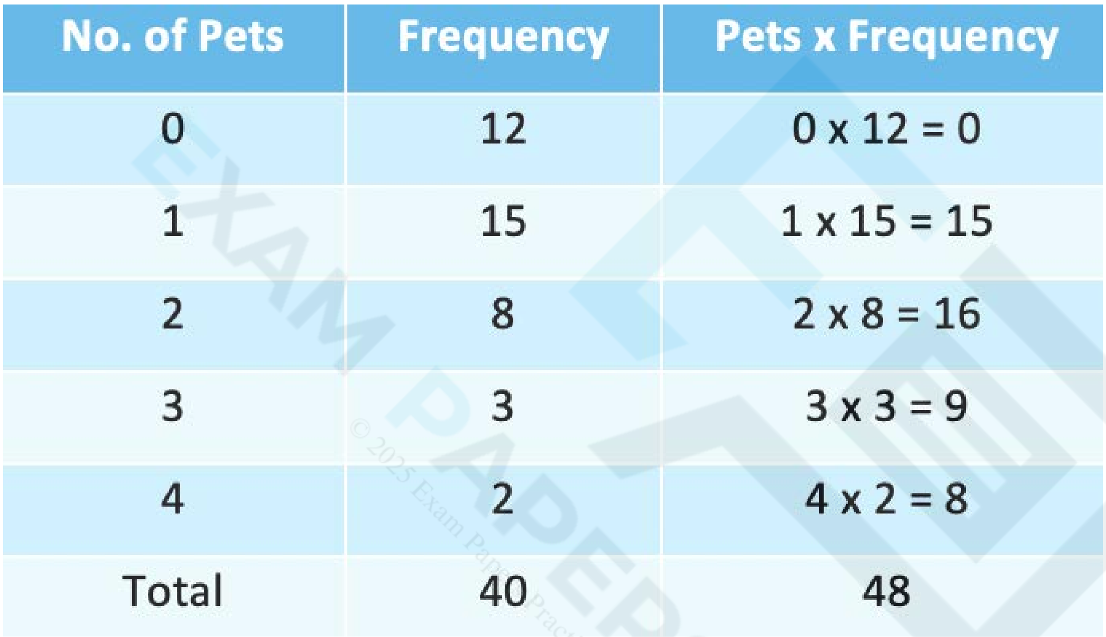
Work out the mean and median number of pets per pupil.

The **mean** can be found as you long as understand what the table is telling you:

It tells you:
12 (of the 40) pupils had no pets; 15 of them had 1 pet; 8 had 2 pets

This means you can add up all the 0's very quickly, all the 1's very quickly etc. using multiplication: 12 x 0 = 0, 15 x 1 = 15, etc.

The easiest way to do this is to add another column to the table and adding a **total row** will prove useful in the next stage too:



No. of Pets	Frequency	Pets x Frequency
0	12	0 x 12 = 0
1	15	1 x 15 = 15
2	8	2 x 8 = 16
3	3	3 x 3 = 9
4	2	4 x 2 = 8
Total	40	48

- You can now see that the total number of pets for the whole of the 40 pupils is 48
- If you hadn't been told in the question, the total of the frequency column would've told you how many pupils there are
- You can now find the mean:

  48 ÷ 40 = 1.2

  The mean number of pets is 1.2 (pets per pupil)
- You may sometimes see the number of pets called x and the frequency f so the last column would be:
  f x (f times x)

9. Statistics

2. Median

- The **median** is a little complicated to understand but easy to work out
- Remember you are looking for the middle value when the data is in order
- For the example above the table is in order (0, then 1, 2, etc.) – so you need to work out which of the 40 values is in the median position
- To do this you add one to the number of values and divide by 2

  (40 + 1) ÷ 2 = 20.5 – so the median is in the "20.5th" position

- The table tells you the first twelve numbers on the list are all 0's, the next 15 are all 1's – so the "20.5th" value in the list must be a 1
  The median number of pets is 1

  (Note that the "20.5th" value is referring to halfway between the 20th and 21st values. Both of these are 1's so median must be 1 (or use (1 + 1) ÷ 2 = 1 if you're not convinced!)

3. Mode (modal value)

- The **mode** (or **modal value**) is simple to identify
- Look for the highest frequency – and then you find the corresponding data value

  In the example above the highest frequency is 15

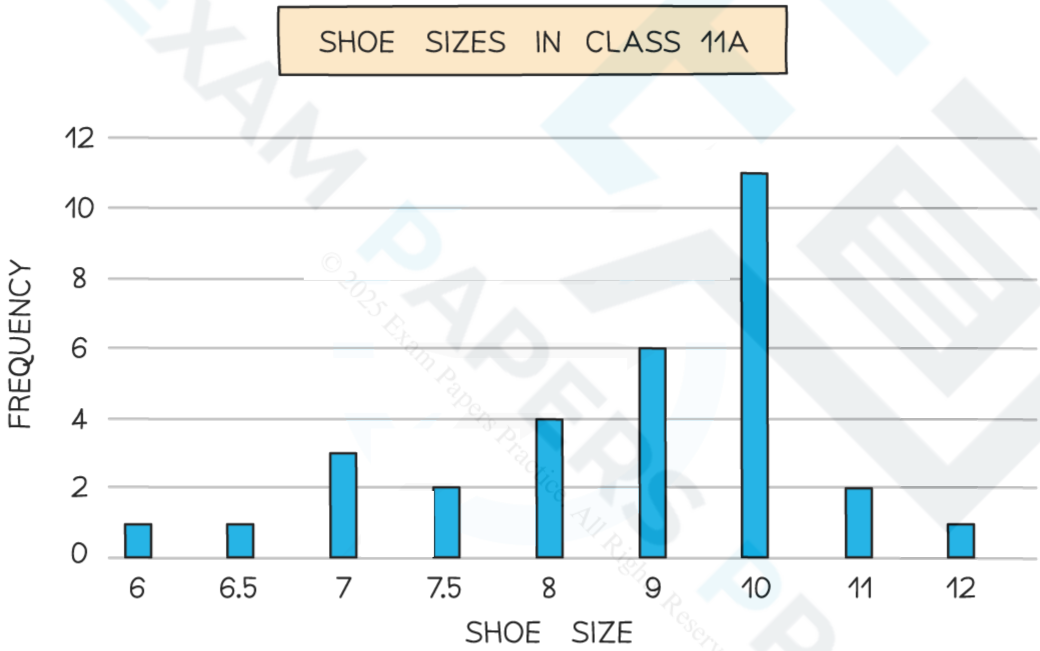
  Modal number of pets = 1

- Don't confuse the number of pets with the frequency

9. Statistics

Worked Example

1. The bar chart shows data about the shoe sizes of pupils in class 11A
   (a) Find the mean shoe size for the class,
   (b) Find the median shoe size,
   (c) Suggest a reason why a shoe shop owner might want to know the modal shoe size of their customers.



9. Statistics

(a)

1 - Although the data is given in a bar chart this is essentially the same as a table You should rewrite it as a table and that allows you to add in that extra column for working out

Shoe size	Frequency	Shoe size x Frequency
6	1	6 x 1 = 6
6.5	1	6.5 x 1 = 6.5
7	3	7 x 3 = 21
7.5	2	7.5 x 2 = 15
8	4	8 x 4 = 32
9	6	9 x 6 = 54
10	11	10 x 11 = 110
11	2	11 x 2 = 22
12	1	12 x 1 = 12
Total	31	278.5

Mean = 278.5 ÷ 31 Be careful to get this the right way round!

Mean = 8.9838...

Mean = 9.0 (to one decimal place) Round to something sensible

(Note that the mean doesn't have to be an actual shoe size)

9. Statistics

(b)

2 - You need to start by finding the position of the median

Position of median = (31 + 1) ÷ 2 = 16th

Median = 9

There are 1 + 1 + 3 + 2 + 4 = 11 values used by the first four rows then the next row with 6 would take you past the 16th value. So the 16th value must be 9.

(c)

A shoe shop manager would want to know the modal size of shoe of his customers as this would be the size of shoes that they are likely to sell most of so he would need to order more of these than the other sizes.

9. Statistics

9.1.3 AVERAGES FROM GROUPED DATA

What is a stem-and-leaf diagram?

- A **stem–and–leaf diagram** is a simple but effective way of showing data
- It puts the data into **order**, puts it into **classes** (**groups**) and we can quickly see patterns
- As the data is in order it is also useful for finding the **median** and **quartiles**

What do I need to know?

- Stem-and-leaf diagrams are particularly useful for two-digit data but can be used for bigger numbers
- Two-digit data could be something like 26 but could also be 2.6, due to this one of the essential things about a stem-and-leaf diagram is that it has a key
- You may also come across back-to-back stem-and-leaf diagrams which are used to compare two sets of data

1. Stem-and-leaf diagrams

9. Statistics

- The digits from the data are split into two – stems and leaves
- As in nature though, a stem can have more than one leaf, so the stems become our classes in our data
- Eg The data value 26 would be split into a stem of 2 and a leaf of 6
  That will then mean the "2" becomes a class interval – ie the 20's
  Any other values in the 20's would join the same class – so a stem of 2 would have two leaves
  Eg. Draw a stem-and-leaf diagram for the following data

26 45 32 27 29 30 40 36 37

As the data is not in order draw a **rough** diagram first to get the data values into the correct format:

Stem	Leaves			
2	6	7	9	
4	5	0		
3	2	0	6	7

Now put the stems and their leaves in order

Stem	Leaves			
2	6	7	9	
3	0	2	6	7
4	0	5		

Key: 2|6 means 26

Add a key so we know what the data is showing

9. Statistics

Worked Example

1. A hospital is trying to compare two different medications that claim to reduce blood pressure. They give one set of patients "Drug 1" and a second set of patients "Drug 2" and three hours later record the amount the blood pressure of every patient is reduced by. The results for both groups are below.

Drug 1

12 31 24 18 21 34 40 19 23 17 16

Drug 2

24 18 29 27 32 36 34 31 28 31

(a) Draw a back-to-back stem-and-leaf diagram to show these results,

(b) Comment briefly on what drug you think is more effective, giving a reason why.

9. Statistics

(a)

Drug 2		Drug 1
8	1	2 8 9 7 6
1 1 4 6 2	3	1 4
8 7 9 4	2	4 1 3
	4	0

Rough:

Notice how the Drug 2 leaves 'grow' from the centre outwards so when you do an ordered diagram the lowest values will be closest to the stems

Final:

Drug 2		**Drug 1**
8	**1**	**2 6 7 8 9**
9 8 7 4	**2**	**1 3 4**
6 4 2 1 1	**3**	**1 4**
	4	**0**

Key: 4|2 means a blood pressure reduction of 42

Take your time to make sure you have all the leaves and that they are in order with the correct leaves (also in order)

Don't forget the key!

(b)

Drug 2 is more effective at reducing blood pressure as it has a median reduction of 30 whereas Drug 1 has a median reduction of 21.

There are a few options here but it is important you give a reason to justify your decision

You could say Drug 2 as it had most values in the 30's whereas drug 1 had most values in the 10's

This option would show you understand how a stem and leaf diagram splits data into classes/groups

9. Statistics

What is grouped data and why use it?

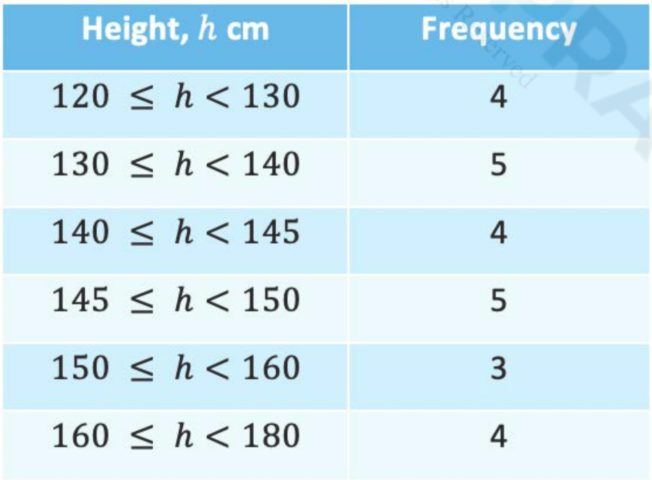
- Some data for a particular scenario can vary a lot
- For example, the heights of people, particularly if you include a mixture of children and adults
- Because data like height is also **continuous** (essentially data that can be measured) it would be difficult, even using a table, to list every height that gets recorded – also, there is little difference between someone who is 176 cm tall and someone who is 177cm tall
- So we often group data into **classes** but that leads to one important point....

What do I need to know?

- When data is grouped we lose the **raw data**
- With height data this means we might know, how many people have a height of between 150 cm and 160 cm but not the specific heights of those 10 people
- This means we cannot find the actual **mean**, **median** and **mode** from their original definitions – but we can **estimate the mean** and we can also talk about the **class the median lies in** and the **modal class**

1. **Estimating the Mean**

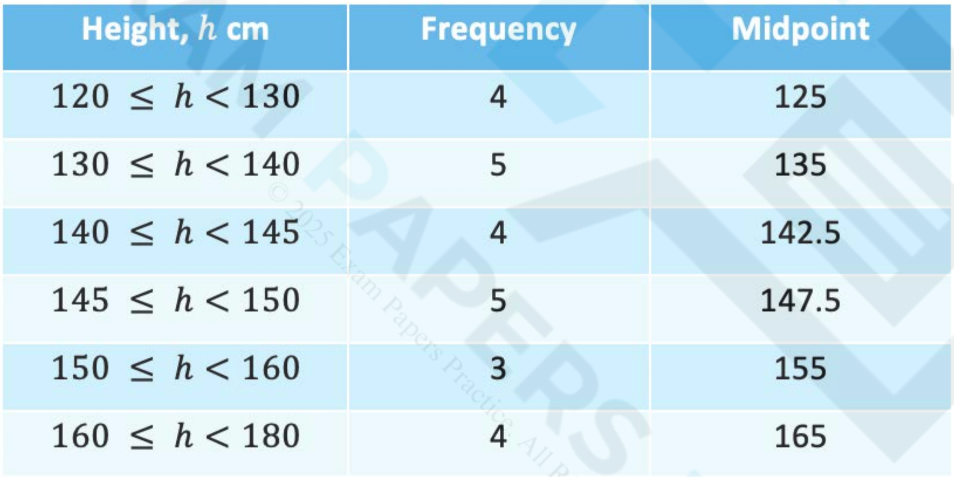
- There is one extra stage to this method compared to finding the mean from tables with discrete data – use the class **midpoints** as our data valueseg. the heights of 25 members of a youth club were recorded and the results are summarised in the table below – estimate the mean height



Height, h cm	Frequency
$120 \leq h < 130$	4
$130 \leq h < 140$	5
$140 \leq h < 145$	4
$145 \leq h < 150$	5
$150 \leq h < 160$	3
$160 \leq h < 180$	4

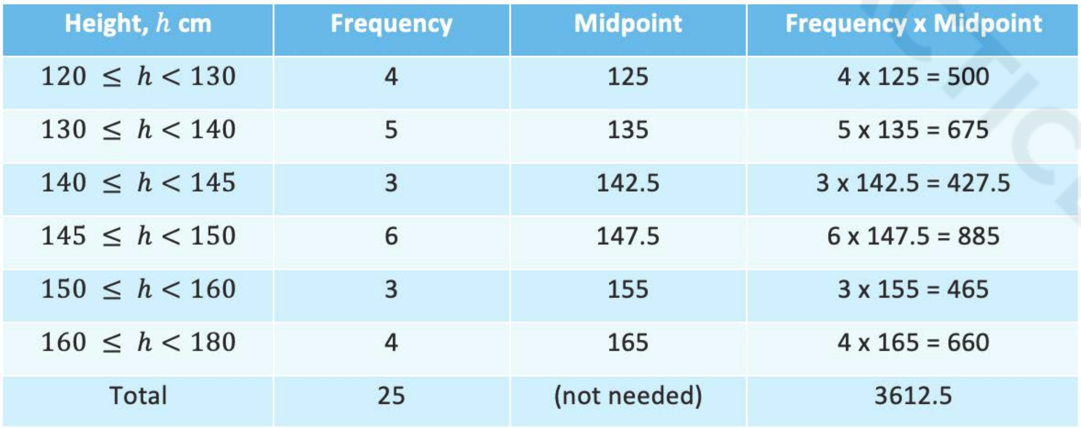
9. Statistics

- Note that the **class widths** (group sizes) are not all equal (this is not a problem so do not let it put you off) and be careful with the inequality signs: a height of exactly 130 cm would be recorded in the second row not the first
- As we don't know the original data we use the midpoint of each group – this is the height that is half way between the start and the end of the group
- Usually these are easy to 'see' but you can always work it out if in doubt (eg halfway between 140 and 145 is (140 + 145) ÷ 2 = 285 ÷ 2 = 142.5)
- We then use theses midpoints as the heights for all the people in the $120 \le h < 130$ class – so we assume that all 4 people in the class will have a height of 125 cm
- This is why the mean will be an estimate – we assume the heights in all the classes 'average out' at the midpoint height



Height, h cm	Frequency	Midpoint
$120 \le h < 130$	4	125
$130 \le h < 140$	5	135
$140 \le h < 145$	4	142.5
$145 \le h < 150$	5	147.5
$150 \le h < 160$	3	155
$160 \le h < 180$	4	165

- Now we can proceed as if it were discrete data and multiply, including a total row as well



Height, h cm	Frequency	Midpoint	Frequency x Midpoint
$120 \le h < 130$	4	125	4 x 125 = 500
$130 \le h < 140$	5	135	5 x 135 = 675
$140 \le h < 145$	3	142.5	3 x 142.5 = 427.5
$145 \le h < 150$	6	147.5	6 x 147.5 = 885
$150 \le h < 160$	3	155	3 x 155 = 465
$160 \le h < 180$	4	165	4 x 165 = 660
Total	25	(not needed)	3612.5

- And finally we can find the mean:
  Mean = 3612.5 ÷ 25 = 144.5

Mean height is 144.5 cm

2. Median

- Rather than find an actual value for the medium you could be asked to find the class in which the median lies
- The process for finding its position is the same as before so for the above example:

  Position of median = (25 + 1) ÷ 2 = 13

  The median is the 13th value

- From looking at the frequency column we can see the 13th value would fall in the 145 ≤ h < 150 class (it is the last value in this class in fact)
  So we would say the **median lies in the 145 ≤ h < 150 class**

3. Modal Class (Mode)

- Similar to finding the median we are only interested in the class the modal value lies within.
- Again using the example above we can see from the table the highest frequency is 6
- So the **modal class is** 145 ≤ h <150

Exam Tip

When presented with data in a table it may not be obvious whether you should use the technique below or the one from the previous notes (see Averages from Tables & Charts) but when you see the phrase “**estimate** the mean” you know that you are in the world of grouped (and usually continuous) data so you know to use the method below.

Worked Example

9. Statistics

1. The weights of 20 three-week-old Labrador puppies were recorded at a vet's clinic.

The results are shown in the table below.

(a) Estimate the mean weight of these puppies

(b) Write down the modal class

Weight, w kg	Frequency
$3 \leq w < 3.5$	3
$3.5 \leq w < 4$	4
$4 \leq w < 4.5$	6
$4.5 \leq w < 5$	5
$5 \leq w < 6$	2

(a)

1 - First thing to do is to add columns as necessary and find the midpoints

We can then proceed as above to complete our table

Height, h cm	Frequency	Midpoint	Frequency x Midpoint
$3 \leq w < 3.5$	3	3.25	3 x 3.25 = 9.75
$3.5 \leq w < 4$	4	3.75	4 x 3.75 = 15
$4 \leq w < 4.5$	6	4.25	6 x 4.25 = 25.5
$4.5 \leq w < 5$	5	4.75	5 x 4.75 = 23.75
$5 \leq w < 6$	2	5.5	2 x 5.5 = 11
Total	20		85

Mean = 85 ÷ 20 = 4.25

The mean weight of the puppies is 4.25 kg

(b)

3 – To find the modal class we look for the highest frequency which is 6

The modal class is $4 \leq w \leq 4.5$

9. Statistics

9.1.4 CALCULATIONS WITH THE MEAN

Solving problems involving the mean

- Because the mean has a formula it means you could be asked questions that use this formula backwards and in other ways
- Since **Mean = Total of values ÷ Number of values** then it is a formula involving 3 quantities
- Therefore, if you know 2 of these you can find the other one

What may I be asked to do?

- Typical questions ask you to work backwards from a known mean or to combine means for two data sets
- But as this is in the area of problem-solving there may be something unusual that you haven't seen before so you will need to make sure you **understand what the mean is, how it works and what it shows**

1. Working backwards

- For example, the mean of the six data values 5, 7, 2x, 6, 8 and 4x have a mean of 5.4
  Find the value of x
  - This is a matter of setting up an equation in x and solving it

$$\frac{5+7+2x+6+8+4x}{6} = 5.4$$

$$\frac{6x+26}{6} = 5.4$$

$$6x + 26 = 5.4 \times 6$$

$$6x + 26 = 32.4$$

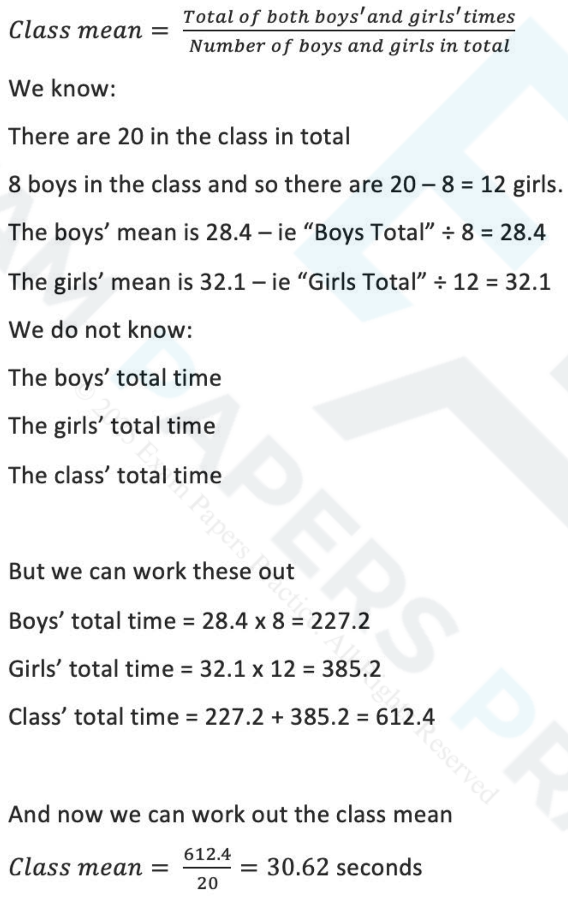
$$6x = 6.4$$

$$x = 1.07 \text{ (to two decimal places)}$$

9. Statistics

2. Combining two sets of data

- These sorts of questions are generally harder and need more thinking about
- For example, a class of 20 ran a 100m race
  The mean time for the 8 boys in the class was 28.4 seconds
  The mean time for the girls was 32.1 seconds
  Find the mean time for the whole class



$$Class\ mean = \frac{Total\ of\ both\ boys'and\ girls'times}{Number\ of\ boys\ and\ girls\ in\ total}$$

We know:

There are 20 in the class in total

8 boys in the class and so there are 20 – 8 = 12 girls.

The boys' mean is 28.4 – ie "Boys Total" ÷ 8 = 28.4

The girls' mean is 32.1 – ie "Girls Total" ÷ 12 = 32.1

We do not know:

The boys' total time

The girls' total time

The class' total time

But we can work these out

Boys' total time = 28.4 x 8 = 227.2

Girls' total time = 32.1 x 12 = 385.2

Class' total time = 227.2 + 385.2 = 612.4

And now we can work out the class mean

$$Class\ mean = \frac{612.4}{20} = 30.62\ \text{seconds}$$

Exam Tip

You have used the mean so often in mathematics that you do not normally think of it as a formula. But it is. And as with other work in using formulas you should write down the information you do know, and the information you are trying to find.

9. Statistics

Worked Example

1. A class of 24 students have a mean height of 1.54 metres.

   Two new students join the class and the mean height of the class increases to 1.56 metres.

   Given that the two new students are of equal height, find their height.

$$Mean\ of\ "original"\ class: \frac{Total\ of\ Heights}{24} = 1.56$$

$$Mean\ of\ "new"\ class: \frac{Total\ of\ Heights\ with\ 2\ new\ students}{24+2} = 1.58$$

2 – Two sets of data – before the new students and after

Writing down what we know and what we don't know – using a combination of words and formulas is fine

"Total of heights" = 1.56 x 24 = 37.44

"Total of heights with 2 new students" = 1.58 x 26 = 41.08

Two new heights combined = 41.08 – 37.44 = 3.64

The difference in these totals will be the total of the two new heights

Height of both new students = 3.64 ÷ 2 = 1.82 metres

As both new students have the same height divide by 2 to find their individual heights

9. Statistics

9.1.5 IQR & RANGE

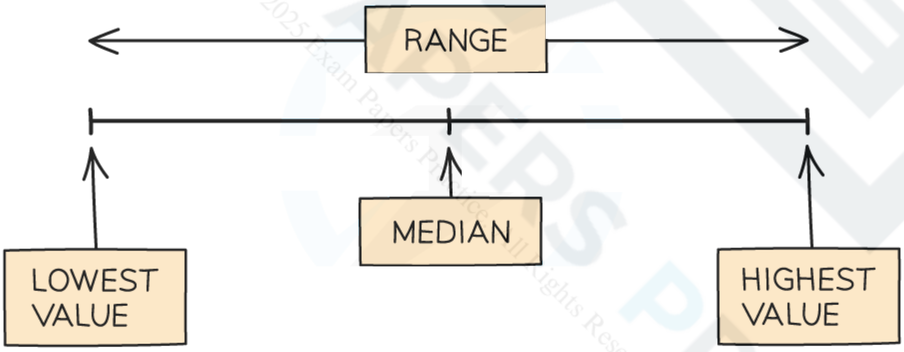
What are IQR and the range?

- The three averages (**mean**, **median** and **mode**) measure what is called **central tendency** – they all give an indication of what is typical about the data, what lies roughly in the middle, etc.
- The **range** and **inter–quartile range** (**IQR**) measure how **spread** out the data is.
- These can only be applied to numerical data, and both are easy to work out!

What do I need to know?

1. Range (Hi-Lo)

- This is the difference between the highest value in the data and the lowest value



- It measures how spread out the data is
- It is simply the **HIGHEST** of all the values **TAKE AWAY** the **LOWEST** of all the values
- For example, find the range of 14, 16, 18, 22

$$\text{Hi} = 22$$

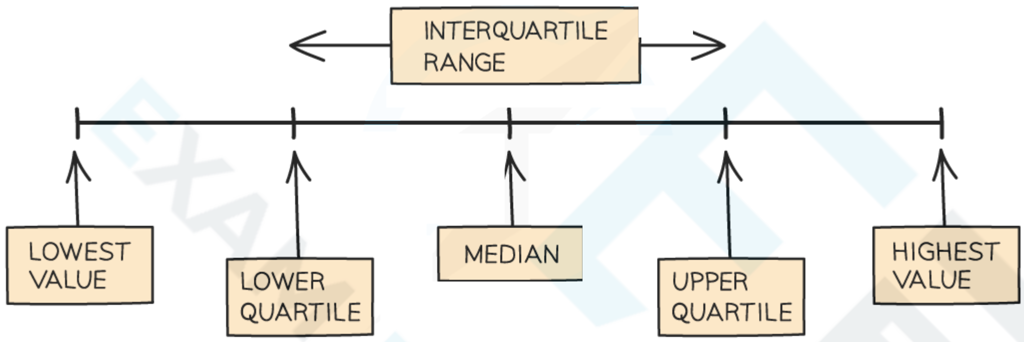
$$\text{Lo} = 14$$

$$\text{Range} = 22 - 14 = 8$$

9. Statistics

2. Inter-Quartile Range (IQR)

- This is the difference between the **upper quartile** and the **lower quartile**
- You know the median splits data into two
- Well as their name suggests, quartiles split the data into four



IQR = UQ - LQ

- The **lower quartile (LQ)** is the value **one quarter** of the way along the data
- To find its position we calculate $\frac{n+1}{4}$, where n is the number of data values
- The **upper quartile (UQ)** is the value **three quarters** of the way along the data
- To find its position we calculate $\frac{3(n+1)}{4}$, where n is the number of data values
- The inter-quartile range is then the difference between these

For example, find the inter-quartile range of the follow data …

20, 23, 32, 35, 37, 38, 43, 45, 47, 49, 52, 56, 58, 58, 59

There are 15 values ($n = 15$)

Position of LQ $= \frac{15+1}{4} = 4^{\text{th}}$

so LQ = 35

Position of UQ $= \frac{3(15+1)}{4} = 12^{\text{th}}$

so UQ = 56

IQR = 56 – 35 = 21

9. Statistics

Exam Tip

Remember with the range that you have to do a calculation (even if it is an easy subtraction). It is not good enough to write something like the range is 14 to 22.

Worked Example

1. (a) Find the range and the inter-quartile range for the following data

3.4	4.2	2.8	3.6	9.2	3.1	2.9	3.4	3.2
3.5	3.7	3.6	3.2	3.1	2.9	4.1	3.6	3.8
3.4	3.2	4.0	3.7	3.6	2.8	3.9	3.1	3.0

(b) Give a reason why, in this case, the inter-quartile range may be a better measure of how spread out the data is than the range.

(a)

2.8	2.8	2.9	2.9	3.0	3.1	3.1	3.1	3.2	3.2
3.2	3.4	3.4	3.4	3.5	3.6	3.6	3.6	3.6	3.7
3.7	3.8	3.9	4.0	4.1	4.2	9.2			

Values need to be in order, lay them out neatly especially if they do not fit on one line and double check you've not missed any out.

Range = 9.2 – 2.8 1 – Range (Hi – Lo)

Range = 6.4

There are 27 values ($n = 27$)

Position of LQ = $\frac{27+1}{4} = 7^{th}$ So LQ = 3.1

Position of UQ = $\frac{3(27+1)}{4} = 21^{st}$ So UQ = 3.7

Notice that if you have worked out the position of the LQ already, the UQ position is just the LQ Position multiplied by 3

IQR = 3.7 – 3.1 2 – IQR (UQ – LQ)

IQR = 0.6

(b)

The IQR would be a better measure of spread for these data as the highest value (9.2) is very far away from the rest of the numbers. It could be an outlier.

9. Statistics

Exam Question: Medium

Bob asked each of 40 friends how many minutes they took to get to work.

The table shows some information about his results.

Time taken (*m* minutes)	**Frequency**
$0 < m \leqslant 10$	3
$10 < m \leqslant 20$	8
$20 < m \leqslant 30$	11
$30 < m \leqslant 40$	9
$40 < m \leqslant 50$	9

Work out an estimate for the mean time taken.

Exam Question: Hard

Hertford Juniors is a basketball team.

At the end of 10 games, their mean score is 35 points per game.
At the end of 11 games, their mean score has gone down to 33 points per game.

How many points did the team score in the 11th game?

9. Statistics

9.2 BOX PLOTS

9.2.1 BOX PLOTS

What are box plots and when should they be used?

- **Box Plots** are also known as **Box–and–Whisker Diagrams** (you'll see why below)
- They are used when we are particularly interested in splitting data up into **quartiles**
- Often, data will contain extreme values – consider the cost of a car: there are far more family cars around than there are expensive sports cars
  So if you had 50 data values about the prices of cars and 49 of them were family cars but 1 was a sports car then the sports car's value does not fit in with the rest of the data
- Using quartiles and drawing a box plot allows us to split the data and so we can see what is happening at the low, middle and high points in the data

What do I need to know?

1. Drawing Box Plots

- You need to know five values to draw a box plot:

  Lowest data value

  Lower quartile

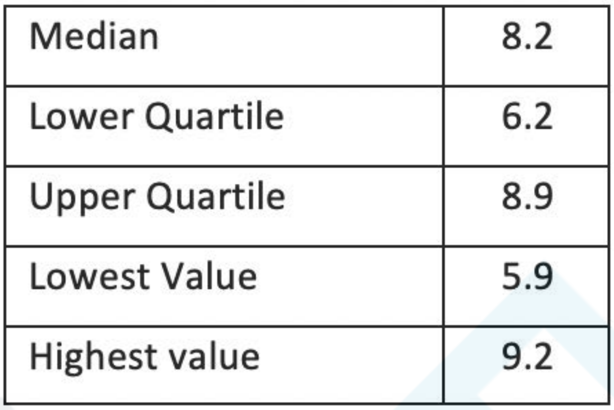
  Median

  Upper quartile

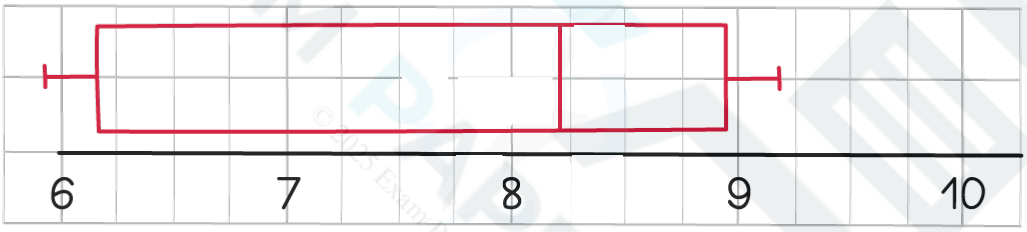
  Highest data value

- Conversely, if you are given a box plot you can work out these five values plus other useful statistics like **range** and **inter–quartile range** (IQR)
- Box plots are normally drawn on square or graph paper so you will need to be accurate

- For example, given the following information draw a box plot on the graph paper provided



Median	8.2
Lower Quartile	6.2
Upper Quartile	8.9
Lowest Value	5.9
Highest value	9.2



- Plot each point first with a small line – it doesn’t matter that they are not listed in order
- The middle three values (lower quartile, median and upper quartile) make the box
- The lowest and highest value make the ‘whiskers’

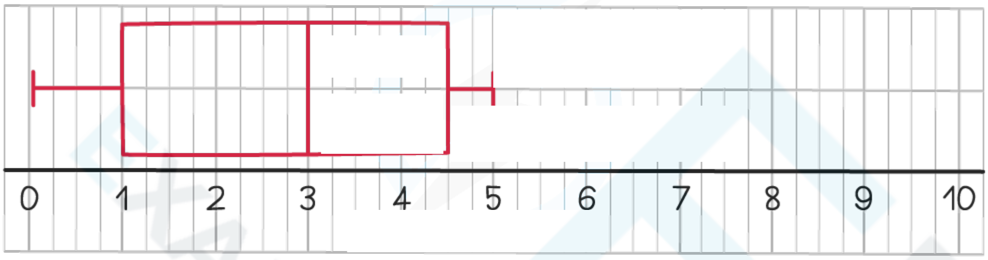
2. Comparing Box Plots

- If you are asked to **compare** box plots (for example between two classes) you should mention at least two things – one about **averages**, ie. **median**, and one about **spread**, ie. **IQR** or **range**

9. Statistics

Worked Example

1. The box plot below shows the number of goals scored per game by Albion Rovers during a football season.



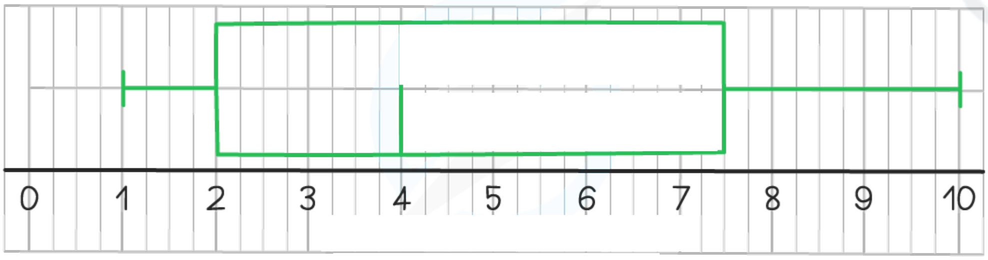
The information below shows the number of goals scored per game by Union Athletic during the same football season.

Median number of goals per game	4
Lower Quartile	2
Upper Quartile	7.5
Lowest number of goals per game	1
Highest number of goals per game	10

(a) Draw a box plot for the Union Athletic data

(b) Compare the number of goals scored per game by the two teams

(a)



9. Statistics

1 – Draw the box plot by first plotting all five points and then drawing the box around the middle three and whiskers to the outer two

(b)

The median number of goals per game is higher for Union Athletic (4) than Albion Rovers (3). This means that on average, Union Athletic scored more goals per game than Albion Rovers.

2 – This is your first comment about **averages**
Do it in two sentences – one that is just about the maths
The second sentence mentions what the data is showing

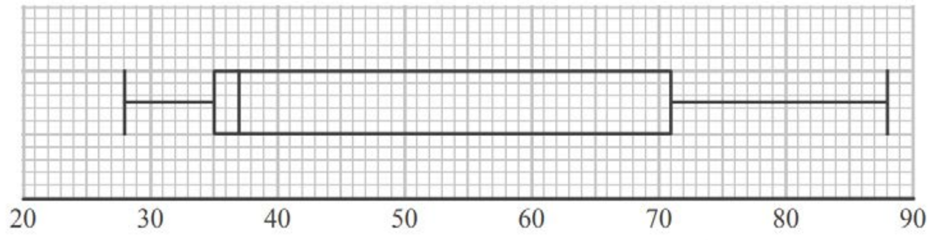
The IQR is higher for Union Athletic (5.5) than Albion Rovers (3.5).
This means that Albion Rovers were more consistent with the number of goals they scored per game.

2 – This is your second comment about **spread**.
You have a choice of range or IQR
Both are larger for Union Athletic
Again use two sentences – remember a small range/IQR means less spread out which can be a good thing if you need **consistency** (think golf!)

Exam Question: Medium

9. Statistics

The box plot shows information about the number of countries competing in each Winter Olympic Games since 1948



Number of countries in the Winter Olympics

(a) Write down the median.

(b) Work out the interquartile range.

The table below shows information about the number of countries competing in each Summer Olympic Games since 1948

	Smallest	**Lower quartile**	**Median**	**Upper quartile**	**Largest**
Number of countries	59	83	121	199	204

*(c) Compare the two distributions.

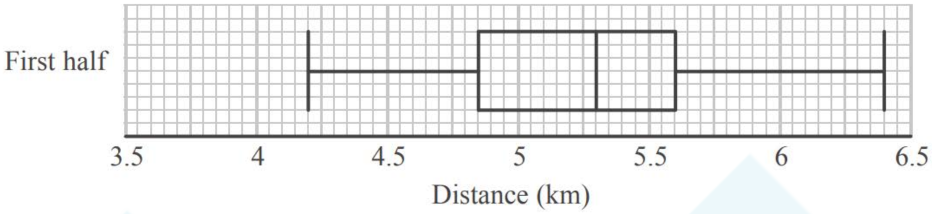
Exam Question: Hard

9. Statistics

Colin took a sample of 80 football players.

He recorded the total distance, in kilometres, each player ran in the first half of their matches on Saturday.

Colin drew this box plot for his results.



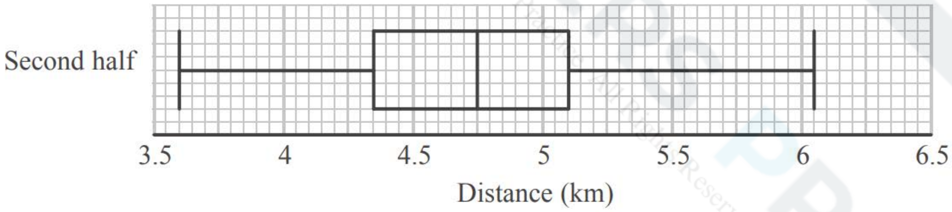
(a) Work out the interquartile range.

There were 80 players in Colin’s sample.

(b) Work out the number of players who ran a distance of more than 5.6 km.

Colin also recorded the total distance each player ran in the second half of their matches.

He drew the box plot below for this information.



(c) Compare the distribution of the distances run in the first half with the distribution of the distances run in the second half.

9. Statistics

9.4 FREQUENCY POLYGONS

9.4.1 FREQUENCY POLYGONS

What are frequency polygons?

- **Frequency polygons** are a very simple way of showing frequencies for **continuous**, **grouped** data and give a quick guide to how frequencies change from one class to the next

What do I need to know?

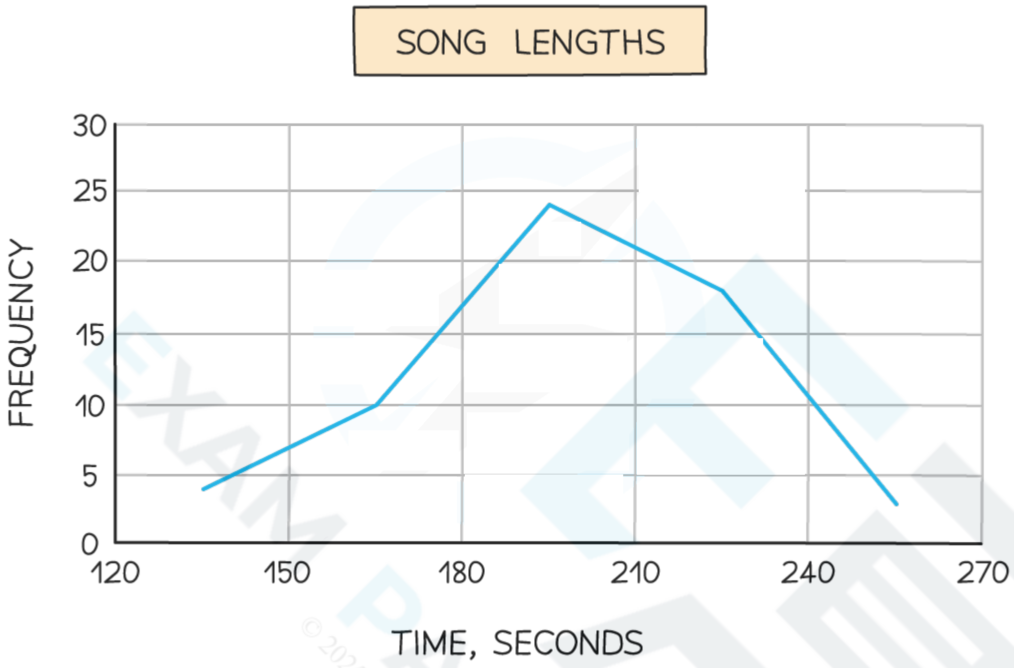
- Apart from plotting and joining up points with straight lines there are 2 rules for frequency polygons:
    - Plot points at the **MIDPOINT** of class intervals
    - Unless one of the frequencies is 0 do not join the frequency polygon to the x-axis, and do not join the first point to the last one
- The result is not actually a polygon but more of an open one that ‘floats’ in mid-air!
- You may be asked to draw a frequency polygon and/or use it to make comments and compare data

1. Drawing

- The lengths of 60 songs, in seconds, are recorded in the table below

Song length, t seconds	Frequency
$120 \leq t < 150$	4
$150 \leq t < 180$	10
$180 \leq t < 210$	24
$210 \leq t < 240$	18
$240 \leq t < 270$	3

Draw a frequency polygon for these data:



2. Using and interpreting

- What can you say about the data above, particularly by looking at the diagram only?
    - The two things to look for are **averages** and **spread**
    - The **modal class** is 180 ≤ t < 210
    - It would be acceptable to say that 195 seconds is the **modal** song length
    - The diagram (rather than the table) shows the **range** of song lengths is 255 – 135 = 120 seconds
    - If 2 frequency polygons are drawn on the same graph comparisons between the 2 sets of data can be made

Exam Tip

Jot down the midpoints next to the frequencies so you are not trying to work them out in your head while also concentrating on actually plotting the points.

9. Statistics

Worked Example

1. A local council ran a campaign to encourage households to waste less food.

To compare the impact of the campaign the council recorded the weight of food waste produced by 30 households in a week both before and after the campaign.

The results are shown in the table below.

Food Waste w kg	Frequency Before Campaign	Frequency After Campaign
$1 \leq w < 1.4$	3	5
$1.4 \leq w < 1.8$	5	8
$1.8 \leq w < 2.2$	8	14
$2.2 \leq w < 2.6$	12	3
$2.6 \leq w < 3$	2	1

(a) On the same diagram, draw two frequency polygons, one for before the council's campaign and one for after.

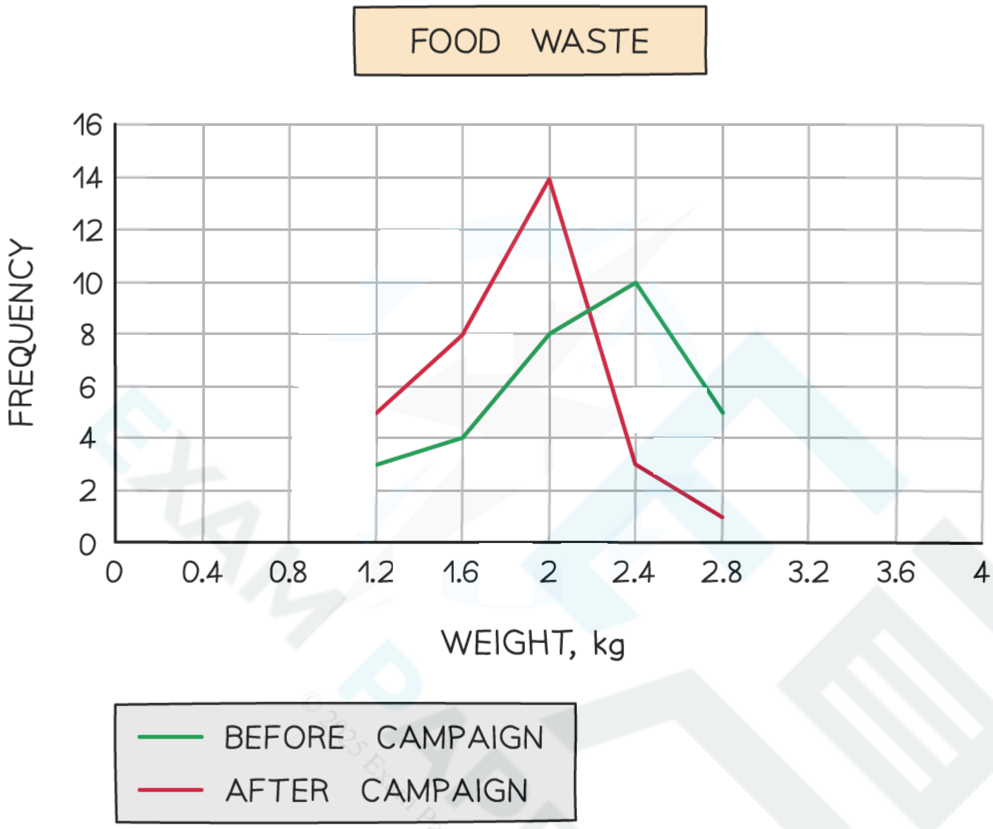
(b) Comment on whether you think the council's campaign has been successful or not and give a reason why.

(a)

Midpoints are: 1.2, 1.6, 2, 2.4 and 2.8

1 - Jot down the midpoints so you can focus on plotting points!

9. Statistics



Remember a key to show which frequency polygon is which

(a)

The council campaign has been successful as the mode amount of waste has reduced from 2.4 kg of food waste per week to 2 kg.

2 - Remember to look for averages and/or spread – in this case the range is the same

Any comment justified by the diagram/maths would be correct

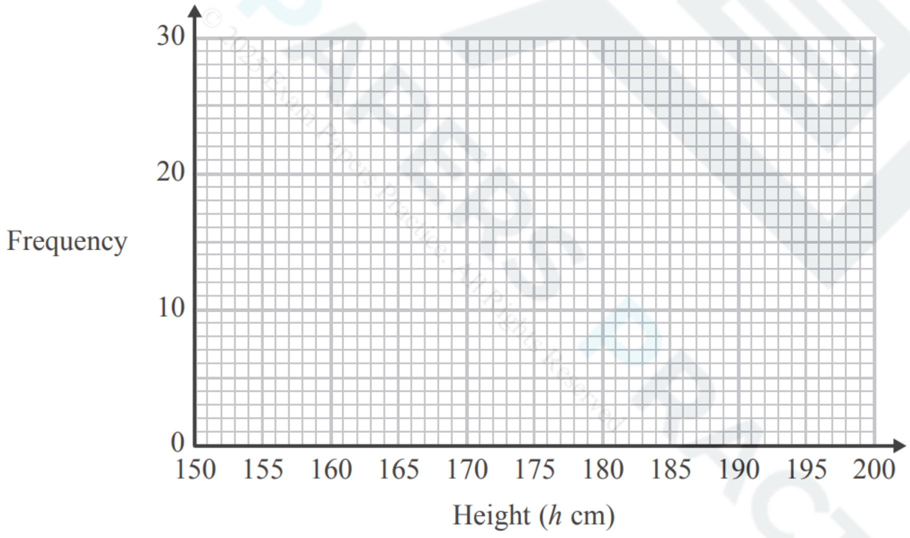
9. Statistics

Exam Question: Easy

The frequency table gives information about the heights of some people.

Height (h cm)	**Frequency**
$160 < h \leqslant 165$	2
$165 < h \leqslant 170$	5
$170 < h \leqslant 175$	10
$175 < h \leqslant 180$	21
$180 < h \leqslant 185$	16
$185 < h \leqslant 190$	4

Draw a frequency polygon for this information.



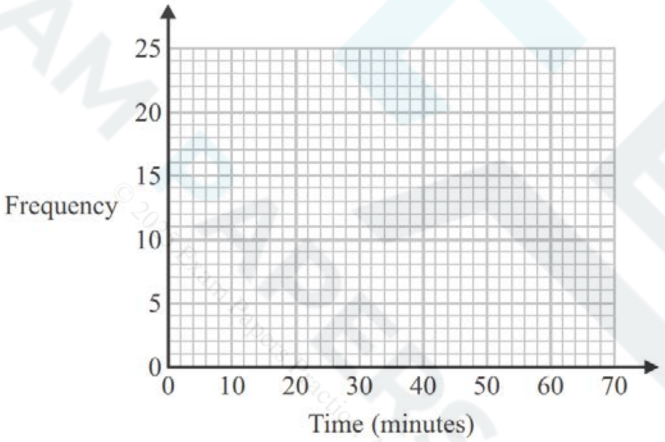
Exam Question: Medium

9. Statistics

The frequency table gives information about the times it took some office workers to get to the office one day.

Time (t minutes)	**Frequency**
$0 < t \leqslant 10$	4
$10 < t \leqslant 20$	8
$20 < t \leqslant 30$	14
$30 < t \leqslant 40$	16
$40 < t \leqslant 50$	6
$50 < t \leqslant 60$	2

(a) Draw a frequency polygon for this information.



(b) Write down the modal class interval.

One of the office workers is chosen at random.

(c) Work out the probability that this office worker took more than 40 minutes to get to the office.

9. Statistics

9.5 SCATTER GRAPHS (INC. TIME SERIES)

9.5.1 SCATTER GRAPHS

What is a scatter graph all about?

- **Scatter graphs** are used to see if there is a connection between two pieces of data
- For example a teacher may want to see if there is a link between grades in mathematics tests and grades in physics tests
- If there is a connection we can use a **line of best fit** to predict one data value (say the physics grade for example) from a known data value (maths grade)

What do I need to know?

- You need to be able to plot and interpret a scatter graph
- They are sometimes called scatter plots or scatter diagrams but these all mean the same thing
- You also need to know about correlation (positive, negative) and how to draw and use a line of best fit

9. Statistics

NEGATIVE CORRELATION

NO CORRELATION

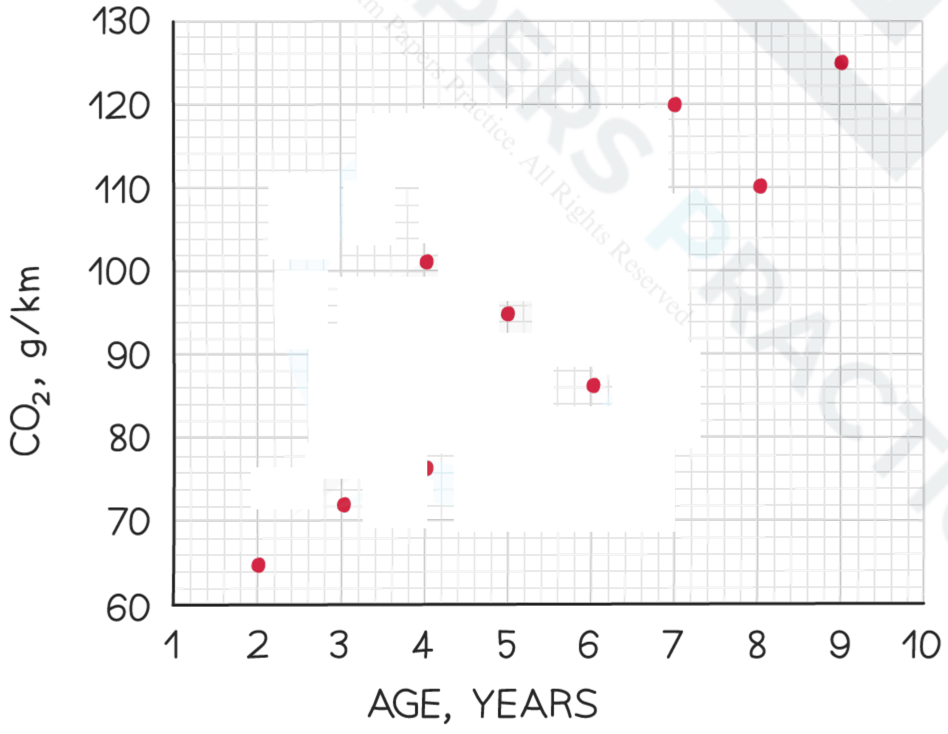
9. Statistics

1. Drawing a scatter graph

- This is simply a matter of plotting points but be very careful which way round you are plotting them particularly when the values are very similar

Age (years)	2	7	4	9	5	6	4	8	3
CO_2 (g/km)	65	120	100	125	94	86	76	110	72

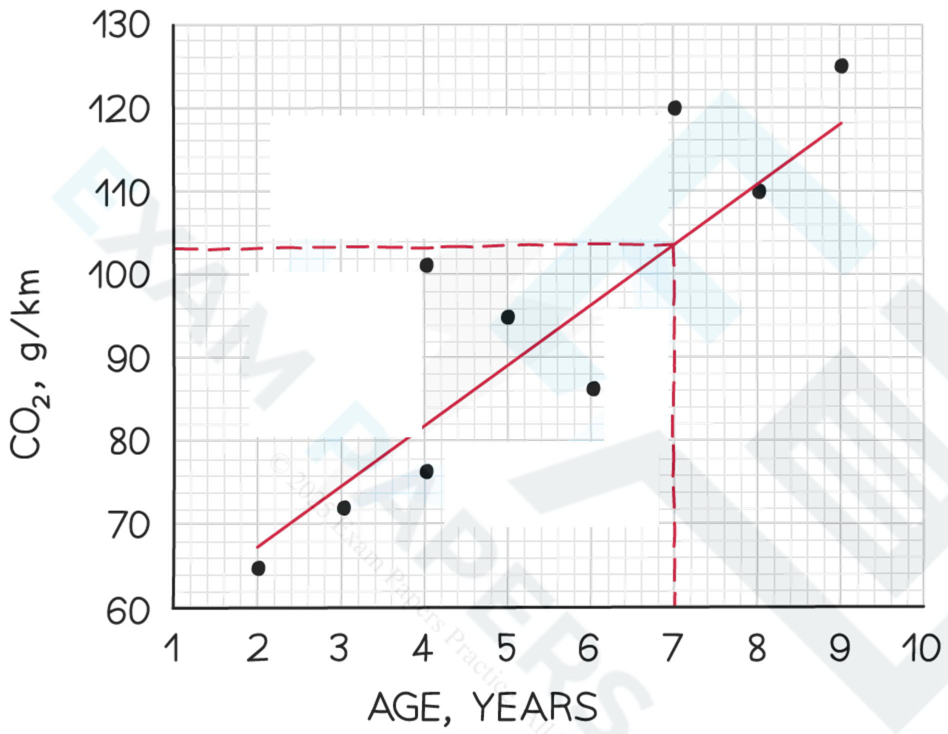
- For example, John is buying a second-hand car but is concerned about carbon dioxide (CO_2) emissions
- He collects data about the age of a car the amount of CO_2 the car emits
- Plot John's results on a scatter graph and decide if there is a connection between age and CO_2 emissions
- If there is use your graph to estimate how old a car emitting 104g/km would be



9. Statistics

2. Using a scatter graph

- There is a **positive correlation** (as one value increases, so does the other) so add in a line of best fit and use this to answer the question



- Notice that the line of best fit does not have to go through any of the data points
- From the line of best fit that a car with emissions of 104 g/km would be about 7 years old

Exam Tip

Watch out for **outliers** on scatter graphs – these are rogue results or values that do not follow the general pattern of the data. You should not consider these points when judging where to draw your line of best fit. Ignore it for this purpose. You'd usually only see one of these in a question, if any.

9. Statistics

Worked Example

1. Sophie is investigating the price of computers to see if the more they cost, the quicker they are. She tests 8 computers and runs the same program on each, measuring how many seconds each takes to complete the program. Sophie's results are shown in the table below.

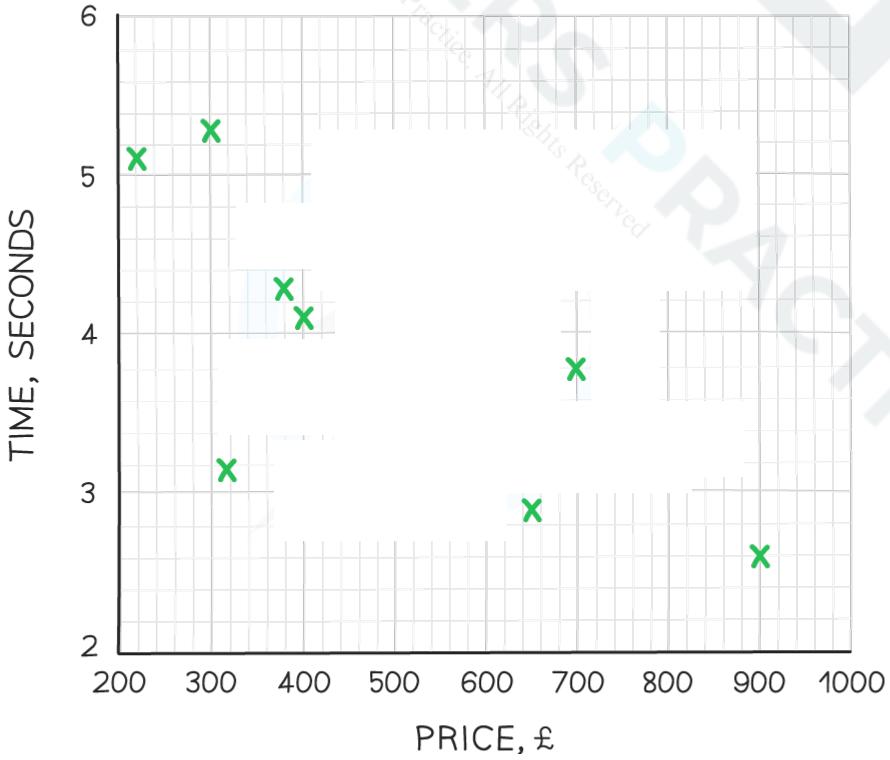
Price (£)	320	300	400	650	250	380	900	700
Time (seconds)	3.2	5.4	4.1	2.8	5.1	4.3	2.6	3.7

(a) Draw a scatter graph to show this information,

(b) Describe the correlation and explain what this means in terms of the question,

(c) Showing your method clearly estimate the price of a computer that completes the task in 3.5 seconds.

(a)



9. Statistics

1 - Plot the points carefully and accurately as to not miss any out

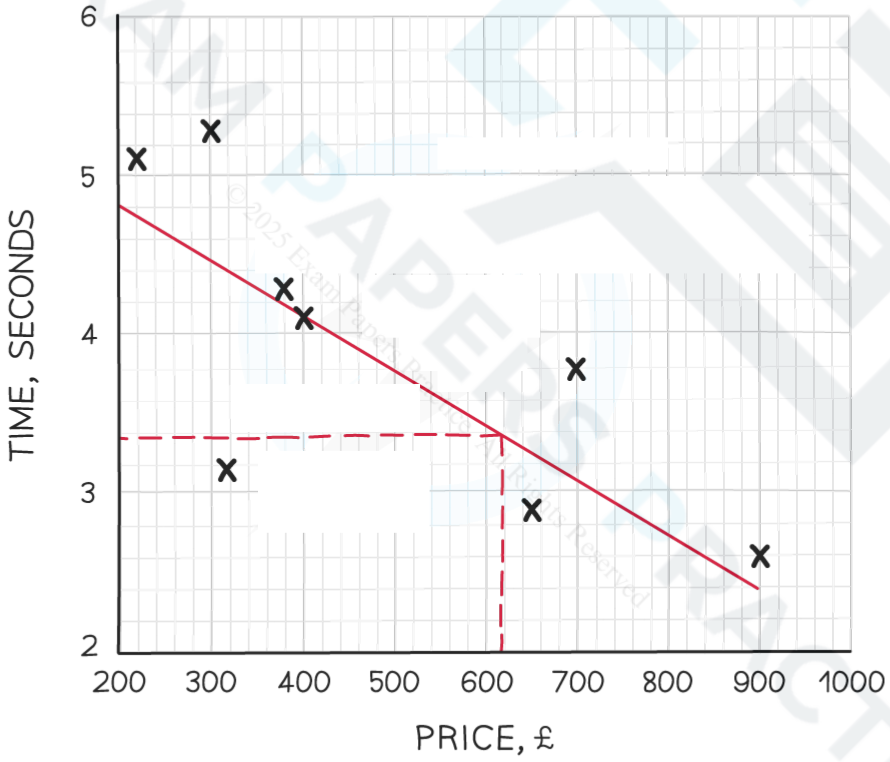
(b)

The graph shows negative correlation.

This means that the more a computer costs, the quicker it is at running the program.

2 - As you were asked to explain what the correlation means in terms of the question you need to mention the connection between cost and speed

(c)



The price of a computer taking 3.5 seconds to run the program should cost around £612.

2 - Depending on the scale/value/etc you may not be able to take an exact reading from your graph, this is fine as a range of answers will be acceptable

9. Statistics

9.5.2 TIME SERIES GRAPHS

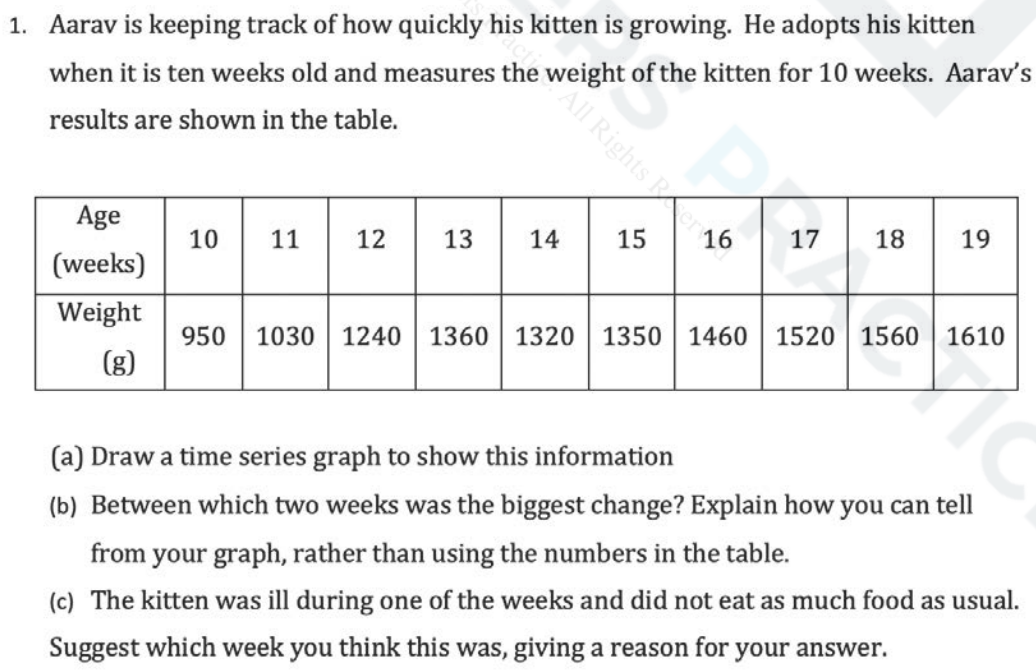
What is meant by time series, these graphs look familiar?

- **Time Series** graphs are often called **line graphs**
- Line graph is a less specific term but they are most commonly used to show **changes over time** – for example how a plant grows over a period of weeks
- Despite the technical-sounding name, Time Series graphs are some of the easiest you will need to draw and use as part of your GCSE Mathematics course

What do I need to know?

- You need to be able to draw and interpret Time Series graphs
- Key points to look for from a graph:
  - Any horizontal line shows no change in value
  - The biggest positive change will come from the steepest line (highest gradient) that goes “uphill” in the left-to-right sense
  - The biggest negative change will come from the steepest line that goes downhill

Worked Example



1. Aarav is keeping track of how quickly his kitten is growing. He adopts his kitten when it is ten weeks old and measures the weight of the kitten for 10 weeks. Aarav’s results are shown in the table.

Age (weeks)	10	11	12	13	14	15	16	17	18	19
Weight (g)	950	1030	1240	1360	1320	1350	1460	1520	1560	1610

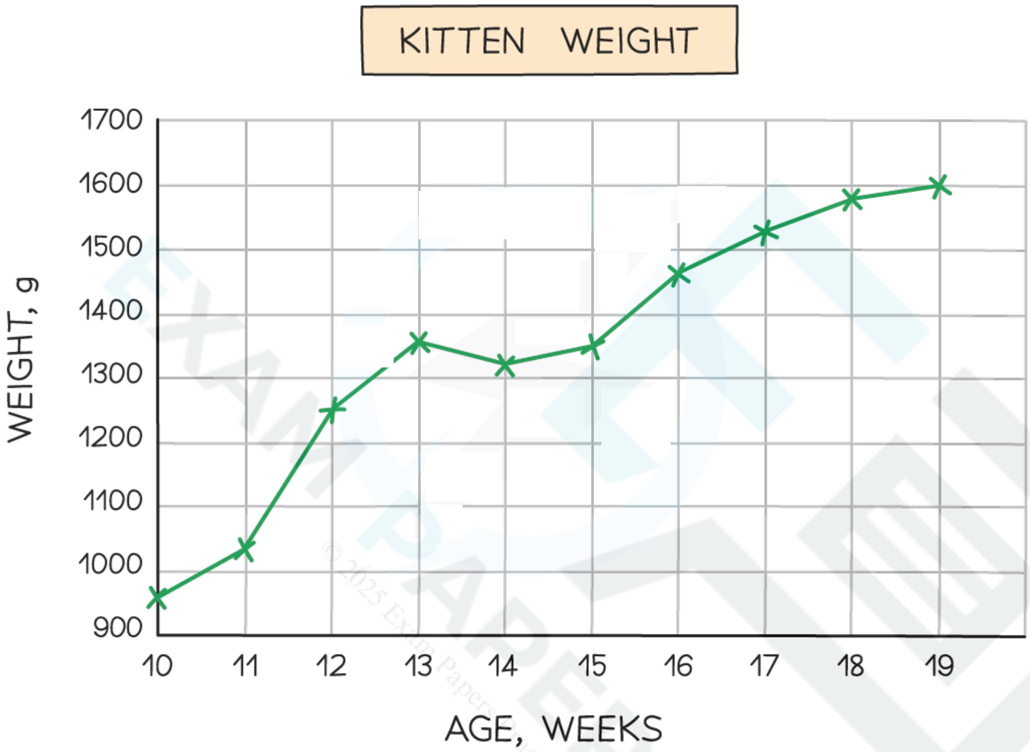
(a) Draw a time series graph to show this information

(b) Between which two weeks was the biggest change? Explain how you can tell from your graph, rather than using the numbers in the table.

(c) The kitten was ill during one of the weeks and did not eat as much food as usual. Suggest which week you think this was, giving a reason for your answer.

9. Statistics

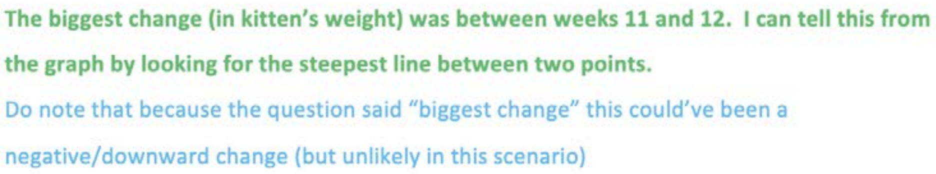
(a)



Do show that you've plotted the points and avoid covering them up when you draw the lines in

9. Statistics

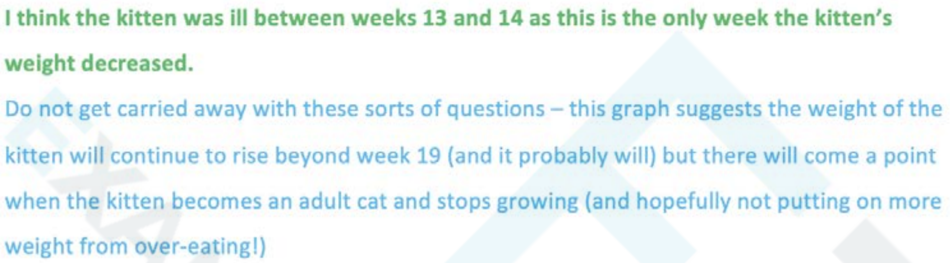
(b)



The biggest change (in kitten's weight) was between weeks 11 and 12. I can tell this from the graph by looking for the steepest line between two points.

Do note that because the question said "biggest change" this could've been a negative/downward change (but unlikely in this scenario)

(c)



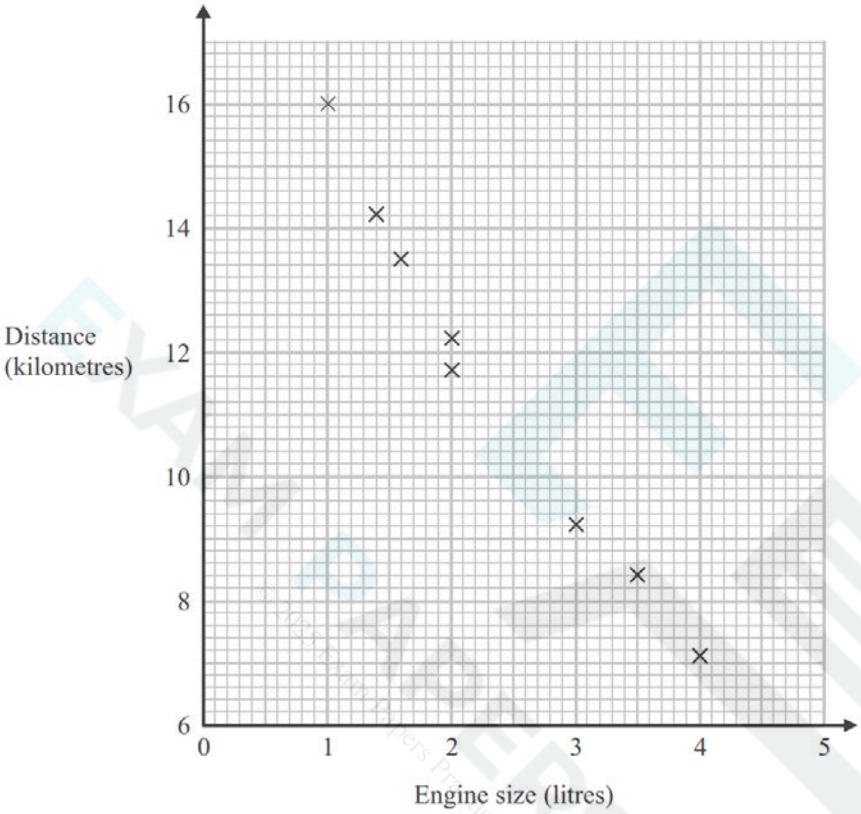
I think the kitten was ill between weeks 13 and 14 as this is the only week the kitten's weight decreased.

Do not get carried away with these sorts of questions – this graph suggests the weight of the kitten will continue to rise beyond week 19 (and it probably will) but there will come a point when the kitten becomes an adult cat and stops growing (and hopefully not putting on more weight from over-eating!)

Exam Question: Easy

9. Statistics

The scatter graph shows some information about 8 cars.
For each car it shows the engine size, in litres, and the distance, in kilometres, the car travels on one litre of petrol.



(a) What type of correlation does the scatter graph show?

A different car of the same type has an engine size of 2.5 litres.

(b) Estimate the distance travelled on one litre of petrol by this car.

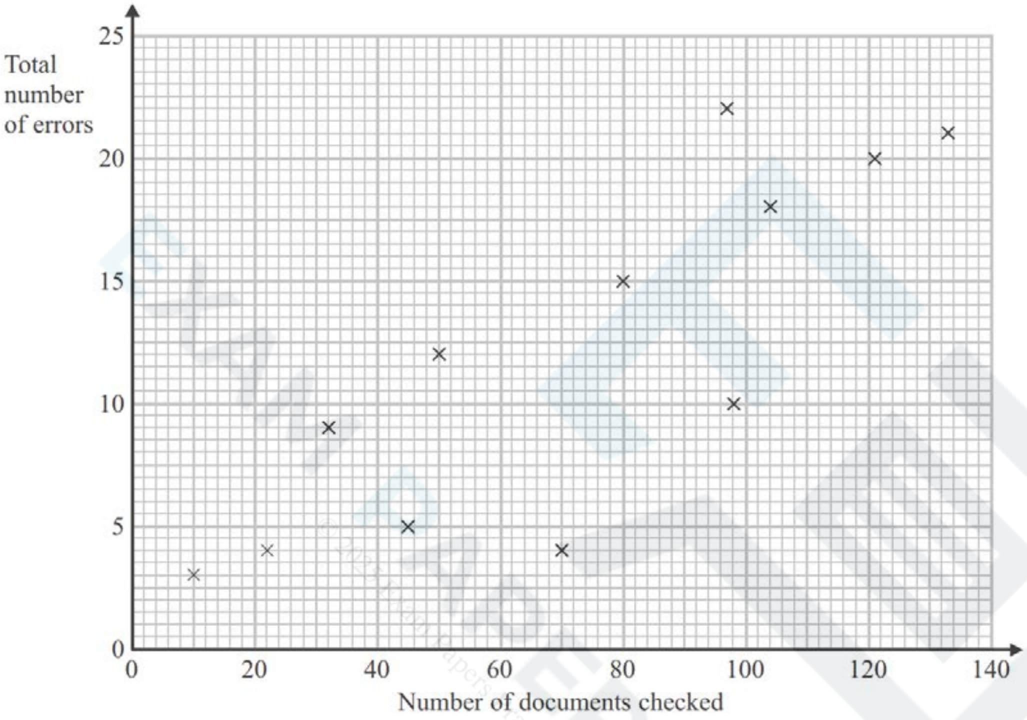
Exam Question: Medium

9. Statistics

A publisher checks documents for errors.

He records the number of documents that are checked each day.
He also records the total number of errors in the documents each day.

The scatter graph shows this information.



On another day 90 documents are checked.
There is a total of 17 errors.

(a) Show this information on the scatter graph.

(b) Describe the correlation between the number of documents checked and the total number of errors.

One day 110 documents are checked.

(c) Estimate the total number of errors in these documents.

9. Statistics

9.6 HISTOGRAMS

9.6.1 HISTOGRAMS

Aren't histograms just really hard bar charts?

- No! There are many mathematical differences that you should be aware of but the key difference between a bar chart and a **histogram** is that with a histogram it is the **area of the bars** that determine the frequency, on a bar chart it's the height (or length) of them
- Digging deeper, histograms are used for **continuous** data (bar charts for discrete) and are particularly useful when data has been **grouped** into **different** sized **classes**
- But the key thing to getting started is that it is the area of the bars that tell us what is happening with the data
- This means, unlike any other graph or chart you have come across, it is very difficult to tell anything from simply looking at a histogram, you have to drill down into the numbers and detail

What do I need to know?

- You need to know how to **draw** a histogram (most questions will get you to finish an incomplete histogram)
- When drawing histograms we will need to use **frequency density** (fd):

$$\text{frequency density} = \frac{frequency}{class\ width}$$

- You'll also need to be able to work backwards from a given histogram to find frequencies and **estimate** the mean

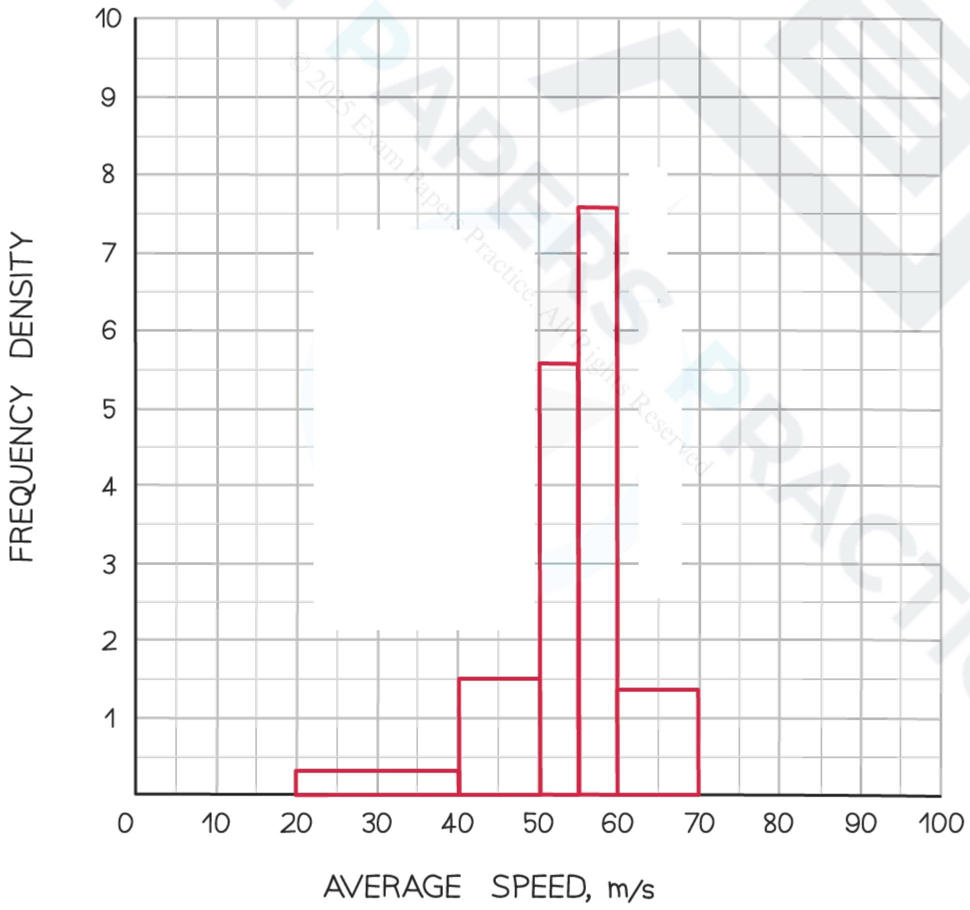
1. Drawing a histogram

- From a given table you need to work out the frequency density for each class
- Then you can plot the data against frequency density with frequency density on the -axis
- For example, plot a histogram for the following data regarding the average speed travelled by trains

Average speed, s m/s	Frequency	Class width	Frequency Density
$20 \le s < 40$	5	40 – 20 = 20	5 ÷ 20 = 0.25
$40 \le s < 50$	15	50 – 40 = 10	15 ÷ 10 = 1.5
$50 \le s < 55$	28	55 – 50 = 5	28 ÷ 5 = 5.6
$55 \le s < 60$	38	60 – 55 = 5	38 ÷ 5 = 7.6
$60 \le s < 70$	14	70 – 60 = 10	14 ÷ 10 = 1.4

- Note that the class width column isn't essential but it is crucial you show the frequency densities

- Now we draw bars (touching, as the data (speed) is continuous) with widths of the class intervals and heights of the frequency densities



9. Statistics

2. Interpreting histograms

- We shall still use the example above here but shall pretend we never had the table of data and were only given the finished histogram
- To **estimate** the **mean**:
  - You need to know the total frequency and what all the data values add up to
  - You can't find the exact total of the data values as this is grouped data but we can estimate it using **midpoints**

Since:

$$\text{frequency density} = \frac{frequency}{class\ width}$$

then it is easy to rearrange to see that:

frequency = frequency density × class width

$20 \leq s < 40$: frequency = 0.25 x 20 = 5 midpoint = 30

$40 \leq s < 50$: frequency = 1.5 x 10 = 15 midpoint = 45

$50 \leq s < 55$: frequency = 5.6 x 5 = 28 midpoint = 52.5

$55 \leq s < 60$: frequency = 7.6 x 5 = 38 midpoint = 57.5

$60 \leq s < 70$: frequency = 1.4 x 10 = 14 midpoint = 65

Total of frequencies = 5 + 15 + 28 + 38 + 14 = 100

- You can draw all of the above in a table if you wish
- Now you can total up (an estimate of) the data values and find the mean:

Total = 5 x 30 + 15 x 45 + 28 x 52.5 + 38 x 57.5 + 14 x 65 = 5390

(Be careful if you type all this into your calculator in one go!)

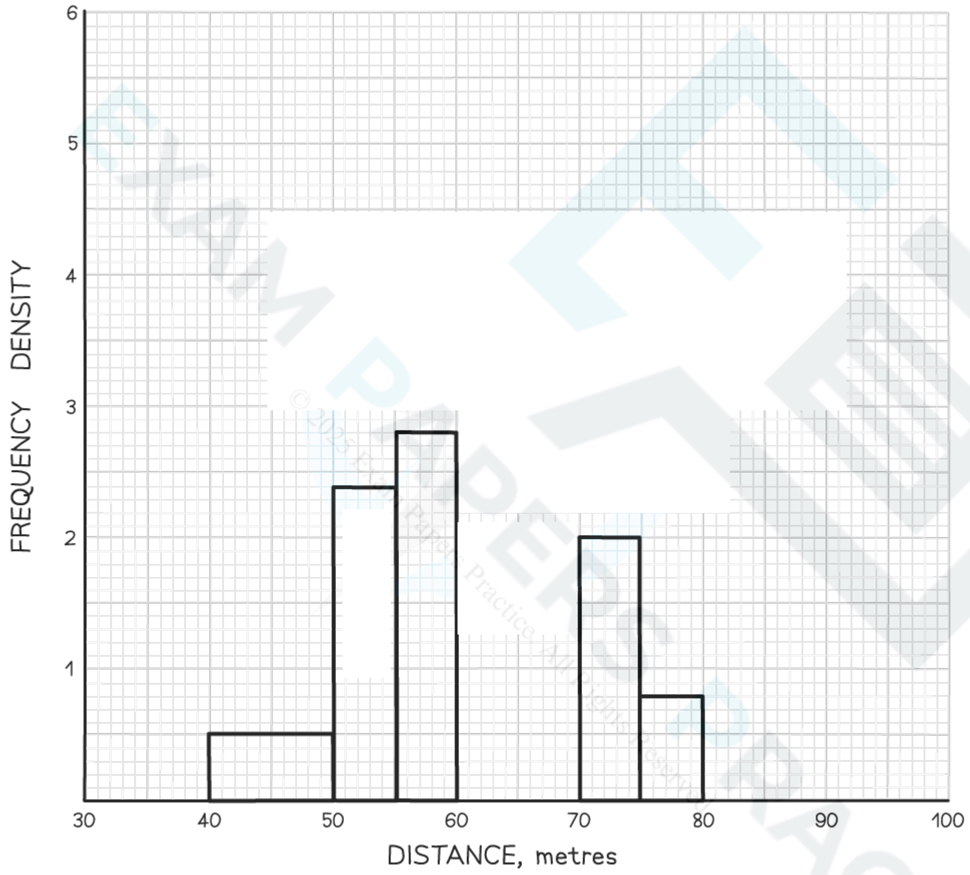
Estimate of Mean = 5390 ÷ 100 = 53.9

Exam Tip

Always work out and write down the frequency densities. Many students lose marks in exams as they go straight to the graph when asked to draw a histogram and they mess up the calculations. Method marks are available for showing you know to use frequency density rather than frequency.

Worked Example

1. A histogram is shown below representing the distances achieved by some athletes throwing a javelin.



9. Statistics

(a) There are two classes missing from the histogram. These are:

Distance, x m	Frequency
$60 \le x < 70$	8
$80 \le x < 100$	2

Add these to the histogram

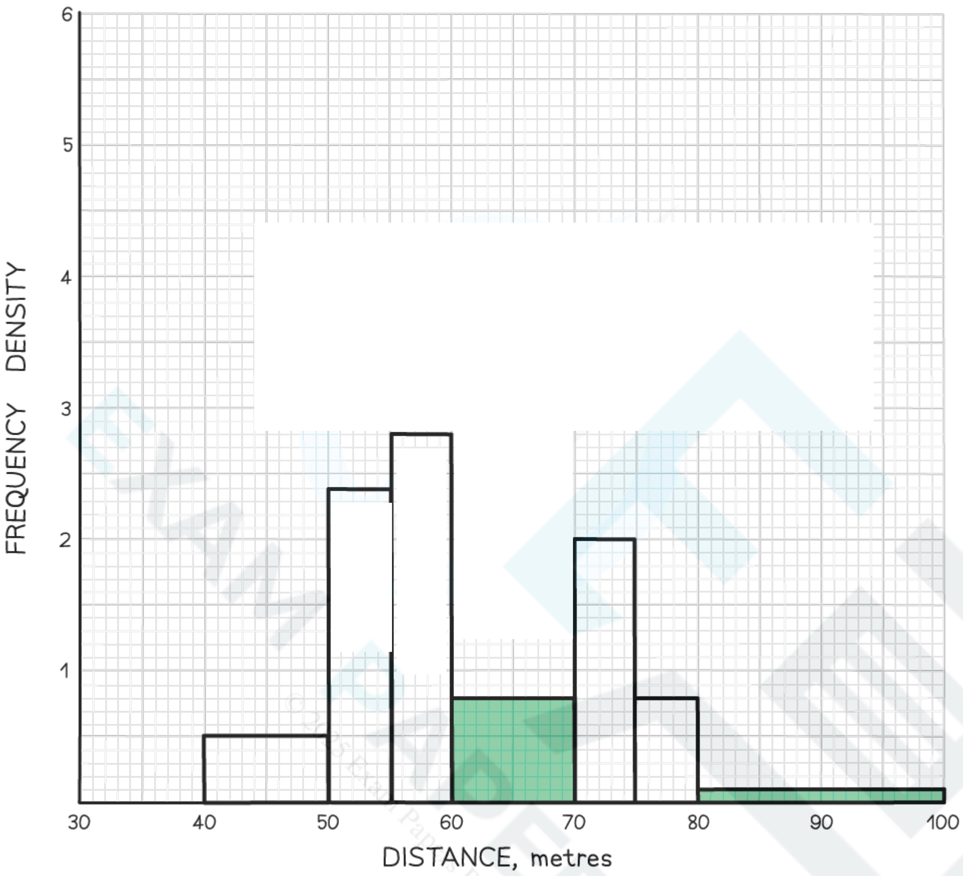
(b) Approximately how many athletes threw the javelin a distance over 75 metres.

Distance, x m	Frequency	Frequency Density
$60 \le x < 70$	8	$8 \div 10 = 0.8$
$80 \le x < 100$	2	$2 \div 20 = 0.1$

(a)

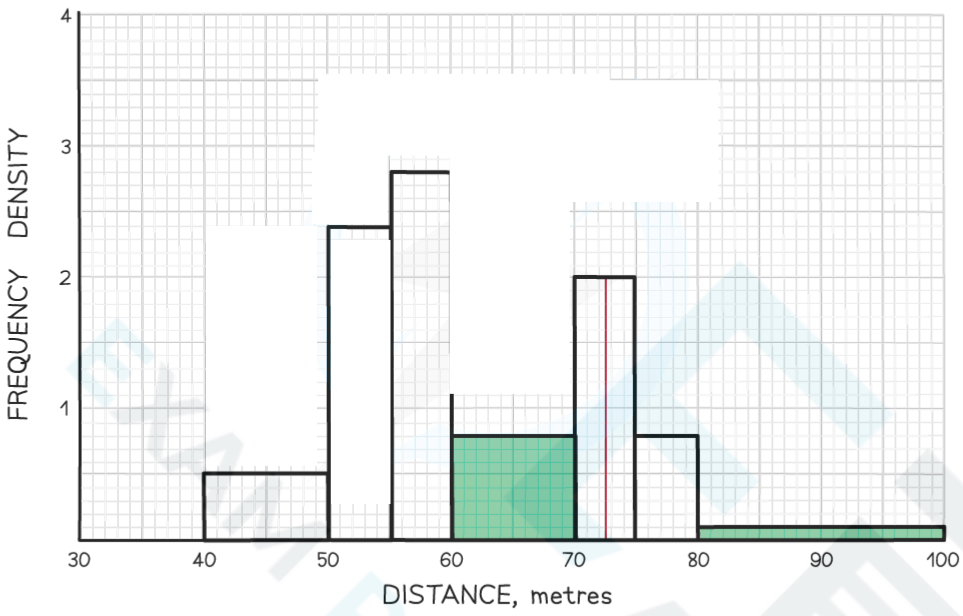
1 - Remember to clearly show you've worked out the missing frequency densities

9. Statistics



9. Statistics

(b)



2 - You need to work out the frequency of the bars that are at 75m or greater

$70 \leq x < 75$: frequency = 5 x 2 = 10

$75 \leq x < 80$: frequency = 5 x 0.8 = 4

However 75 falls in the middle of one of the groups – so you would take half of that frequency

½ x 10 + 4 + 2 = 11

We know already that the frequency for $80 \leq x < 100$ is 2

Number of athletes throwing over 75 m is 11

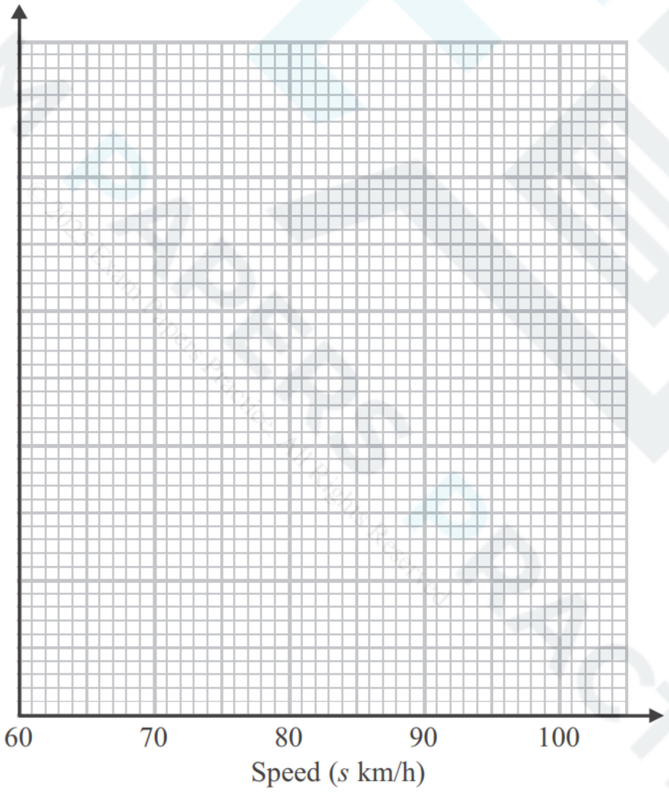
Exam Question: Medium

9. Statistics

The table gives some information about the speeds, in km/h, of 100 cars.

Speed (s km/h)	**Frequency**
$60 < s \leqslant 65$	15
$65 < s \leqslant 70$	25
$70 < s \leqslant 80$	36
$80 < s \leqslant 100$	24

(a) On the grid, draw a histogram for the information in the table.



(b) Work out an estimate for the number of cars with a speed of more than 85 km/h.

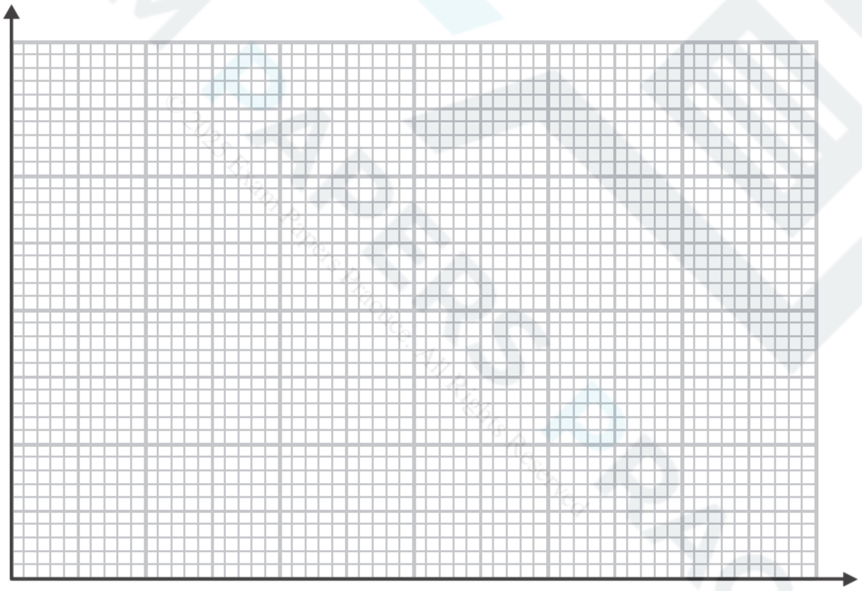
9. Statistics

Exam Question: Hard

The table gives information about the heights, h metres, of trees in a wood.

Height (h metres)	**Frequency**
$0 < h \leqslant 2$	7
$2 < h \leqslant 4$	14
$4 < h \leqslant 8$	18
$8 < h \leqslant 16$	24
$16 < h \leqslant 20$	10

Draw a histogram to show this information.



9. Statistics

9.7 CUMULATIVE FREQUENCY GRAPHS

9.7.1 CUMULATIVE FREQUENCY

What does cumulative frequency mean?

- **Cumulative** basically means “adding up as you go along”
- So cumulative frequency means all of the frequencies for the different classes, totalled
- This allows us to draw a cumulative frequency graph, from which we can find useful information such as the **median** and **quartiles**

What do I need to know?

- You will either be asked to **draw** a cumulative frequency graph or **analyse** one
- It is also possible to draw a box plot from a cumulative frequency graph and you could be asked to **compare** the data with another set of data (see Box Plots)

1. Drawing a cumulative frequency graph

- This is best explained with an example
- For example, the times taken to complete a short general knowledge quiz taken by 50 students are shown in the table below:

Time taken, s seconds	Frequency
$25 \leq s < 30$	3
$30 \leq s < 35$	8
$35 \leq s < 40$	17
$40 \leq s < 45$	12
$45 \leq s < 50$	7
$50 \leq s < 55$	3
Total	50

- When asked to find the cumulative frequencies you may be given another table – or the table above will have an extra column on it
- You should be aware of both possibilities although the process is the same

Version 1 – an extra column in the original table:

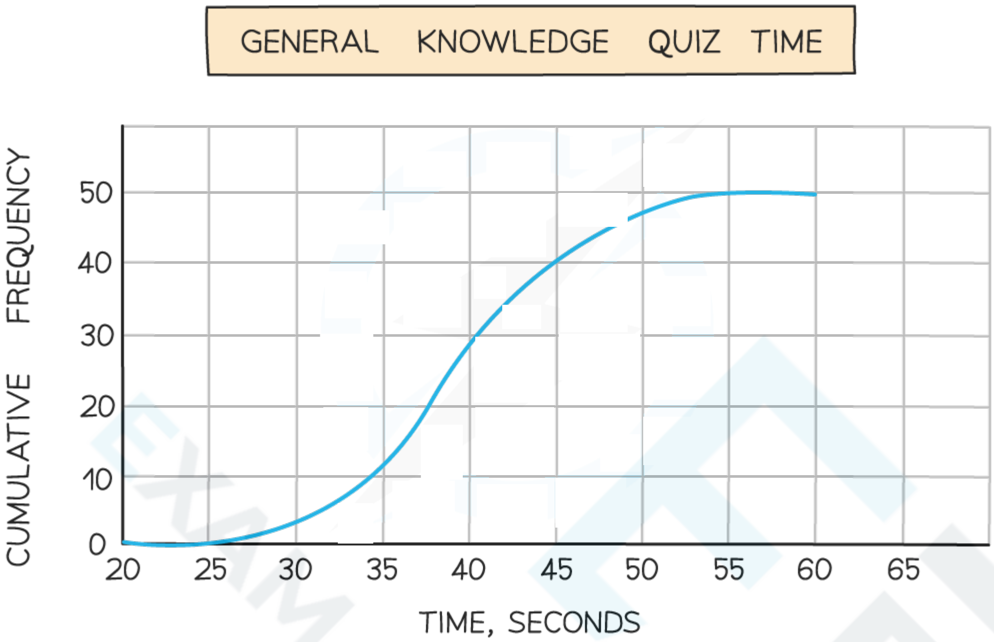
Time taken, s seconds	Frequency	Cumulative Frequency
$25 \leq s < 30$	3	3
$30 \leq s < 35$	8	3 + 8 = 11
$35 \leq s < 40$	17	11 + 17 = 28
$40 \leq s < 45$	12	28 + 12 = 40
$45 \leq s < 50$	7	40 + 7 = 47
$50 \leq s < 55$	3	47 + 3 = 50
Total	50	(not needed)

Version 2 – a new table with the class intervals all starting at the same value – the lowest in the data – here this is 25, but will often be 0:

Time taken, s seconds	Cumulative Frequency
$25 \leq s < 30$	3
$25 \leq s < 35$	3 + 8 = 11
$25 \leq s < 40$	3 + 8 + 17 = 28
$25 \leq s < 45$	3 + 8 + 17 + 12 = 40
$25 \leq s < 50$	3 + 8 + 17 + 12 + 7 = 47
$25 \leq s < 55$	3 + 8 + 17 + 12 + 7 + 3 = 50

- Note the cumulative frequencies are found in exactly the same way whichever version you may come across but we've shown two different ways of thinking about it
- Now you have your cumulative frequencies you can draw a graph from it
- The key here is that cumulative frequencies are plotted against the **end** (**upper bound**) of the class interval
- This is because you can't say, for example, you have covered all 11 students (2nd row of table) until you have reached 35 seconds
- After 30 seconds (start) only 3 people had finished the quiz and you cannot tell how many had finished by 32.5 seconds (midpoint)

9. Statistics

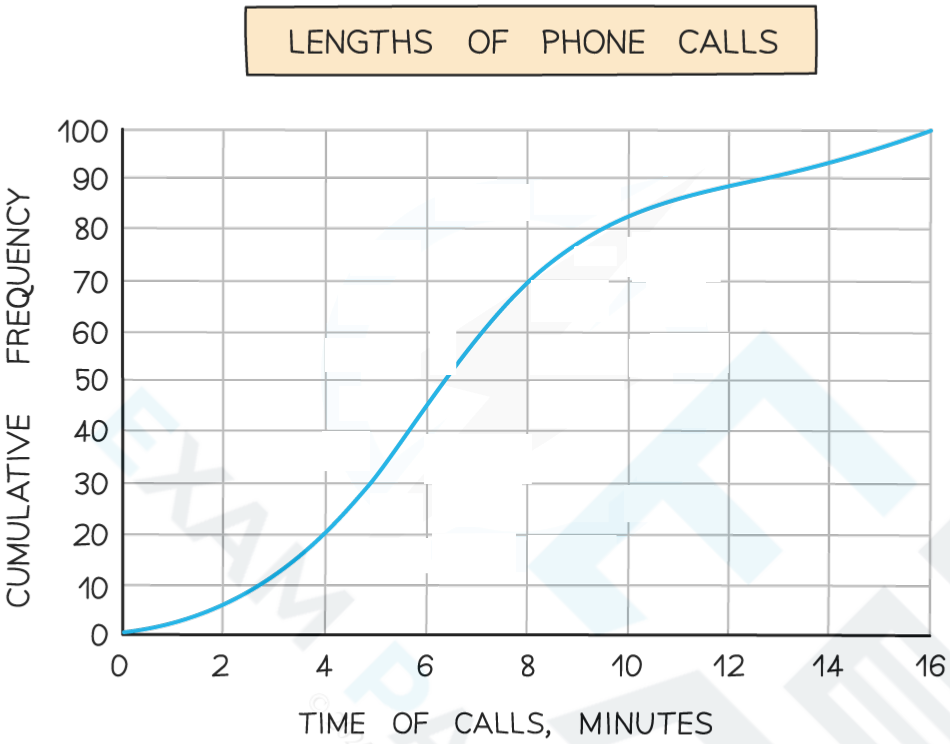


Worked Example

1. A company is investigating the length of telephone calls customers make to its help centre. The company randomly selects 100 phone calls from a particular day and the results are displayed in the cumulative frequency graph below.

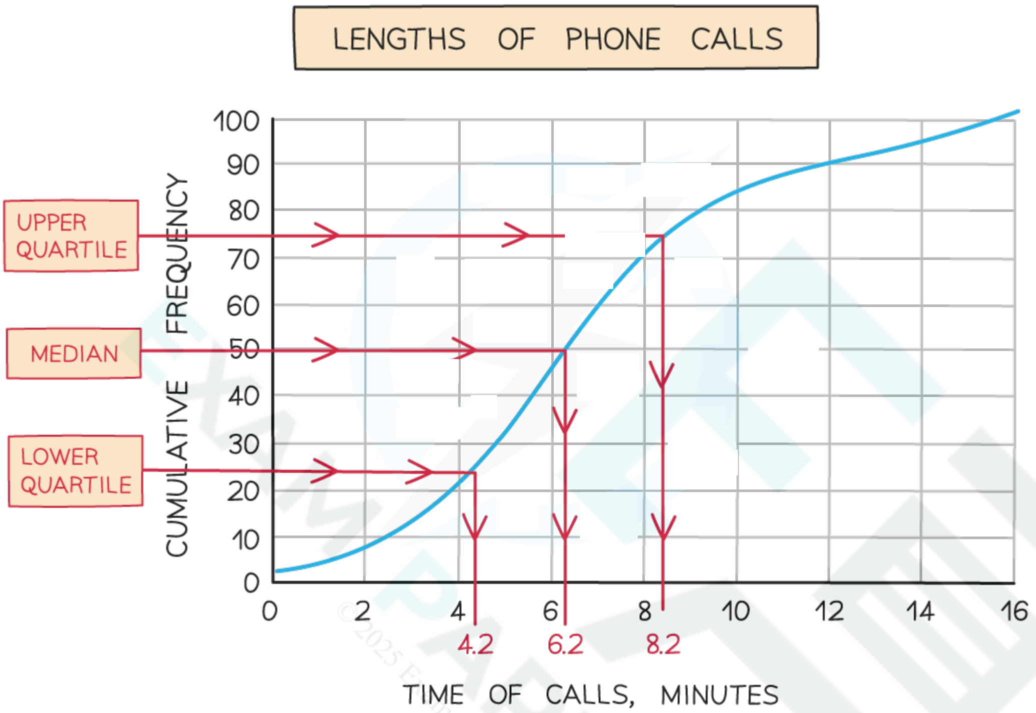
   (a) Estimate the median, lower quartile and upper quartile

   (b) The company is thinking of putting an upper limit of 12 minutes on a call – how many of the 100 phone calls would have been beyond this limit?


LENGTHS OF PHONE CALLS
CUMULATIVE FREQUENCY
100
90
80
70
60
50
40
30
20
10
0
0
2
4
6
8
10
12
14
16
TIME OF CALLS, MINUTES

9. Statistics

(a)



There are 100 pieces of data so to find the median you need the half of this, 50

Starting at 50 on the cumulative frequency axes draw a line across to the graph and down to the time axis and take a reading

Similarly for the lower quartile draw a line from 100 ÷ 4 = 25 and for the upper quartile draw a line across from 100 ÷ 4 x 3 = 75

Median = 6.2 minutes (6m 12s)

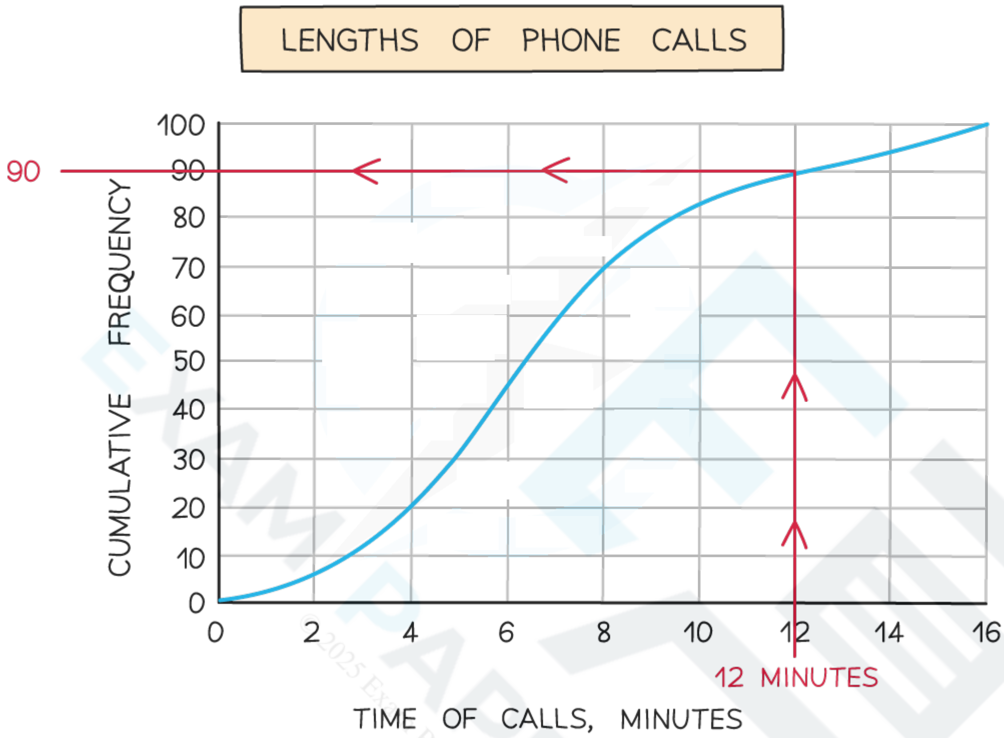
Lower Quartile = 4.2 minutes (4m 12s)

Upper Quartile = 8.2 minutes (8m 12s)

There's no need to convert to minutes and seconds unless asked to by the question

9. Statistics

(b)



Draw a line up from the 12 minute mark on the time axis

Draw a line across and take a reading – in this case 90

100 – 90 = 10

10 calls, out of the 100 the company used, would have been longer than the upper limit of 12 minutes

As the company are cutting off calls **greater** than 12 minutes you need to work out how many calls took longer than this

Exam Question: Easy

9. Statistics

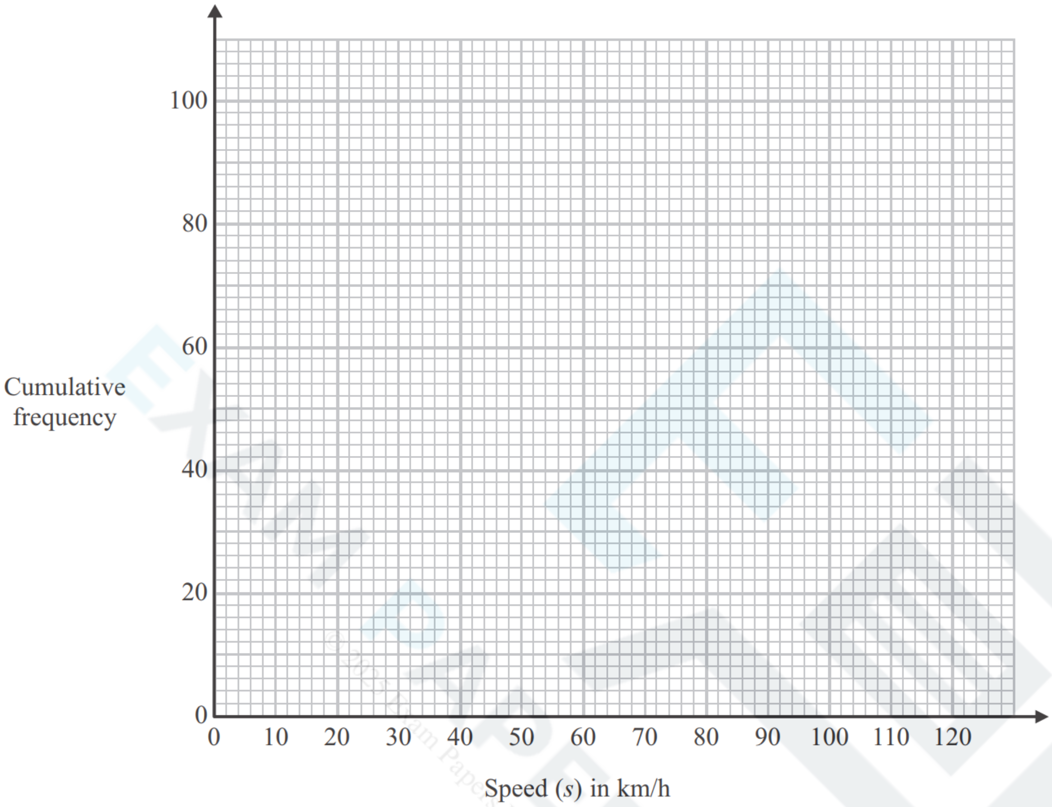
The table shows information about the speeds of 100 lorries.

Speed (s) in km/h	**Frequency**
$0 < s \leqslant 20$	2
$20 < s \leqslant 40$	9
$40 < s \leqslant 60$	23
$60 < s \leqslant 80$	31
$80 < s \leqslant 100$	27
$100 < s \leqslant 120$	8

(a) Complete the cumulative frequency table for this information.

Speed (s) in km/h	**Cumulative frequency**
$0 < s \leqslant 20$	2
$0 < s \leqslant 40$	
$0 < s \leqslant 60$	
$0 < s \leqslant 80$	
$0 < s \leqslant 100$	
$0 < s \leqslant 120$	

(b) On the grid, draw a cumulative frequency graph for your table.



(c) Find an estimate for the number of lorries with a speed of more than 90 km/h.

Exam Question: Medium

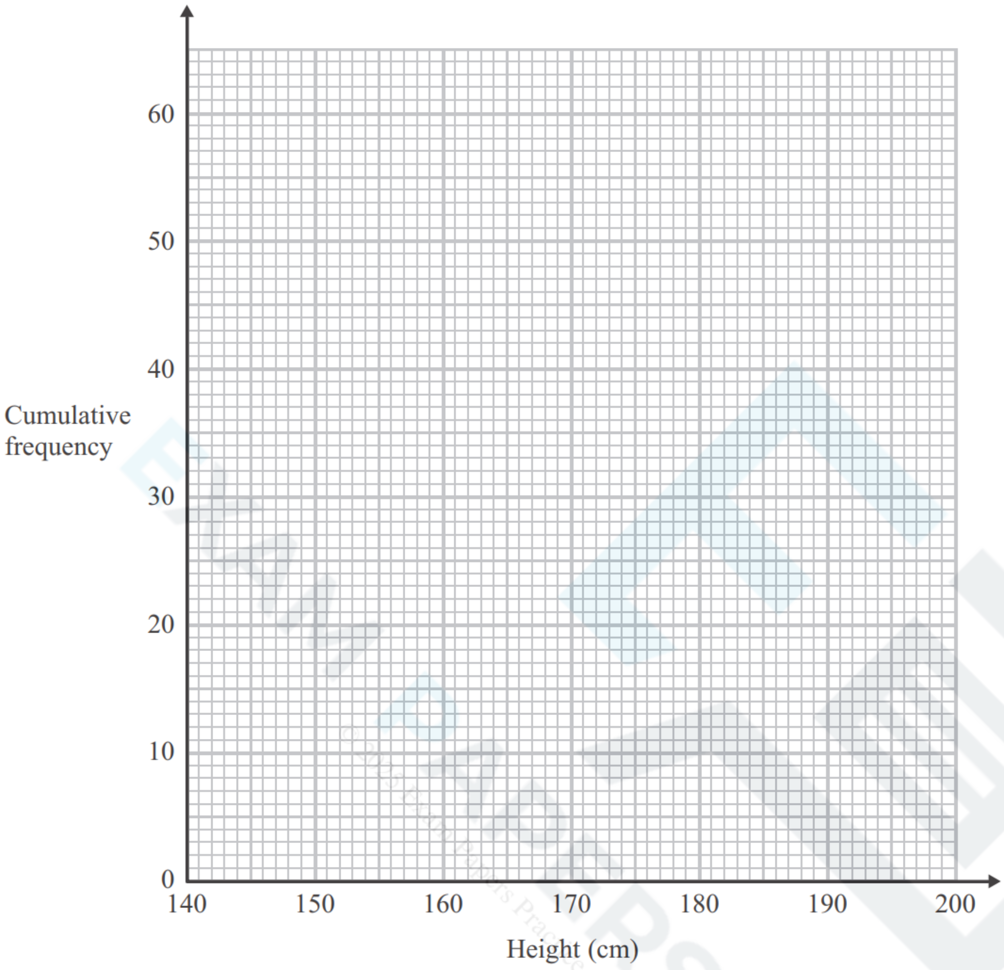
9. Statistics

The table below shows information about the heights of 60 students.

Height (x cm)	**Number of students**
$140 < x \leqslant 150$	4
$150 < x \leqslant 160$	5
$160 < x \leqslant 170$	16
$170 < x \leqslant 180$	27
$180 < x \leqslant 190$	5
$190 < x \leqslant 200$	3

(a) On the grid opposite, draw a cumulative frequency graph for the information in the table.

9. Statistics



(b) Find an estimate

(i) for the median,

(ii) for the interquartile range.

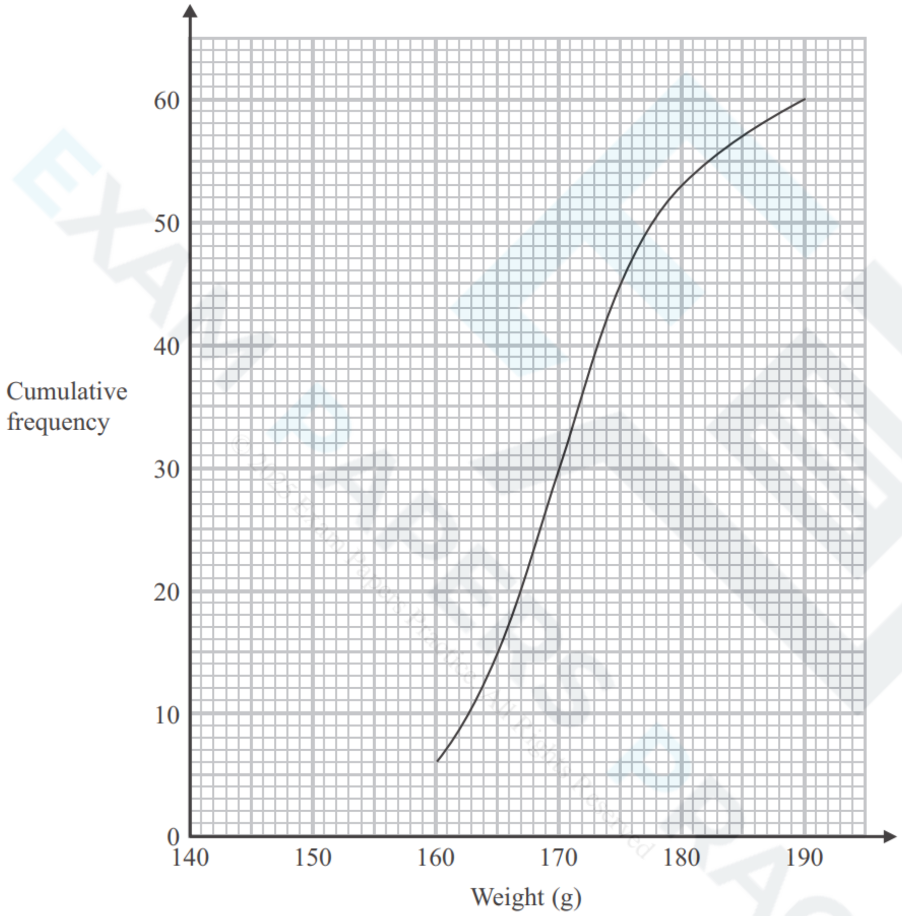
Exam Question: Hard

9. Statistics

Harry grows tomatoes.
This year he put his tomato plants into two groups, group A and group B.

Harry gave fertiliser to the tomato plants in group A.
He did not give fertiliser to the tomato plants in group B.

Harry weighed 60 tomatoes from group A.
The cumulative frequency graph shows some information about these weights.

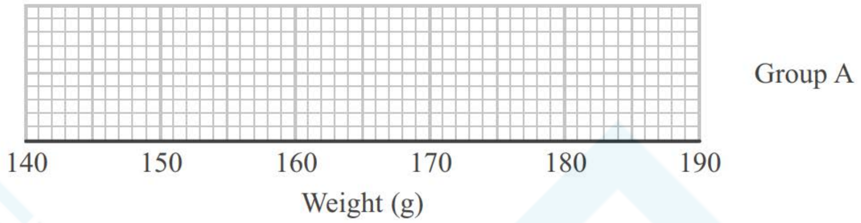


(a) Use the graph to find an estimate for the median weight.

9. Statistics

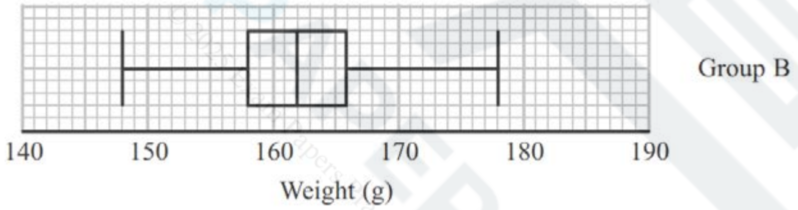
The 60 tomatoes from group A
had a minimum weight of 153 grams
and a maximum weight of 186 grams.

(b) Use this information and the cumulative frequency graph to draw a box plot for the 60 tomatoes from group A.



Harry did not give fertiliser to the tomato plants in group B.

Harry weighed 60 tomatoes from group B.
He drew this box plot for his results.



(c) Compare the distribution of the weights of the tomatoes from group A with the distribution of the weights of the tomatoes from group B.