Question 1

(a) Write down the matrix that represents a rotation of 225° clockwise about the origin. [2]

(b) Write down the matrix that represents a reflection in the y-axis. [2]

(c) Find a single matrix that represents the composite transformation consisting of the transformation in part (a) followed by the transformation in part (b). [2]

(d) Hence find the coordinates of the image of the point $(4, -1)$ after a rotation of 225° clockwise about the origin followed by a reflection in the y-axis. [2]

Transformation matrices	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$, reflection in the line $y = (\tan\theta)x$
	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, horizontal stretch / stretch parallel to x-axis with a scale factor of k
	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, vertical stretch/ stretch parallel to y-axis with a scale factor of k
	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, enlargement, with a scale factor of k, centre $(0, 0)$
	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, anticlockwise/counter-clockwise rotation of angle θ about the origin $(\theta > 0)$
	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, clockwise rotation of angle θ about the origin $(\theta > 0)$

a) Use the formula from the formula booklet.

$\cos 225° = -\frac{\sqrt{2}}{2}$ $\sin 225° = -\frac{\sqrt{2}}{2}$

$$\boxed{\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}}$$

(a) Write down the matrix that represents a rotation of 225° clockwise about the origin. [2]

$$\boxed{\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}}$$

(b) Write down the matrix that represents a reflection in the y-axis. [2]

(c) Find a single matrix that represents the composite transformation consisting of the transformation in part (a) followed by the transformation in part (b). [2]

(d) Hence find the coordinates of the image of the point $(4, -1)$ after a rotation of 225° clockwise about the origin followed by a reflection in the y-axis. [2]

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	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, clockwise rotation of angle θ about the origin $(\theta > 0)$

b) Use the formula from the formula booklet.

For the reflection formula, $\theta = 0°$ for reflection in the x-axis, and $\theta = 90°$ for reflection in the y-axis.

$\theta = 90° \Rightarrow 2\theta = 180°$

$\cos 180° = -1$ $\sin 180° = 0$

$$\boxed{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}$$

(a) Write down the matrix that represents a rotation of 225° clockwise about the origin.

$\boxed{\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}}$ [2]

(b) Write down the matrix that represents a reflection in the y-axis.

$\boxed{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}$ [2]

(c) Find a single matrix that represents the composite transformation consisting of the transformation in part (a) followed by the transformation in part (b).

[2]

(d) Hence find the coordinates of the image of the point $(4, -1)$ after a rotation of 225° clockwise about the origin followed by a reflection in the y-axis.

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Transformation matrices	
	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$, reflection in the line $y = (\tan\theta)x$
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	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, anticlockwise/counter-clockwise rotation of angle θ about the origin $(\theta > 0)$
	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, clockwise rotation of angle θ about the origin $(\theta > 0)$

c) Put matrices in proper order and multiply.

Remember, the order of transformations is from right to left!

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} =$$

$$\boxed{\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}}$$

Note: This is also the matrix for reflection in the line $y = (\tan 22.5°)x$.

(a) Write down the matrix that represents a rotation of 225° clockwise about the origin.

[2]

(b) Write down the matrix that represents a reflection in the y-axis.

[2]

(c) Find a single matrix that represents the composite transformation consisting of the transformation in part (a) followed by the transformation in part (b).

[2]

(d) Hence find the coordinates of the image of the point $(4, -1)$ after a rotation of 225° clockwise about the origin followed by a reflection in the y-axis.

[2]

$$\boxed{\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}} \Big\} \text{ from part (c)}$$

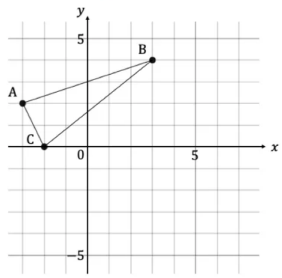
d) Write the coordinates in vector form and multiply (from the left!) by the transformation matrix:

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \end{pmatrix}$$

$$\boxed{\left(\frac{3\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)}$$

Question 2

The diagram below shows a triangle ABC.



(a) Write down the position matrix of the triangle ABC.

[2]

The triangle ABC is to be mapped to triangle A′B′C′ by a single transformation defined by the transformation matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

(b) Find the position matrix of the mapped image and draw triangle A′B′C′ on the diagram.

[3]

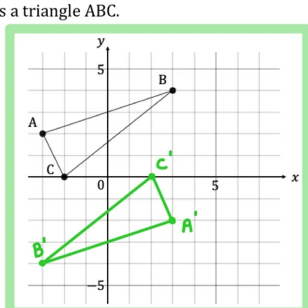
(c) Describe fully the transformation that triangle ABC has undergone.

[2]

a) $A(-3, 2)$ $B(3\ 4)$ $C(-2, 0)$

$$\begin{array}{c} \boxed{\begin{pmatrix} -3 & 3 & -2 \\ 2 & 4 & 0 \end{pmatrix}} \\ \begin{array}{ccc} A & B & C \end{array} \end{array}$$

The diagram below shows a triangle ABC.



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(b) Find the position matrix of the mapped image and draw triangle A′B′C′ on the diagram.

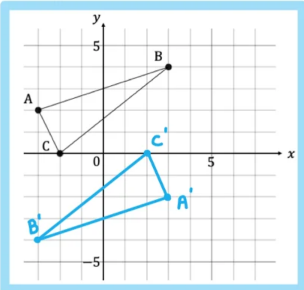
[3]

(c) Describe fully the transformation that triangle ABC has undergone.

[2]

b) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{array}{c} \begin{pmatrix} -3 & 3 & -2 \\ 2 & 4 & 0 \end{pmatrix} \\ \begin{array}{ccc} A & B & C \end{array} \end{array} = \begin{array}{c} \boxed{\begin{pmatrix} 3 & -3 & 2 \\ -2 & -4 & 0 \end{pmatrix}} \\ \begin{array}{ccc} A' & B' & C' \end{array} \end{array}$

The diagram below shows a triangle ABC.



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	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, clockwise rotation of angle θ about the origin $(\theta > 0)$

$\cos 180° = -1 \quad \sin 180° = 0$

$\boxed{\text{Rotation by } 180° \text{ about the origin.}}$

Note: For a 180° rotation there is no difference between clockwise and anticlockwise.

Question 3

Points in a plane are subjected to a transformation T that transforms a point (x, y) to the point (x', y'), where T is defined by

$$T: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Describe in words the transformation T.

[2]

(b) Find the matrix T^{-1}.

[2]

(c) Hence find the coordinates of the point (x, y) if $(x', y') = (12, -4)$.

[2]

Transformation matrices	
	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$, reflection in the line $y = (\tan\theta)x$
	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, horizontal stretch / stretch parallel to x-axis with a scale factor of k
	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, vertical stretch/ stretch parallel to y-axis with a scale factor of k
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	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, clockwise rotation of angle θ about the origin $(\theta > 0)$

a) $\boxed{\text{A horizontal stretch with scale factor 4, combined with a vertical stretch with scale factor } \tfrac{1}{2}}$

Note that for these particular transformations, the order doesn't matter:

$$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 0.5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

Points in a plane are subjected to a transformation T that transforms a point (x, y) to the point (x', y'), where T is defined by

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(a) Describe in words the transformation T.

[2]

(b) Find the matrix T^{-1}.

[2]

(c) Hence find the coordinates of the point (x, y) if $(x', y') = (12, -4)$.

[2]

b) Use inverse and determinant formulae from the formula booklet.

$$T = \begin{pmatrix} 4 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$\det T = (4)(0.5) - (0)(0) = 2$$

$$T^{-1} = \frac{1}{2}\begin{pmatrix} 0.5 & 0 \\ 0 & 4 \end{pmatrix} = \boxed{\begin{pmatrix} 0.25 & 0 \\ 0 & 2 \end{pmatrix}}$$

Inverse of a 2×2 matrix $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad \neq bc$

Determinant of a 2×2 matrix $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = \lvert A \rvert = ad - bc$

Points in a plane are subjected to a transformation T that transforms a point (x, y) to the point (x', y'), where T is defined by

$$T: \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Describe in words the transformation T.

[2]

(b) Find the matrix T^{-1}. $\boxed{\begin{pmatrix} 0.25 & 0 \\ 0 & 2 \end{pmatrix}}$

[2]

(c) Hence find the coordinates of the point (x, y) if $(x', y') = (12, -4)$.

[2]

c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = T^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.25 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 12 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \end{pmatrix}$$

$$\boxed{(3, -8)}$$

Question 4

A quadrilateral has vertices (2, 1), (4, 4), (7, 1) and (9, 4).

(a) Find the area of the quadrilateral.

[2]

The quadrilateral undergoes a transformation represented by the matrix $\begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$.

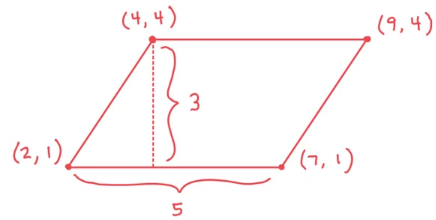
(b) Find the determinant of the transformation matrix.

[2]

(c) Hence find the area of the image.

[2]

a) A quick sketch will show that the quadrilateral is a parallelogram with base 5 and height 3 :



Area of parallelogram = base × height

Area = 5 × 3 = $\boxed{15 \text{ units}^2}$

A quadrilateral has vertices (2, 1), (4, 4), (7, 1) and (9, 4).

(a) Find the area of the quadrilateral. $\boxed{15 \text{ units}^2}$

[2]

The quadrilateral undergoes a transformation represented by the matrix $\begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$.

(b) Find the determinant of the transformation matrix.

[2]

(c) Hence find the area of the image.

[2]

b) Use determinant formula from the formula booklet.

$\begin{vmatrix} -2 & 1 \\ -4 & 3 \end{vmatrix} = (-2)(3) - (1)(-4) = \boxed{-2}$

Determinant of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = \lvert A \rvert = ad - bc$

A quadrilateral has vertices $(2,1), (4,4), (7,1)$ and $(9,4)$.

(a) Find the area of the quadrilateral. 15 units^2

[2]

The quadrilateral undergoes a transformation represented by the matrix $\begin{pmatrix}-2 & 1\\ -4 & 3\end{pmatrix}$.

(b) Find the determinant of the transformation matrix. -2

[2]

(c) Hence find the area of the image.

[2]

c) The modulus of the determinant is the area scale factor.

Area $= |-2| \times 15 =$ 30 units^2

Question 5

An object undergoes a vertical stretch with scale factor 2 followed by a reflection in the x-axis. The position matrix of the image is $\begin{pmatrix}2 & 5 & 4\\ 3 & -2 & -1\end{pmatrix}$.

Use a matrix method to find the coordinates of the object before the transformation.

[5]

Transformation matrices	$\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$,	reflection in the line $y = (\tan\theta)x$
	$\begin{pmatrix}k & 0\\ 0 & 1\end{pmatrix}$,	horizontal stretch / stretch parallel to x-axis with a scale factor of k
	$\begin{pmatrix}1 & 0\\ 0 & k\end{pmatrix}$,	vertical stretch/ stretch parallel to y-axis with a scale factor of k
	$\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}$,	enlargement, with a scale factor of k, centre $(0, 0)$
	$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$,	anticlockwise/counter-clockwise rotation of angle θ about the origin $(\theta > 0)$
	$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$,	clockwise rotation of angle θ about the origin $(\theta > 0)$

Inverse of a 2×2 matrix	$A = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow A^{-1} = \dfrac{1}{\det A}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}, ad \neq bc$

Determinant of a 2×2 matrix	$A = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow \det A = \lvert A\rvert = ad - bc$

Use the formulae from the formula booklet.

For the reflection formula, $\theta = 0°$ for reflection in the x-axis, and $\theta = 90°$ for reflection in the y-axis.

Remember, the order of transformations is from right to left!

$$T = \underbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}_{\text{reflection}}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}}_{\text{stretch}} = \begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}$$

$$T^{-1} = \frac{1}{-2}\begin{pmatrix}-2 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -\frac{1}{2}\end{pmatrix}$$

Use the inverse to undo the transformation:

$$\begin{pmatrix}1 & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}2 & 5 & 4\\ 3 & -2 & -1\end{pmatrix} = \begin{pmatrix}2 & 5 & 4\\ -\frac{3}{2} & 1 & \frac{1}{2}\end{pmatrix}$$

The original coordinates are

$(2, -\frac{3}{2})$ $(5, 1)$ $(4, \frac{1}{2})$

Question 6

The points $A(7,-3)$ and $B(2,6)$ are transformed to become the points $A'(18,8)$ and $B'(12,16)$ respectively.

(a) Find the 2×2 matrix T that represents the linear transformation.

[5]

(b) Given that point $C(-4,5)$ is transformed by T^2, find the coordinates of the image point C'.

[3]

a) Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 7a-3b \\ 7c-3d \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2a+6b \\ 2c+6d \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$$

This gives two sets of simultaneous equations:

$$\left.\begin{array}{l} 7a-3b=18 \\ 2a+6b=12 \end{array}\right\} \Longrightarrow a=3,\ b=1$$

Solve in GDC, by hand, or by inspection

$$\left.\begin{array}{l} 7c-3d=8 \\ 2c+6d=16 \end{array}\right\} \Longrightarrow c=2,\ d=2$$

$$\boxed{T = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}}$$

The points $A(7,-3)$ and $B(2,6)$ are transformed to become the points $A'(18,8)$ and $B'(12,16)$ respectively.

(a) Find the 2×2 matrix T that represents the linear transformation.

$$\boxed{T = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}}$$

[5]

(b) Given that point $C(-4,5)$ is transformed by T^2, find the coordinates of the image point C'.

[3]

b) $T^2 = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 11 & 5 \\ 10 & 6 \end{pmatrix}$

$C' = \begin{pmatrix} 11 & 5 \\ 10 & 6 \end{pmatrix}\begin{pmatrix} -4 \\ 5 \end{pmatrix} = \begin{pmatrix} -19 \\ -10 \end{pmatrix}$

$\boxed{C' \text{ is the point } (-19,-10)}$

Question 7

An object is reflected in the line $y = \frac{\sqrt{3}}{3}x$.

(a) Write down the matrix that represents the transformation. [3]

P is a vertex of the object that is being reflected.

(b) Find the coordinates of P if the coordinates of its image P′ are $(2, 2\sqrt{3})$. [3]

a) Use the formula from the formula booklet.

$\tan\theta = \frac{\sqrt{3}}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = 30° \Rightarrow 2\theta = 60°$

$\cos 60° = \frac{1}{2} \quad \sin 60° = \frac{\sqrt{3}}{2}$

$$\boxed{\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}}$$

Transformation matrices	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$, reflection in the line $y = (\tan\theta)x$
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	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, enlargement, with a scale factor of k, centre $(0, 0)$
	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, anticlockwise/counter-clockwise rotation of angle θ about the origin $(\theta > 0)$
	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, clockwise rotation of angle θ about the origin $(\theta > 0)$

An object is reflected in the line $y = \frac{\sqrt{3}}{3}x$.

(a) Write down the matrix that represents the transformation. [3]

$$\boxed{\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}}$$

P is a vertex of the object that is being reflected.

(b) Find the coordinates of P if the coordinates of its image P′ are $(2, 2\sqrt{3})$. [3]

You could also GDC to find inverse

b) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}^{-1} = \frac{1}{-1}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

Note that the inverse is the same as the original matrix. This shouldn't be a surprise – doing a reflection a second time undoes the original reflection!

Use inverse matrix to undo the transformation:

$P = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

$\boxed{\text{P is the point } (4, 0)}$

Inverse of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $ad \neq bc$

Determinant of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = \lvert A \rvert = ad - bc$

Question 8

The triangle PQR, with vertices P(−1, 1), Q(5, 3) and R(−9, −2), is translated by the vector $\begin{pmatrix}-7\\2\end{pmatrix}$ and then enlarged by a scale factor of 3 with the centre of enlargement at the origin.

(a) Find a single transformation in the form $AX + b$ that maps PQR onto P′Q′R′.

[2]

(b) Hence determine the coordinates of P′Q′R′.

[3]

a) The translation turns $\begin{pmatrix}x\\y\end{pmatrix}$ into $\begin{pmatrix}x-7\\y+2\end{pmatrix}$.

Then the enlargement turns $\begin{pmatrix}x-7\\y+2\end{pmatrix}$ into:

$$\begin{pmatrix}3&0\\0&3\end{pmatrix}\begin{pmatrix}x-7\\y+2\end{pmatrix} = \begin{pmatrix}3x-21\\3y+6\end{pmatrix}$$

$$= \begin{pmatrix}3x+0y\\0x+3y\end{pmatrix} + \begin{pmatrix}-21\\6\end{pmatrix}$$

$$\boxed{= \begin{pmatrix}3&0\\0&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-21\\6\end{pmatrix}}$$

Note that this is the same as doing the enlargement first, and then translating by $\begin{pmatrix}-21\\6\end{pmatrix}$.

Transformation matrices	$\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$, reflection in the line $y = (\tan\theta)x$
	$\begin{pmatrix}k & 0\\ 0 & 1\end{pmatrix}$, horizontal stretch / stretch parallel to x-axis with a scale factor of k
	$\begin{pmatrix}1 & 0\\ 0 & k\end{pmatrix}$, vertical stretch/ stretch parallel to y-axis with a scale factor of k
	$\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}$, enlargement, with a scale factor of k, centre $(0, 0)$
	$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$, anticlockwise/counter-clockwise rotation of angle θ about the origin $(\theta > 0)$
	$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$, clockwise rotation of angle θ about the origin $(\theta > 0)$

The triangle PQR, with vertices P(−1, 1), Q(5, 3) and R(−9, −2), is translated by the vector $\begin{pmatrix}-7\\2\end{pmatrix}$ and then enlarged by a scale factor of 3 with the centre of enlargement at the origin.

(a) Find a single transformation in the form $AX + b$ that maps PQR onto P′Q′R′.

$\boxed{\begin{pmatrix}3&0\\0&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-21\\6\end{pmatrix}}$

[2]

(b) Hence determine the coordinates of P′Q′R′.

[3]

b) Write PQR as a position matrix and then have the transformation act on that:

$$\begin{pmatrix}3&0\\0&3\end{pmatrix}\begin{pmatrix}-1&5&-9\\1&3&-2\\ \uparrow&\uparrow&\uparrow\\ P&Q&R\end{pmatrix} + \begin{pmatrix}-21\\6\end{pmatrix} = \begin{pmatrix}-24&-6&-48\\9&15&0\\ \uparrow&\uparrow&\uparrow\\ P'&Q'&R'\end{pmatrix}$$

$\boxed{\text{The coordinates are } P'(-24, 9),\ Q'(-6, 15) \text{ and } R'(-48, 0)}$

Question 9

(a) Find the 2×2 transformation matrices that represent the following transformations:

(i) R, a rotation of $\frac{\pi}{4}$ radians anti-clockwise

(ii) S, a reflection in the line $y = x$

(iii) T, a stretch with scale factor 5 parallel to the y-axis.

[3]

(b) Find a single transformation matrix that represents the composite transformation

(i) RT^3

(ii) R^8STS

[4]

(c) Find the coordinates of the image of the point $A(3, -1)$ after it has undergone the composite transformation specified in part (b)(i).

[2]

(d) State the name of the single transformation that is equivalent to the composite transformation specified in part (b)(ii).

[1]

a) Use the formulae from the formula booklet.

(i) $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$\boxed{R = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}}$$

(ii) $y = (1)x \quad \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$2\theta = \frac{\pi}{2}, \quad \cos\frac{\pi}{2} = 0, \quad \sin\frac{\pi}{2} = 1$

$$\boxed{S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

(iii) $\boxed{T = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}}$

Transformation matrices	
	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$, reflection in the line $y = (\tan\theta)x$
	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, horizontal stretch / stretch parallel to x-axis with a scale factor of k
	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, vertical stretch/ stretch parallel to y-axis with a scale factor of k
	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, enlargement, with a scale factor of k, centre $(0, 0)$
	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, anticlockwise/counter-clockwise rotation of angle θ about the origin $(\theta > 0)$
	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, clockwise rotation of angle θ about the origin $(\theta > 0)$

(a) Find the 2×2 transformation matrices that represent the following transformations:

(i) R, a rotation of $\frac{\pi}{4}$ radians anti-clockwise

(ii) S, a reflection in the line $y = x$

(iii) T, a stretch with scale factor 5 parallel to the y-axis.

$\boxed{R = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}}$ $\boxed{S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$ $\boxed{T = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}}$

[3]

(b) Find a single transformation matrix that represents the composite transformation

(i) RT^3

(ii) R^8STS

[4]

(c) Find the coordinates of the image of the point $A(3, -1)$ after it has undergone the composite transformation specified in part (b)(i).

[2]

(d) State the name of the single transformation that is equivalent to the composite transformation specified in part (b)(ii).

[1]

b) Your GDC can calculate these matrix products.

(i) $RT^3 = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}^3 \qquad \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 125 \end{pmatrix}$

$$\boxed{RT^3 = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{125\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{125\sqrt{2}}{2} \end{pmatrix}}$$

(ii) $R^8STS = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}^8 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$R^8 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad STS = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$

$$\boxed{R^8STS = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}}$$

(a) Find the 2×2 transformation matrices that represent the following transformations:

(i) R, a rotation of $\frac{\pi}{4}$ radians anti-clockwise

(ii) S, a reflection in the line $y = x$

(iii) T, a stretch with scale factor 5 parallel to the y-axis.

[3]

(b) Find a single transformation matrix that represents the composite transformation

(i) RT^3

(ii) R^8STS

$$\boxed{RT^3 = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{125\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{125\sqrt{2}}{2} \end{pmatrix}} \quad \boxed{R^8STS = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}}$$

[4]

(c) Find the coordinates of the image of the point $A(3, -1)$ after it has undergone the composite transformation specified in part (b)(i).

[2]

(d) State the name of the single transformation that is equivalent to the composite transformation specified in part (b)(ii).

[1]

c) $\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{125\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{125\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 64\sqrt{2} \\ -61\sqrt{2} \end{pmatrix}$

$$\boxed{(64\sqrt{2}, -61\sqrt{2})}$$

(a) Find the 2×2 transformation matrices that represent the following transformations:

(i) R, a rotation of $\frac{\pi}{4}$ radians anti-clockwise

(ii) S, a reflection in the line $y = x$

(iii) T, a stretch with scale factor 5 parallel to the y-axis.

[3]

(b) Find a single transformation matrix that represents the composite transformation

(i) RT^3

(ii) R^8STS

$$\boxed{RT^3 = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{125\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{125\sqrt{2}}{2} \end{pmatrix}} \quad \boxed{R^8STS = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}}$$

[4]

(c) Find the coordinates of the image of the point $A(3, -1)$ after it has undergone the composite transformation specified in part (b)(i).

[2]

(d) State the name of the single transformation that is equivalent to the composite transformation specified in part (b)(ii).

[1]

d) Compare the matrix for R^8STS to the formulae from the formula booklet.

Stretch parallel to the x-axis with scale factor 5

Or 'Horizontal stretch with scale factor 5'

Transformation matrices	
	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$, reflection in the line $y = (\tan\theta)x$
	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, horizontal stretch / stretch parallel to x-axis with a scale factor of k
	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, vertical stretch/ stretch parallel to y-axis with a scale factor of k
	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, enlargement, with a scale factor of k, centre $(0, 0)$
	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, anticlockwise/counter-clockwise rotation of angle θ about the origin ($\theta > 0$)
	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, clockwise rotation of angle θ about the origin ($\theta > 0$)

Question 10

The triangle PQR with position matrix T_0 has vertices $\mathrm{P}(5,2)$, $\mathrm{Q}(-3,1)$ and $\mathrm{R}(-5,-4)$.

The triangle is transformed by the transformation matrix $M = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$.

T_n denotes the position matrix of the image triangle after PQR has been transformed n times by matrix M.

(a) By multiplying two appropriate single transformation matrices together, verify that the matrix M is an enlargement by scale factor $\frac{1}{2}$ followed by a 90° clockwise rotation.

[3]

(b) Explain why the area of the triangle with position matrix T_1 will be $\frac{1}{4}$ of the area of triangle PQR.

[2]

(c) Find T_2, and hence the coordinates of the image triangle after triangle PQR is transformed twice by matrix M.

[3]

a) Use the formulae from the formula booklet.

Remember, the order of transformations is from right to left!

$$\boxed{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}}$$

$\cos 90° = 0$, $\sin 90° = 1$ $\qquad$ $k = \frac{1}{2}$

Transformation matrices	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$, reflection in the line $y = (\tan\theta)x$	
	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$,	horizontal stretch / stretch parallel to x-axis with a scale factor of k
	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$,	vertical stretch/ stretch parallel to y-axis with a scale factor of k
	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$,	enlargement, with a scale factor of k, centre $(0, 0)$
	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$,	anticlockwise/counter-clockwise rotation of angle θ about the origin $(\theta > 0)$
	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$,	clockwise rotation of angle θ about the origin $(\theta > 0)$

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[3]

(b) Explain why the area of the triangle with position matrix T_1 will be $\frac{1}{4}$ of the area of triangle PQR.

[2]

(c) Find T_2, and hence the coordinates of the image triangle after triangle PQR is transformed twice by matrix M.

[3]

Determinant of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = \lvert A \rvert = ad - bc$

b) $\det M = \begin{vmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{vmatrix} = (0)(0) - (\frac{1}{2})(-\frac{1}{2}) = \frac{1}{4}$

So the modulus of M's determinant is $\lvert \frac{1}{4} \rvert = \frac{1}{4}$.

The modulus of a transformation matrix's determinant is the area scale factor of the corresponding transformation.

The triangle with position matrix T_1 has been transformed one time by M, so its area will be $\frac{1}{4}$ of the area of the original triangle.

EXAM PAPERS PRACTICE

The triangle PQR with position matrix T_0 has vertices $P(5,2)$, $Q(-3,1)$ and $R(-5,-4)$.

The triangle is transformed by the transformation matrix $M = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$.

T_n denotes the position matrix of the image triangle after PQR has been transformed n times by matrix M.

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[3]

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[2]

(c) Find T_2, and hence the coordinates of the image triangle after triangle PQR is transformed twice by matrix M.

[3]

c) $T_0 = \begin{array}{ccc} P & Q & R \\ \downarrow & \downarrow & \downarrow \end{array}$ $\begin{pmatrix} 5 & -3 & -5 \\ 2 & 1 & -4 \end{pmatrix}$

$M^2 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$

$T_2 = M^2 T_0 = \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}\begin{pmatrix} 5 & -3 & -5 \\ 2 & 1 & -4 \end{pmatrix} = \begin{array}{ccc} P' & Q' & R' \\ \downarrow & \downarrow & \downarrow \end{array}$ $\begin{pmatrix} -\frac{5}{4} & \frac{3}{4} & \frac{5}{4} \\ -\frac{1}{2} & -\frac{1}{4} & 1 \end{pmatrix}$

$\boxed{\text{The coordinates are } P'\left(-\frac{5}{4}, -\frac{1}{2}\right), Q'\left(\frac{3}{4}, -\frac{1}{4}\right), \text{ and } R'\left(\frac{5}{4}, 1\right).}$