Question 1

Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -1 & a & 3 \end{pmatrix} \quad B = \begin{pmatrix} b & 2 \\ 1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & c & -3 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 3 & d & 1 \\ 0 & 3 & 0 \end{pmatrix} \quad E = \begin{pmatrix} -1 & 4 & e \\ -2 & 1 & 0 \end{pmatrix} \quad F = \begin{pmatrix} f \\ 2 \\ -1 \end{pmatrix}$$

where $a, b, c, d, e, f \in \mathbb{R}$ are constants.

(a) Find each of the following matrix sums or differences, or if that is not possible then explain why:

(i) $A + B$ (ii) $A + E$ (iii) $E - 3A$

[4]

(b) Find each of the following matrix products, or if that is not possible then explain why:

(i) BE (ii) EB (iii) CD (iv) FC (v) AE

[5]

(c) List any other matrix products of two of the above matrices that it *would* be possible to find, other than the ones included in part (b). You do not need to find the products, but do specify what the order of each of those products would be.

[3]

(d) Find each of the following:

(i) $B(A + E)$ (ii) $BA + BE$

[3]

a) (i) A is a 2×3 matrix, and B is a 2×2 matrix.

A and B do not have the same order, so they cannot be added.

(ii) Just add the individual elements

$$\boxed{A + E = \begin{pmatrix} 0 & 4 & e-2 \\ -3 & a+1 & 3 \end{pmatrix}}$$

(iii) $$E - 3A = \begin{pmatrix} -1 & 4 & e \\ -2 & 1 & 0 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & -2 \\ -1 & a & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 4 & e \\ -2 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 & -6 \\ -3 & 3a & 9 \end{pmatrix}$$

$$\boxed{E - 3A = \begin{pmatrix} -4 & 4 & e+6 \\ 1 & 1-3a & -9 \end{pmatrix}}$$

Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -1 & a & 3 \end{pmatrix} \quad B = \begin{pmatrix} b & 2 \\ 1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & c & -3 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 3 & d & 1 \\ 0 & 3 & 0 \end{pmatrix} \quad E = \begin{pmatrix} -1 & 4 & e \\ -2 & 1 & 0 \end{pmatrix} \quad F = \begin{pmatrix} f \\ 2 \\ -1 \end{pmatrix}$$

where $a, b, c, d, e, f \in \mathbb{R}$ are constants.

(a) Find each of the following matrix sums or differences, or if that is not possible then explain why:

(i) $A + B$ (ii) $A + E$ (iii) $E - 3A$

[4]

(b) Find each of the following matrix products, or if that is not possible then explain why:

(i) BE (ii) EB (iii) CD (iv) FC (v) AE

[5]

(c) List any other matrix products of two of the above matrices that it *would* be possible to find, other than the ones included in part (b). You do not need to find the products, but do specify what the order of each of those products would be.

[3]

(d) Find each of the following:

(i) $B(A + E)$ (ii) $BA + BE$

[3]

b) (i) $$\boxed{BE = \begin{pmatrix} -b-4 & 4b+2 & be \\ 1 & 3 & e \end{pmatrix}}$$

(ii) Number of columns in E doesn't match number of rows in B, so they can't be multiplied in that order.

Note that BE exists, but EB doesn't.

(iii) $$\boxed{CD = \begin{pmatrix} 3c & cd-9 & c \end{pmatrix}}$$

(iv) $$\boxed{FC = \begin{pmatrix} 0 & cf & -3f \\ 0 & 2c & -6 \\ 0 & -c & 3 \end{pmatrix}}$$

(v) Number of columns in A doesn't match number of rows in E, so they can't be multiplied in that order.

EA doesn't exist either. Note that A and E can be added or subtracted, but not multiplied.

Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -1 & a & 3 \end{pmatrix} \quad B = \begin{pmatrix} b & 2 \\ 1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & c & -3 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 3 & d & 1 \\ 0 & 3 & 0 \end{pmatrix} \quad E = \begin{pmatrix} -1 & 4 & e \\ -2 & 1 & 0 \end{pmatrix} \quad F = \begin{pmatrix} f \\ 2 \\ -1 \end{pmatrix}$$

where $a, b, c, d, e, f \in \mathbb{R}$ are constants.

(a) Find each of the following matrix sums or differences, or if that is not possible then explain why:

(i) $A + B$ (ii) $A + E$ (iii) $E - 3A$

[4]

(b) Find each of the following matrix products, or if that is not possible then explain why:

(i) BE (ii) EB (iii) CD (iv) FC (v) AE

[5]

(c) List any other matrix products of two of the above matrices that it *would* be possible to find, other than the ones included in part (b). You do not need to find the products, but do specify what the order of each of those products would be.

[3]

(d) Find each of the following:

(i) $B(A + E)$ (ii) $BA + BE$

[3]

c) The product of a $p \times q$ matrix and a $q \times r$ matrix (in that order!) is a $p \times r$ matrix.

Remember that for the product to exist, the number of columns in the first matrix must match the number of rows in the second matrix.

AD (2×3)	AF (2×1)
BA (2×3)	BB (2×2)
CF (1×1)	
DD (3×3)	DF (3×1)
ED (2×3)	EF (2×1)

Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -1 & a & 3 \end{pmatrix} \quad B = \begin{pmatrix} b & 2 \\ 1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & c & -3 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 3 & d & 1 \\ 0 & 3 & 0 \end{pmatrix} \quad E = \begin{pmatrix} -1 & 4 & e \\ -2 & 1 & 0 \end{pmatrix} \quad F = \begin{pmatrix} f \\ 2 \\ -1 \end{pmatrix}$$

where $a, b, c, d, e, f \in \mathbb{R}$ are constants.

(a) Find each of the following matrix sums or differences, or if that is not possible then explain why:

(i) $A + B$ (ii) $A + E$ (iii) $E - 3A$

$$A + E = \begin{pmatrix} 0 & 4 & e-2 \\ -3 & a+1 & 3 \end{pmatrix}$$

[4]

(b) Find each of the following matrix products, or if that is not possible then explain why:

(i) BE (ii) EB (iii) CD (iv) FC (v) AE

$$BE = \begin{pmatrix} -b-4 & 4b+2 & be \\ 1 & 3 & e \end{pmatrix}$$

[5]

(c) List any other matrix products of two of the above matrices that it *would* be possible to find, other than the ones included in part (b). You do not need to find the products, but do specify what the order of each of those products would be.

[3]

(d) Find each of the following:

(i) $B(A + E)$ (ii) $BA + BE$

[3]

d) (i) $B(A+E) = \begin{pmatrix} b & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 4 & e-2 \\ -3 & a+1 & 3 \end{pmatrix}$

$$\boxed{B(A+E) = \begin{pmatrix} -6 & 2a+4b+2 & be-2b+6 \\ 3 & 3-a & e-5 \end{pmatrix}}$$

(ii) $BA = \begin{pmatrix} b-2 & 2a & -2b+6 \\ 2 & -a & -5 \end{pmatrix}$

$BA + BE = \begin{pmatrix} b-2 & 2a & -2b+6 \\ 2 & -a & -5 \end{pmatrix} + \begin{pmatrix} -b-4 & 4b+2 & be \\ 1 & 3 & e \end{pmatrix}$

$$\boxed{BA + BE = \begin{pmatrix} -6 & 2a+4b+2 & be-2b+6 \\ 3 & 3-a & e-5 \end{pmatrix}}$$

Note that $B(A+E) = BA + BE$.

This illustrates the distributive property of matrix multiplication.

Question 2

Consider the matrices:

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ 4 & -1 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ -4 & k \end{pmatrix}$$

where $k \in \mathbb{R}$ is a constant.

Let $\boldsymbol{I}$ be the 2×2 identity matrix, and let $\boldsymbol{0}$ be the 2×2 zero matrix. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(a) Find the following:

(i) $\boldsymbol{B}^2$ (ii) $\boldsymbol{A} - 2\boldsymbol{I}$ (iii) $\boldsymbol{AB}$ (iv) $\boldsymbol{BA}$ (v) $\boldsymbol{A}^{-1}\boldsymbol{B}$

[7]

(b) Find the following:

(i) $\det \boldsymbol{A}$ (ii) $\det \boldsymbol{A}^{-1}$ (iii) $\det \boldsymbol{B}$ (iv) $\det \boldsymbol{AB}$

[4]

$\boldsymbol{C}$ and $\boldsymbol{D}$ are matrices such that $\boldsymbol{A} + \boldsymbol{C} = \boldsymbol{0}$ and $\boldsymbol{B}^{-1} + \boldsymbol{D} = \boldsymbol{0}$.

(c) (i) Write down matrix $\boldsymbol{C}$.

(ii) Find matrix $\boldsymbol{D}$.

[4]

Determinant of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = \lvert A \rvert = ad - bc$
Inverse of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad \neq bc$

a) (i) $B^2 = \begin{pmatrix} 1 & 1 \\ -4 & k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -4 & k \end{pmatrix} \Rightarrow \boxed{B^2 = \begin{pmatrix} -3 & k+1 \\ -4-4k & k^2-4 \end{pmatrix}}$

(ii) $A - 2I = \begin{pmatrix} 2 & -1 \\ 4 & -1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\boxed{A - 2I = \begin{pmatrix} 0 & -1 \\ 4 & -3 \end{pmatrix}}$

(iii) $\boxed{AB = \begin{pmatrix} 6 & 2-k \\ 8 & 4-k \end{pmatrix}}$ (iv) $\boxed{BA = \begin{pmatrix} 6 & -2 \\ 4k-8 & 4-k \end{pmatrix}}$

Note that if $k = 4$, then $AB = BA$. For any other value of k, however, $AB \neq BA$.

In general for two matrices A and B, $AB \neq BA$. We describe this by saying that matrix multiplication is non-commutative.

(v) $A^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -4 & 2 \end{pmatrix} \Rightarrow A^{-1}B = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -4 & k \end{pmatrix}$

$\boxed{A^{-1}B = \frac{1}{2}\begin{pmatrix} -5 & k-1 \\ -12 & 2k-4 \end{pmatrix}} = \begin{pmatrix} -\frac{5}{2} & \frac{k-1}{2} \\ -6 & k-2 \end{pmatrix}$

$\det A = -2-(-4) = 2$

Consider the matrices:

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ 4 & -1 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ -4 & k \end{pmatrix}$$

where $k \in \mathbb{R}$ is a constant.

Let $\boldsymbol{I}$ be the 2×2 identity matrix, and let $\boldsymbol{0}$ be the 2×2 zero matrix.

(a) Find the following:

(i) $\boldsymbol{B}^2$ (ii) $\boldsymbol{A} - 2\boldsymbol{I}$ (iii) $\boldsymbol{AB}$ (iv) $\boldsymbol{BA}$ (v) $\boldsymbol{A}^{-1}\boldsymbol{B}$

$\boxed{AB = \begin{pmatrix} 6 & 2-k \\ 8 & 4-k \end{pmatrix}}$

[7]

(b) Find the following:

(i) $\det \boldsymbol{A}$ (ii) $\det \boldsymbol{A}^{-1}$ (iii) $\det \boldsymbol{B}$ (iv) $\det \boldsymbol{AB}$

[4]

$\boldsymbol{C}$ and $\boldsymbol{D}$ are matrices such that $\boldsymbol{A} + \boldsymbol{C} = \boldsymbol{0}$ and $\boldsymbol{B}^{-1} + \boldsymbol{D} = \boldsymbol{0}$.

(c) (i) Write down matrix $\boldsymbol{C}$.

(ii) Find matrix $\boldsymbol{D}$.

[4]

Determinant of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = \lvert A \rvert = ad - bc$
Inverse of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad \neq bc$

b) (i) $\begin{vmatrix} 2 & -1 \\ 4 & -1 \end{vmatrix} = -2-(-4) = \boxed{2}$

(ii) $A^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -2 & 1 \end{pmatrix}$

Note that $\det A^{-1} = \dfrac{1}{\det A}$

$\begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ -2 & 1 \end{vmatrix} = -\frac{1}{2} - (-1) = \boxed{\frac{1}{2}}$

(iii) $\begin{vmatrix} 1 & 1 \\ -4 & k \end{vmatrix} = k-(-4) = \boxed{k+4}$

Note that $\det AB = \det A \times \det B$

(iv) $\begin{vmatrix} 6 & 2-k \\ 8 & 4-k \end{vmatrix} = (24-6k)-(16-8k) = \boxed{2k+8}$

Consider the matrices:

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ -4 & k \end{pmatrix}$$

where $k \in \mathbb{R}$ is a constant.

Let I be the 2×2 identity matrix, and let 0 be the 2×2 zero matrix. $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(a) Find the following:

(i) B^2 (ii) $A - 2I$ (iii) AB (iv) BA (v) $A^{-1}B$

[7]

(b) Find the following:

(i) $\det A$ (ii) $\det A^{-1}$ (iii) $\det B$ (iv) $\det AB$

$\boxed{\det B = k + 4}$

[4]

C and D are matrices such that $A + C = 0$ and $B^{-1} + D = 0$.

(c) (i) Write down matrix C.

(ii) Find matrix D.

[4]

Inverse of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $ad \neq bc$

c) (i) $\boxed{C = \begin{pmatrix} -2 & 1 \\ -4 & 1 \end{pmatrix}}$ Note that $C = -A$

(ii) $B^{-1} = \dfrac{1}{k+4}\begin{pmatrix} k & -1 \\ 4 & 1 \end{pmatrix}$

$\boxed{D = \dfrac{1}{k+4}\begin{pmatrix} -k & 1 \\ -4 & -1 \end{pmatrix}} = \begin{pmatrix} -\frac{k}{k+4} & \frac{1}{k+4} \\ -\frac{4}{k+4} & -\frac{1}{k+4} \end{pmatrix}$

Question 3

Consider the matrix

$$A = \begin{pmatrix} 3 & -8 \\ p & 7 \end{pmatrix}$$

where $p \in \mathbb{R}$ is a constant.

(a) Given that $\det A = -3$, find the value of p.

[3]

Consider the two matrices

$$B = \begin{pmatrix} 4 & q \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} r & 1 \\ 1 & 3 \end{pmatrix}$$

where $q, r \in \mathbb{R}$ are constants.

(b) Given that $BC = CB$, find the values of q and r.

[4]

Determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = |A| = ad - bc$

a) $\begin{vmatrix} 3 & -8 \\ p & 7 \end{vmatrix} = 21 - (-8p) = 8p + 21$

So $8p + 21 = -3$

$8p = -24$

$\boxed{p = -3}$

Consider the matrix

$$A = \begin{pmatrix} 3 & -8 \\ p & 7 \end{pmatrix}$$

where $p \in \mathbb{R}$ is a constant.

(a) Given that $\det A = -3$, find the value of p.

[3]

Consider the two matrices

$$B = \begin{pmatrix} 4 & q \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} r & 1 \\ 1 & 3 \end{pmatrix}$$

where $q, r \in \mathbb{R}$ are constants.

(b) Given that $BC = CB$, find the values of q and r.

[4]

In general for two matrices A and B, $AB \neq BA$. We describe this by saying that matrix multiplication is non-commutative.

b) $BC = \begin{pmatrix} 4 & q \\ -1 & 3 \end{pmatrix}\begin{pmatrix} r & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} q+4r & 3q+4 \\ 3-r & 8 \end{pmatrix}$

$CB = \begin{pmatrix} r & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4 & q \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 4r-1 & qr+3 \\ 1 & q+9 \end{pmatrix}$

For two matrices to be equal, all their elements must be equal. Therefore if $BC = CB$:

$3 - r = 1 \implies r = 2$

$q + 9 = 8 \implies q = -1$

Check that these values work for the other elements:

If $q = -1$, $r = 2$, then

$q + 4r = 7 \quad 4r - 1 = 7$ ✓

$3q + 4 = 1 \quad qr + 3 = 1$ ✓

$\boxed{q = -1 \quad r = 2}$

So $B = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, and $BC = CB = \begin{pmatrix} 7 & 1 \\ 1 & 8 \end{pmatrix}$

For any other values of q and r, $BC \neq CB$.

Question 4

Consider the matrices $P = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}$.

(a) Find

(i) the determinant of P

(ii) P^{-1}.

[2]

Consider the matrix product PDP^{-1}.

(b) (i) Show that $(PDP^{-1})^2 = PD^2P^{-1}$ and $(PDP^{-1})^3 = PD^3P^{-1}$.

(ii) Use the results of part (b)(i) to suggest an expression for $(PDP^{-1})^n$ in terms of P, D and P^{-1}.

[3]

The transition matrix of a dynamic system is $T = \begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}$.

(c) Find T^5.

[1]

(d) (i) Show that $T = PDP^{-1}$.

(ii) Hence use the answer to part (c) to confirm the validity of your expression from part (b)(ii) for $n = 5$.

[4]

a) (i) $\boxed{\det P = (2)(1) - (-1)(3) = 5}$

(ii) $\boxed{P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}}$

Determinant of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = \lvert A \rvert = ad - bc$

Inverse of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad \neq bc$

Consider the matrices $\boldsymbol{P} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}$.

(a) Find

(i) the determinant of $\boldsymbol{P}$

(ii) $\boldsymbol{P}^{-1}$.

$$P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}$$

[2]

Consider the matrix product $\boldsymbol{PDP}^{-1}$.

(b) (i) Show that $(\boldsymbol{PDP}^{-1})^2 = \boldsymbol{PD}^2\boldsymbol{P}^{-1}$ and $(\boldsymbol{PDP}^{-1})^3 = \boldsymbol{PD}^3\boldsymbol{P}^{-1}$.

(ii) Use the results of part (b)(i) to suggest an expression for $(\boldsymbol{PDP}^{-1})^n$ in terms of $\boldsymbol{P}$, $\boldsymbol{D}$ and $\boldsymbol{P}^{-1}$.

[3]

The transition matrix of a dynamic system is $\boldsymbol{T} = \begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}$.

(c) Find $\boldsymbol{T}^5$.

[1]

(d) (i) Show that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$.

(ii) Hence use the answer to part (c) to confirm the validity of your expression from part (b)(ii) for $n = 5$.

[4]

b) (i) Use GDC to calculate matrix products:

$$\left(\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}\right)^2 = \begin{pmatrix} 0.784 & 0.144 \\ 0.216 & 0.856 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}^2\frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 0.784 & 0.144 \\ 0.216 & 0.856 \end{pmatrix}$$

$$\left(\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}\right)^3 = \begin{pmatrix} 0.7072 & 0.1952 \\ 0.2928 & 0.8048 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}^3\frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 0.7072 & 0.1952 \\ 0.2928 & 0.8048 \end{pmatrix}$$

(ii) $\boxed{(PDP^{-1})^n = PD^nP^{-1}}$

Consider the matrices $\boldsymbol{P} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}$.

(a) Find

(i) the determinant of $\boldsymbol{P}$

(ii) $\boldsymbol{P}^{-1}$.

[2]

Consider the matrix product $\boldsymbol{PDP}^{-1}$.

(b) (i) Show that $(\boldsymbol{PDP}^{-1})^2 = \boldsymbol{PD}^2\boldsymbol{P}^{-1}$ and $(\boldsymbol{PDP}^{-1})^3 = \boldsymbol{PD}^3\boldsymbol{P}^{-1}$.

(ii) Use the results of part (b)(i) to suggest an expression for $(\boldsymbol{PDP}^{-1})^n$ in terms of $\boldsymbol{P}$, $\boldsymbol{D}$ and $\boldsymbol{P}^{-1}$.

[3]

The transition matrix of a dynamic system is $\boldsymbol{T} = \begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}$.

(c) Find $\boldsymbol{T}^5$.

[1]

(d) (i) Show that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$.

(ii) Hence use the answer to part (c) to confirm the validity of your expression from part (b)(ii) for $n = 5$.

[4]

c) Calculate matrix power in GDC:

$$\begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}^5 = \boxed{\begin{pmatrix} 0.596608 & 0.268928 \\ 0.403392 & 0.731072 \end{pmatrix}}$$

Consider the matrices $\boldsymbol{P} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}$.

(a) Find

(i) the determinant of $\boldsymbol{P}$

(ii) $\boldsymbol{P}^{-1}$.

$$\boxed{P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}}$$

[2]

Consider the matrix product $\boldsymbol{PDP}^{-1}$.

(b) (i) Show that $(\boldsymbol{PDP}^{-1})^2 = \boldsymbol{PD}^2\boldsymbol{P}^{-1}$ and $(\boldsymbol{PDP}^{-1})^3 = \boldsymbol{PD}^3\boldsymbol{P}^{-1}$.

(ii) Use the results of part (b)(i) to suggest an expression for $(\boldsymbol{PDP}^{-1})^n$ in terms of $\boldsymbol{P}$, $\boldsymbol{D}$ and $\boldsymbol{P}^{-1}$.

$$\boxed{(PDP^{-1})^n = PD^nP^{-1}}$$

[3]

The transition matrix of a dynamic system is $\boldsymbol{T} = \begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}$.

(c) Find $\boldsymbol{T}^5$.

$$\boxed{\begin{pmatrix} 0.596608 & 0.268928 \\ 0.403392 & 0.731072 \end{pmatrix}}$$

[1]

(d) (i) Show that $\boldsymbol{T} = \boldsymbol{PDP}^{-1}$.

(ii) Hence use the answer to part (c) to confirm the validity of your expression from part (b)(ii) for $n = 5$.

[4]

d) (i) Use GDC to calculate matrix product:

$$\boxed{\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 0.88 & 0.08 \\ 0.12 & 0.92 \end{pmatrix}}$$

(ii) From parts (d)(i) and (c) we know that

$$(PDP^{-1})^5 = T^5 = \begin{pmatrix} 0.596608 & 0.268928 \\ 0.403392 & 0.731072 \end{pmatrix}$$

So now use calculator to calculate PD^5P^{-1}:

$$\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.8 \end{pmatrix}^5\frac{1}{5}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 0.596608 & 0.268928 \\ 0.403392 & 0.731072 \end{pmatrix}$$

$$\boxed{\text{So } (PDP^{-1})^5 = PD^5P^{-1}}$$

Question 5

A family are buying burgers for dinner and wish to order three veggie burgers, one beef burger and two chicken burgers.

From burger store A three veggie burgers would cost a total of \$15.45, one beef burger would cost \$6.15, and 2 chicken burgers would cost a total of \$11.90.

(a) Write down

(i) a row matrix, $\boldsymbol{Q}$, to represent the quantities of each type of burger that the family wishes to purchase

(ii) a column matrix, $\boldsymbol{P}_A$, to represent the cost of each type of burger at store A.

[3]

Burger store B sells their veggie burger, beef burger and chicken burger for \$4.75, \$5.85 and \$5.50, respectively.

(b) (i) Write down a column matrix, $\boldsymbol{P}_B$, to represent the cost of each type of burger at store B.

(ii) Hence write down a cost matrix, $\boldsymbol{C}$, to represent the costs for both stores.

[3]

(c) By first calculating the matrix $\boldsymbol{QC}$, compare the total cost of the family's dinner at stores A and B.

[2]

a) (i) $\boxed{Q = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix}}$

$$\begin{array}{ccc} \text{veggie} & \text{beef} & \text{chicken} \end{array}$$

(ii) $15.45 \div 3 = 5.15$ veggie

$11.90 \div 2 = 5.95$ chicken

$$\boxed{P_A = \begin{pmatrix} 5.15 \\ 6.15 \\ 5.95 \end{pmatrix}} \begin{array}{l} \text{veggie} \\ \text{beef} \\ \text{chicken} \end{array}$$

A family are buying burgers for dinner and wish to order three veggie burgers, one beef burger and two chicken burgers.

From burger store A three veggie burgers would cost a total of $15.45, one beef burger would cost $6.15, and 2 chicken burgers would cost a total of $11.90.

(a) Write down

(i) a row matrix, $\boldsymbol{Q}$, to represent the quantities of each type of burger that the family wishes to purchase

(ii) a column matrix, $\boldsymbol{P}_A$, to represent the cost of each type of burger at store A.

[3]

Burger store B sells their veggie burger, beef burger and chicken burger for $4.75, $5.85 and $5.50, respectively.

(b) (i) Write down a column matrix, $\boldsymbol{P}_B$, to represent the cost of each type of burger at store B.

(ii) Hence write down a cost matrix, $\boldsymbol{C}$, to represent the costs for both stores.

[3]

(c) By first calculating the matrix $\boldsymbol{QC}$, compare the total cost of the family's dinner at stores A and B.

[2]

$$\boxed{P_A = \begin{pmatrix} 5.15 \\ 6.15 \\ 5.95 \end{pmatrix}}$$ from part (a)

b) (i)

$$\boxed{P_B = \begin{pmatrix} 4.75 \\ 5.85 \\ 5.50 \end{pmatrix}} \begin{array}{l} \leftarrow \text{veggie} \\ \leftarrow \text{beef} \\ \leftarrow \text{chicken} \end{array}$$

(ii)

$$\boxed{C = \begin{pmatrix} 5.15 & 4.75 \\ 6.15 & 5.85 \\ 5.95 & 5.50 \end{pmatrix}} \begin{array}{l} \leftarrow \text{veggie} \\ \leftarrow \text{beef} \\ \leftarrow \text{chicken} \end{array}$$

Store A Store B

A family are buying burgers for dinner and wish to order three veggie burgers, one beef burger and two chicken burgers.

From burger store A three veggie burgers would cost a total of $15.45, one beef burger would cost $6.15, and 2 chicken burgers would cost a total of $11.90.

(a) Write down

(i) a row matrix, $\boldsymbol{Q}$, to represent the quantities of each type of burger that the family wishes to purchase

(ii) a column matrix, $\boldsymbol{P}_A$, to represent the cost of each type of burger at store A.

[3]

Burger store B sells their veggie burger, beef burger and chicken burger for $4.75, $5.85 and $5.50, respectively.

(b) (i) Write down a column matrix, $\boldsymbol{P}_B$, to represent the cost of each type of burger at store B.

(ii) Hence write down a cost matrix, $\boldsymbol{C}$, to represent the costs for both stores.

[3]

(c) By first calculating the matrix $\boldsymbol{QC}$, compare the total cost of the family's dinner at stores A and B.

[2]

$$\boxed{Q = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix}}$$

$$\boxed{C = \begin{pmatrix} 5.15 & 4.75 \\ 6.15 & 5.85 \\ 5.95 & 5.50 \end{pmatrix}}$$ from part (b)

c) Use GDC to calculate matrix product:

$$QC = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 5.15 & 4.75 \\ 6.15 & 5.85 \\ 5.95 & 5.50 \end{pmatrix} = \begin{pmatrix} 33.50 & 31.10 \end{pmatrix}$$

Store A Store B

The dinner would be $33.50 at Store A and $31.10 at Store B. So Store B would be $2.40 cheaper.

Question 6

A café is looking to hire a duty manager, two baristas, a dishwasher, and three waiters. They decide to advertise the jobs on social media, as well as using a hiring agency. They receive 123 applications from advertising the job on social media and 57 applications from the hiring agency.

(a) Write down a column matrix, $\boldsymbol{C}$, to represent the number of applications they received from advertising the job on social media and from using the hiring agency.

[1]

Overall, 22% of the applicants applied for the duty manager job, 28% applied for the barista job, 18% applied for the dishwasher job, and the rest applied for the waiter job. Note that every applicant was only allowed to apply for one of the available jobs.

(b) Write down a row matrix, $\boldsymbol{R}$, to represent the percentages of the applicants that applied for each of the different jobs.

[1]

(c) (i) Calculate the product $\boldsymbol{P} = \boldsymbol{CR}$.

(ii) Use the elements of the matrix $\boldsymbol{P}$ to work out the total number of applicants for each of the positions, giving your answers to the nearest integer.

[3]

Once the café has selected the right candidate for each job, each new employee will work a total of 40 hours per week. The hourly wage for the duty manager and barista jobs is \$20.00 per hour, and the hourly wage for the dishwasher and waiter jobs is \$17.25.

(d) Calculate the café's weekly wage expenses for the new employees.

[2]

a) $$\boxed{C = \begin{pmatrix} 123 \\ 57 \end{pmatrix}} \begin{array}{l} \leftarrow \text{social media} \\ \leftarrow \text{hiring agency} \end{array}$$

A café is looking to hire a duty manager, two baristas, a dishwasher, and three waiters. They decide to advertise the jobs on social media, as well as using a hiring agency. They receive 123 applications from advertising the job on social media and 57 applications from the hiring agency.

(a) Write down a column matrix, $\boldsymbol{C}$, to represent the number of applications they received from advertising the job on social media and from using the hiring agency.

$\boxed{C = \begin{pmatrix} 123 \\ 57 \end{pmatrix}}$

[1]

Overall, 22% of the applicants applied for the duty manager job, 28% applied for the barista job, 18% applied for the dishwasher job, and the rest applied for the waiter job. Note that every applicant was only allowed to apply for one of the available jobs.

(b) Write down a row matrix, $\boldsymbol{R}$, to represent the percentages of the applicants that applied for each of the different jobs.

b) $$\boxed{R = \begin{pmatrix} 0.22 & 0.28 & 0.18 & 0.32 \end{pmatrix}}$$
$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \text{duty manager} & \text{barista} & \text{dishwasher} & \text{waiter} \end{array}$$

[1]

(c) (i) Calculate the product $\boldsymbol{P} = \boldsymbol{CR}$.

(ii) Use the elements of the matrix $\boldsymbol{P}$ to work out the total number of applicants for each of the positions, giving your answers to the nearest integer.

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Once the café has selected the right candidate for each job, each new employee will work a total of 40 hours per week. The hourly wage for the duty manager and barista jobs is \$20.00 per hour, and the hourly wage for the dishwasher and waiter jobs is \$17.25.

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(a) Write down a column matrix, $\boldsymbol{C}$, to represent the number of applications they received from advertising the job on social media and from using the hiring agency.

$\boxed{C = \begin{pmatrix} 123 \\ 57 \end{pmatrix}}$

[1]

Overall, 22% of the applicants applied for the duty manager job, 28% applied for the barista job, 18% applied for the dishwasher job, and the rest applied for the waiter job. Note that every applicant was only allowed to apply for one of the available jobs.

(b) Write down a row matrix, $\boldsymbol{R}$, to represent the percentages of the applicants that applied for each of the different jobs.

$\boxed{R = \begin{pmatrix} 0.22 & 0.28 & 0.18 & 0.32 \end{pmatrix}}$

[1]

(c) (i) Calculate the product $\boldsymbol{P} = \boldsymbol{CR}$.

(ii) Use the elements of the matrix $\boldsymbol{P}$ to work out the total number of applicants for each of the positions, giving your answers to the nearest integer.

[3]

Once the café has selected the right candidate for each job, each new employee will work a total of 40 hours per week. The hourly wage for the duty manager and barista jobs is \$20.00 per hour, and the hourly wage for the dishwasher and waiter jobs is \$17.25.

(d) Calculate the café's weekly wage expenses for the new employees.

[2]

c) (i) Use GDC to calculate matrix product:

$$\boxed{CR = \begin{pmatrix} 27.06 & 34.44 & 22.14 & 39.36 \\ 12.54 & 15.96 & 10.26 & 18.24 \end{pmatrix}}$$

$$\begin{array}{cccc} \text{duty manager} & \text{barista} & \text{dishwasher} & \text{waiter} \end{array}$$

(ii) Add columns of CR to get overall totals:

$27.06 + 12.54 = 39.6$
$34.44 + 15.96 = 50.4$
$22.14 + 10.26 = 32.4$
$39.36 + 18.24 = 57.6$

40 duty manager
50 barista
32 dishwasher
58 waiter

Be careful here! You couldn't use, say, the first column of CR to determine that 27 manager applicants came from social media and 13 came from the agency, because we don't know the percentage breakdowns for the two hiring methods individually. But

$$27.06 + 12.54 = 123(0.22) + 57(0.22) = 0.22(123 + 57)$$

which is 22% of all applicants. So the column sums give the correct overall figures.

A café is looking to hire a duty manager, two baristas, a dishwasher, and three waiters. They decide to advertise the jobs on social media, as well as using a hiring agency. They receive 123 applications from advertising the job on social media and 57 applications from the hiring agency.

(a) Write down a column matrix, $\boldsymbol{C}$, to represent the number of applications they received from advertising the job on social media and from using the hiring agency.

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[1]

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(ii) Use the elements of the matrix $\boldsymbol{P}$ to work out the total number of applicants for each of the positions, giving your answers to the nearest integer.

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Once the café has selected the right candidate for each job, each new employee will work a total of 40 hours per week. The hourly wage for the duty manager and barista jobs is \$20.00 per hour, and the hourly wage for the dishwasher and waiter jobs is \$17.25.

(d) Calculate the café's weekly wage expenses for the new employees.

[2]

d) You don't have to use a matrix method to get the marks here, but using matrices can sometimes save a lot of time in the GDC.

$$(40)\begin{pmatrix} 1 & 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 20.00 \\ 20.00 \\ 17.25 \\ 17.25 \end{pmatrix} = \begin{pmatrix} 5160 \end{pmatrix}$$

$\boxed{\$\,5160}$

Question 7

Amanda is a landlord who has new sets of tenants moving into three new unfurnished homes. Amanda receives a special deal at a small local furniture store and so she offers her tenants that she will buy some tables, chairs, and/or sofas on their behalf, given that they reimburse her for the cost. The quantities of different items ordered for each house are shown in the table below.

	Tables	Chairs	Sofas
House 1	2	5	2
House 2	1	3	0
House 3	1	4	1

(a) (i) Write down a 3×3 matrix, $\boldsymbol{F}$, to represent the furniture orders for the three houses.

(ii) Write down a 1×3 row matrix, $\boldsymbol{S}$, to represent Amanda's total shopping list at the furniture store.

[2]

The price Amanda pays for each item is shown in the table below.

	Tables	Chairs	Sofas
Price	$72.00	$14.50	$47.50

(b) (i) By performing an appropriate matrix multiplication with matrix $\boldsymbol{F}$, find the total amount owed to Amanda by each house.

(ii) By performing an appropriate matrix multiplication with matrix $\boldsymbol{S}$, find the total amount paid by Amanda to the furniture store.

[5]

a) (i)

$$\boxed{F = \begin{array}{c} \\ \begin{pmatrix} 2 & 5 & 2 \\ 1 & 3 & 0 \\ 1 & 4 & 1 \end{pmatrix} \\ \text{tables} \quad \text{chairs} \quad \text{sofas} \end{array} \begin{array}{l} \leftarrow \text{House 1} \\ \leftarrow \text{House 2} \\ \leftarrow \text{House 3} \end{array}}$$

(ii) Just add up the columns of matrix F.

$$\boxed{S = (4 \quad 12 \quad 3)}$$

tables chairs sofas

Amanda is a landlord who has new sets of tenants moving into three new unfurnished homes. Amanda receives a special deal at a small local furniture store and so she offers her tenants that she will buy some tables, chairs, and/or sofas on their behalf, given that they reimburse her for the cost. The quantities of different items ordered for each house are shown in the table below.

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(a) (i) Write down a 3×3 matrix, $\boldsymbol{F}$, to represent the furniture orders for the three houses.

(ii) Write down a 1×3 row matrix, $\boldsymbol{S}$, to represent Amanda's total shopping list at the furniture store.

$$\boxed{F = \begin{pmatrix} 2 & 5 & 2 \\ 1 & 3 & 0 \\ 1 & 4 & 1 \end{pmatrix}} \qquad \boxed{S = (4 \quad 12 \quad 3)}$$

[2]

The price Amanda pays for each item is shown in the table below.

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(ii) By performing an appropriate matrix multiplication with matrix $\boldsymbol{S}$, find the total amount paid by Amanda to the furniture store.

[5]

b) Write the prices as a column matrix, and use GDC to calculate matrix products:

(i)

$$\begin{pmatrix} 2 & 5 & 2 \\ 1 & 3 & 0 \\ 1 & 4 & 1 \end{pmatrix} \times \begin{pmatrix} 72.00 \\ 14.50 \\ 47.50 \end{pmatrix} = \begin{pmatrix} 311.5 \\ 115.5 \\ 177.5 \end{pmatrix} \begin{array}{l} \leftarrow \text{House 1} \\ \leftarrow \text{House 2} \\ \leftarrow \text{House 3} \end{array}$$

$$\boxed{\begin{array}{l} \text{House 1: } \$311.50 \\ \text{House 2: } \$115.50 \\ \text{House 3: } \$177.50 \end{array}}$$

(ii)

$$(4 \quad 12 \quad 3) \times \begin{pmatrix} 72.00 \\ 14.50 \\ 47.50 \end{pmatrix} = (604.5)$$

$$\boxed{\$604.50}$$

Question 8

The bus fare a person pays in a city is dependent on whether the person is a student, an adult or a pensioner.

The total amount taken in on a particular day by three different buses, along with the numbers of each type of fare paid, are shown in the table below.

	Student	Adult	Pensioner	Total
Bus A	91	82	13	$348
Bus B	102	80	4	$355
Bus C	71	54	11	$247

Let s, a and p represent the amount paid by a student, an adult, and a pensioner respectively.

(a) Write down a system of three linear equations in terms of s, a and p that represent the information shown in the table above.

[2]

(b) Find the values of s, a and p using appropriate matrices and matrix inverses.

[2]

Bus D finished the day having sold 112 student tickets, 91 adult tickets and 22 pensioner tickets.

(c) Calculate the total amount taken in by Bus D. Give your answer to 2 decimal places.

[2]

a)

$$\boxed{\begin{aligned} 91s + 82a + 13p &= 348 \\ 102s + 80a + 4p &= 355 \\ 71s + 54a + 11p &= 247 \end{aligned}}$$

The bus fare a person pays in a city is dependent on whether the person is a student, an adult or a pensioner.

The total amount taken in on a particular day by three different buses, along with the numbers of each type of fare paid, are shown in the table below.

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[2]

(b) Find the values of s, a and p using appropriate matrices and matrix inverses.

[2]

Bus D finished the day having sold 112 student tickets, 91 adult tickets and 22 pensioner tickets.

(c) Calculate the total amount taken in by Bus D. Give your answer to 2 decimal places.

[2]

b) Write system in matrix form:

$$\begin{pmatrix} 91 & 82 & 13 \\ 102 & 80 & 4 \\ 71 & 54 & 11 \end{pmatrix} \begin{pmatrix} s \\ a \\ p \end{pmatrix} = \begin{pmatrix} 348 \\ 355 \\ 247 \end{pmatrix}$$

Use matrix inverse to solve system in GDC:

$$\begin{pmatrix} s \\ a \\ p \end{pmatrix} = \begin{pmatrix} 91 & 82 & 13 \\ 102 & 80 & 4 \\ 71 & 54 & 11 \end{pmatrix}^{-1} \begin{pmatrix} 348 \\ 355 \\ 247 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 2.5 \\ 0.5 \end{pmatrix}$$

$$\boxed{s = \$1.50 \quad a = \$2.50 \quad p = \$0.50}$$

EXAM PAPERS PRACTICE

The bus fare a person pays in a city is dependent on whether the person is a student, an adult or a pensioner.

The total amount taken in on a particular day by three different buses, along with the numbers of each type of fare paid, are shown in the table below.

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Let s, a and p represent the amount paid by a student, an adult, and a pensioner respectively.

(a) Write down a system of three linear equations in terms of s, a and p that represent the information shown in the table above.

[2]

(b) Find the values of s, a and p using appropriate matrices and matrix inverses.

$\boxed{s = \$1.50 \quad a = \$2.50 \quad p = \$0.50}$

[2]

Bus D finished the day having sold 112 student tickets, 91 adult tickets and 22 pensioner tickets.

(c) Calculate the total amount taken in by Bus D. Give your answer to 2 decimal places.

[2]

c) You don't have to use a matrix method to get the marks here, but using matrices can sometimes save a lot of time in the GDC.

$$\begin{pmatrix} 112 & 91 & 22 \end{pmatrix}\begin{pmatrix} 1.50 \\ 2.50 \\ 0.50 \end{pmatrix} = \begin{pmatrix} 406.5 \end{pmatrix}$$

$\boxed{\$ 406.50}$

Question 9

The number of times a gamer logged in to Call of Duty, FIFA, or Assassin's Creed over 3 weeks is shown in the table below.

	Call of Duty	FIFA	Assassin's Creed
Week 1	5	4	3
Week 2	7	2	4
Week 3	3	5	6

The total number of hours the gamer spent playing each week is shown in the table below.

	Week 1	Week 2	Week 3
Total hours	12.35	13.84	14.16

The gamer was never logged in to more than one game at the same time.

The gamer believes that, for each game, the average amount of time spent playing per log-in session was consistent over the three weeks.

(a) Assuming that the gamer's belief is true, use matrix multiplication to find the average number of hours and minutes per log-in session that the gamer spent playing each game.

[3]

(b) Write down a system of linear equations that could be used to find the answers in part (a).

[2]

a) Let x be the average time for Call of Duty, y the average time for FIFA, and z the average time for Assassin's Creed.

Represent system in matrix form:

$$\begin{pmatrix} 5 & 4 & 3 \\ 7 & 2 & 4 \\ 3 & 5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12.35 \\ 13.84 \\ 14.16 \end{pmatrix}$$

Use matrix inverse to solve system in GDC:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 & 4 & 3 \\ 7 & 2 & 4 \\ 3 & 5 & 6 \end{pmatrix}^{-1}\begin{pmatrix} 12.35 \\ 13.84 \\ 14.16 \end{pmatrix} = \begin{pmatrix} 1.12 \\ 0.9 \\ 1.05 \end{pmatrix}$$

Convert solutions to hours and minutes:

Call of Duty : 1 h 7.2 m per session
FIFA : 54 m per session
Assassin's Creed : 1 h 3 m per session

EXAM PAPERS PRACTICE

The number of times a gamer logged in to Call of Duty, FIFA, or Assassin's Creed over 3 weeks is shown in the table below.

	Call of Duty	FIFA	Assassin's Creed
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(a) Assuming that the gamer's belief is true, use matrix multiplication to find the average number of hours and minutes per log-in session that the gamer spent playing each game.

[3]

(b) Write down a system of linear equations that could be used to find the answers in part (a).

[2]

b)

$$\boxed{\begin{aligned} 5x + 4y + 3z &= 12.35 \\ 7x + 2y + 4z &= 13.84 \\ 3x + 5y + 6z &= 14.16 \end{aligned}}$$

Question 10

The graph of the quadratic function $f(x) = ax^2 + bx + c$ passes through the points $(-1, 11)$, $(3, -5)$ and $(6, 4)$.

(a) Show that a, b and c must satisfy the following system of linear equations:

$$a - b + c = 11$$
$$9a + 3b + c = -5$$
$$36a + 6b + c = 4$$

[2]

(b) Represent the system of equations in part (a) in matrix form.

[1]

(c) Hence use a matrix method to find the values of a, b and c.

[3]

a) The three points give us $(x, f(x))$ for 3 values of x.

When $x = -1$,

$a(-1)^2 + b(-1) + c = 11$

$\Rightarrow \boxed{a - b + c = 11}$

When $x = 3$,

$a(3)^2 + b(3) + c = -5$

$\Rightarrow \boxed{9a + 3b + c = -5}$

When $x = 6$,

$a(6)^2 + b(6) + c = 4$

$\Rightarrow \boxed{36a + 6b + c = 4}$

EXAM PAPERS PRACTICE

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[2]

(b) Represent the system of equations in part (a) in matrix form.

[1]

(c) Hence use a matrix method to find the values of a, b and c.

[3]

b) $$\boxed{\begin{pmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 36 & 6 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \\ 4 \end{pmatrix}}$$

The graph of the quadratic function $f(x) = ax^2 + bx + c$ passes through the points $(-1, 11)$, $(3, -5)$ and $(6, 4)$.

(a) Show that a, b and c must satisfy the following system of linear equations:

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[2]

(b) Represent the system of equations in part (a) in matrix form.

[1]

(c) Hence use a matrix method to find the values of a, b and c.

[3]

$$\boxed{\begin{pmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 36 & 6 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \\ 4 \end{pmatrix}}$$

c) Use matrix inverse to solve system in GDC:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 36 & 6 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 11 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix}$$

$$\boxed{a = 1 \quad b = -6 \quad c = 4}$$

So $f(x) = x^2 - 6x + 4$

Question 11

The graph of the function $f(x) = ax^2 + bx + c$ passes through the points $(-1, 5)$, $(3, 1)$ and $(4, -5)$.

(a) Write down a system of linear equations that a, b and c must satisfy.

[2]

(b) Hence, using a matrix method or otherwise, determine the values of a, b and c.

[4]

a) The three points give us $(x, f(x))$ for 3 values of x.

$$a(-1)^2 + b(-1) + c = 5$$

$$a(3)^2 + b(3) + c = 1$$

$$a(4)^2 + b(4) + c = -5$$

$$\boxed{\begin{aligned} a - b + c &= 5 \\ 9a + 3b + c &= 1 \\ 16a + 4b + c &= -5 \end{aligned}}$$

The graph of the function $f(x) = ax^2 + bx + c$ passes through the points $(-1, 5)$, $(3, 1)$ and $(4, -5)$.

(a) Write down a system of linear equations that a, b and c must satisfy.

[2]

(b) Hence, using a matrix method or otherwise, determine the values of a, b and c.

[4]

$$\boxed{\begin{aligned} a - b + c &= 5 \\ 9a + 3b + c &= 1 \\ 16a + 4b + c &= -5 \end{aligned}}$$

b) Write system in matrix form:

$$\begin{pmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -5 \end{pmatrix}$$

Use matrix inverse to solve system in GDC:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 7 \end{pmatrix}$$

$$\boxed{a = -1 \quad b = 1 \quad c = 7}$$

So $f(x) = -x^2 + x + 7$

Question 12

The amounts of wheat, soybeans and sugar produced by three different farms in a given week, along with the respective total revenues for each farm, are shown in the table below.

	Wheat, kg	Soybeans, kg	Sugar, kg	Revenue, $
Farm A	820	532	535	835.54
Farm B	1210	641	274	948.75
Farm C	922	211	503	716.11

Let x, y and z represent the prices, in \$/kg, for wheat, soybeans and sugar respectively.

(a) (i) Write down a system of linear equations that represents the information in the table above.

(ii) Solve the system of linear equations using matrices.

[4]

In the same week, Farm D produced a fifth of the amount of wheat as Farm A, a quarter of the amount of soybeans as Farm B, and half the amount of sugar as Farm C.

(b) Calculate the revenue made by Farm D from selling these crops. Give your answer correct to two decimal places.

[3]

a) (i)

$$\boxed{\begin{aligned} 820x + 532y + 535z &= 835.54 \\ 1210x + 641y + 274z &= 948.75 \\ 922x + 211y + 503z &= 716.11 \end{aligned}}$$

(ii) Write system in matrix form:

$$\begin{pmatrix} 820 & 532 & 535 \\ 1210 & 641 & 274 \\ 922 & 211 & 503 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 835.54 \\ 948.75 \\ 716.11 \end{pmatrix}$$

Use matrix inverse to solve system in GDC:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 820 & 532 & 535 \\ 1210 & 641 & 274 \\ 922 & 211 & 503 \end{pmatrix}^{-1}\begin{pmatrix} 835.54 \\ 948.75 \\ 716.11 \end{pmatrix} = \begin{pmatrix} 0.44 \\ 0.47 \\ 0.42 \end{pmatrix}$$

wheat : \$ 0.44 / kg

soybeans : \$ 0.47 / kg

sugar : \$ 0.42 / kg

The amounts of wheat, soybeans and sugar produced by three different farms in a given week, along with the respective total revenues for each farm, are shown in the table below.

	Wheat, kg	Soybeans, kg	Sugar, kg	Revenue, $
Farm A	820	532	535	835.54
Farm B	1210	641	274	948.75
Farm C	922	211	503	716.11

Let x, y and z represent the prices, in \$/kg, for wheat, soybeans and sugar respectively.

(a) (i) Write down a system of linear equations that represents the information in the table above.

(ii) Solve the system of linear equations using matrices.

[4]

In the same week, Farm D produced a fifth of the amount of wheat as Farm A, a quarter of the amount of soybeans as Farm B, and half the amount of sugar as Farm C.

(b) Calculate the revenue made by Farm D from selling these crops. Give your answer correct to two decimal places.

[3]

b) You don't <u>have</u> to use a matrix method to get the marks here, but using matrices can sometimes save a lot of time in the GDC.

$$\begin{pmatrix} \frac{820}{5} & \frac{641}{4} & \frac{503}{2} \end{pmatrix}\begin{pmatrix} 0.44 \\ 0.47 \\ 0.42 \end{pmatrix} = (253.1075)$$

$$\boxed{\$\ 253.11\ (2\text{ d.p.})}$$

Question 13

Grace has decided that she wants to invest $10 000 split between three companies: company A, company B, and company C. She creates three different portfolio options based on risk levels, and calculates what each option's value would be today if the identical amounts had been invested one year ago.

	Company A	Company B	Company C	Value
Safe	$1500	$8000	$500	$10,620.00
Middle	$2000	$6750	$1250	$10,827.50
Risky	$2500	$2500	$5000	$11,725.00

(a) Use a matrix method to find the annual percentage return (i.e., the percentage increase or decrease of an investment in the company) for the previous year for each company.

[6]

Grace hears some good news about the growth of company A before she invests her money, and so she decides to put 78% of it into company A and split the rest evenly between company B and company C.

Compared with the previous year, the annual percentage return for company A for the coming year is expected to increase by 26 percentage points (so if the previous year's return was x%, then the return is expected to be $(x + 26)$% for the coming year). For company B the return is expected to remain the same, while for company C it is expected to decrease by 8 percentage points.

(b) Find the expected value of Grace's investment at the end of the coming year.

[2]

a) Let x, y and z be the multipliers for investments in companies A, B and C respectively.

Represent system in matrix form:

$$\begin{pmatrix} 1500 & 8000 & 500 \\ 2000 & 6750 & 1250 \\ 2500 & 2500 & 5000 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10620.00 \\ 10827.50 \\ 11725.00 \end{pmatrix}$$

Use matrix inverse to solve system in GDC:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1500 & 8000 & 500 \\ 2000 & 6750 & 1250 \\ 2500 & 2500 & 5000 \end{pmatrix}^{-1}\begin{pmatrix} 10620.00 \\ 10827.50 \\ 11725.00 \end{pmatrix} = \begin{pmatrix} 1.11 \\ 1.04 \\ 1.27 \end{pmatrix}$$

Intepret answers as percentage increases:

The percentage returns are:

Company A: 11%
Company B: 4%
Company C: 27%

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(b) Find the expected value of Grace's investment at the end of the coming year.

[2]

The percentage returns are:
Company A: 11%
Company B: 4%
Company C: 27%

from part (a)

b) $0.78 \times 10000 = 7800$ (Company A)

$\frac{1}{2} \times (10000 - 7800) = 1100$ (Companies B, C)

$11 + 26 = 37\%$ (new Company A return)

$27 - 8 = 19\%$ (new Company C return)

You don't have to use a matrix method to get the marks here, but using matrices can sometimes save a lot of time in the GDC.

$$\begin{pmatrix} 7800 & 1100 & 1100 \end{pmatrix}\begin{pmatrix} 1.37 \\ 1.04 \\ 1.19 \end{pmatrix} = \begin{pmatrix} 13139 \end{pmatrix}$$

$$\boxed{\$\,13139}$$