Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
## Higher level
## Paper 1

14 May 2026

Zone A afternoon | Zone B afternoon | Zone C afternoon

Session number

2 hours

### Instructions to students

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.



## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 5]

Consider the function $f(x) = 3(x-2)(x-q)$, where $x \in \mathbb{R}$ and $q$ is a real constant.

The axis of symmetry of the graph of $f$ has equation $x = 4$.

(a) Show that $q = 6$. [2]

(b) Find the coordinates of the vertex of the graph of $f$. [2]

(c) Hence, write down the range of $f$. [1]

a) $\frac{2+q}{2} = 4$

$2+q = 8$

$q = 6$

b) $f(4) = 3\times(4-2)\times(4-6) = 3\times 2\times(-2) = -12$

$(4, -12)$

c) $y \geq -12$

**2.** [Maximum mark: 5]

A particle $P$ is travelling along a straight line, with displacement measured relative to a point O on the line. The velocity, $v\ \mathrm{m\,s^{-1}}$, of the particle at time $t$ seconds is given by

$$v = \frac{1}{t+1} + e^{-t}, \text{ where } t \geq 0.$$

(a) Find $\int \left( \frac{1}{t+1} + e^{-t} \right) dt$. [2]

Initially the particle is at O.

(b) Find an expression for the displacement, $s$ metres, in terms of $t$. [3]

a) $\int \frac{1}{t+1} + e^{-t}\, dt$

$= \ln|t+1| - e^{-t} + C$

b) $S = \int v\, dt = \ln|t+1| - e^{-t} + C$

When $t=0$, $S=0$, $0 = \ln 1 - e^{0} + C$

$C = 1$

$\therefore S = \ln(t+1) - e^{-t} + 1$

**3.** [Maximum mark: 6]

Consider a geometric sequence, $u_n$, with common ratio $r$, where each term in the sequence is positive.

It is given that $u_6 = 1.6 \times 10^5$ and $u_8 = 6.4 \times 10^5$.

(a) Find the value of $r$. [3]

(b) Find the value of $u_1$, giving your answer in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$. [3]

a) $r^2 = \frac{u_8}{u_6} = \frac{6.4 \times 10^5}{1.6 \times 10^5} = 4$

$r = 2$ $(r > 0)$

b) $u_6 = u_1 \times r^5$

$1.6 \times 10^5 = u_1 \times 2^5$

$u_1 = \frac{16 \times 10^4}{32} = 0.5 \times 10^4 = 5 \times 10^3$

**4.** [Maximum mark: 7]

(a) Show that $\log_{25}(81x^2) = \log_5 9 + \log_5 x$ for $x > 0$. [4]

(b) Hence, solve the equation $\log_5 \sqrt[3]{x} = \log_{25}(81x^2) - \log_5 4$, where $x > 0$. [3]

a) $LHS = \log_{25} 81 + \log_{25} x^2$

$= \frac{\log_5 81}{\log_5 25} + \frac{\log_5 x^2}{\log_5 25}$

$= \frac{\log_5 (9^2)}{2} + \frac{2\log_5 x}{2}$

$= \frac{2\log_5 9}{2} + \frac{2\log_5 x}{2}$

$= \log_5 9 + \log_5 x$ $= RHS$

b) $\log_5 (x^{\frac{1}{3}}) = \log_5 9 + \log_5 x - \log_5 4$

$\frac{1}{3}\log_5 x = \log_5 \frac{9}{4} + \log_5 x$

$-\frac{2}{3}\log_5 x = \log_5 \frac{9}{4}$

$\log_5 x^{-\frac{2}{3}} = \log_5 \frac{9}{4}$

$x^{-\frac{2}{3}} = \frac{9}{4}$

$x^{\frac{2}{3}} = \frac{4}{9}$ $(x>0)$

$x^{\frac{1}{3}} = \frac{2}{3}$

$x = \frac{8}{27}$

**5.** [Maximum mark: 9]

Consider two events, $A$ and $B$.

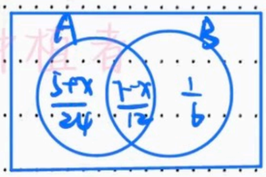
It is given that $\mathrm{P}(A \cap B') = \frac{5+x}{24}$, $\mathrm{P}(A \cap B) = \frac{7-x}{12}$ and $\mathrm{P}(A' \cap B) = \frac{1}{6}$, where $0 \le x \le 7$.

(a) Show that $\mathrm{P}(A) = \frac{19-x}{24}$. [3]

It is given that $A$ and $B$ are independent.

(b) Find the possible values of $x$. [6]

a) $P(A) = P(A \cap B') + P(A \cap B)$




$= \frac{5+x}{24} + \frac{7-x}{12}$

$= \frac{5+x+14-2x}{24}$

$= \frac{19-x}{24}$

b) $P(A \cap B) = P(A) \times P(B)$

$P(B) = \frac{7-x}{12} + \frac{1}{6} = \frac{7-x+2}{12} = \frac{9-x}{12}$

$\frac{7-x}{\cancel{12}} = \frac{19-x}{24} \times \frac{9-x}{\cancel{12}}$

$24(7-x) = (19-x)(9-x)$

$168 - 24x = 171 - 28x + x^2$

$0 = x^2 - 4x + 3$

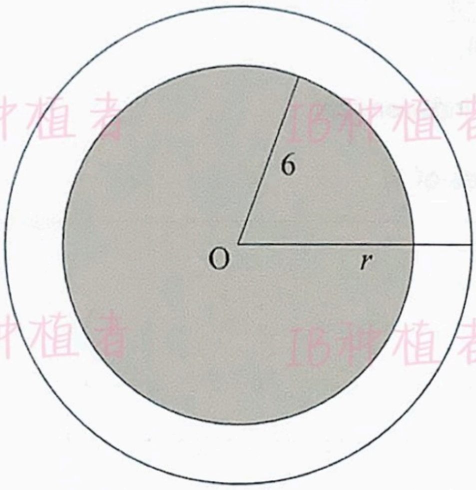
$0 = (x-1)(x-3)$

$x = 1$ or $3$

**6.** [Maximum mark: 4]

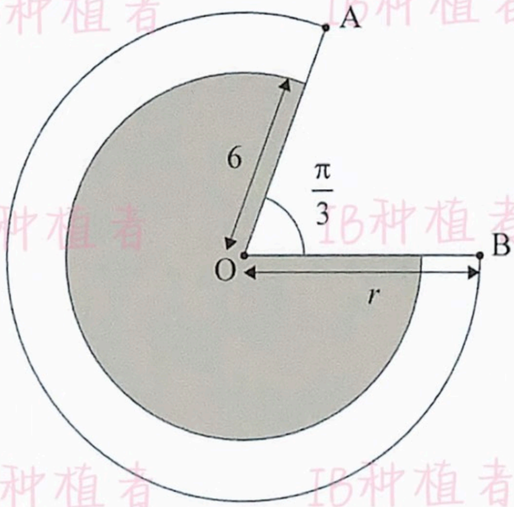
Consider a piece of paper in the shape of a circle with centre O and radius $r$ cm, where $r > 6$. A second circle with centre O and radius 6 cm is drawn inside the first and shaded, as shown in the following diagram.

**diagram not to scale**




The points A and B are on the circumference of the larger circle such that the acute angle $\text{A}\hat{\text{O}}\text{B} = \frac{\pi}{3}$. The paper is cut along the lines AO and BO and the sector AOB with a central angle of $\frac{\pi}{3}$ is removed as shown on the following diagram.

**diagram not to scale**




**(This question continues on the following page)**

(Question 6 continued)

After removing the sector, it is now given that

$$\frac{\text{the area of the unshaded region}}{\text{the area of the shaded region}} = \frac{2}{3}$$

Find the value of $r$, giving your answer in the form $\sqrt{k}$, where $k \in \mathbb{Z}$.

$A_S = \frac{1}{2} \times 6^2 \times \left(2\pi - \frac{\pi}{3}\right) = 18 \times \frac{5}{3}\pi = 30\pi$

$A_{OAB} = \frac{1}{2} r^2 \times \frac{5}{3}\pi = \frac{5}{6} r^2 \pi$

$A_{blank} = \frac{5}{6} r^2 \pi - 30\pi$

$\therefore \frac{\frac{5}{6} r^2 \pi - 30\pi}{30\pi} = \frac{2}{3}$

$\frac{5}{2} r^2 \pi - 90\pi = 60\pi$

$\frac{5}{2} r^2 \cancel{\pi} = 150 \cancel{\pi}$

$r^2 = 60$

$r = \sqrt{60}$

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**7.** [Maximum mark: 6]

Reyn is investigating the fertility of a group of adult female foxes.

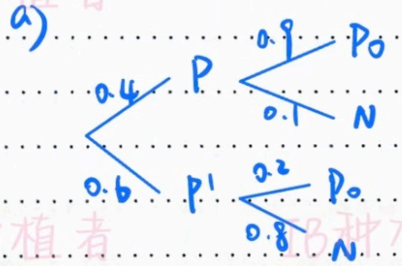
He has developed a clinical test that aims to identify whether or not a given fox is pregnant, and he wants to evaluate the reliability of this test.

Reyn knows that
- $40\%$ of the foxes in the group are pregnant
- if a fox is pregnant, the probability that it will test positive is $90\%$
- if a fox is not pregnant, the probability it will test positive is $20\%$.

(a) Find the probability that a randomly selected fox from the group will test positive. [3]

(b) Find the probability that a randomly selected fox from the group is pregnant, given that it tested positive. [3]

a)




$0.4\times 0.9 + 0.6\times 0.2 = 0.36 + 0.12 = 0.48$

b) $P(P \mid P_o) = \dfrac{P(P\cap P_o)}{P(P_o)}$

$= \dfrac{0.4\times 0.9}{0.48} = \dfrac{0.36}{0.48} = \dfrac{3}{4}$

**8.** [Maximum mark: 8]

Consider the polynomial $p(z) = 3z^5 - 7z^4 + cz^3 + dz^2 + ez - 26$, where $z \in \mathbb{C}$ and $c, d, e \in \mathbb{R}$.

Three of the roots of $p(z)$ are $\frac{1}{3}$, $a - \mathrm{i}$, $2 + b\mathrm{i}$, where $a, b \in \mathbb{R}$, $b > 1$.

(a) By considering the sum and the product of the roots of $p(z)$,

(i) show that $a = -1$;

(ii) find the value of $b$. [5]

(b) Show that $p(z) = (Az + B)(z^2 + Cz + D)(z^2 + Ez + F)$, where $A, B, C, D, E, F$ are integers to be determined. [3]

a/ i) Sum $= -\frac{-7}{3} = \frac{1}{3} + a - i + a + i + 2 + bi + 2 - bi$

$\frac{7}{3} = \frac{1}{3} + 2a + 4$

$2 = 2a + 4$

$2a = -2$

$a = -1$

2) product $= (-1)^5 \frac{-26}{3} = \frac{1}{3} \times (-1-i) \times (-1+i) \times (2+bi)(2-bi)$

$\frac{26}{\cancel{3}} = \frac{1}{\cancel{3}} \times (1+1) \times (4+b^2)$

$26 = 2 \times (4 + b^2)$

$13 = 4 + b^2$

$b = \pm 3$

$\because b > 1 \quad \therefore b = 3$

b) Let $P(z) = a(z - \frac{1}{3})(z - (-1-i))(z - (-1+i))(z - (2+3i))(z - (2-3i))$

$= 3(z - \frac{1}{3})(z + 1 + i)(z + 1 - i)(z - 2 - 3i)(z - 2 + 3i)$

$= (3z - 1)[(z+1)^2 + 1][(z-2)^2 + 9]$

$= (3z - 1)(z^2 + 2z + 2)(z^2 - 4z + 13)$

**9.** [Maximum mark: 8]

Consider non-zero vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in three-dimensional space.

The non-zero vector $\boldsymbol{v}$ is defined as $\boldsymbol{v} = (\boldsymbol{a} \bullet \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \bullet \boldsymbol{b})\boldsymbol{c}$.

(a) Show that $\boldsymbol{v} \bullet \boldsymbol{a} = 0$. [2]

It is given that $\boldsymbol{b} \times \boldsymbol{c}$ and $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ are non-zero vectors.

(b) (i) Justify why $\boldsymbol{b} \bullet (\boldsymbol{b} \times \boldsymbol{c}) = 0$.

(ii) Show that $\boldsymbol{v} \bullet (\boldsymbol{b} \times \boldsymbol{c}) = 0$. [4]

(c) Hence, state the geometrical relationship between $\boldsymbol{v}$ and $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$, briefly justifying your answer. [2]

a) $\vec{v} \cdot \vec{a} = \left[(\vec{a} \cdot \vec{c}) \cdot \vec{b} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}\right] \cdot \vec{a}$

$= (\vec{a} \cdot \vec{c}) \cdot \vec{b} \cdot \vec{a} - (\vec{a} \cdot \vec{b}) \cdot \vec{c} \cdot \vec{a}$

$= (\vec{a} \cdot \vec{c}) \cdot \vec{a} \cdot \vec{b} - (\vec{a} \cdot \vec{b}) \cdot \vec{a} \cdot \vec{c} = 0$ (Commutative)

$\therefore \vec{v} \cdot \vec{a} = 0$

b)1) $\vec{b} \times \vec{c}$ is perpendicular to $\vec{b}$, hence the angle between $(\vec{b} \times \vec{c})$ and $\vec{b}$ is $90°$

$\therefore \vec{b} \cdot (\vec{b} \times \vec{c}) = |\vec{b}| \cdot |\vec{b} \times \vec{c}| \cdot \cos 90° = 0$

2) $\left[(\vec{a} \cdot \vec{c}) \cdot \vec{b} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}\right] \cdot (\vec{b} \times \vec{c})$

$= (\vec{a} \cdot \vec{c}) \cdot \vec{b} \cdot (\vec{b} \times \vec{c}) - (\vec{a} \cdot \vec{b}) \cdot \vec{c} \cdot (\vec{b} \times \vec{c})$

From b)1) $\vec{b} \cdot (\vec{b} \times \vec{c}) = 0$ and $\vec{c} \cdot (\vec{b} \times \vec{c}) = 0$

$\therefore \vec{v} \cdot (\vec{b} \times \vec{c}) = 0$

c) $\vec{v} \perp \vec{a}$

$\vec{v} \perp \vec{b} \times \vec{c}$

$\Rightarrow \vec{a} \times (\vec{b} \times \vec{c})$ is parallel to $\vec{v}$

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 15]

Consider the function $f(x) = ax^3 + bx^2 + cx + d$, where $x \in \mathbb{R}$ and where $a$, $b$, $c$ and $d$ are real constants.

The graph of $f$ has a point of inflexion at $(-2, -2c + d + 24)$.

(a) Show that $a = \frac{3}{2}$ and $b = 9$. [6]

It is given that the tangents to the graph of $f$ at $x = -3$ and $x = k$ are horizontal.

(b) (i) Show that $c = \frac{27}{2}$.

(ii) Find the value of $k$.

(iii) State whether $f$ has a local maximum or a local minimum at $x = k$, justifying your answer. [7]

The graph of $f$ intersects the $y$-axis at the point P.

(c) Show that the tangent to the graph of $f$ at $x = -3$ passes through P. [2]

a) $-\cancel{2c}+\cancel{d}+24=-8a+4b-\cancel{2c}+\cancel{d} \Rightarrow 24=-8a+4b$ ①

$f'(x)=3ax^2+2bx+c$

$f''(x)=6ax+2b$

$f''(-2)=0$

$-12a+2b=0$ ②

$$\begin{cases} 6=-2a+b \\ -6a+b=0 \Rightarrow b=6a \end{cases}$$

$6=-2a+6a$

$a=\frac{6}{4}=\frac{3}{2}$

$b=6a=6\times\frac{3}{2}=9$

b/1) $f'(-3)=0$

$3a\times 9-6b+c=0$

$27\times\frac{3}{2}-6\times 9+c=0$

$c=54-\frac{81}{2}$

$c=\frac{108-81}{2}$

$c=\frac{27}{2}$

2) $f'(x) = 0$

$\frac{9}{2}x^2 + 18x + \frac{27}{2} = 0$

$\frac{1}{2}x^2 + 2x + \frac{3}{2} = 0$

$x^2 + 4x + 3 = 0$

$(x+1)(x+3) = 0$

$x = -1$ or $-3$

$k = -1$

3) $f''(-1) = 6 \times \frac{3}{2} \times (-1) + 2 \times 9$

$= -9 + 18 = 9 > 0 \Rightarrow$ concave up

$\therefore x = -1$ is a local min

c) $P(0, d)$

$f(-3) = \frac{3}{2} \times (-3)^3 + 9 \times (-3)^2 + \frac{27}{2} \times (-3) + d$

$= \frac{3}{2} \times (-27) + 81 - \frac{81}{2} + d$

$= \cancel{\frac{-81}{2}} + \cancel{81} - \cancel{\frac{81}{2}} + d = d$

$\therefore$ Tangent at $x = -3$ passes point $P$.

Do **not** write solutions on this page.

**11.** [Maximum mark: 19]

(a) Prove the identity $\cot 2\theta \equiv \frac{1}{2}\left(\cot\theta - \frac{1}{\cot\theta}\right)$. [3]

(b) Hence, solve the equation

$$\cot 2\theta = \frac{3}{2}\cot\theta\left(\cot^2\theta - 1\right) \text{ for } 0 < \theta < \pi \text{ where } \theta \neq \frac{\pi}{2}.$$ [6]

(c) By differentiating both sides of the identity in part (a), show that

$$4\operatorname{cosec}^2 2\theta \equiv \operatorname{cosec}^2\theta + \sec^2\theta.$$ [5]

Consider the right-angled triangle $\Delta\text{PQR}$, where $\text{R}\hat{\text{P}}\text{Q} = \frac{\pi}{2}$.

$\text{PR} = \operatorname{cosec}\theta$, $\text{QR} = 2\operatorname{cosec} 2\theta$, where $0 < \theta < \frac{\pi}{2}$ and all lengths are given in metres.

This is shown in the following diagram.

**diagram not to scale**


QR has length $L$ metres and $\Delta\text{PQR}$ has area $A$ square metres.

(d) Calculate the ratio $L : A$ in the form $n : 1$, where $n \in \mathbb{Z}^+$. [5]

a) $\text{LHS} = \dfrac{1}{\tan(2\theta)} = \dfrac{1}{\dfrac{2\tan\theta}{1-\tan^2\theta}}$

$= \dfrac{1-\tan^2\theta}{2\tan\theta} = \dfrac{1}{2}\left(\dfrac{1-\tan^2\theta}{\tan\theta}\right)$

$= \dfrac{1}{2}\left(\dfrac{1}{\tan\theta} - \tan\theta\right)$

$= \dfrac{1}{2}\left(\cot\theta - \dfrac{1}{\cot\theta}\right) = \text{RHS}.$

b) $\dfrac{1}{2}\left(\cot\theta - \dfrac{1}{\cot\theta}\right) = \dfrac{3}{2}\cot\theta\left(\cot^2\theta - 1\right)$

$\cot\theta - \dfrac{1}{\cot\theta} = 3\cot\theta\left(\cot^2\theta - 1\right)$

Let $t = \cot\theta$

$t - \dfrac{1}{t} = 3t(t^2-1)$

$t^2 - 1 = 3t^2(t^2-1)$

$0 = (3t^2-1)(t^2-1)$

$\therefore 3t^2 - 1 = 0$ or $t^2 - 1 = 0$

$\therefore t = \pm\sqrt{\frac{1}{3}}$ or $t = \pm 1$

$\therefore \cot\theta = \pm\sqrt{\frac{1}{3}}$ or $\pm 1$

$\therefore \tan\theta = \pm\sqrt{3}$ or $\pm 1$

$\therefore \theta = \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$ or $\dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$

c)

$$-\operatorname{cosec}^2(2\theta)\times 2 \equiv \frac{1}{2}\left(-\operatorname{cosec}^2\theta-\sec^2\theta\right)$$

$$-4\operatorname{cosec}^2(2\theta) \equiv -\operatorname{cosec}^2\theta-\sec^2\theta$$

$$4\operatorname{cosec}^2(2\theta) = \operatorname{cosec}^2\theta+\sec^2\theta$$

d)

$$\theta P = \sqrt{4\operatorname{cosec}^2 2\theta-\operatorname{cosec}^2\theta}$$

From c

$$= \sec\theta$$

$$A = \frac{1}{2}\times\operatorname{cosec}\theta\times\sec\theta$$

$$L:A = 2\operatorname{cosec}2\theta : \frac{1}{2}\operatorname{cosec}\theta\times\sec\theta$$

$$= \frac{4\operatorname{cosec}2\theta}{\operatorname{cosec}\theta\times\sec\theta}$$

$$= \frac{4\times\frac{1}{\sin 2\theta}}{\frac{1}{\sin\theta}\times\frac{1}{\cos\theta}}$$

$$= \frac{4\times\frac{1}{\sin 2\theta}}{\frac{1}{\frac{1}{2}\sin 2\theta}}$$

$$= \frac{\frac{4}{\sin 2\theta}}{\frac{2}{\sin 2\theta}}$$

$$= 2$$

$$= 2:1 \Rightarrow n=2$$

Do **not** write solutions on this page.

**12.** [Maximum mark: 18]

Rowan is investigating the spread of an infection through a population of rabbits.

He models $N$, the number (in thousands) of infected rabbits in the population, at time $t$ days.

Rowan models the spread of the infection by the differential equation

$$\frac{dN}{dt} = k\left(-3+4N-N^2\right), \text{ where } t \geq 0 \text{ and } k \in \mathbb{R}^+.$$

Initially, $1500$ rabbits are infected. $N=1.5$

(a) Express $\frac{1}{-3+4x-x^2}$ in the form $\frac{A}{x-1}+\frac{B}{3-x}$, where $A, B \in \mathbb{R}$. [3]

(b) Hence, show that $\ln\left(\frac{N-1}{3-N}\right) = 2kt - \ln p$, where $p \in \mathbb{R}^+$ is a constant to be determined. [8]

(c) Find an expression for $N$ in the form $N = \frac{a+b\mathrm{e}^{-2kt}}{1+c\mathrm{e}^{-2kt}}$, where $a, b, c \in \mathbb{Z}$. [5]

According to Rowan's model, the number of infected rabbits approaches a limit, $L$, over the long term.

(d) Find the value of $L$. [2]

a)

Let $\frac{1}{(x-1)(3-x)} \equiv \frac{A}{x-1} + \frac{B}{3-x}$

$\equiv \frac{A(3-x)+B(x-1)}{(x-1)(3-x)}$

$\Rightarrow 1 \equiv A(3-x)+B(x-1)$

when $x=3$, $1 = B\times 2$ $\therefore B=\frac{1}{2}$

when $x=1$, $1 = 2A$ $\therefore A=\frac{1}{2}$

$\frac{1}{(x-1)(3-x)} = \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{3-x}$

b) $\int \frac{1}{-3+4N-N^2}\,dN = \int k\,dt$

$\int \frac{\frac{1}{2}}{N-1} + \frac{\frac{1}{2}}{3-N}\,dN = kt + C$

$\frac{1}{2}\ln|N-1| - \frac{1}{2}\ln|3-N| = kt + C$

when $t=0$, $N=1.5$

$\frac{1}{2}\ln|0.5| - \frac{1}{2}\ln|1.5| = C$

$\frac{1}{2}\ln\frac{1}{3} = C$

$\Rightarrow \frac{1}{2}\ln\left|\frac{N-1}{3-N}\right| = kt + \frac{1}{2}\ln\frac{1}{3}$

$\ln\left(\frac{N-1}{3-N}\right) = 2kt + \ln\frac{1}{3}$

$\ln\left(\frac{N-1}{3-N}\right) = 2kt - \ln 3$

c) $\dfrac{N-1}{3-N} = e^{2kt-\ln 3}$

$N-1 = (3-N)\,e^{2kt-\ln 3}$

$N-1 = (3-N)\,\dfrac{e^{2kt}}{e^{\ln 3}}$

$N-1 = (3-N)\,\dfrac{1}{3}e^{2kt}$

$N-1 = e^{2kt} - \dfrac{1}{3}e^{2kt}\cdot N$

$\left(1+\dfrac{1}{3}e^{2kt}\right)N = e^{2kt}+1$

$N = \dfrac{e^{2kt}+1}{1+\frac{1}{3}e^{2kt}} = \dfrac{3e^{2kt}+3}{3+e^{2kt}}$

$N = \dfrac{3+3e^{-2kt}}{3e^{-2kt}+1}$

d)

as $t \to \infty$, $N \longrightarrow \dfrac{3+3\times 0}{1+3\times 0} = \dfrac{3}{1} = 3$

Hence, L is 3000.