IB Maths: AA HL

MacLaurin Series

Topic Questions

These practice questions can be used by students and teachers and is Suitable for IB Maths AA HL Topic Questions

Course	IB Maths
Section	5. Calculus
Topic	5.11 MacLaurin Series
Difficulty	Medium

Level: IB Maths

Subject: IB Maths AA HL

Board: IB Maths

Topic: MacLaurin Series

Question 1

Consider the general Maclaurin series formula

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

(where $f^{(n)}$ indicates the n^{th} derivative of f).

a)
Use the formula to find the first five terms of the Maclaurin series for e^{2x}.

[4 marks]

b)
Hence approximate the value of e^{2x} when $x = 1$.

[2 marks]

c)
(i)
Compare the approximation found in part (b) to the exact value of e^{2x} when $x = 1$.

(ii)
Explain how the accuracy of the Maclaurin series approximation could be improved.

[3 marks]

d)
Use the general Maclaurin series formula to show that the general term of the Maclaurin series for e^{2x} is

$$\frac{(2x)^n}{n!}$$

[2 marks]

Question 2

a)
Use substitution into the Maclaurin series for $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

to find the first four terms of the Maclaurin series for $\sin\left(\frac{x}{2}\right)$.

[3 marks]

b)

Hence approximate the value of $\sin\frac{\pi}{2}$ and compare this approximation to the exact value.

[3 marks]

c)
Without performing any additional calculations, explain whether the answer to part (a) would be expected to give an approximation of $\sin\frac{\pi}{4}$ that is more accurate or less accurate than its approximation for $\sin\frac{\pi}{2}$.

[2 marks]

Question 3

The Maclaurin series for e^x and $\sin x$ are

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

a)
Find the Maclaurin series for $e^x sin\ x$ up to and including the term in x^4.

[4 marks]

b)

Use the Maclaurin series for $\sin x$, along with the fact that $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$, to find the first four terms of the Maclaurin series for $\cos x$.

[3 marks]

Question 4

a)

Use the general Maclaurin series formula to find the first four terms of the Maclaurin series for $\frac{1}{1+x}$.

[4 marks]

b)

Confirm that the answer to part (a) matches the first four terms of the binomial theorem expansion of $\frac{1}{1+x}$.

[3 marks]

The Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

c)

Differentiate the Maclaurin series for $\ln(1+x)$ up to its fourth term and compare this to the answer from part (a). Give an explanation for any similarities that are found.

[2 marks]

Question 5

a)
Use the Maclaurin series for $\sin x$ and $\cos x$ to find a Maclaurin series approximation for $2\sin x\cos x$ up until the term in x^4.

[3 marks]

The double angle identity for sine tells us that

$$\sin 2x = 2\sin x\cos x$$

b)
Use substitution into the Maclaurin series for $\sin x$ to find a Maclaurin series approximation for $\sin 2x$ up until the term in x^4, and confirm that this matches the answer to part (a).

[3 marks]

Question 6

a)
Use the Binomial theorem to find a Maclaurin series for the function f defined by

$$f(x) = \sqrt{1-2x^2}$$

Give the series up to and including the term in x^6.

[4 marks]

b)
State any limitations on the validity of the series expansion found in part (a).

[2 marks]

c)
Use the answer to part (a) to estimate the value of $\sqrt{0.5}$, and compare the accuracy of that estimated value to the actual value of $\sqrt{0.5}$.

[4 marks]

Question 7

Consider the differential equation

$$y' = 2y^2 + x$$

together with the initial condition $y(0) = 1$.

a)

(i)

Show that $y'' = 4yy' + 1$.

(ii)

Use an equivalent method to find expressions for y''', $y^{(4)}$ and $y^{(5)}$. Each should be given in terms of y and of lower-order derivatives of y.

[4 marks]

b)

Using the boundary condition above, calculate the values of $y'(0)$, $y''(0)$, $y'''(0)$, $y^{(4)}(0)$ and $y^{(5)}(0)$.

[3 marks]

Let $f(x)$ be the solution to the differential equation above with the given boundary condition, so that $y = f(x)$.

c)

Using the answers to part (b), find the first six terms of the Maclaurin series for $f(x)$.

[4 marks]

d)

Hence approximate the value of to 4 d.p. when $x = 0.1$.

[2 marks]

Question 8

Consider the differential equation

$$y' = 2xy^2$$

with the initial condition $y(0) = 1$.

a)
(i)
Find y''.

(ii)
Hence show that $y''' = 8yy' + 4x(y')^2 + 4xyy''$ and

$$y^{(4)} = 12(y')^2 + 12yy'' + 12xy'y'' + 4xyy'''$$

[5 marks]

b)
Use the results from part (a) along with the given initial condition to find a Maclaurin series to approximate the solution of the differential equation, giving the approximation up to the term in x^4.

[4 marks]

c)
Use separation of variables to show that the exact solution of the differential equation with the given initial condition is

$$y = \frac{1}{1-x^2}$$

[4 marks]

d)
Use the binomial theorem to find an approximation for $\frac{1}{1-x^2}$ up to the term in x^4, and verify that it matches the answer to part (b).

[3 marks]