Question 1

Consider the general Maclaurin series formula

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

(where $f^{(n)}$ indicates the n^{th} derivative of f).

(a) Use the formula to find the first five terms of the Maclaurin series for e^{2x}.

[4]

(b) Hence approximate the value of e^{2x} when $x = 1$.

[2]

(c) (i) Compare the approximation found in part (b) to the exact value of e^{2x} when $x = 1$.

(ii) Explain how the accuracy of the Maclaurin series approximation could be improved.

[3]

(d) Use the general Maclaurin series formula to show that the general term of the Maclaurin series for e^{2x} is

$$\frac{(2x)^n}{n!}$$

[2]

(a) $f(x) = e^{2x} \Rightarrow f(0) = e^{2(0)} = 1$

$f'(x) = 2e^{2x} \Rightarrow f'(0) = 2e^{2(0)} = 2$

$f''(x) = 4e^{2x} \Rightarrow f''(0) = 4e^{2(0)} = 4$

$f'''(x) = 8e^{2x} \Rightarrow f'''(0) = 8e^{2(0)} = 8$

$f^4(x) = 16e^{2x} \Rightarrow f^4(0) = 16e^{2(0)} = 16$

Put the values of the successive derivatives into the formula

$$e^{2x} \approx 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!}$$

$$\boxed{e^{2x} \approx 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4}$$

Consider the general Maclaurin series formula

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

(where $f^{(n)}$ indicates the n^{th} derivative of f).

(a) Use the formula to find the first five terms of the Maclaurin series for e^{2x}.

$$\boxed{e^{2x} \approx 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4}$$

[4]

(b) Hence approximate the value of e^{2x} when $x = 1$.

[2]

(c) (i) Compare the approximation found in part (b) to the exact value of e^{2x} when $x = 1$.

(ii) Explain how the accuracy of the Maclaurin series approximation could be improved.

[3]

(d) Use the general Maclaurin series formula to show that the general term of the Maclaurin series for e^{2x} is

$$\frac{(2x)^n}{n!}$$

[2]

(b) Substitute $x=1$ into answer from (a)

$$f(1) \approx 1 + 2(1) + 2(1)^2 + \frac{4}{3}(1)^3 + \frac{2}{3}(1)^4$$

$$\boxed{f(1) \approx 7}$$

EXAM PAPERS PRACTICE

Consider the general Maclaurin series formula

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

(where $f^{(n)}$ indicates the n^{th} derivative of f).

(a) Use the formula to find the first five terms of the Maclaurin series for e^{2x}.

[4]

(b) Hence approximate the value of e^{2x} when $x = 1$.

$\boxed{f(1) \approx 7}$

[2]

(c) (i) Compare the approximation found in part (b) to the exact value of e^{2x} when $x = 1$.

(ii) Explain how the accuracy of the Maclaurin series approximation could be improved.

[3]

(d) Use the general Maclaurin series formula to show that the general term of the Maclaurin series for e^{2x} is

$$\frac{(2x)^n}{n!}$$

[2]

(c) (i) Using the GDC

$$f(1) = e^{2(1)} = 7.3905609\dots$$

Calculate the percentage error

$$\varepsilon = \left|\frac{e^2 - 7}{e^2}\right| \times 100\% = 5.26530173\dots$$

$$\boxed{\varepsilon = 5.27\% \ (3\text{sf})}$$

(ii) The accuracy of the approximation can be improved by adding more terms to the series

Consider the general Maclaurin series formula

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

(where $f^{(n)}$ indicates the n^{th} derivative of f).

(a) Use the formula to find the first five terms of the Maclaurin series for e^{2x}.

[4]

(b) Hence approximate the value of e^{2x} when $x = 1$.

[2]

(c) (i) Compare the approximation found in part (b) to the exact value of e^{2x} when $x = 1$.

(ii) Explain how the accuracy of the Maclaurin series approximation could be improved.

[3]

(d) Use the general Maclaurin series formula to show that the general term of the Maclaurin series for e^{2x} is

$$\frac{(2x)^n}{n!}$$

[2]

(d) Each time the function is differentiated, it is essentially just multiplied by 2

$$f(x) = e^{2x} \qquad f^n(x) = 2^n e^{2x}$$

Each term in the Maclaurin series

$$\frac{x^n}{n!}f^n(0) = \frac{x^n}{n!}2^n e^{2(0)}$$

$$= \frac{2^n x^n}{n!}$$

$$\boxed{\frac{(2x)^n}{n!}}$$

EXAM PAPERS PRACTICE

Question 2

(a) Use substitution into the Maclaurin series for $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

to find the first four terms of the Maclaurin series for $\sin\left(\frac{x}{2}\right)$.

[3]

(b) Hence approximate the value of $\sin\frac{\pi}{2}$ and compare this approximation to the exact value.

[3]

(c) Without performing any additional calculations, explain whether the answer to part (a) would be expected to give an approximation of $\sin\frac{\pi}{4}$ that is more accurate or less accurate than its approximation for $\sin\frac{\pi}{2}$.

[2]

(a) Replace x with $\frac{x}{2}$ in the expansion

$$\sin\left(\frac{x}{2}\right) \approx \left(\frac{x}{2}\right) - \frac{\left(\frac{x}{2}\right)^3}{3!} + \frac{\left(\frac{x}{2}\right)^5}{5!} - \frac{\left(\frac{x}{2}\right)^7}{7!}$$

$$\boxed{\sin\left(\frac{x}{2}\right) \approx \frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{3840} - \frac{x^7}{645120}}$$

(a) Use substitution into the Maclaurin series for $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

to find the first four terms of the Maclaurin series for $\sin\left(\frac{x}{2}\right)$.

[3]

(b) Hence approximate the value of $\sin\frac{\pi}{2}$ and compare this approximation to the exact value.

[3]

(c) Without performing any additional calculations, explain whether the answer to part (a) would be expected to give an approximation of $\sin\frac{\pi}{4}$ that is more accurate or less accurate than its approximation for $\sin\frac{\pi}{2}$.

[2]

(c) The Maclaurin series gives an exact value for a function for $x=0$, then gets less accurate as you move away from $x=0$. For $\sin\frac{\pi}{4}$ we would be substituting $x=\frac{\pi}{2}$ into the truncated series and since $0<\frac{\pi}{2}<\pi$, we would expect the approximation for $\sin\frac{\pi}{4}$ to be even more accurate than the one for $\sin\frac{\pi}{2}$.

Question 3

The Maclaurin series for e^x and $\sin x$ are

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

(a) Find the Maclaurin series for $e^x \sin x$ up to and including the term in x^4.

[4]

(b) Use the Maclaurin series for $\sin x$, along with the fact that $\frac{d}{dx}(\sin x) = \cos x$, to find the first four terms of the Maclaurin series for $\cos x$.

[3]

(a) $e^x \sin x \approx \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(x - \frac{x^3}{3!}\right)$

Note:
The expansion for e^x is only required up to x^3, as the smallest term it will be multiplied by from $\sin x$ is x, bringing it up to x^4.
The expansion for $\sin x$ is also only required up to x^3 as the next power (x^5) is higher than required.

$e^x \sin x \approx \left(x + x^2 + \frac{x^3}{2} + \frac{x^4}{6}\right) - \left(\frac{x^3}{6} + \frac{x^4}{6} + \text{higher powers not required}\right)$

$$\boxed{e^x \sin x \approx x + x^2 + \frac{x^3}{3}}$$

The Maclaurin series for e^x and $\sin x$ are

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots \quad \text{and} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

(a) Find the Maclaurin series for $e^x \sin x$ up to and including the term in x^4.

[4]

(b) Use the Maclaurin series for $\sin x$, along with the fact that $\frac{d}{dx}(\sin x) = \cos x$, to find the first four terms of the Maclaurin series for $\cos x$.

[3]

(b) Differentiate each term of the series for $\sin x$ to find the series for $\cos x$

$$\cos x \approx 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!}$$

Remember $\frac{3}{3!} = \frac{\cancel{3}}{\cancel{3} \times 2 \times 1} = \frac{1}{2!}$, so $\frac{n}{n!} = \frac{1}{(n-1)!}$

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\boxed{\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}}$$

EXAM PAPERS PRACTICE

Question 4

(a) Use the general Maclaurin series formula to find the first four terms of the Maclaurin series for $\frac{1}{1+x}$.

[4]

(b) Confirm that the answer to part (a) matches the first four terms of the binomial theorem expansion of $\frac{1}{1+x}$.

[3]

The Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

(c) Differentiate the Maclaurin series for $\ln(1+x)$ up to its fourth term and compare this to the answer from part (a). Give an explanation for any similarities that are found.

[2]

(a) $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$ ← Formula booklet

$f(x) = (1+x)^{-1} \Rightarrow f(0) = (1+0)^{-1} = 1$

$f'(x) = -(1+x)^{-2} \Rightarrow f'(0) = -(1+0)^{-2} = -1$

$f''(x) = 2(1+x)^{-3} \Rightarrow f''(0) = 2(1+0)^{-3} = 2$

$f'''(x) = -6(1+x)^{-4} \Rightarrow f'''(0) = -6(1+0)^{-4} = -6$

$f(x) = 1 + x(-1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(-6)$

$$\boxed{f(x) = 1 - x + x^2 - x^3}$$

(a) Use the general Maclaurin series formula to find the first four terms of the Maclaurin series for $\frac{1}{1+x}$.

[4]

(b) Confirm that the answer to part (a) matches the first four terms of the binomial theorem expansion of $\frac{1}{1+x}$.

[3]

The Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

(c) Differentiate the Maclaurin series for $\ln(1+x)$ up to its fourth term and compare this to the answer from part (a). Give an explanation for any similarities that are found.

[2]

(b) Using the binomial theorem, for any value of n:

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

$\frac{1}{1+x} = (1+x)^{-1}$ ← n

$= 1 + (-1)x + \frac{(-1)(-1-1)}{2 \times 1}x^2 + \frac{(-1)(-1-1)(-1-2)}{3 \times 2 \times 1}x^3$

$= 1 - x + x^2 - x^3 + \ldots$

First 4 terms: $1 - x + x^2 - x^3$

These match the terms generated by the Maclaurin series in (a)

(a) Use the general Maclaurin series formula to find the first four terms of the Maclaurin series for $\frac{1}{1+x}$.

$$\boxed{f(x) = 1 - x + x^2 - x^3}$$

[4]

(b) Confirm that the answer to part (a) matches the first four terms of the binomial theorem expansion of $\frac{1}{1+x}$.

[3]

The Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

(c) Differentiate the Maclaurin series for $\ln(1+x)$ up to its fourth term and compare this to the answer from part (a). Give an explanation for any similarities that are found.

[2]

(c) Continue the Maclaurin series up to the 4th term

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

Differentiate

$$\frac{d}{dx}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right) \approx 1 - x + x^2 - x^3$$

The derivative of the Maclaurin series for $\ln(1+x)$ is the same as the Maclaurin series for $\frac{1}{1+x}$, this is because the derivative of $\ln(1+x)$ is $\frac{1}{1+x}$

Question 5

(a) Use the Maclaurin series for $\sin x$ and $\cos x$ to find a Maclaurin series approximation for $2\sin x\cos x$ up until the term in x^4.

[3]

The double angle identity for sine tells us that

$$\sin 2x = 2\sin x\cos x$$

(b) Use substitution into the Maclaurin series for $\sin x$ to find a Maclaurin series approximation for $\sin 2x$ up until the term in x^4, and confirm that this matches the answer to part (a).

[3]

(a) $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots$ ← Formula booklet

Exact trig values: $\sin(0) = 0$ $\cos(0) = 1$

$$\sin(x) \approx \sin(0) + x\cos(0) - \frac{x^2}{2!}\sin(0) - \frac{x^3}{3!}\cos(0)$$

$$\approx x - \frac{x^3}{6}$$

The next non-zero term would be x^5, which is not required

$$\cos(x) \approx \cos(0) - x\sin(0) - \frac{x^2}{2!}\cos(0) + \frac{x^3}{3!}\sin(0)$$

$$\approx 1 - \frac{x^2}{2}$$

The next non-zero term would be x^4, which when multiplied by the smallest term from the $\sin x$ expansion would generate an x^5 term

$$2\sin x\cos x \approx 2\left(x - \frac{x^3}{6}\right)\left(1 - \frac{x^2}{2}\right)$$

$$\approx 2\left(x - \frac{x^3}{2} - \frac{x^3}{6} + \frac{x^5}{12}\right)$$

Ignore this term

$$\boxed{2\sin x\cos x \approx 2x - \frac{4}{3}x^3}$$

(a) Use the Maclaurin series for $\sin x$ and $\cos x$ to find a Maclaurin series approximation for $2\sin x\cos x$ up until the term in x^4.

$\sin(x) \approx x - \frac{x^3}{6}$ $\quad$ $2\sin x\cos x \approx 2x - \frac{4}{3}x^3$

[3]

The double angle identity for sine tells us that

$$\sin 2x = 2\sin x\cos x$$

(b) Use substitution into the Maclaurin series for $\sin x$ to find a Maclaurin series approximation for $\sin 2x$ up until the term in x^4, and confirm that this matches the answer to part (a).

[3]

(b) Replace x with $2x$ in the Maclaurin series for $\sin x$

$$\sin(2x) \approx (2x) - \frac{(2x)^3}{6}$$

$$\approx 2x - \frac{8x^3}{6}$$

$\sin(2x) \approx 2x - \frac{4}{3}x^3$

This matches the expansion for $2\sin x\cos x$ in part (a)

Question 6

(a) Use the Binomial theorem to find a Maclaurin series for the function f defined by

$$f(x) = \sqrt{1-2x^2}$$

Give the series up to and including the term in x^6.

[4]

(b) State any limitations on the validity of the series expansion found in part (a).

[2]

(c) Use the answer to part (a) to estimate the value of $\sqrt{0.5}$, and compare the accuracy of that estimated value to the actual value of $\sqrt{0.5}$.

[4]

(a) Using the binomial theorem, for any value of n:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$\sqrt{1-2x^2} = (1\overbrace{-2x^2}^{x})^{\frac{1}{2}} \leftarrow n$$

$$f(x) = 1 + \left(\tfrac{1}{2}\right)(-2x^2) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)(-2x^2)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(-2x^2)^3}{3!} \ldots$$

$$= 1 - x^2 + \frac{\left(-\frac{1}{4}\right)(4x^4)}{2} + \frac{\left(\frac{3}{8}\right)(-8x^6)}{6} \ldots$$

$$f(x) \approx 1 - x^2 - \frac{1}{2}x^4 - \frac{1}{2}x^6$$

(a) Use the Binomial theorem to find a Maclaurin series for the function f defined by

$$f(x) = \sqrt{1 - 2x^2}$$

Give the series up to and including the term in x^6.

[4]

(b) State any limitations on the validity of the series expansion found in part (a).

[2]

(c) Use the answer to part (a) to estimate the value of $\sqrt{0.5}$, and compare the accuracy of that estimated value to the actual value of $\sqrt{0.5}$.

[4]

(b) The binomial expansion for $(1+x)^\alpha$ is an infinite series when $-1 < x < 1$

$$-1 < (-2x^2) < 1$$

$$\Rightarrow \quad \frac{1}{2} > x^2 > -\frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2} > x^2 > 0 \qquad x^2 \text{ can't be negative}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} > |x|$$

$$\Rightarrow \quad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

$$\boxed{-\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}}$$

(a) Use the Binomial theorem to find a Maclaurin series for the function f defined by

$$f(x) = \sqrt{1 - 2x^2}$$

Give the series up to and including the term in x^6.

$$\boxed{f(x) \approx 1 - x^2 - \frac{1}{2}x^4 - \frac{1}{2}x^6}$$

[4]

(b) State any limitations on the validity of the series expansion found in part (a).

[2]

(c) Use the answer to part (a) to estimate the value of $\sqrt{0.5}$, and compare the accuracy of that estimated value to the actual value of $\sqrt{0.5}$.

[4]

(c) $\sqrt{1 - 2x^2} = \sqrt{0.5}$

$$1 - 2x^2 = \frac{1}{2} \quad \Rightarrow \quad x = \frac{1}{2}$$

Substitute $x = \frac{1}{2}$ into the expansion from (a)

$$f\left(\frac{1}{2}\right) \approx 1 - \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 - \frac{1}{2}\left(\frac{1}{2}\right)^6$$

$$\boxed{f\left(\frac{1}{2}\right) \approx \frac{91}{128}}$$

Find the percentage error

$$\varepsilon = \left| \frac{\frac{91}{128} - \sqrt{0.5}}{\sqrt{0.5}} \right| \times 100\% = 0.5417454\ldots$$

$$\boxed{\varepsilon = 0.542\% \ (3sf)}$$

Question 7

Consider the differential equation

$$y' = 2y^2 + x$$

together with the initial condition $y(0) = 1$.

(a) (i) Show that $y'' = 4yy' + 1$.

(ii) Use an equivalent method to find expressions for y''', $y^{(4)}$ and $y^{(5)}$. Each should be given in terms of y and of lower-order derivatives of y.

[4]

(b) Using the boundary condition above, calculate the values of $y'(0)$, $y''(0)$, $y'''(0)$, $y^{(4)}(0)$ and $y^{(5)}(0)$.

[3]

Let $f(x)$ be the solution to the differential equation above with the given boundary condition, so that $y = f(x)$.

(c) Using the answers to part (b), find the first six terms of the Maclaurin series for $f(x)$.

[4]

(d) Hence approximate the value of y to 4 d.p. when $x = 0.1$.

[2]

(a) (i) Use implicit differentiation

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(2y^2 + x)$$

$$= \frac{d}{dx}(2y^2) + 1$$

Let $u = 2y^2 \Rightarrow \frac{du}{dy} = 4y$

Chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ ← Formula booklet

$\Rightarrow \frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$

$$\Rightarrow \frac{du}{dx} = 4y \times y'$$

$$\frac{d}{dx}(2y^2) = 4yy'$$

$$\Rightarrow \frac{d}{dx}(2y^2 + x) = 4yy' + 1$$

$$\boxed{y'' = 4yy' + 1}$$

(ii) $y''' = \frac{d}{dx}y'' = \frac{d}{dx}(4yy' + 1) = \frac{d}{dx}(4yy')$

$$= 4\frac{d}{dx}(yy')$$

Using the product rule $y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ ← Formula booklet

Let $u = y$, $v = y'$

$\Rightarrow \frac{du}{dy} = y'$, $\frac{dv}{dy} = y''$

$$\Rightarrow 4\frac{d}{dx}(yy') = 4yy'' + 4y'y'$$

$$\boxed{y''' = 4yy'' + 4(y')^2}$$

$$y^{4} = \frac{d}{dx} y''' = \frac{d}{dx}\left(4yy'' + 4(y')^2\right)$$

$$\frac{d}{dx}(4yy'') = 4yy''' + 4y'y''$$ ← Using the product rule as before

$$\frac{d}{dx}\left(4(y')^2\right) = \frac{d}{dx}(4y'y')$$

$$= 8y'y''$$ ← Using the product rule as before

$$y^{(4)} = \frac{d}{dx}\left(4yy'' + 4(y')^2\right) = 4yy''' + 4y'y'' + 8y'y''$$

$$\boxed{y^{(4)} = 12y'y'' + 4yy'''}$$

$$y^{(5)} = \frac{d}{dx}\left(y^{(4)}\right) = \frac{d}{dx}\left(12y'y'' + 4yy'''\right)$$ ← Using the product rule as before

$$= \left(12y'y''' + 12(y'')^2\right) + \left(4yy^{(4)} + 4y'y'''\right)$$

$$\boxed{y^{(5)} = 16y'y''' + 12(y'')^2 + 4yy^{(4)}}$$

Consider the differential equation

$$y' = 2y^2 + x$$

together with the initial condition $y(0) = 1$.

$y'' = 4yy' + 1$

$y''' = 4yy'' + 4(y')^2$

$y^{(4)} = 12y'y'' + 4yy'''$

$y^{(5)} = 16y'y''' + 12(y'')^2 + 4yy^{(4)}$

(a) (i) Show that $y'' = 4yy' + 1$.

(ii) Use an equivalent method to find expressions for y''', $y^{(4)}$ and $y^{(5)}$. Each should be given in terms of y and of lower-order derivatives of y.

[4]

(b) Using the boundary condition above, calculate the values of $y'(0)$, $y''(0)$, $y'''(0)$, $y^{(4)}(0)$ and $y^{(5)}(0)$.

[3]

Let $f(x)$ be the solution to the differential equation above with the given boundary condition, so that $y = f(x)$.

(c) Using the answers to part (b), find the first six terms of the Maclaurin series for $f(x)$.

[4]

(d) Hence approximate the value of y to 4 d.p. when $x = 0.1$.

[2]

(b) $y'(0) = 2(1)^2 + (0)$ $\boxed{y'(0) = 2}$

$y''(0) = 4(1)(2) + 1$ $\boxed{y''(0) = 9}$

$y'''(0) = 4(1)(9) + 4(2)^2$ $\boxed{y'''(0) = 52}$

$y^{(4)}(0) = 12(2)(9) + 4(1)(52)$ $\boxed{y^{(4)} = 424}$

$y^{(5)}(0) = 16(2)(52) + 12(9)^2 + 4(1)(424)$

$\boxed{y^{(5)} = 4332}$

Consider the differential equation

$$y' = 2y^2 + x$$

together with the initial condition $y(0) = 1$.

(a) (i) Show that $y'' = 4yy' + 1$.

(ii) Use an equivalent method to find expressions for y''', $y^{(4)}$ and $y^{(5)}$. Each should be given in terms of y and of lower-order derivatives of y.

[4]

(b) Using the boundary condition above, calculate the values of $y'(0)$, $y''(0)$, $y'''(0)$, $y^{(4)}(0)$ and $y^{(5)}(0)$.

$\boxed{y'(0) = 2}$ $\boxed{y''(0) = 9}$ $\boxed{y'''(0) = 52}$ $\boxed{y^{(4)} = 424}$ $\boxed{y^{(5)} = 4332}$ [3]

Let $f(x)$ be the solution to the differential equation above with the given boundary condition, so that $y = f(x)$.

(c) Using the answers to part (b), find the first six terms of the Maclaurin series for $f(x)$.

[4]

(d) Hence approximate the value of y to 4 d.p. when $x = 0.1$.

[2]

(c) $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$ Formula booklet

$$f(x) \approx 1 + (2)x + \frac{(9)x^2}{2!} + \frac{(52)x^3}{3!} + \frac{(424)x^4}{4!} + \frac{(4332)x^5}{5!}$$

$$\boxed{f(x) \approx 1 + 2x + \frac{9}{2}x^2 + \frac{26}{3}x^3 + \frac{53}{3}x^4 + \frac{361}{10}x^5}$$

Consider the differential equation

$$y' = 2y^2 + x$$

together with the initial condition $y(0) = 1$.

(a) (i) Show that $y'' = 4yy' + 1$.

(ii) Use an equivalent method to find expressions for y''', $y^{(4)}$ and $y^{(5)}$. Each should be given in terms of y and of lower-order derivatives of y.

[4]

(b) Using the boundary condition above, calculate the values of $y'(0)$, $y''(0)$, $y'''(0)$, $y^{(4)}(0)$ and $y^{(5)}(0)$.

[3]

Let $f(x)$ be the solution to the differential equation above with the given boundary condition, so that $y = f(x)$.

(c) Using the answers to part (b), find the first six terms of the Maclaurin series for $f(x)$.

$$\boxed{f(x) = 1 + 2x + \frac{9}{2}x^2 + \frac{26}{3}x^3 + \frac{53}{3}x^4 + \frac{361}{10}x^5}$$

[4]

(d) Hence approximate the value of y to 4 d.p. when $x = 0.1$.

[2]

(d) Substitute $x = 0.1$ into $f(x)$ from (c)

$$f(0.1) \approx 1 + 2(0.1) + \frac{9}{2}(0.1)^2 + \frac{26}{3}(0.1)^3 + \frac{53}{3}(0.1)^4 + \frac{361}{10}(0.1)^5$$

$$\approx 1.2557943\ldots$$

$$\boxed{f(0.1) \approx 1.256 \ (4 \text{ sf})}$$

Question 8

Consider the differential equation

$$y' = 2xy^2$$

with the initial condition $y(0) = 1$.

(a) (i) Find y''.

(ii) Hence show that $y''' = 8yy' + 4x(y')^2 + 4xyy''$ and

$$y^{(4)} = 12(y')^2 + 12yy'' + 12xy'y'' + 4xyy'''$$

[5]

(b) Use the results from part (a) along with the given initial condition to find a Maclaurin series to approximate the solution of the differential equation, giving the approximation up to the term in x^4.

[4]

(c) Use separation of variables to show that the exact solution of the differential equation with the given initial condition is

$$y = \frac{1}{1-x^2}$$

[4]

(d) Use the binomial theorem to find an approximation for $\frac{1}{1-x^2}$ up to the term in x^4, and verify that it matches the answer to part (b).

[3]

(a) (i) Use implicit differentiation

$$\frac{d}{dx}(y') = \frac{d}{dx}(2xy^2) = 2\frac{d}{dx}(xy^2)$$

Use the chain rule to differentiate (y^2)

Let $u = y^2 \Rightarrow \frac{du}{dy} = 2y$

$\Rightarrow \frac{du}{dx} = 2yy'$ $\quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ ← Formula booklet

Use the product rule to now differentiate (xy^2)

Let $u = x \quad v = y^2$

$u' = 1 \quad v' = 2yy'$ $\quad y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ ← Formula booklet

$$\Rightarrow 2\frac{d}{dx}(xy^2) = 2(y^2 + 2xyy')$$

$$\boxed{y'' = 2y^2 + 4xyy'}$$

(ii) $y''' = \frac{d}{dx}(2y^2 + 4xyy')$

Same as for (i)

Use the product rule first for yy' then for $x(yy')$

$$\frac{d}{dx}(2y^2 + 4xyy') = 4yy' + \frac{d}{dx}(4xyy')$$

$\frac{d}{dx}(yy')$ Let $u = y \quad v = y'$

$u' = y' \quad v' = y'' \quad \Rightarrow \quad \frac{d}{dx}(yy') = yy'' + (y')^2$

$\frac{d}{dx}(4x(yy'))$ Let $u = 4x \quad v = yy'$

$u' = 4 \quad v' = yy'' + (y')^2$

$$\Rightarrow \frac{d}{dx}(xyy') = 4x(yy'' + (y')^2) + 4yy'$$

$$= 4xyy'' + 4x(y')^2 + 4yy'$$

$$\frac{d}{dx}(2y^2 + 4xyy') = 4yy' + 4xyy'' + 4x(y')^2 + 4yy'$$

$$\boxed{y''' = 8yy' + 4x(y')^2 + 4xyy''}$$

$y^{(4)} = \frac{d}{dx}(8yy' + 4x(y')^2 + 4xyy'')$

$\frac{d}{dx}(8yy')$ $\quad 8yy'' + 8(y')^2$ ← Product rule

$\frac{d}{dx}(4x(y')^2)$

Let $u = 4x \quad v = (y')^2$

$u' = 4 \quad v' = 2y'y''$ ← Same as for $\frac{d}{dx}(y^2)$ in (i)

$\Rightarrow \frac{d}{dx}(4x(y')^2) = 8xy'y'' + 4(y')^2$

$\frac{d}{dx}(4xyy'')$ $\quad 4xyy''' + 4xy'y'' + 4yy''$ ← same as for $\frac{d}{dx}(4xyy')$ in (i)

$\frac{d}{dx}(8yy' + 4x(y')^2 + 4xyy'')$

$= 8yy'' + 8(y')^2 + 8xy'y'' + 4(y')^2 + 4xyy''' + 4xy'y'' + 4yy''$

$$\boxed{y^{(4)} = 12(y')^2 + 12yy'' + 12xy'y'' + 4xyy'''}$$

Consider the differential equation

$$y' = 2xy^2$$

with the initial condition $y(0) = 1$.

(a) (i) Find y''. $\boxed{y'' = 2y^2 + 4xyy'}$

(ii) Hence show that $y''' = 8yy' + 4x(y')^2 + 4xyy''$ and

$$y^{(4)} = 12(y')^2 + 12yy'' + 12xy'y'' + 4xyy'''$$

[5]

(b) Use the results from part (a) along with the given initial condition to find a Maclaurin series to approximate the solution of the differential equation, giving the approximation up to the term in x^4.

[4]

(c) Use separation of variables to show that the exact solution of the differential equation with the given initial condition is

$$y = \frac{1}{1 - x^2}$$

[4]

(d) Use the binomial theorem to find an approximation for $\frac{1}{1-x^2}$ up to the term in x^4, and verify that it matches the answer to part (b).

[3]

(b) Find $f'(0), f''(0), f'''(0), f^{(4)}(0)$

$f'(0) = 2(0)(1)^2 = 0$

$f''(0) = 2(1)^2 + 4(0)(1)(0) = 2$

$f'''(0) = 8(1)(0) + 4(0)(0)^2 + 4(0)(1)(2) = 0$

$f^{(4)}(0) = 12(0)^2 + 12(1)(2) + 12(0)(0)(2) + 4(0)(1)(0) = 24$

Substitute the values into the Maclaurin series formula

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$ ← Formula booklet

$f(x) = 1 + x(0) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(24) \ldots$

$$\boxed{f(x) \approx 1 + x^2 + x^4}$$

EXAM PAPERS PRACTICE

Consider the differential equation

$$y' = 2xy^2$$

with the initial condition $y(0) = 1$.

(a) (i) Find y''.

(ii) Hence show that $y''' = 8yy' + 4x(y')^2 + 4xyy''$ and

$$y^{(4)} = 12(y')^2 + 12yy'' + 12xy'y'' + 4xyy'''$$

[5]

(b) Use the results from part (a) along with the given initial condition to find a Maclaurin series to approximate the solution of the differential equation, giving the approximation up to the term in x^4.

$\boxed{f(x) \approx 1 + x^2 + x^4}$

[4]

(c) Use separation of variables to show that the exact solution of the differential equation with the given initial condition is

$$y = \frac{1}{1 - x^2}$$

[4]

(d) Use the binomial theorem to find an approximation for $\frac{1}{1-x^2}$ up to the term in x^4, and verify that it matches the answer to part (b).

[3]

(c) $\frac{dy}{dx} = 2xy^2$

Separate the variables and integrate both sides

$$\int \frac{1}{y^2}\,dy = \int 2x\,dx$$

$$-\frac{1}{y} = x^2 + c$$

Initial condition: $y(0) = 1$

$$-\frac{1}{1} = (0)^2 + c \quad \Rightarrow c = -1$$

Rearrange

$$-\frac{1}{y} = x^2 - 1$$

$$-\frac{1}{x^2 - 1} = y$$

Note: $-\frac{1}{x^2-1} = \frac{1}{-(x^2-1)} = \frac{1}{-x^2+1}$

$$\boxed{y = \frac{1}{1-x^2}}$$

Consider the differential equation

$$y' = 2xy^2$$

with the initial condition $y(0) = 1$.

(a) (i) Find y''.

(ii) Hence show that $y''' = 8yy' + 4x(y')^2 + 4xyy''$ and

$$y^{(4)} = 12(y')^2 + 12yy'' + 12xy'y'' + 4xyy'''$$

[5]

(b) Use the results from part (a) along with the given initial condition to find a Maclaurin series to approximate the solution of the differential equation, giving the approximation up to the term in x^4.

[4]

(c) Use separation of variables to show that the exact solution of the differential equation with the given initial condition is

$$y = \frac{1}{1 - x^2}$$

[4]

(d) Use the binomial theorem to find an approximation for $\frac{1}{1-x^2}$ up to the term in x^4, and verify that it matches the answer to part (b).

[3]

(d) Using the binomial theorem, for any value of n:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1 \underbrace{-x^2}_{x})^{\overbrace{-1}^{n}} \approx 1 + (-1)(-x^2) + \frac{(-1)(-1-1)}{2!}(-x^2)^2$$

$$\boxed{\begin{array}{c}(1-x^2)^{-1} \approx 1 + x^2 + x^4 \\ \text{This matches the result from part (b)}\end{array}}$$