Applied Math

Lesson 5

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Vector functions

Some physical quantities are entirely defined by a numerical value and are called scalar quantities or scalars. Examples of scalars include time, mass, temperature, energy and volume. Other physical quantities are defined by both a numerical value and a direction in space and these are called vector quantities or vectors. Examples of vectors include force, velocity, moment and displacement.

Vector addition

A vector may be represented by a straight line, the length of line being directly proportional to the magnitude of the quantity and the direction of the line being in the same direction as the line of action of the quantity. An arrow is used to denote the sense of the vector, that is, for a horizontal vector, say, whether it acts from left to right or vice-versa. Figure 1 shows a velocity of 20 m/s at an angle of 45◦ to the horizontal and may be depicted by ***oa***=20 m/s at 45◦ to the horizontal.

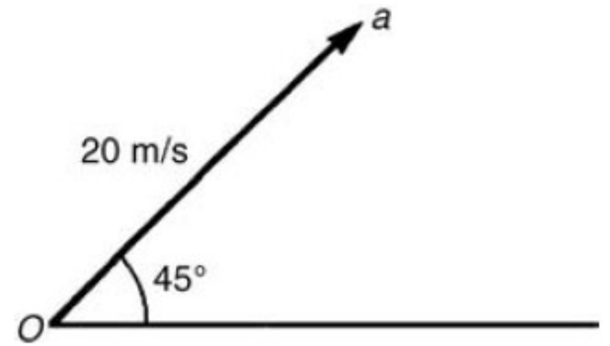


Figure 1

(Obtained from J. O. Bird 2017)

To distinguish between vector and scalar quantities, various ways are used. These include: (i) two capital letters with an arrow above them to denote the sense of direction, e.g. $\overrightarrow{AB}$ where *A* is the starting point and *B* the end point of the vector, (ii) a line over the top of letters $\overline{AB}$ or $\bar{a}$

(iii) letters with an arrow above, e.g $\vec{a}, \vec{A}$

(iv) xi+jy, where *i* and *j* are axes at right-angles to each other; for example, *3i+4j* means 3 units in the *i* direction and 4 units in the *j* direction, as shown in Fig 2

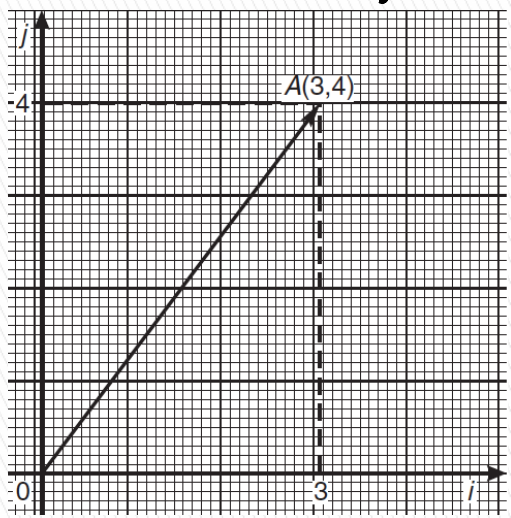


Fig 2 (Obtained from J. O. Bird 2017)

(vii) a column matrix $\begin{pmatrix} a \\ b \end{pmatrix}$; for example, the vector **OA** shown in Fig. 21.2 could be represented by $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Thus, in Fig.2

$$\boldsymbol{OA} \equiv \overrightarrow{OA} \equiv \overline{OA} \equiv 3i + 4j \equiv \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

The resultant of adding two vectors together, say V_1 at an angle θ_1 and V_2 at angle ($-\theta_2$), as shown in Fig. 3a, can be obtained by drawing ***oa*** to represent V_1 and then drawing ***ar*** to represent V_2. The resultant of $V_1 + V_2$ is given by ***or***. This is shown in Fig. 3b, the vector equation being ***oa+ar=or***. This is called the '**nose-to-tail**' method of vector addition.

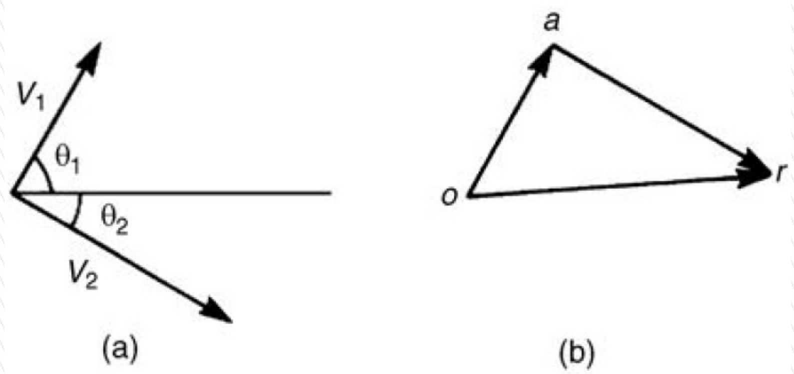
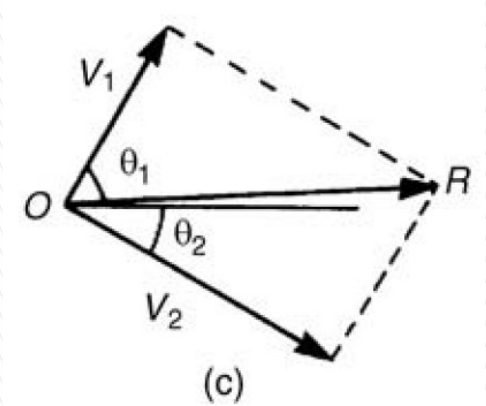


Fig 3. (Obtained from J. O. Bird 2017)

Alternatively, by drawing lines parallel to V_1 and V_2 from the noses of V_2 and V_1, respectively, and letting the point of intersection of these parallel lines be R, gives ***OR*** as the magnitude and direction of the resultant of adding V_1 and V_2, as shown in Fig.3c. This is called the '**parallelogram**' method of vector addition.



Problem 1. A force of 4N is inclined at an angle of 45◦ to a second force of 7 N, both forces acting at a point. Find the magnitude of the resultant of these two forces and the direction of the resultant with respect to the 7N force by both the 'triangle' and the 'parallelogram' methods?

Solution:

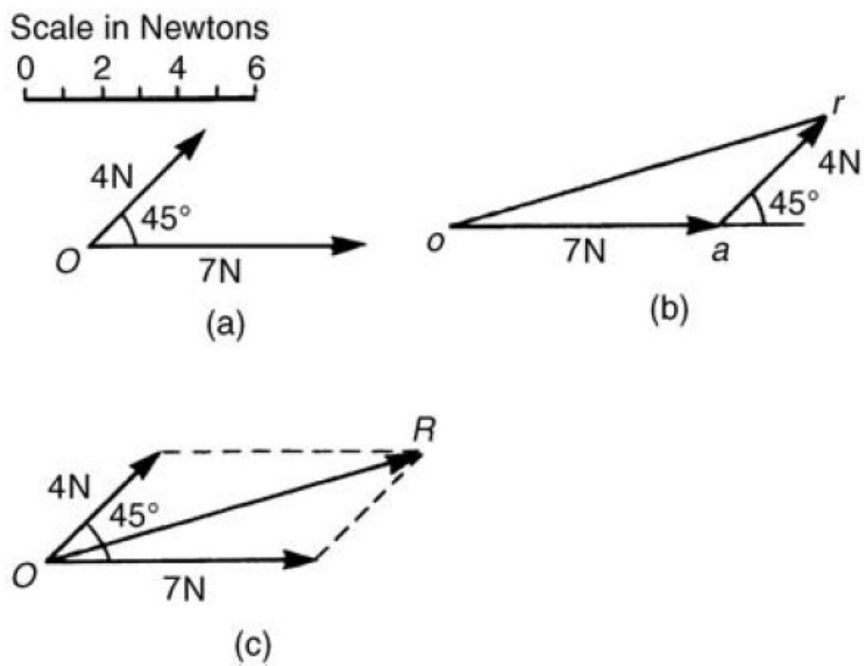


Fig 4 (J. O. Bird 2017)

Using the '**nose-to-tail' method**, a line 7 units long is drawn horizontally to give vector ***oa*** in Fig 4(b). To the nose of this vector ***ar*** is drawn 4 units long at an angle of 45◦ to ***oa***. The resultant of vector addition is ***or*** and by measurement is 10.2 units long and at an angle of 16◦ to the 7N force.

Fig 4c uses the '**parallelogram**' method in which lines are drawn parallel to the 7N and 4N forces from the noses of the 4N and 7N forces, respectively. These intersect at *R*. Vector ***OR*** gives the magnitude and direction of the resultant of vector addition and as obtained by the '**nose-to-tail'method** is 10.2 units long at an angle of 16◦ to the 7N force.

Resolution of vectors

A vector can be resolved into two component parts such that the vector addition of the component parts is equal to the original vector. The two components usually taken are a horizontal component and a vertical component. For the vector shown as ***F*** in Fig. 5, the horizontal component is *F cos θ* and the vertical component is *F sin θ*.

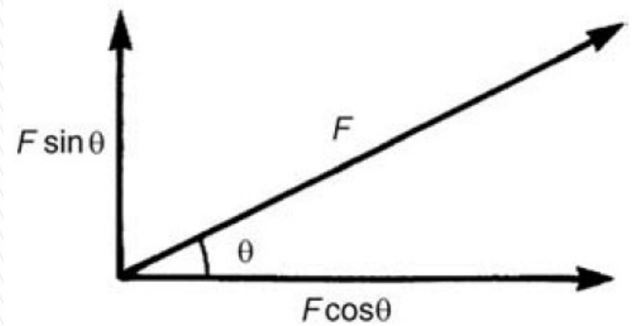


Fig 5 (Obtained from J. O. Bird 2017)

For the vectors *F*1 and *F*2 shown in Fig. 6, the horizontal component of vector addition is:

$$H = F_1 \cos\theta_1 + F_2 \cos\theta_2$$

the vertical component of vector addition is:

$$V = F_1 \sin\theta_1 + F_2 \sin\theta_2$$

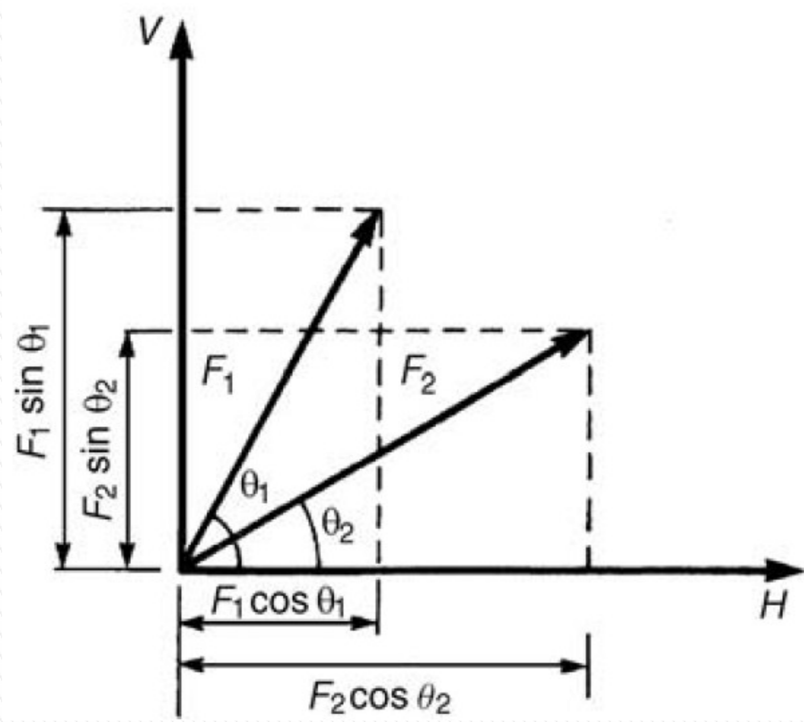


Fig 6

Having obtained H and V, the magnitude of the resultant vector R is given by $\sqrt{(H^2 + V^2)}$ and its angle to the horizontal is given by $\tan^{-1}(V/H)$.

Problem 2. Resolve the acceleration vector of 17 m/s^2 at an angle of 120° to the horizontal into a horizontal and a vertical component?

Solution:

Using Fig7, for a vector A at angle θ to the horizontal, the horizontal component is given by $A \cos \theta$ and the vertical component by $A \sin \theta$. Any convention of signs may be adopted, in this case horizontally from left to right is taken as positive and vertically upwards is taken as positive.

Horizontal component:

$H = 17 \cos 120^\circ = -8.5$ m/s^2,

acting from left to right Vertical component:

$V = 17 \sin 120^\circ = 14.72 m/s^2$, acting vertically upwards.

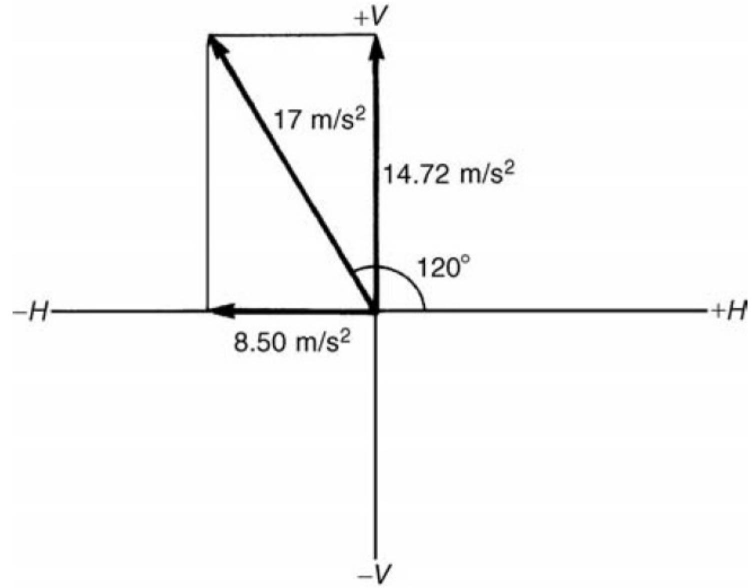


Fig 7 (J. O. Bird 2017)

Problem 3. Calculate the resultant force of the two forces given in Problem 1?
Solution:

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$$H = 7\cos 0^\circ + 4\cos 45^\circ = 7 + 2.828 = \mathbf{9.828\,N}$$

Vertical component of force,

$$V = 7\sin 0^\circ + 4\sin 45^\circ = 0 + 2.828 = \mathbf{2.828\,N}$$

The magnitude of the resultant of vector addition

$$= \sqrt{(H^2 + V^2)} = \sqrt{(9.828^2 + 2.828^2)}$$
$$= \sqrt{(104.59)} = \mathbf{10.23\,N}$$

The direction of the resultant of vector addition

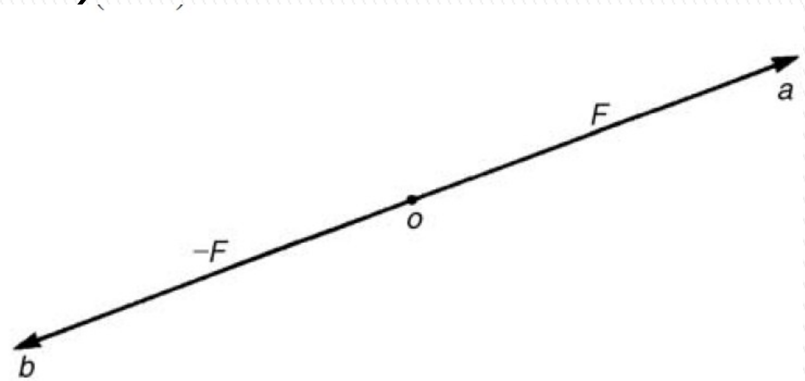
$$= \tan^{-1}\left(\frac{V}{H}\right) = \tan^{-1}\left(\frac{2.828}{9.828}\right) = \mathbf{16.05^\circ}$$

Thus, the resultant of the two forces is a single vector of 10.23 N at 16.05° to the 7 N vector.

Vector subtraction

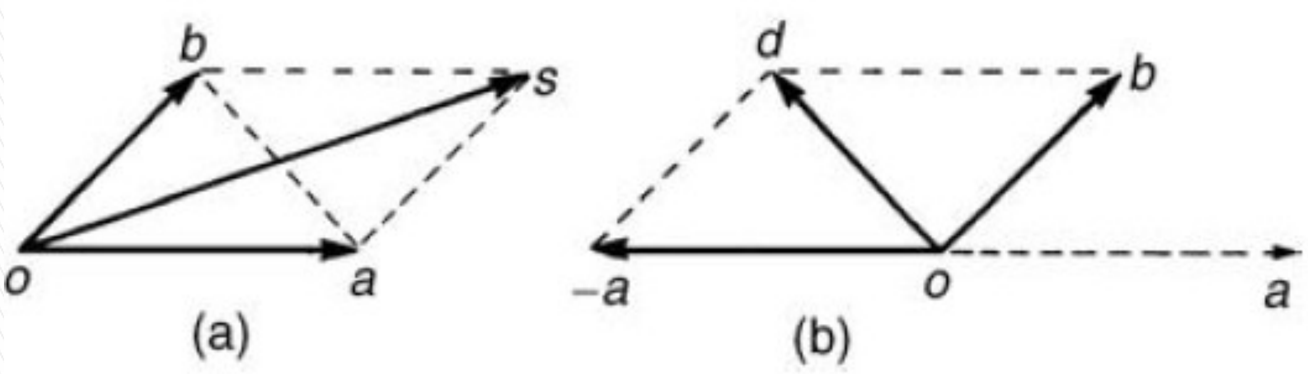
In Fig. 21.11, a force vector ***F*** is represented by ***oa***.
The vector (−***oa***) can be obtained by drawing a vector from *o* in the opposite sense to ***oa*** but having the same magnitude, shown as ***ob*** in Fig 8, i.e. ***ob=(−oa)***

Fig 8



For two vectors acting at a point, as shown in Fig 9a, the resultant of vector addition is ***os=oa+ob***. Fig 9b shows vectors ***ob+(−oa)***, that is, ***ob−oa*** and the vector equation is ***ob−oa=od***. Comparing ***od*** in Fig 9b with the broken line ab in Fig 9a shows that the second diagonal of the '**parallelogram' method** of vector addition gives the magnitude and direction of vector subtraction of ***oa*** from ***ob***.

Fig 9

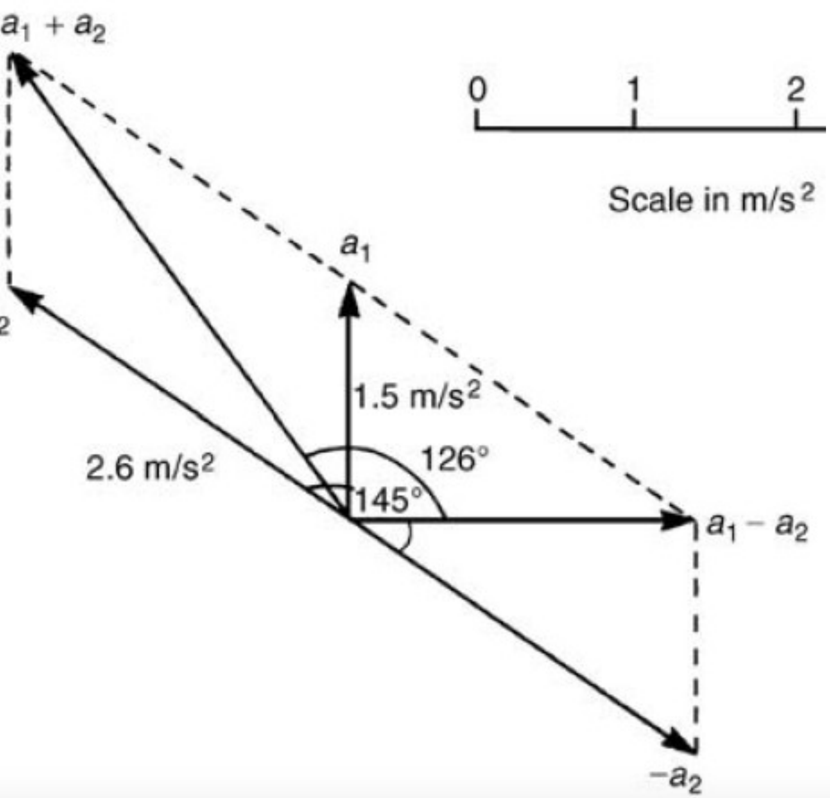


Problem 4. Accelerations of ***a1*** =1.5 m/s2 at 90◦ and ***a2*** =2.6 m/s2 at 145◦ act at a point. Find ***a1* + *a2*** and ***a1−a2*** by (i) drawing a scale vector diagram and (ii) by calculation.
Solution:

(i) The scale vector diagram is shown in Fig. By measurement,

$$\boldsymbol{a_1} + \boldsymbol{a_2} = \mathbf{3.7\,m/s^2 \text{ at } 126°}$$

$$\boldsymbol{a_1} - \boldsymbol{a_2} = \mathbf{2.1\,m/s^2 \text{ at } 0°}$$



(ii) Resolving horizontally and vertically gives:

Horizontal component of $\boldsymbol{a_1} + \boldsymbol{a_2}$,

$$\boldsymbol{H} = 1.5 \cos 90° + 2.6 \cos 145° = -2.13$$

Vertical component of $\boldsymbol{a_1} + \boldsymbol{a_2}$,

$$V = 1.5 \sin 90° + 2.6 \sin 145° = 2.99$$

$$\text{Magnitude of } \boldsymbol{a_1} + \boldsymbol{a_2} = \sqrt{(-2.13^2 + 2.99^2)} = \mathbf{3.67\,m/s^2}$$

$$\text{Direction of } \boldsymbol{a_1} + \boldsymbol{a_2} = \tan^{-1}\left(\frac{2.99}{-2.13}\right)$$

and must lie in the second quadrant since H is negative and V is positive.

Solution:

$\text{Tan}^{-1}\left(\frac{2.99}{-2.13}\right) = -54.53°$, and for this to be in the second quadrant, the true angle is 180° displaced, i.e. 180° – 54.53° or 125.47°.

Thus $\boldsymbol{a_1 + a_2 = 3.67\ m/s^2}$ **at 125.47°**.

Horizontal component of $\boldsymbol{a_1 - a_2}$, that is,

$$\boldsymbol{a_1} + (-\boldsymbol{a_2}) = 1.5\cos 90° + 2.6\cos(145° - 180°)$$
$$= 2.6\cos(-35°) = 2.13$$

Vertical component of $\boldsymbol{a_1 - a_2}$, that is,

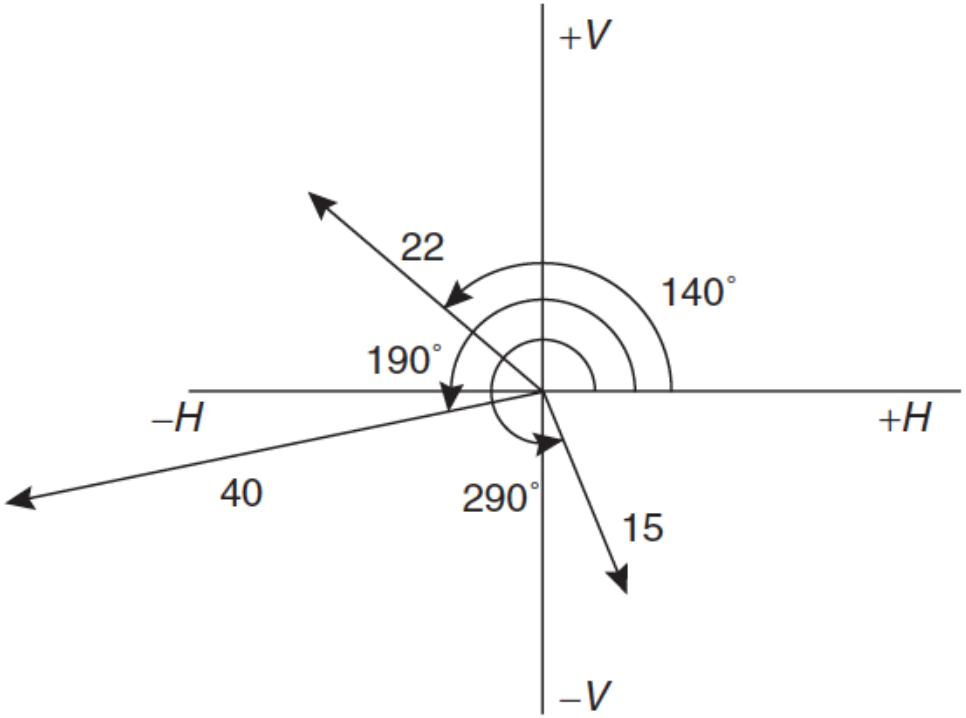
$$\boldsymbol{a_1} + (-\boldsymbol{a_2}) = 1.5\sin 90° + 2.6\sin(-35°) = 0$$

$$\text{Magnitude of } \boldsymbol{a_1 - a_2} = \sqrt{(2.13^2 + 0^2)} = 2.13\ \text{m/s}^2$$

$$\text{Direction of } \boldsymbol{a_1 - a_2} = \tan^{-1}\left(\frac{0}{2.13}\right) = 0°$$

Thus $\boldsymbol{a_1 - a_2 = 2.13\ m/s^2}$ **at 0°**.

Problem 5. Calculate the resultant of (i) ***v1*** **−*v2*** **+*v3*** and (ii) ***v2*** **−*v1*** **−*v3*** when ***v1*** =22 units at 140◦, ***v2*** =40 units at 190◦ and ***v3*** =15 units at 290◦.



Solution:

The horizontal component of $\boldsymbol{v_1} - \boldsymbol{v_2} + \boldsymbol{v_3}$

$$= (22 \cos 140°) - (40 \cos 190°) + (15 \cos 290°)$$

$$= (-16.85) - (-39.39) + (5.13)$$

$$= \mathbf{27.67\ units}$$

The vertical component of $\boldsymbol{v_1 - v_2 + v_3}$

$$= (22 \sin 140°) - (40 \sin 190°) + (15 \sin 290°)$$

$$= (14.14) - (-6.95) + (-14.10)$$

$$= \mathbf{6.99\ units}$$

The magnitude of the resultant, R, which can be represented by the mathematical symbol for 'the **modulus** of' as $|v_1 - v_2 + v_3|$ is given by:

$$|R| = \sqrt{(27.67^2 + 6.99^2)} = 28.54 \text{ units}$$

The direction of the resultant, $\boldsymbol{R}$, which can be represented by the mathematical symbol for 'the **argument** of' as $\arg(v_1 - v_2 + v_3)$ is given by:

$$\mathbf{arg}\ \boldsymbol{R} = \tan^{-1}\left(\frac{6.99}{27.67}\right) = 14.18°$$

Thus $\boldsymbol{v_1 - v_2 + v_3} = \mathbf{28.54\ units\ at\ 14.18°}$.

(ii) The horizontal component of $\boldsymbol{v_2 - v_1 - v_3}$

$$= (40\cos 190°) - (22\cos 140°) - (15\cos 290°)$$

$$= (-39.39) - (-16.85) - (5.13)$$

$$= \mathbf{-27.67\ units}$$

The vertical component of $\boldsymbol{v_2 - v_1 - v_3}$

$$= (40\sin 190°) - (22\sin 140°) - (15\sin 290°)$$

$$= (-6.95) - (14.14) - (-14.10)$$

$$= \mathbf{-6.99\ units}$$

Let $\boldsymbol{R} = v_2 - v_1 - v_3$

then $|R| = \sqrt{[(-27.67)^2 + (-6.99)^2]}$

$$= 28.54 \text{ units}$$

and $\mathbf{arg}\ \boldsymbol{R} = \tan^{-1}\left(\frac{-6.99}{-27.67}\right)$

and must lie in the third quadrant since both H and V are negative quantities.

$\text{Tan}^{-1}\left(\frac{-6.99}{-27.67}\right) = 14.18°$, hence the required angle is $180° + 14.18° = 194.18°$.

Thus $\boldsymbol{v_2 - v_1 - v_3} = \mathbf{28.54\ units\ at\ 194.18°}$.

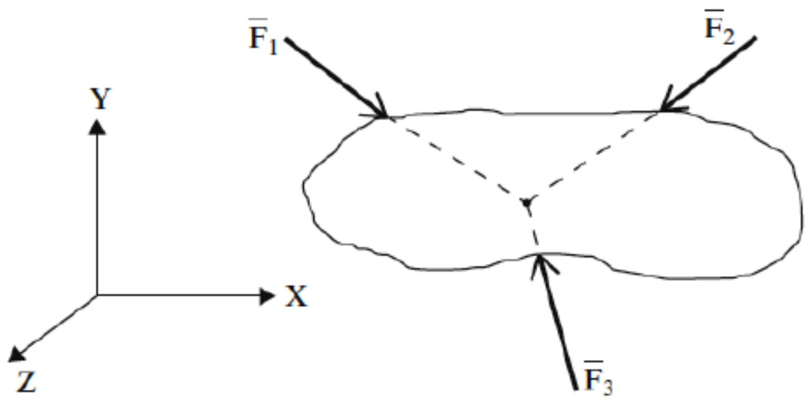
This result is as expected, since

$$\boldsymbol{v_2 - v_1 - v_3 = -(v_1 - v_2 + v_3)}$$

and the vector 28.54 units at 194.18° is minus times the vector 28.54 units at 14.18°.

Concept of Equilibrium-Concurrent Force System:

For static equilibrium, the resultant of the force system must be a null vector.



$$\overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \overrightarrow{0}$$ Eq(1)

$$\sum_{i=1}^{3} F_{i,x} = F_{1,x} + F_{2,x} + F_{3,x} = 0$$

$$\sum_{i=1}^{3} F_{i,y} = F_{1,y} + F_{2,y} + F_{3,y} = 0$$ Eq (2)

$$\sum_{i=1}^{3} F_{i,z} = F_{1,z} + F_{2,z} + F_{3,z} = 0$$

(Connor and Faraji 2012)

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Concept of Equilibrium: Non-concurrent Force System

For static equilibrium, Here the forces tend to rotate the body as well as translate it. Static equilibrium requires the resultant force vector to vanish and, in addition, the resultant moment vector about an arbitrary point to vanish

$$\overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \overrightarrow{0}$$

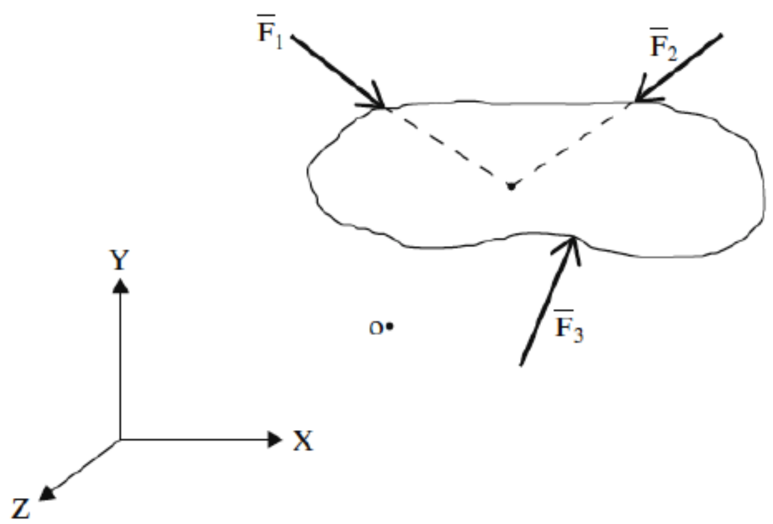
$$\overrightarrow{M_0} = \overrightarrow{0}$$

$$\sum_{i=1}^{3} F_{i,x} = 0$$

$$\sum_{i=1}^{3} F_{i,Y} = 0$$

$$\sum M_0 = 0$$

(Connor and Faraji 2012)



(Connor and Faraji 2012)

Example 1.2 Equilibrium equations

Given: The rigid body and force system shown in Fig. E1.2a. Forces A_x, A_Y, and B_Y are unknown.

a

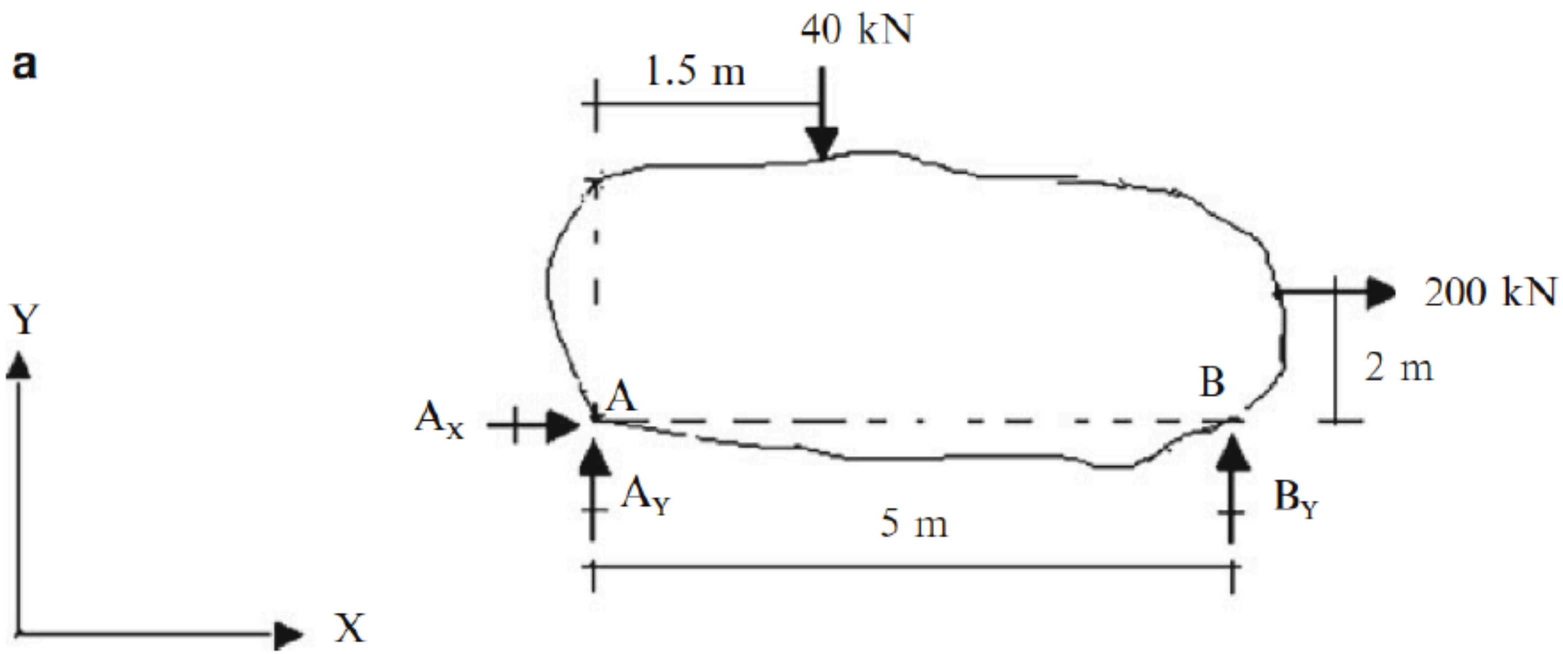


Fig. E1.2a

Determine: The forces A_x, A_Y, and B_Y

(Connor and Faraji 2012)

Solution: We sum moments about A, and solve for B_Y

$$\sum M_A = -40(1.5) - 200(2) + B_Y(5) = 0$$

$$B_Y = +92 \Rightarrow B_Y = 92\,\text{kN} \uparrow$$

Next, summing forces in the X and Y directions leads to (Fig. E1.2b)

$$\sum F_x \rightarrow^{+} = A_x + 200 = 0 \Rightarrow A_x = -200 \Rightarrow A_x = 200\text{ kN} \leftarrow$$

$$\sum F_y \uparrow^{+} = A_Y + 92 - 40 = 0 \Rightarrow A_Y = -52 \Rightarrow A_Y = 52\text{ kN} \downarrow$$

(Connor and Faraji 2012)

The unit triad

In general, the unit vector for ***oa*** is, $\frac{oa}{|oa|}$,

the ***oa*** being a vector and having both magnitude and direction and |***oa***| being the magnitude of the vector only

One method of completely specifying the direction of a vector in space relative to some reference point is to use three unit vectors, mutually at right angles to each other, as shown in Fig 12 Such a system is called a unit triad

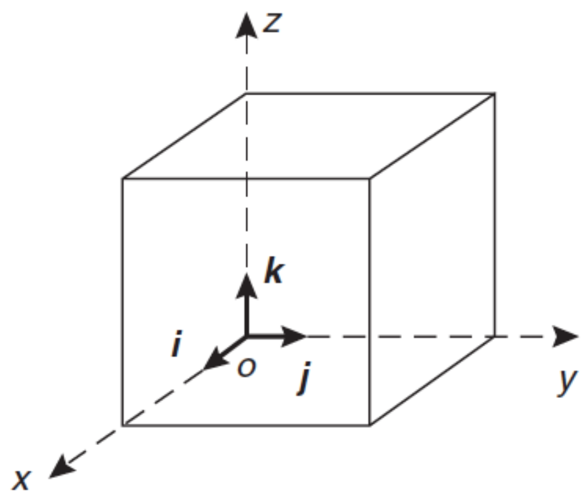


Fig 12 (J. O. Bird 2017)

In Fig. 13, one way to get from *o* to *r* is to move *x* units along ***i*** to point *a*, then y units in direction ***j*** to get to *b* and finally *z* units in direction ***k*** to get to *r*. The vector or is specified as:

$$or = xi + yj + zk$$

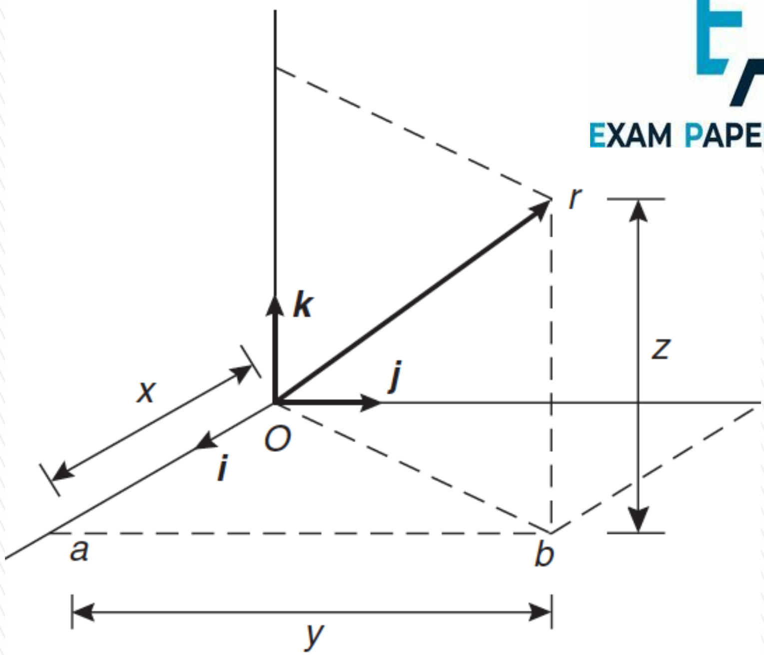
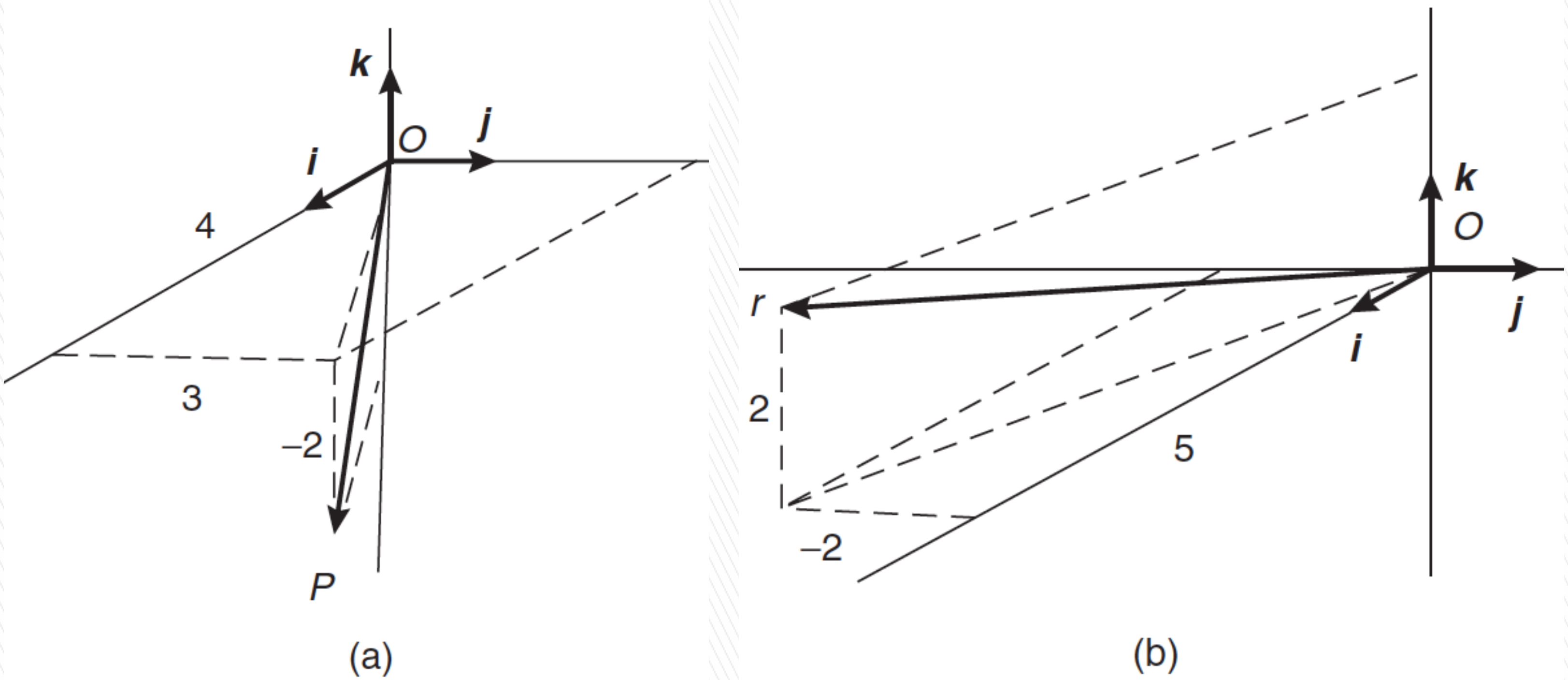


Fig 13

Problem 6. With reference to three axes drawn mutually at right angles, depict the vectors (i) ***op=4i+3j −2k*** and (ii) ***or=5i−2j +2k***.
Solution: The required vectors are depicted in Fig. 14, ***op*** being shown in Fig. 14a and ***or*** in Fig 14b.

Fig 14



(Obtained from J. O. Bird 2017)

The scalar product of two vectors

When vector $\boldsymbol{oa}$ is multiplied by a scalar quantity, say k, the magnitude of the resultant vector will be k times the magnitude of $\boldsymbol{oa}$ and its direction will remain the same. Thus $2 \times (5\,\text{N at } 20^\circ)$ results in a vector of magnitude 10 N at 20°.

One of the products of two vector quantities is called the **scalar** or **dot product** of two vectors and is defined as the product of their magnitudes multiplied by the cosine of the angle between them. The scalar product of $\boldsymbol{oa}$ and $\boldsymbol{ob}$ is shown as $\boldsymbol{oa} \bullet \boldsymbol{ob}$. For vectors $\boldsymbol{oa} = oa$ at θ_1, and $\boldsymbol{ob} = ob$ at θ_2 where $\theta_2 > \theta_1$, **the scalar product is**:

$$\boldsymbol{oa} \bullet \boldsymbol{ob} = oa\, ob\, \cos(\theta_2 - \theta_1)$$

For vectors ***v1*** and ***v2*** shown in Fig. 15, the scalar product is:

$$\boldsymbol{v}_1 \bullet \boldsymbol{v}_2 = v_1 v_2 \cos\theta$$

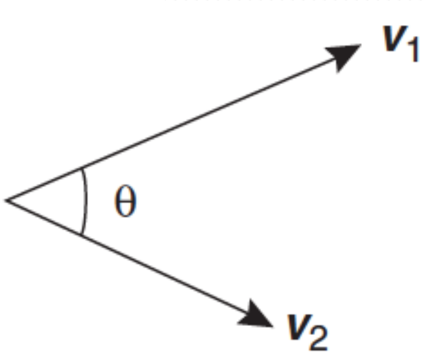


Fig 15

The angle between two vectors can be expressed in terms of the vector constants as follows: Because $\boldsymbol{a \bullet b} = a\, b \cos\theta$,

$$\text{then} \quad \boxed{\cos\theta = \frac{a \bullet b}{ab}} \qquad (1)$$

Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

$$\boldsymbol{a} \bullet \boldsymbol{b} = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \bullet (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k})$$

Multiplying out the brackets gives:

$$\begin{aligned}\boldsymbol{a} \bullet \boldsymbol{b} = {} & a_1b_1\boldsymbol{i}\bullet\boldsymbol{i} + a_1b_2\boldsymbol{i}\bullet\boldsymbol{j} + a_1b_3\boldsymbol{i}\bullet\boldsymbol{k} \\ & + a_2b_1\boldsymbol{j}\bullet\boldsymbol{i} + a_2b_2\boldsymbol{j}\bullet\boldsymbol{j} + a_2b_3\boldsymbol{j}\bullet\boldsymbol{k} \\ & + a_3b_1\boldsymbol{k}\bullet\boldsymbol{i} + a_3b_2\boldsymbol{k}\bullet\boldsymbol{j} + a_3b_3\boldsymbol{k}\bullet\boldsymbol{k}\end{aligned}$$

However, the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ all have a magnitude of 1 and $\boldsymbol{i}\bullet\boldsymbol{i} = (1)(1)\cos 0° = 1$, $\boldsymbol{i}\bullet\boldsymbol{j} = (1)(1)\cos 90° = 0$, $\boldsymbol{i}\bullet\boldsymbol{k} = (1)(1)\cos 90° = 0$ and similarly $\boldsymbol{j}\bullet\boldsymbol{j} = 1$, $\boldsymbol{j}\bullet\boldsymbol{k} = 0$ and $\boldsymbol{k}\bullet\boldsymbol{k} = 1$. Thus, only terms containing $\boldsymbol{i}\bullet\boldsymbol{i}$, $\boldsymbol{j}\bullet\boldsymbol{j}$ or $\boldsymbol{k}\bullet\boldsymbol{k}$ in the expansion above will not be zero.

Thus, the scalar product

$$\boxed{\boldsymbol{a}\bullet\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3} \qquad (2)$$

Both *a* and *b* in equation (1) can be expressed in terms of *a1, b1, a2, b2, a3* and *b3*.

From the geometry of Fig. 16, the length of diagonal *OP* in terms of side lengths *a, b* and *c* can be obtained from Pythagoras' theorem as follows:

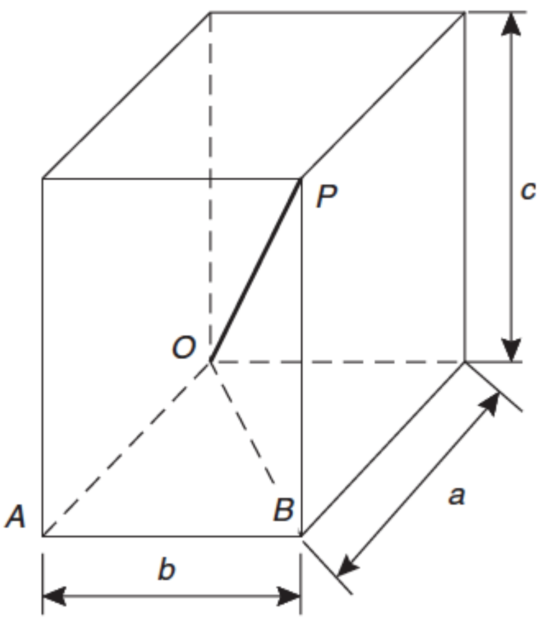


Fig 16 (J. O. Bird 2017)

$$OP^2 = OB^2 + BP^2 \text{ and}$$
$$OB^2 = OA^2 + AB^2$$

Thus, $OP^2 = OA^2 + AB^2 + BP^2$

$$= a^2 + b^2 + c^2,$$

in terms of side lengths

Thus, the **length** or **modulus** or **magnitude** or **norm of vector *OP*** is given by:

$$OP = \sqrt{(a^2 + b^2 + c^2)} \qquad (3)$$

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Relating this result to the two vectors $a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, gives:

$$a = \sqrt{(a_1^2 + a_2^2 + a_3^2)}$$

and $$b = \sqrt{(b_1^2 + b_2^2 + b_3^2)}.$$

That is, from equation (1),

$$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)}\sqrt{(b_1^2 + b_2^2 + b_3^2)}} \quad (4)$$

(Obtained from J. O. Bird 2017)

Problem 7. Find vector a joining points P and Q where point P has co-ordinates (4, −1, 3) and point Q has co-ordinates (2, 5, 0). Also, find $|a|$, the magnitude or norm of $\boldsymbol{a}$?

Solution:

Let O be the origin, i.e. its co-ordinates are (0, 0, 0).
The position vector of P and Q are given by:

$$\boldsymbol{OP} = 4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k} \text{ and } \boldsymbol{OQ} = 2\boldsymbol{i} + 5\boldsymbol{j}$$

By the addition law of vectors $\boldsymbol{OP} + \boldsymbol{PQ} = \boldsymbol{OQ}$.

Hence $\boldsymbol{a} = \boldsymbol{PQ} = \boldsymbol{OQ} - \boldsymbol{OP}$

i.e. $\boldsymbol{a} = \boldsymbol{PQ} = (2\boldsymbol{i} + 5\boldsymbol{j}) - (4i - \boldsymbol{j} + 3\boldsymbol{k})$

$$= -2\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$$

From equation (3), the magnitude or norm of $\boldsymbol{a}$,

$$|a| = \sqrt{(a^2 + b^2 + c^2)}$$

$$= \sqrt{[(-2)^2 + 6^2 + (-3)^2]} = \sqrt{49} = \mathbf{7}$$

Problem 8. If $p=2i+j-k$ and $q=i-3j+2k$ determine:

(i) $\boldsymbol{p} \bullet \boldsymbol{q}$ (ii) $\boldsymbol{p}+\boldsymbol{q}$

(iii) $|\boldsymbol{p}+\boldsymbol{q}|$ (iv) $|\boldsymbol{p}|+|\boldsymbol{q}|$

Solution:

(i) From equation (2),

$$\text{if} \quad \boldsymbol{p} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$$

$$\text{and} \quad \boldsymbol{q} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$$

$$\text{then} \quad \boldsymbol{p} \bullet \boldsymbol{q} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\text{When} \quad \boldsymbol{p} = \boldsymbol{2i + j - k},$$

$$a_1 = 2,\ a_2 = 1 \text{ and } a_3 = -1$$

$$\text{and when} \quad \boldsymbol{q} = \boldsymbol{i - 3j + 2k},$$

$$b_1 = 1,\ b_2 = -3 \text{ and } b_3 = 2$$

$$\text{Hence} \quad \boldsymbol{p} \bullet \boldsymbol{q} = (2)(1) + (1)(-3) + (-1)(2)$$

$$\text{i.e.} \quad \boldsymbol{p} \bullet \boldsymbol{q} = \boldsymbol{-3}$$

(ii) $\boldsymbol{p}+\boldsymbol{q} = (2\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}) + (\boldsymbol{i}-3\boldsymbol{j}+2\boldsymbol{k})$

$$= \boldsymbol{3i - 2j + k}$$

Solution:

(iii) $|\boldsymbol{p}+\boldsymbol{q}| = |3i - 2j + k|$

From equation (3),

$$|\boldsymbol{p}+\boldsymbol{q}| = \sqrt{[3^2 + (-2)^2 + 1^2]} = \sqrt{\mathbf{14}}$$

(iv) From equation (3),

$$|\boldsymbol{p}| = |2i + j - k|$$
$$= \sqrt{[2^2 + 1^2 + (-1)^2]} = \sqrt{6}$$

Similarly,

$$|\boldsymbol{q}| = |i - 3j + 2k|$$
$$= \sqrt{[1^2 + (-3)^2 + 2^2]} = \sqrt{14}$$

Hence $|\boldsymbol{p}| + |\boldsymbol{q}| = \sqrt{6} + \sqrt{14} = \mathbf{6.191}$, correct to 3 decimal places.

Problem 9. Determine the angle between vectors ***oa*** and ***ob*** when

$$\boldsymbol{oa} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$$

Solution:

and $\boldsymbol{ob} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$.

An equation for $\cos\theta$ is given in equation (4)

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)}\sqrt{(b_1^2 + b_2^2 + b_3^2)}}$$

Since $\boldsymbol{oa} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$,

$$a_1 = 1, a_2 = 2 \text{ and } a_3 = -3$$

Since $\boldsymbol{ob} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$,

$$b_1 = 2, b_2 = -1 \text{ and } b_3 = 4$$

Thus,

$$\cos\theta = \frac{(1 \times 2) + (2 \times -1) + (-3 \times 4)}{\sqrt{(1^2 + 2^2 + (-3)^2)}\sqrt{(2^2 + (-1)^2 + 4^2)}}$$

$$= \frac{-12}{\sqrt{14}\sqrt{21}} = -0.6999$$

i.e. $\theta = 134.4^\circ$ or 225.6°.

Vector products

A second product of two vectors is called the **vector or cross product** and is defined in terms of its modulus and the magnitudes of the two vectors and the sine of the angle between them. The vector product of vectors ***oa*** and ***ob*** is written as ***oa x ob*** and is defined by:

$$|\boldsymbol{oa} \times \boldsymbol{ob}| = oa\ ob \sin\theta$$

where θ is the angle between the two vectors. The direction of ***oa x ob*** is perpendicular to both ***oa*** and ***ob***, as shown in Fig 17

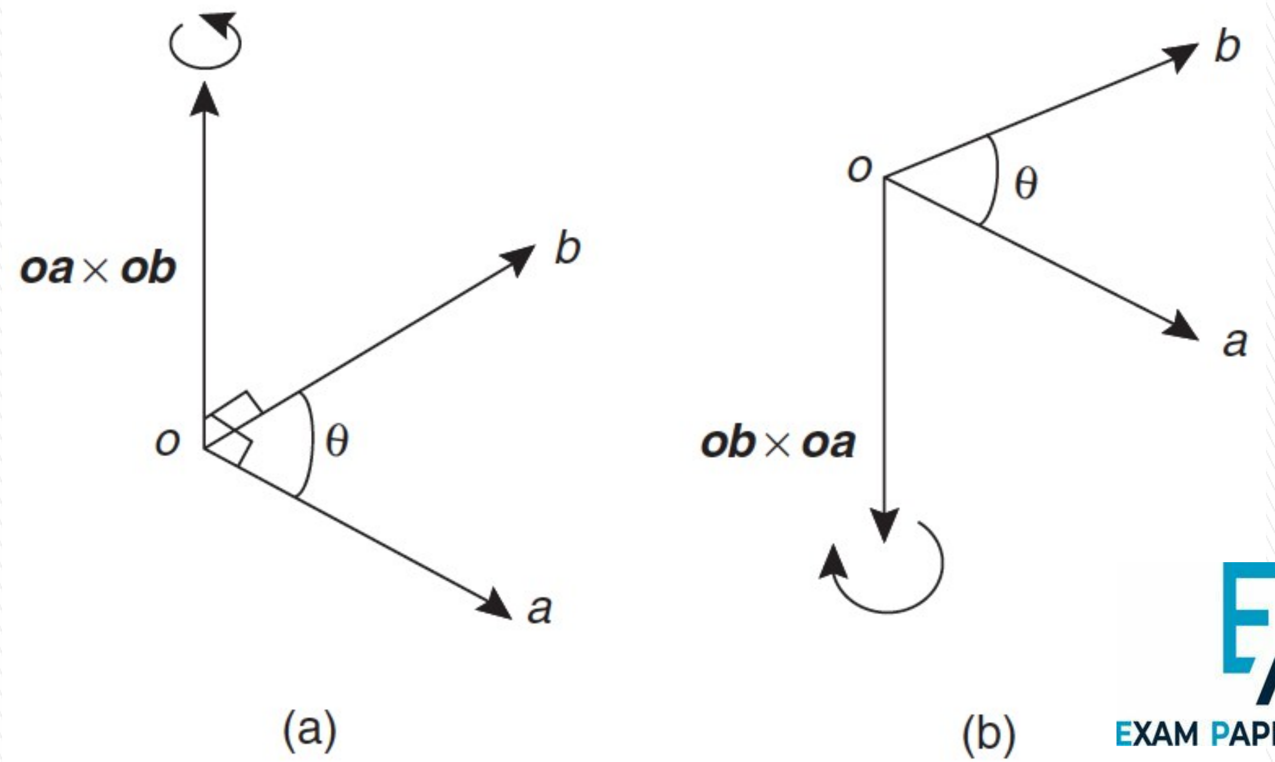

Fig 17
(J. O. Bird 2017)

The vector product of two vectors may be expressed in terms of the unit vectors. Let two vectors, a and b, be such that:

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \text{ and}$$

$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$$

Then,

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \times (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}) \\ &= a_1b_1\boldsymbol{i} \times \boldsymbol{i} + a_1b_2\boldsymbol{i} \times \boldsymbol{j} \\ &\quad + a_1b_3\boldsymbol{i} \times \boldsymbol{k} + a_2b_1\boldsymbol{j} \times \boldsymbol{i} + a_2b_2\boldsymbol{j} \times \boldsymbol{j} \\ &\quad + a_2b_3\boldsymbol{j} \times \boldsymbol{k} + a_3b_1\boldsymbol{k} \times \boldsymbol{i} + a_3b_2\boldsymbol{k} \times \boldsymbol{j} \\ &\quad + a_3b_3\boldsymbol{k} \times \boldsymbol{k}\end{aligned}$$

But by the definition of a vector product,

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k},\ \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i} \text{ and } \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$

Also $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = (1)(1)\sin 0^\circ = 0$.

Remembering that $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$ gives:

$$\boldsymbol{a} \times \boldsymbol{b} = a_1 b_2 \boldsymbol{k} - a_1 b_3 \boldsymbol{j} - a_2 b_1 \boldsymbol{k} + a_2 b_3 \boldsymbol{i} + a_3 b_1 \boldsymbol{j} - a_3 b_2 \boldsymbol{i}$$

Grouping the $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ terms together, gives:

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\boldsymbol{i} + (a_3 b_1 - a_1 b_3)\boldsymbol{j} + (a_1 b_2 - a_2 b_1)\boldsymbol{k}$$

The vector product can be written in determinant form as:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad (5)$$

The 3×3 determinant $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ is evaluated as:

$$\boldsymbol{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

where

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2 b_3 - a_3 b_2,$$

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = a_1 b_3 - a_3 b_1 \text{ and}$$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Note:

$$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{[(\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{b})^2]}$$

Problem 10. Problem 7. For the vectors *a*=*i*+4*j* −2*k* and *b*=2*i*−*j* +3*k* find ***a* x *b*** and **| *a* x *b* |**?

(i) From equation (5),

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \boldsymbol{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Hence

(*p*

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} i & j & k \\ 1 & 4 & -2 \\ 2 & -1 & 3 \end{vmatrix}$$

(b) (2*r*

$$= \boldsymbol{i}\begin{vmatrix} 4 & -2 \\ -1 & 3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}$$

$$+ \boldsymbol{k}\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}$$

$$= \boldsymbol{i}(12-2) - \boldsymbol{j}(3+4) + \boldsymbol{k}(-1-8)$$

$$= \mathbf{10\boldsymbol{i} - 7\boldsymbol{j} - 9\boldsymbol{k}}$$

Hence

Solution:

(ii) From equation (7)

$$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{[(\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{b})^2]}$$

Now $\boldsymbol{a} \cdot \boldsymbol{a} = (1)(1) + (4 \times 4) + (-2)(-2)$

$= 21$

$\boldsymbol{b} \cdot \boldsymbol{b} = (2)(2) + (-1)(-1) + (3)(3)$

$= 14$

and $\boldsymbol{a} \cdot \boldsymbol{b} = (1)(2) + (4)(-1) + (-2)(3)$

$= -8$

Thus $|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{(21 \times 14 - 64)}$

$= \sqrt{230} = \mathbf{15.17}$

Problem 11. Find the moment and the magnitude of the moment of a force of (i+2j −3k) newtons about point B having co-ordinates (0, 1, 1), when the force acts on a line through A whose co-ordinate are (1, 3, 4)?

Solution:

The moment $\boldsymbol{M}$ about point B of a force vector $\boldsymbol{F}$ which has a position vector of $\boldsymbol{r}$ from A is given by:

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$$

$\boldsymbol{r}$ is the vector from B to A, i.e. $\boldsymbol{r} = \boldsymbol{BA}$.
But $\boldsymbol{BA} = \boldsymbol{BO} + \boldsymbol{OA} = \boldsymbol{OA} - \boldsymbol{OB}$ (see Problem 8, Chapter 21), that is:

$$\boldsymbol{r} = (\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}) - (\boldsymbol{j} + \boldsymbol{k})$$
$$= \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$$

Moment,

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) \times (\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k})$$
$$= \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 2 & -3 \end{vmatrix}$$
$$= \boldsymbol{i}(-6 - 6) - \boldsymbol{j}(-3 - 3) + \boldsymbol{k}(2 - 2)$$
$$= \mathbf{-12\boldsymbol{i} + 6\boldsymbol{j}\ Nm}$$

The magnitude of $\boldsymbol{M}$,

$$
\begin{aligned}
|\boldsymbol{M}| &= |\boldsymbol{r} \times \boldsymbol{F}| \\
&= \sqrt{[(\boldsymbol{r} \bullet \boldsymbol{r})(\boldsymbol{F} \bullet \boldsymbol{F}) - (\boldsymbol{r} \bullet \boldsymbol{F})^2]} \\
\boldsymbol{r} \bullet \boldsymbol{r} &= (1)(1) + (2)(2) + (3)(3) = 14 \\
\boldsymbol{F} \bullet \boldsymbol{F} &= (1)(1) + (2)(2) + (-3)(-3) = 14 \\
\boldsymbol{r} \bullet \boldsymbol{F} &= (1)(1) + (2)(2) + (3)(-3) = -4 \\
|\boldsymbol{M}| &= \sqrt{[14 \times 14 - (-4)^2]} \\
&= \sqrt{180}\,\text{Nm} = \mathbf{13.42\,Nm}
\end{aligned}
$$

Vector equation of a line

In fig 18 If ***r=a+ AP*** and ***AP =λb***, where ***λ*** is a scalar quantity Hence ***r=a+λ b***

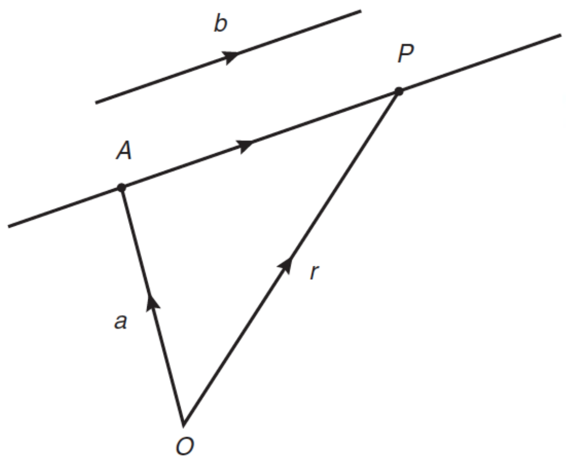

Fig 18

If, say, $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then from equation (8),

$$x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) + \lambda(b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k})$$

Hence $x = a_1 + \lambda b_1$, $y = a_2 + \lambda b_2$ and $z = a_3 + \lambda b_3$. Solving for λ gives:

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} = \lambda \qquad (9)$$

Equation (9) is the standard Cartesian form for the vector equation of a straight line.

Problem 11. (a) Determine the vector equation of the line through the point with position vector $2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$ which is parallel to the vector $\boldsymbol{i}-2\boldsymbol{i}+3\boldsymbol{k}$.

(b) Find the point on the line corresponding to $\lambda=3$ in the resulting equation of part (a).

(c) Express the vector equation of the line in standard Cartesian form.

(a) From equation (8),

$$\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{b}$$

i.e. $\boldsymbol{r}=(2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k})+\lambda(\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k})$

or $\boldsymbol{r}=(2+\lambda)\boldsymbol{i}+(3-2\lambda)\boldsymbol{j}+(3\lambda-1)\boldsymbol{k}$

which is the vector equation of the line.

(b) When $\lambda=3$, $\boldsymbol{r}=5\boldsymbol{i}-3\boldsymbol{j}+8\boldsymbol{k}$.

(c) From equation (9),

$$\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}=\lambda$$

Since $\boldsymbol{a}=2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$, then $a_1=2$,

$a_2=3$ and $a_3=-1$ and

$\boldsymbol{b}=\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$, then

$b_1=1, b_2=-2$ and $b_3=3$

Hence, the Cartesian equations are:

$$\frac{x-2}{1}=\frac{y-3}{-2}=\frac{z-(-1)}{3}=\lambda$$

i.e. $$x-2=\frac{3-y}{2}=\frac{z+1}{3}=\lambda$$

References:

1. J. O. Bird (2017), Higher engineering mathematics Eighth ed.
2. K. A. Stroud (1995) , Engineering mathematics Fourth ed.
3. Fundamentals of Structural Engineering. Jerome J. Connor , Susan Faraji. Springer New York Heidelberg Dordrecht London. (2012).