Applied Math

Lesson 4

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Integral calculus

The process of integration reverses the process of differentiation.
In differentiation, if *f (x)*= $2x^2$ then $f'(x)$ =*4x*. Thus the integral of 4x is $2x^2$
i.e. integration is the process of moving from $f'(x)$ to $f(x)$. Hence:

In integration the variable of integration is shown by adding *d* (the variable) after the function to be integrated

Thus $\int 4x\,dx$ means 'the integral of $4x$ with respect to x'

(J. O. Bird 2017)

the differential coefficient of $2x^2$+7 is also 4x. To allow for the possible presence of a constant, whenever the process of integration is performed, a constant '*c*' is added to the result.

$$\text{Thus} \quad \int 4x\,dx = 2x^2 + c$$

The general solution of integrals of the form ax^n

The general solution of integrals of the form $\int ax^n \mathrm{d}x$, where a and n are constants is given by:

$$\int ax^n \,\mathrm{d}x = \frac{ax^{n+1}}{n+1} + c$$

Using this rule gives:

(i) $\displaystyle\int 3x^4 \,\mathrm{d}x = \frac{3x^{4+1}}{4+1} + c = \frac{3}{5}x^5 + c$

(ii) $\displaystyle\int \frac{2}{x^2} \,\mathrm{d}x = \int 2x^{-2} \,\mathrm{d}x = \frac{2x^{-2+1}}{-2+1} + c$

$\displaystyle = \frac{2x^{-1}}{-1} + c = \frac{-2}{x} + c$, and

(iii) $\displaystyle\int \sqrt{x} \,\mathrm{d}x = \int x^{\frac{1}{2}} \,\mathrm{d}x = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$\displaystyle = \frac{2}{3}\sqrt{x^3} + c$

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When a sum of several terms is integrated the result is the sum of the integrals of the separate terms. For example:

$$\int (3x + 2x^2 - 5)\,dx$$

$$= \int 3x\,dx + \int 2x^2\,dx - \int 5\,dx$$

$$= \frac{3x^2}{2} + \frac{2x^3}{3} - 5x + c$$

(Obtained from J. O. Bird 2017)

Integration is the reverse process differentiation the standard integrals listed in below table

(i) $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$ (except when $n = -1$)

(ii) $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

(iii) $\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$

(iv) $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c$

(v) $\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + c$

(vi) $\int \operatorname{cosec} ax \cot ax\,dx = -\frac{1}{a}\operatorname{cosec} ax + c$

(vii) $\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + c$

(viii) $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$

(ix) $\int \frac{1}{x}\,dx = \ln x + c$

Problem 1. Determine $\int \left(4 + \frac{3}{7}x - 6x^2\right) dx$

Solution:

$$= 4x + \left(\frac{3}{7}\right)\frac{x^{1+1}}{1+1} - (6)\frac{x^{2+1}}{2+1} + c$$

$$= 4x + \left(\frac{3}{7}\right)\frac{x^2}{2} - (6)\frac{x^3}{3} + c$$

(Obtained from J. O. Bird 2017)

Problem 2. Determine $\int (1-t)^2 \, dt$

Solution:

$$\int (1 - 2t + t^2)\, dt = t - \frac{2t^{1+1}}{1+1} + \frac{t^{2+1}}{2+1} + c$$

$$= t - \frac{2t^2}{2} + \frac{t^3}{3} + c$$

Problem 3. Determine (a) $\int 3\sqrt{x}\,dx$ (b) $\int \frac{-5}{9\sqrt[4]{t^3}}\,dt$

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Solution (a):

$$\int 3\sqrt{x}\,dx = \int 3x^{\frac{1}{2}}\,dx = \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = 2\sqrt{x^3} + c$$

Solution (b):

$$\int \frac{-5}{9\sqrt[4]{t^3}}\,dt = \int \frac{-5}{9t^{\frac{3}{4}}}\,dt = \int \left(-\frac{5}{9}\right) t^{-\frac{3}{4}}\,dt$$

$$= \left(-\frac{5}{9}\right) \frac{t^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} + c$$

$$= \left(-\frac{5}{9}\right) \frac{t^{\frac{1}{4}}}{\frac{1}{4}} + c = \left(-\frac{5}{9}\right)\left(\frac{4}{1}\right) t^{\frac{1}{4}} + c$$

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(Obtained from J. O. Bird 2017)

Problem 4. Determine (a) $\int 4\cos 3x\,dx$ (b) $\int 5\sin 2\theta\,d\theta$

Solution: (a) From Table 37.1(ii),

$$\int 4\cos 3x\,dx = (4)\left(\frac{1}{3}\right)\sin 3x + c$$

$$= \frac{4}{3}\sin 3x + c$$

(b) From Table 37.1(iii),

$$\int 5\sin 2\theta\,d\theta = (5)\left(-\frac{1}{2}\right)\cos 2\theta + c$$

$$= -\frac{5}{2}\cos 2\theta + c$$

(Obtained from J. O. Bird 2017)

Problem 5. Determine (a) $\int 5\,e^{3x}\,dx$ (b) $\int \frac{2}{3\,e^{4t}}\,dt$

Solution: (a) From Table 37.1(viii),

$$\int 5\,e^{3x}\,dx = (5)\left(\frac{1}{3}\right)e^{3x} + c = \mathbf{\frac{5}{3}\,e^{3x} + c}$$

(b) $$\int \frac{2}{3\,e^{4t}}\,dt = \int \frac{2}{3}\,e^{-4t}\,dt = \left(\frac{2}{3}\right)\left(-\frac{1}{4}\right)e^{-4t} + c$$

$$= -\frac{1}{6}\,e^{-4t} + c = \mathbf{-\frac{1}{6e^{4t}} + c}$$

Problem 6. Determine (a) $\int \frac{3}{5x}\,dx$ (b) $\int \left(\frac{2m^2+1}{m}\right)dm$

Solution:

(a) $$\int \frac{3}{5x}\,dx = \int \left(\frac{3}{5}\right)\left(\frac{1}{x}\right)dx = \mathbf{\frac{3}{5}\ln x + c}$$

(from Table 37.1(ix))

(b) $$\int \left(\frac{2m^2+1}{m}\right)dm = \int \left(\frac{2m^2}{m} + \frac{1}{m}\right)dm$$

$$= \int \left(2m + \frac{1}{m}\right)dm$$

$$= \frac{2m^2}{2} + \ln m + c$$

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Definite integrals

Integrals containing an arbitrary constant c in their results are called indefinite integrals since their precise value cannot be determined without further information. Definite integrals are those in which limits are applied. If an expression is written as $[x]_a^b$, 'b' is called the upper limit and 'a' the lower limit. The operation of applying the limits is defined as $[x]_a^b = (b) - (a)$.

The increase in the value of the integral x^2 as x increases from 1 to 3 is written as $\int_1^3 x^2 \, dx$.

Applying the limits gives:

$$\int_1^3 x^2 \, dx = \left[\frac{x^3}{3} + c\right]_1^3 = \left(\frac{3^3}{3} + c\right) - \left(\frac{1^3}{3} + c\right)$$

$$= (9 + c) - \left(\frac{1}{3} + c\right) = \mathbf{8\frac{2}{3}}$$

(Obtained from J. O. Bird 2017)

Problem 7. Determine $\int_0^{\frac{\pi}{2}} 3\sin 2x\,dx$

Solution:

$$= \left[(3)\left(-\frac{1}{2}\right)\cos 2x\right]_0^{\frac{\pi}{2}} = \left[-\frac{3}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left\{-\frac{3}{2}\cos 2\left(\frac{\pi}{2}\right)\right\} - \left\{-\frac{3}{2}\cos 2(0)\right\}$$

$$= \left\{-\frac{3}{2}\cos \pi\right\} - \left\{-\frac{3}{2}\cos 0\right\}$$

$$= \left\{-\frac{3}{2}(-1)\right\} - \left\{-\frac{3}{2}(1)\right\} = \frac{3}{2} + \frac{3}{2} = \mathbf{3}$$

(Obtained from J. O. Bird 2017)

Algebraic substitutions

With algebraic substitutions, the substitution usually made is to let *u* be equal to *f(x)* such that *f(u)du* is a standard integral.
It is found that integrals of the forms,

$$k\int [f(x)]^n f'(x)\,dx \text{ and } k\int \frac{f'(x)}{[f(x)]^n}\,dx$$

(where *k* and *n* are constants) can both be integrated by substituting *u* for *f (x)*.

Problem 8. Determine $\int \cos(3x+7)\,dx$

Solution: $\int \cos(3x+7)\,dx$ is not a standard integral

Let $u = 3x + 7$ then $\frac{du}{dx} = 3$ and rearranging gives $dx = \frac{du}{3}$. Hence,

$$\int \cos(3x+7)\,dx = \int (\cos u)\frac{du}{3} = \int \frac{1}{3}\cos u\,du,$$

which is a standard integral

$$= \frac{1}{3}\sin u + c$$

Rewriting u as $(3x+7)$ gives:

$$\int \cos(3x+7)\,dx = \mathbf{\frac{1}{3}\sin(3x+7)+c},$$

Problem 9. Find $\int (2x-5)^7\,dx$

Solution: Let $u=(2x-5)$ then $\frac{du}{dx}=2$ and $dx=\frac{du}{2}$

Hence

$$\int (2x-5)^7\,dx=\int u^7\frac{du}{2}=\frac{1}{2}\int u^7\,du$$

$$=\frac{1}{2}\left(\frac{u^8}{8}\right)+c=\frac{1}{16}u^8+c$$

Rewriting u as $(2x-5)$ gives:

$$\int (2x-5)^7\,dx=\frac{1}{16}(2x-5)^8+c$$

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Problem 10. Find $\int \frac{4}{(5x-3)}\,dx$

Solution:

Let $u = (5x-3)$ then $\frac{du}{dx} = 5$ and $dx = \frac{du}{5}$

Hence

$$\int \frac{4}{(5x-3)}\,dx = \int \frac{4}{u}\frac{du}{5} = \frac{4}{5}\int \frac{1}{u}\,du$$

$$= \frac{4}{5}\ln u + c = \mathbf{\frac{4}{5}\ln(5x-3)+c}$$

Problem 11. Evaluate $\int_0^1 2e^{6x-1}\,dx$

Solution:

Let $u = 6x-1$ then $\frac{du}{dx} = 6$ and $dx = \frac{du}{6}$

Hence

$$\int 2e^{6x-1}\,dx = \int 2e^{u}\frac{du}{6} = \frac{1}{3}\int e^{u}\,du$$

$$= \frac{1}{3}e^{u} + c = \frac{1}{3}e^{6x-1} + c$$

Thus

$$\int_0^1 2e^{6x-1}\,dx = \frac{1}{3}[e^{6x-1}]_0^1 = \frac{1}{3}[e^5 - e^{-1}] = \mathbf{49.35},$$

Problem 12. Evaluate $\int_0^{\frac{\pi}{6}} 24\sin^5\theta\cos\theta\,d\theta$.

Solution: Let $u = \sin\theta$ then $\frac{du}{d\theta} = \cos\theta$ and $d\theta = \frac{du}{\cos\theta}$

Hence $\int 24\sin^5\theta\cos\theta\,d\theta = \int 24u^5\cos\theta\frac{du}{\cos\theta}$

$= 24\int u^5\,du$, by cancelling

$= 24\frac{u^6}{6} + c = 4u^6 + c = 4(\sin\theta)^6 + c$

$= 4\sin^6\theta + c$

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Thus $\int_0^{\frac{\pi}{6}} 24\sin^5\theta\cos\theta\,d\theta = [4\sin^6\theta]_0^{\frac{\pi}{6}}$

$= 4\left[\left(\sin\frac{\pi}{6}\right)^6 - (\sin 0)^6\right]$

$= 4\left[\left(\frac{1}{2}\right)^6 - 0\right] = \mathbf{\frac{1}{16}}$ or $\mathbf{0.0625}$

Change of limits

When evaluating definite integrals involving substitutions it is sometimes more convenient to change the limits of the integral as shown in Problems 13

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Evaluate $\int_1^3 5x\sqrt{(2x^2+7)}\,dx$, Solution:

Let $u = 2x^2 + 7$, then $\frac{du}{dx} = 4x$ and $dx = \frac{du}{4x}$

It is possible in this case to change the limits of integration. Thus when $x = 3$, $u = 2(3)^2 + 7 = 25$ and when $x = 1$, $u = 2(1)^2 + 7 = 9$.

Hence

$$\int_{x=1}^{x=3} 5x\sqrt{(2x^2+7)}\,dx = \int_{u=9}^{u=25} 5x\sqrt{u}\frac{du}{4x}$$

$$= \frac{5}{4}\int_9^{25}\sqrt{u}\,du$$

$$= \frac{5}{4}\int_9^{25} u^{\frac{1}{2}}\,du$$

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Thus the limits have been changed, and it is unneces sary to change the integral back in terms of x.

Thus $$\int_{x=1}^{x=3} 5x\sqrt{(2x^2+7)}\,dx = \frac{5}{4}\left[\frac{u^{\frac{3}{2}}}{3/2}\right]_9^{25}$$

$$= \frac{5}{6}\left[\sqrt{u^3}\right]_9^{25} = \frac{5}{6}\left[\sqrt{25^3} - \sqrt{9^3}\right]$$

$$= \frac{5}{6}(125 - 27) = \mathbf{81\frac{2}{3}}$$

Integration using trigonometric and hyperbolic substitutions

Following Table gives a summary of the integrals that require the use of trigonometric and hyperbolic substitutions

$f(x)$	$\int f(x)\mathrm{d}x$
1. $\cos^2 x$	$\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+c$
2. $\sin^2 x$	$\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)+c$
3. $\tan^2 x$	$\tan x - x + c$
4. $\cot^2 x$	$-\cot x - x + c$
5. $\cos^m x \sin^n x$	(a) If either m or n is odd (but not both), use $\cos^2 x + \sin^2 x = 1$ (b) If both m and n are even, use either $\cos 2x = 2\cos^2 x - 1$ or $\cos 2x = 1 - 2\sin^2 x$

(Obtained from J. O. Bird 2017)

10.	$\dfrac{1}{\sqrt{(a^2 - x^2)}}$	$\sin^{-1}\dfrac{x}{a} + c$
11.	$\sqrt{(a^2 - x^2)}$	$\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} + \dfrac{x}{2}\sqrt{(a^2 - x^2)} + c$
12.	$\dfrac{1}{a^2 + x^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + c$
13.	$\dfrac{1}{\sqrt{(x^2 + a^2)}}$	$\sinh^{-1}\dfrac{x}{a} + c$ or $\ln\left\{\dfrac{x + \sqrt{(x^2 + a^2)}}{a}\right\} + c$
14.	$\sqrt{(x^2 + a^2)}$	$\dfrac{a^2}{2}\sinh^{-1}\dfrac{x}{a} + \dfrac{x}{2}\sqrt{(x^2 + a^2)} + c$
15.	$\dfrac{1}{\sqrt{(x^2 - a^2)}}$	$\cosh^{-1}\dfrac{x}{a} + c$ or $\ln\left\{\dfrac{x + \sqrt{(x^2 - a^2)}}{a}\right\} + c$
16.	$\sqrt{(x^2 - a^2)}$	$\dfrac{x}{2}\sqrt{(x^2 - a^2)} - \dfrac{a^2}{2}\cosh^{-1}\dfrac{x}{a} + c$

(J. O. Bird 2017)

Problem 14. Evaluate $\int_0^{\frac{\pi}{4}} 2\cos^2 4t \, dt$.

Solution: Since $\cos 2t = 2\cos^2 t - 1$

then $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$ and

$$\cos^2 4t = \frac{1}{2}(1 + \cos 8t)$$

Hence $\int_0^{\frac{\pi}{4}} 2\cos^2 4t \, dt$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 8t)\, dt$$

$$= \left[t + \frac{\sin 8t}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4} + \frac{\sin 8\left(\frac{\pi}{4}\right)}{8}\right] - \left[0 + \frac{\sin 0}{8}\right] = \boldsymbol{\frac{\pi}{4}} \text{ or } \mathbf{0.7854}$$

(J. O. Bird 2017)

Problem 15. Find $3\int \tan^2 4x \, dx$

Solution: Since $1 + \tan^2 x = \sec^2 x$, then $\tan^2 x = \sec^2 x - 1$ and $\tan^2 4x = \sec^2 4x - 1$.

$$\text{Hence } 3\int \tan^2 4x \, dx = 3\int (\sec^2 4x - 1)\, dx$$

$$= 3\left(\frac{\tan 4x}{4} - x\right) + c$$

Problem 16. Determine $\int \sin^5 \theta \, d\theta$.

Solution: Since $\cos^2\theta + \sin^2\theta = 1$ then $\sin^2\theta = (1 - \cos^2\theta)$.

$$\text{Hence } \int \sin^5\theta \, d\theta$$

$$= \int \sin\theta (\sin^2\theta)^2 \, d\theta = \int \sin\theta (1 - \cos^2\theta)^2 \, d\theta$$

$$= \int \sin\theta (1 - 2\cos^2\theta + \cos^4\theta)\, d\theta$$

$$= \int (\sin\theta - 2\sin\theta\cos^2\theta + \sin\theta\cos^4\theta)\, d\theta$$

$$= -\cos\theta + \frac{2\cos^3\theta}{3} - \frac{\cos^5\theta}{5} + c$$

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Problem 17. Determine $\int \sin 3t \cos 2t \, dt$.

Solution:

$\sin A \cos B$ Use $\frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\cos A \sin B$ Use $\frac{1}{2}[\sin(A+B) - \sin(A-B)]$

$$= \int \frac{1}{2}[\sin(3t + 2t) + \sin(3t - 2t)] \, dt,$$

$$= \frac{1}{2}\int (\sin 5t + \sin t) \, dt$$

$$= \frac{1}{2}\left(\frac{-\cos 5t}{5} - \cos t\right) + c$$

(Obtained from J. O. Bird 2017)

Problem 18. Determine $\int \frac{1}{\sqrt{(a^2 - x^2)}} \mathrm{d}x$

Solution:

Let $x = a \sin\theta$, then $\frac{\mathrm{d}x}{\mathrm{d}\theta} = a \cos\theta$ and $\mathrm{d}x = a \cos\theta \, \mathrm{d}\theta$.

Hence $\int \frac{1}{\sqrt{(a^2 - x^2)}} \mathrm{d}x$

$$= \int \frac{1}{\sqrt{(a^2 - a^2 \sin^2\theta)}} a \cos\theta \, \mathrm{d}\theta$$

$$= \int \frac{a \cos\theta \, \mathrm{d}\theta}{\sqrt{[a^2(1 - \sin^2\theta)]}}$$

$$= \int \frac{a \cos\theta \, \mathrm{d}\theta}{\sqrt{(a^2 \cos^2\theta)}}, \quad \text{since } \sin^2\theta + \cos^2\theta = 1$$

$$= \int \frac{a \cos\theta \, \mathrm{d}\theta}{a \cos\theta} = \int \mathrm{d}\theta = \theta + c$$

Since $x = a \sin\theta$, then $\sin\theta = \frac{x}{a}$ and $\theta = \sin^{-1} \frac{x}{a}$.

Hence $\int \frac{1}{\sqrt{(a^2 - x^2)}} \mathrm{d}x = \boldsymbol{\sin^{-1} \frac{x}{a} + c}$

Problem 19. Evaluate $\int_0^3 \frac{1}{\sqrt{(9-x^2)}}\,dx$.

Solution: From Problem 18

$$= \left[\sin^{-1}\frac{x}{3}\right]_0^3, \quad \text{since } a = 3$$

$$= (\sin^{-1} 1 - \sin^{-1} 0) = \frac{\pi}{2} \text{ or } \mathbf{1.5708}$$

Problem 20. Determine $\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx$.

Solution:

$$= \boldsymbol{\sinh^{-1}\frac{x}{a} + c}, \text{ since } x = a\sinh\theta$$

Problem 21. Evaluate $\int_0^2 \frac{1}{\sqrt{(x^2+4)}}\,dx$,

Solution: $= \left[\sinh^{-1}\frac{x}{2}\right]_0^2$ or

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$$\left[\ln\left\{\frac{x+\sqrt{(x^2+4)}}{2}\right\}\right]_0^2$$

$$= \left[\ln\left(\frac{2+\sqrt{8}}{2}\right) - \ln\left(\frac{0+\sqrt{4}}{2}\right)\right]$$

$$= \ln 2.4142 - \ln 1 = \mathbf{0.8814},$$

Integration by parts

From the product rule of differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x},$$

where u and v are both functions of x.

Rearranging gives: $u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - v\frac{\mathrm{d}u}{\mathrm{d}x}$

Integrating both sides with respect to x gives:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\,\mathrm{d}x - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

i.e.

$$\boxed{\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x}$$

or

$$\boxed{\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u}$$

Problem 22 Determine $\int x \cos x \, dx$.

Solution:

From the integration by parts formula,

$$\int u \, dv = uv - \int v \, du$$

Let $u = x$, from which $\frac{du}{dx} = 1$, i.e. $du = dx$ and let $dv = \cos x \, dx$, from which $v = \int \cos x \, dx = \sin x$.

Expressions for u, du and v are now substituted into the 'by parts' formula as shown below.

$$\begin{array}{llcllcll} \int u & dv & = & u & v & - & \int v & du \\ \int x & \cos x \, dx & = & (x) & (\sin x) & - & \int (\sin x) & (dx) \end{array}$$

i.e. $\int x \cos x \, dx = x \sin x - (-\cos x) + c$

$$= \mathbf{x \sin x + \cos x + c}$$

Obtained from J. O. Bird 2017

Problem 23 Find $\int 3te^{2t}\,dt$.

Solution: Let $u = 3t$, from which, $\frac{du}{dt} = 3$, i.e. $du = 3\,dt$ and

let $dv = e^{2t}\,dt$, from which, $v = \int e^{2t}\,dt = \frac{1}{2}e^{2t}$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int 3te^{2t}\,dt = (3t)\left(\frac{1}{2}e^{2t}\right) - \int\left(\frac{1}{2}e^{2t}\right)(3\,dt)$$

$$= \frac{3}{2}te^{2t} - \frac{3}{2}\int e^{2t}\,dt$$

$$= \frac{3}{2}te^{2t} - \frac{3}{2}\left(\frac{e^{2t}}{2}\right) + c$$

Hence

$$\int 3t\,e^{2t}\,dt = \frac{3}{2}e^{2t}\left(t - \frac{1}{2}\right) + c,$$

which may be checked by differentiating.

Problem 24. Find $\int x \ln x \, dx$.

The logarithmic function is chosen as the 'u part'.

Thus when $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$, i.e. $du = \frac{dx}{x}$

Letting $dv = x\,dx$ gives $v = \int x\,dx = \frac{x^2}{2}$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int x \ln x \, dx = (\ln x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right) \frac{dx}{x}$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x\,dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \left(\frac{x^2}{2}\right) + c$$

Hence $\int x \ln x \, dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$ or

$$\frac{x^2}{4}(2 \ln x - 1) + c$$

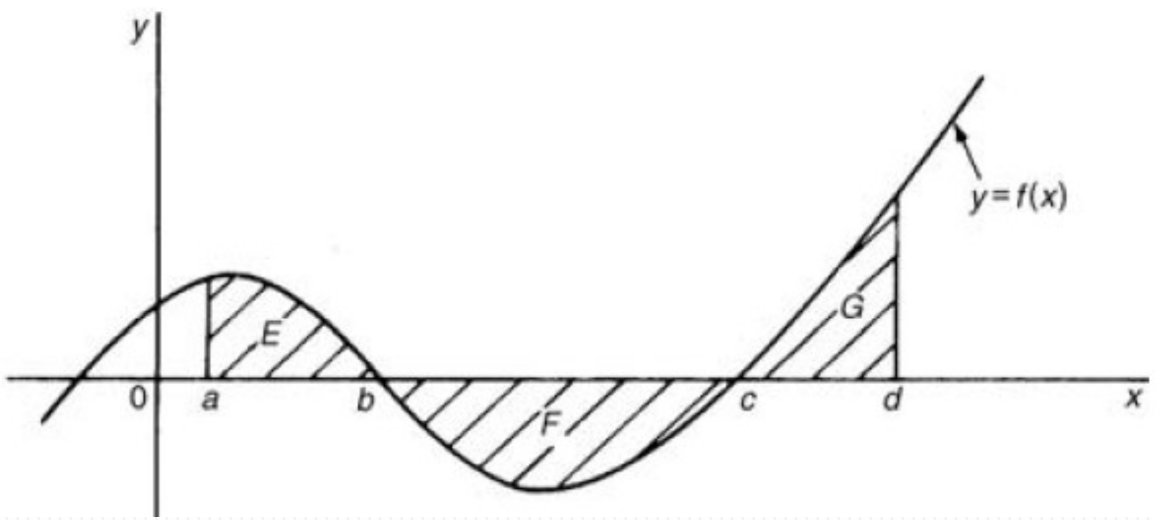
Some applications of integration

There are a number of applications of integral calculus in engineering. The determination of areas, mean and r.m.s. values, volumes, centroids and second moments of area.

Areas under and between curves

In Fig 1

$$\text{total shaded area} = \int_a^b f(x)\,dx - \int_b^c f(x)\,dx + \int_c^d f(x)\,dx$$



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Problem 25. Determine the area between the curve $y = x^3 - 2x^2 - 8x$ and the x-axis.

$$y = x^3 - 2x^2 - 8x = x(x^2 - 2x - 8) = x(x+2)(x-4)$$

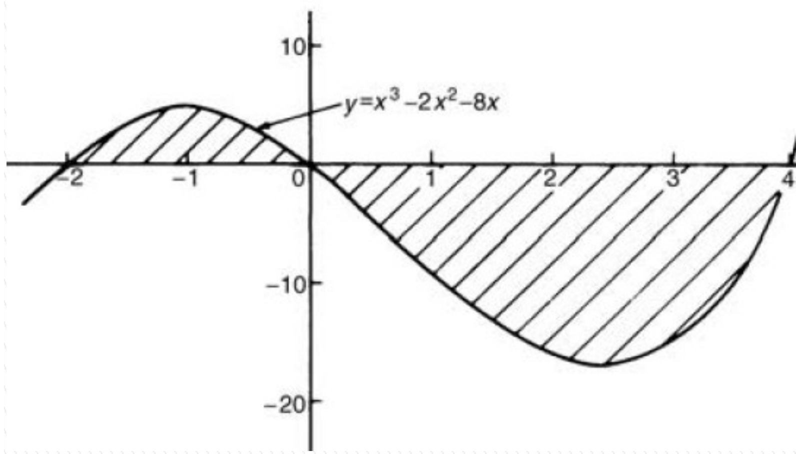
When $y = 0$, $x = 0$ or $(x + 2) = 0$ or $(x - 4) = 0$, i.e. when $y = 0$, $x = 0$ or -2 or 4, which means that the curve crosses the x-axis at 0, -2, and 4. Since the curve is a continuous function, only one other co-ordinate value needs to be calculated before a sketch of the curve can be produced. When $x = 1$, $y = -9$, showing that the part of the curve between $x = 0$ and $x = 4$ is negative. A sketch of $y = x^3 - 2x^2 - 8x$ is shown in Fig. 38.2. (Another method of sketching Fig. 38.2 would have been to draw up a table of values).

Shaded area

$$= \int_{-2}^{0} (x^3 - 2x^2 - 8x)\mathrm{d}x - \int_{0}^{4} (x^3 - 2x^2 - 8x)\mathrm{d}x$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{8x^2}{2}\right]_{-2}^{0} - \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{8x^2}{2}\right]_{0}^{4}$$

$$= \left(6\frac{2}{3}\right) - \left(-42\frac{2}{3}\right) = \mathbf{49\frac{1}{3}\ square\ units}$$



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1. J. O. Bird (2017), Higher engineering mathematics Eighth ed.
2. K. A. Stroud (1995) , Engineering mathematics Fourth ed.