LITE Regal EDUCATION

EXAM PAPERS PRACTICE

Applied Math

Lesson 1

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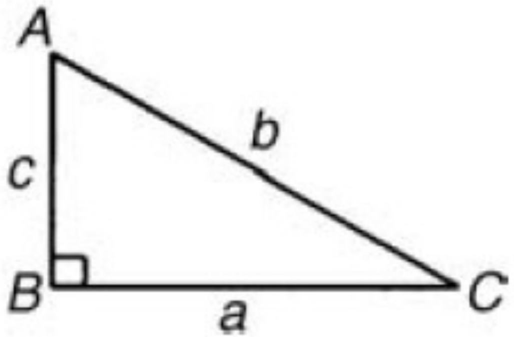
Trigonometry

Trigonometry is the branch of mathematics which deals with the measurement of sides and angles of triangles, and their relationship with each other. There are many applications in engineering where a knowledge of trigonometry is needed.

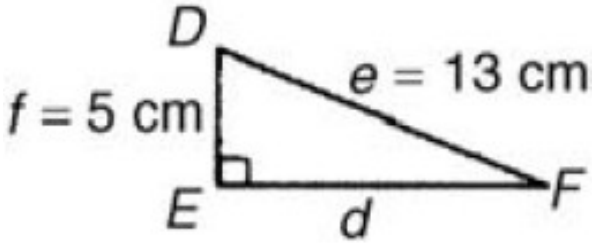
The theorem of Pythagoras

OIn any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.'

$$\text{Hence } b^2 = a^2 + c^2$$



Problem 1. In Fig. 1, find the length of EF



By Pythagoras' theorem:

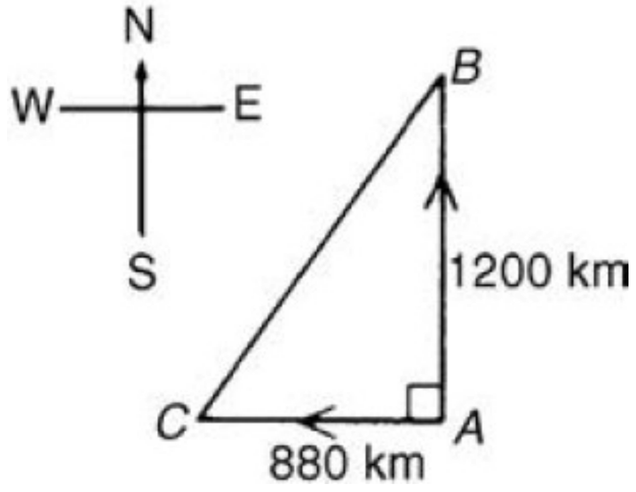
$$e^2 = d^2 + f^2$$

$$\text{Hence} \quad 13^2 = d^2 + 5^2$$

$$169 = d^2 + 25$$

$$d^2 = 169 - 25 = 144$$

Problem 2. Two aircraft leave an airfield at the same time. One travels due north at an average peed of 300 km/h and the other due west at an average speed of 220 km/h. Calculate their distance apart after 4 hours? Solution: After 4 hours, the first aircraft has travelled 4x300=1200 km, due north, and the second aircraft has travelled 4x220=880 km due west



From Pythagoras' theorem:

$$BC^2 = 1200^2 + 880^2 = 1\,440\,000 + 774\,400$$

$$\text{and} \quad BC = \sqrt{(2\,214\,400)}$$

Hence distance apart after 4 hours = 1488 km.

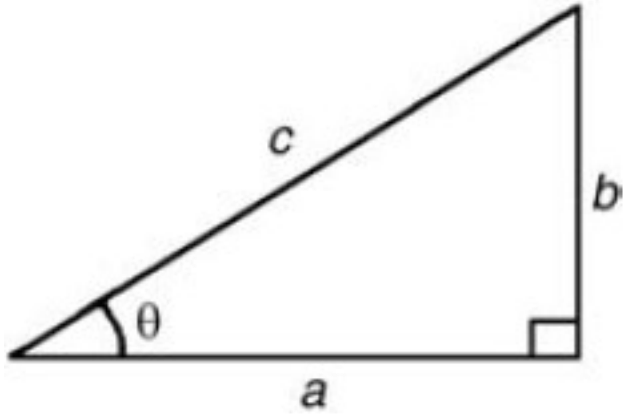
Trigonometric ratios

A trigonometric ratio (trig. ratio for short) is the ratio of the lengths of two sides of a right-angled triangle. The basic trig. ratios are **sine**, **cosine** and **tangent**. We usually write them as **sin**, **cos** and **tan**.

We can define trig ratios in two ways:

- using a right-angled triangle. This method defines trig ratios **only** for angles greater than 0° and less than 90°.
- using a unit circle. This method defines trig ratios for **any** angle.

With reference to the right-angled triangle shown in the next Fig.



(i) $$\text{sine}\,\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

i.e. $$\boldsymbol{\sin\theta = \frac{b}{c}}$$

(ii) $$\text{cosine}\,\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

i.e. $$\boldsymbol{\cos\theta = \frac{a}{c}}$$

(iii) $$\text{tangent}\,\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

i.e. $$\boldsymbol{\tan\theta = \frac{b}{a}}$$

(iv) $$\text{secant}\,\theta = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

i.e. $$\boldsymbol{\sec\theta = \frac{c}{a}}$$

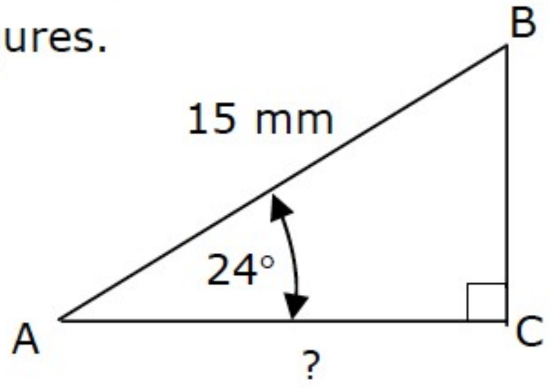
(v) $$\text{cosecant}\,\theta = \frac{\text{hypotenuse}}{\text{opposite side}}$$

i.e. $$\boldsymbol{\text{cosec}\,\theta = \frac{c}{b}}$$

(vi) $$\text{cotangent}\,\theta = \frac{\text{adjacent side}}{\text{opposite side}}$$

i.e. $$\boldsymbol{\cot\theta = \frac{a}{b}}$$

Problem 3. Find the length of the side AC in the right-angled triangle below giving your answer correct to 3 significant figures.



Solution:

Step 1: Substitute into cosine ratio:

$$\cos 24^o = \frac{AC}{15}$$

Step 2: Use the cosine function on your calculator to find the cosine of 24°:

$$0.9135 = \frac{AC}{15}$$

Step 3: Cross-multiply to find AC:

$$AC = 0.9135 \times 15 = 13.7025$$

So, the length of the side AC is 13.7 mm (to 3 s. f.)

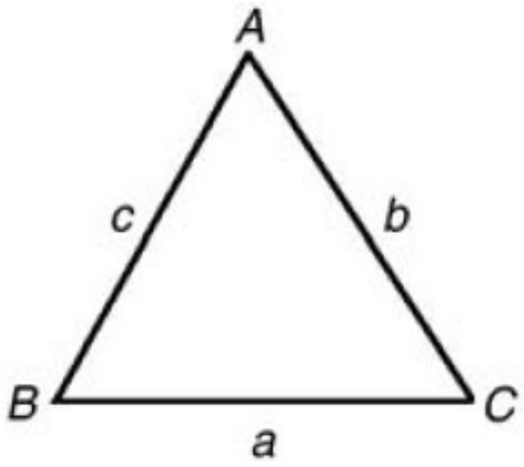
Sine and cosine rules

To ‘solve a triangle’ means ‘to find the values of unknown sides and angles’. If a triangle is right angled, trigonometric ratios and the theorem of Pythagoras may be used for its solution, as shown in previous section. However, for a non-right-angled triangle, trigonometric ratios and Pythagoras’ theorem cannot be used. Instead, two rules, called the sine rule and the cosine rule, are used.

Sine rule

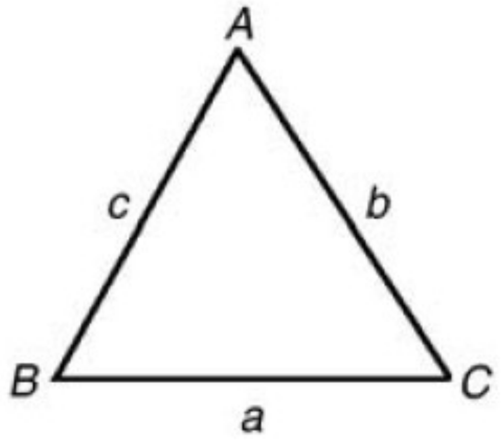
With reference to triangle ABC of below Fig the sine rule states:

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$



Cosine rule

With reference to triangle ABC of next Fig. the cosine rule states:



$$
\begin{aligned}
& a^2 = b^2 + c^2 - 2bc\cos A \\
\text{or} \quad & b^2 = a^2 + c^2 - 2ac\cos B \\
\text{or} \quad & c^2 = a^2 + b^2 - 2ab\cos C
\end{aligned}
$$

The rule may be used only when:

(i) 2 sides and the included angle are initially given,
or
(ii) 3 sides are initially given.

Area of any triangle:

(i) $\frac{1}{2} \times$ base $\times$ perpendicular height, or

(ii) $\frac{1}{2}ab\sin C$ or $\frac{1}{2}ac\sin B$ or $\frac{1}{2}bc\sin A$, or

(iii) $\sqrt{[s(s-a)(s-b)(s-c)]}$, where

$$s = \frac{a+b+c}{2}$$

Problem 21. In a triangle XYZ, $\angle X = 51°$, $\angle Y = 67°$ and $YZ = 15.2$ cm. Solve the triangle and find its area.

The triangle XYZ is shown in Fig. 12.20. Since the angles in a triangle add up to 180°, then $\mathbf{Z} = 180° - 51° - 67° = \mathbf{62°}$. Applying the sine rule:

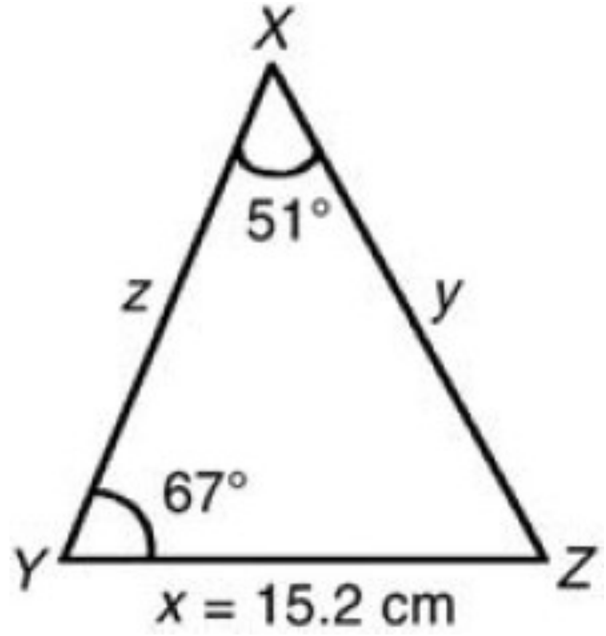
$$\frac{15.2}{\sin 51°} = \frac{y}{\sin 67°} = \frac{z}{\sin 62°}$$

Using $\dfrac{15.2}{\sin 51°} = \dfrac{y}{\sin 67°}$ and transposing gives:

$$y = \frac{15.2 \sin 67°}{\sin 51°} = \mathbf{18.00\ cm} = \boldsymbol{XZ}$$

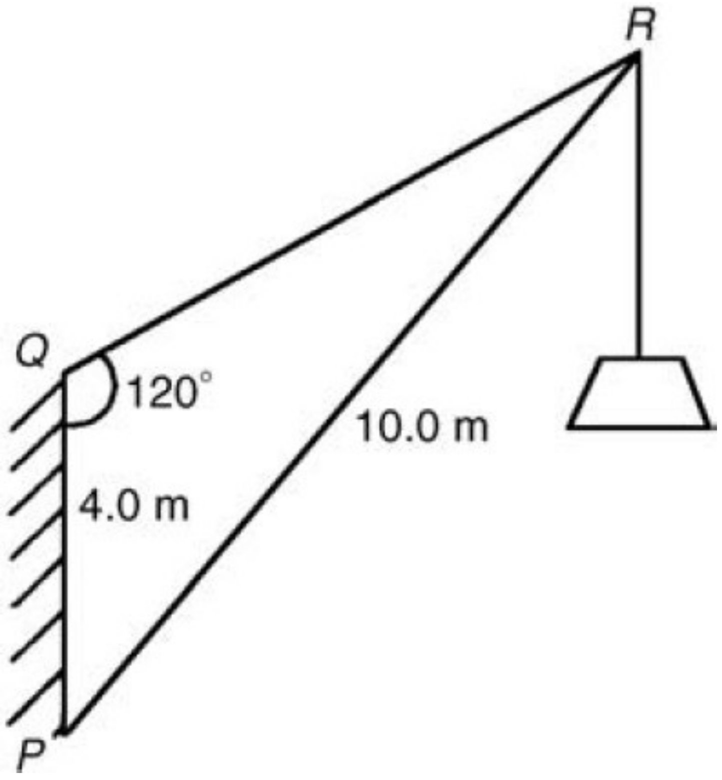
Using $\dfrac{15.2}{\sin 51°} = \dfrac{z}{\sin 62°}$ and transposing gives:

$$z = \frac{15.2 \sin 62°}{\sin 51°} = \mathbf{17.27\ cm} = \boldsymbol{XY}$$



Area of triangle $\boldsymbol{XYZ} = \frac{1}{2} xy \sin Z$
$= \frac{1}{2}(15.2)(18.00) \sin 62° = \mathbf{120.8\ cm^2}$ (or area
$= \frac{1}{2} xz \sin Y = \frac{1}{2}(15.2)(17.27) \sin 67° = \mathbf{120.8\ cm^2}$

Problem 5. In the next Fig. PR represents the inclined jib of a crane and is 10.0m long. PQ is 4.0m long. Determine the inclination of the jib to the vertical and the length of tie QR?



Applying the sine rule:

$$\frac{PR}{\sin 120°} = \frac{PQ}{\sin R}$$

from which,

$$\sin R = \frac{PQ \sin 120°}{PR} = \frac{(4.0)\sin 120°}{10.0}$$
$$= 0.3464$$

Hence $\angle R = \sin^{-1} 0.3464 = 20°16'$ (or $159°44'$, which is impossible in this case).
$\angle \boldsymbol{P} = 180° - 120° - 20°16' = \mathbf{39°44'}$, **which is the inclination of the jib to the vertical**.

Applying the sine rule:

$$\frac{10.0}{\sin 120°} = \frac{QR}{\sin 39°44'}$$

from which, **length of tie**,

$$\boldsymbol{QR} = \frac{10.0 \sin 39°44'}{\sin 120°} = \mathbf{7.38\,m}$$

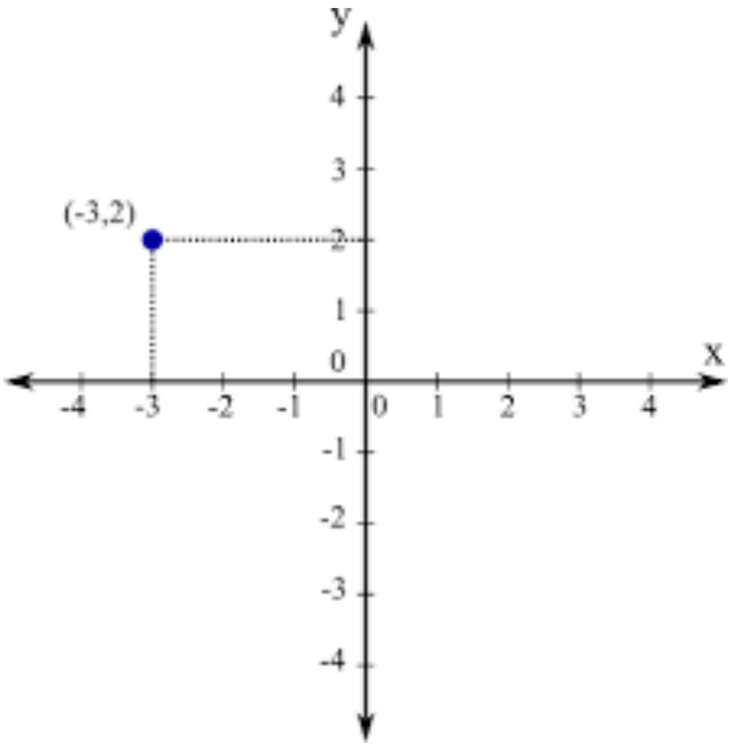
Coordinate Systems:

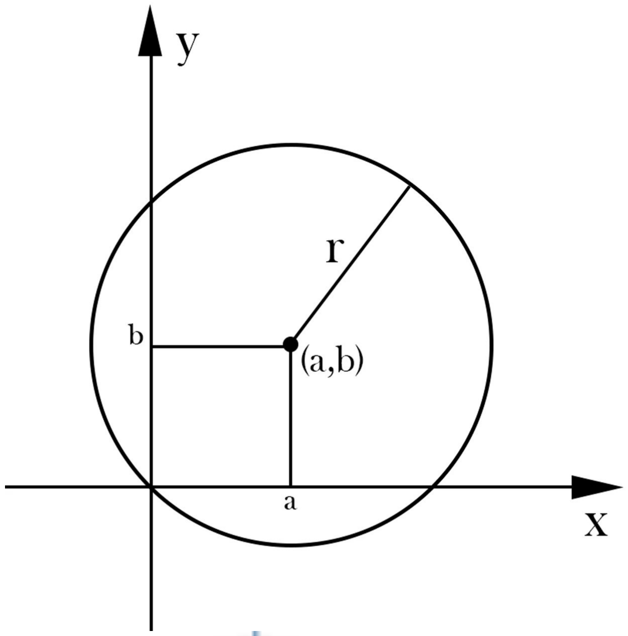
Cartesian Coordinate Systems: This type of coordinate system is the most commonly used coordinate system in different calculations. These cartesian coordinate systems can be used in both 2D and 3D positioning.

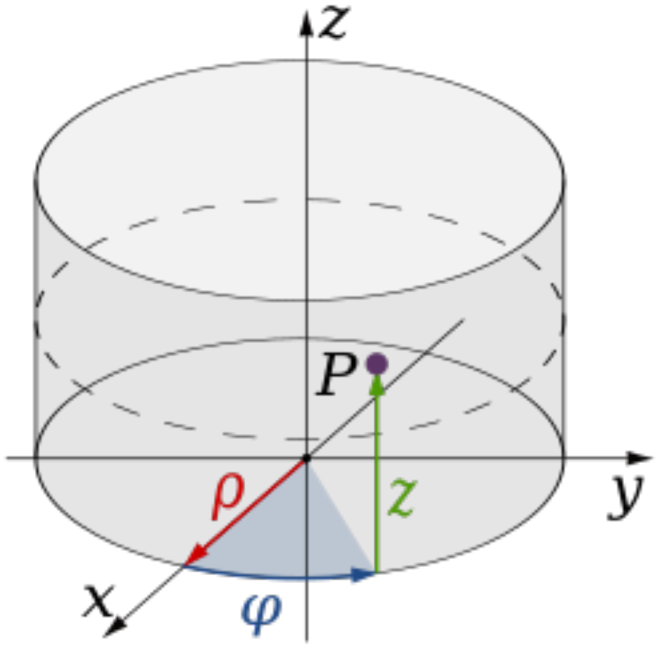
Polar Coordinate Systems: Polar coordinate systems are suitable for the 2D positionings. It has a different logic other than cartesian coordinate systems. The values to define the position of a point are angle and radius. Relative and absolute positionings are possible for polar coordinate systems also.

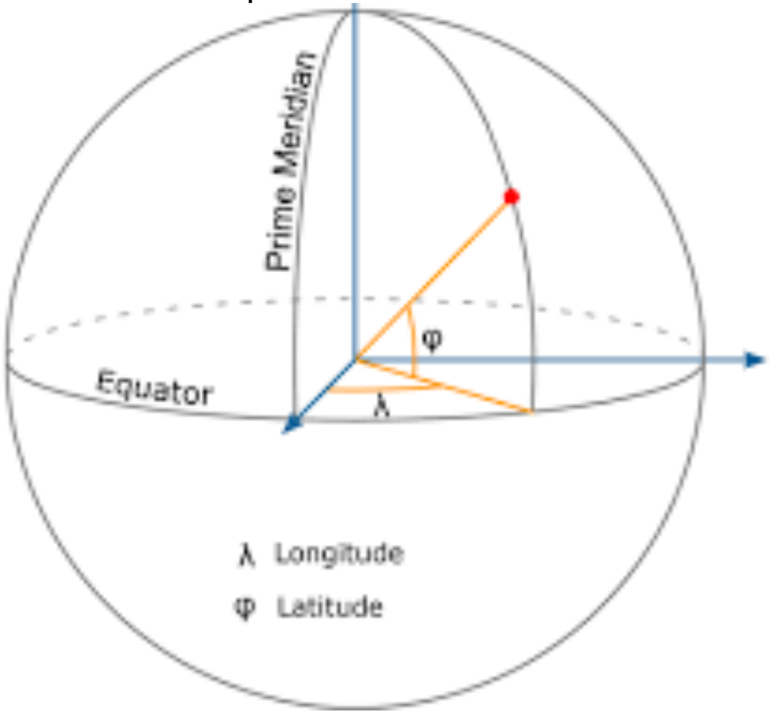
Cylindrical Coordinate Systems: Cylindrical coordinate system is the same as the polar coordinate system. But the cylindrical coordinate system is produced for 3D positioning and it includes the radius and two angle values.

Spherical Coordinate Systems: Spherical coordinate system is another type of coordinate system developed for 3D space positioning. It includes radius and two angle values like cylindrical coordinate systems. But it has a different notation from that. Coordinate systems are very common in CAD environments also.


y
x
(-3,2)
4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4


y
x
r
b
a
(a,b)

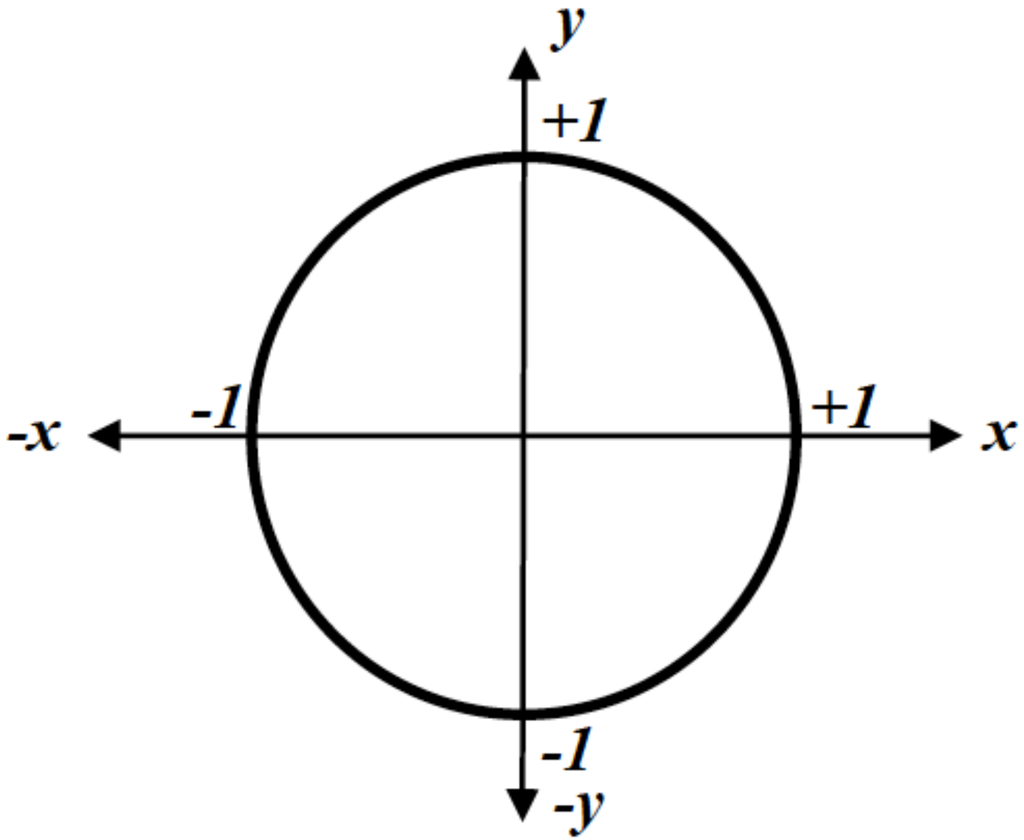

z
P
ρ
z
y
x
φ


Prime Meridian
Equator
φ
λ
λ Longitude
φ Latitude

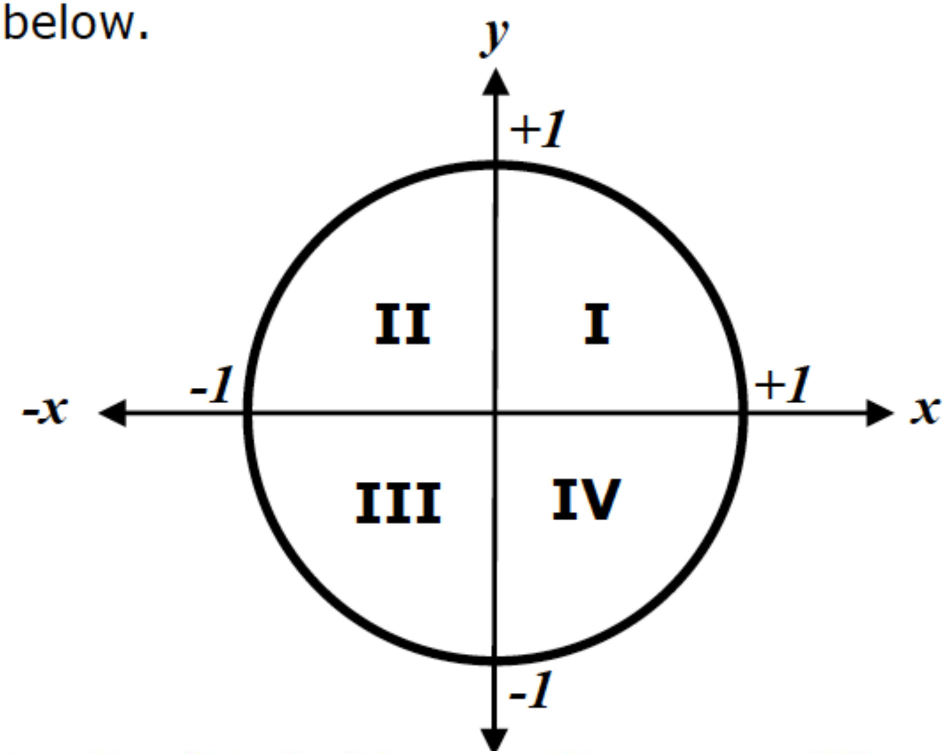
Unite Circle

A unit circle is a circle that has a radius of one (unit radius). It is called a unit circle because it has a unit radius.

In trigonometry, the unit circle has its centre at the origin (0,0).



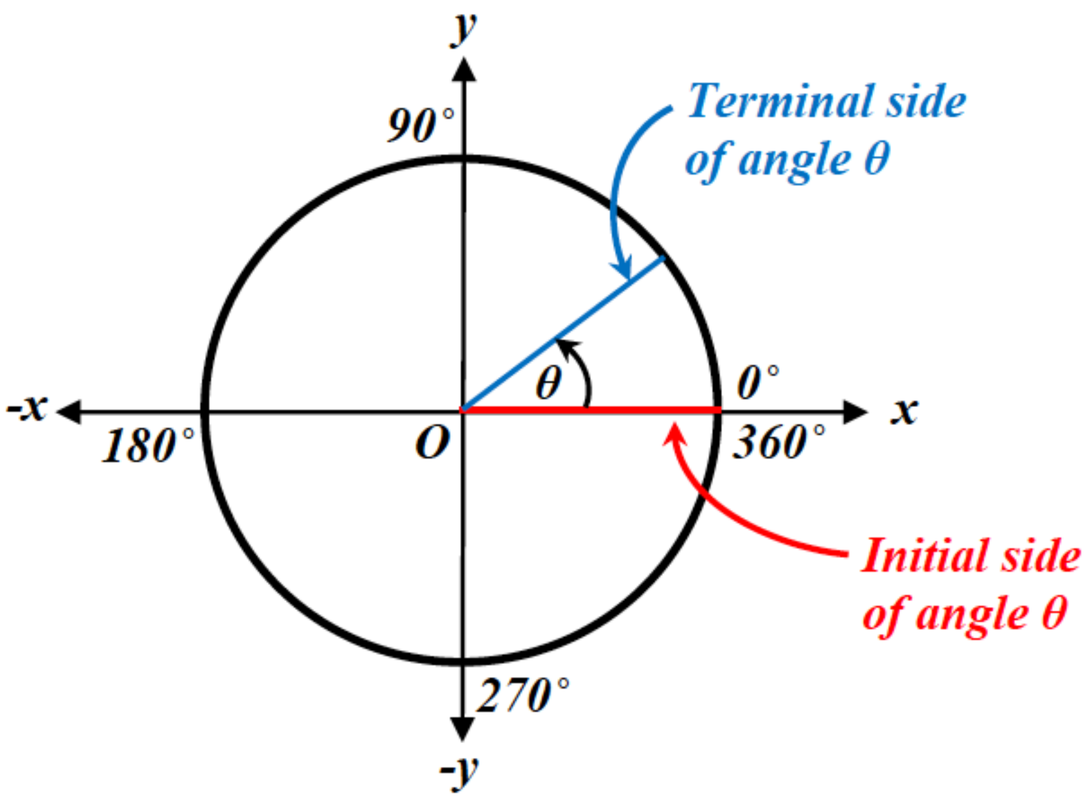
The unit circle is divided into four quadrants as shown below. The quadrants are numbered I, II, III and IV in an anti-clockwise direction, starting in the upper right quadrant as shown below.



Note:

- In Quadrant I, both x and y are positive.
- In Quadrant II, x is negative, but y is positive
- In Quadrant III, both x and y are negative
- In Quadrant IV, x is positive, but y is negative

In the unit circle, angles are made up of an initial side and a terminal side. The initial side is always on the positive side of the x-axis. The terminal side will go anti-clockwise around the circle.

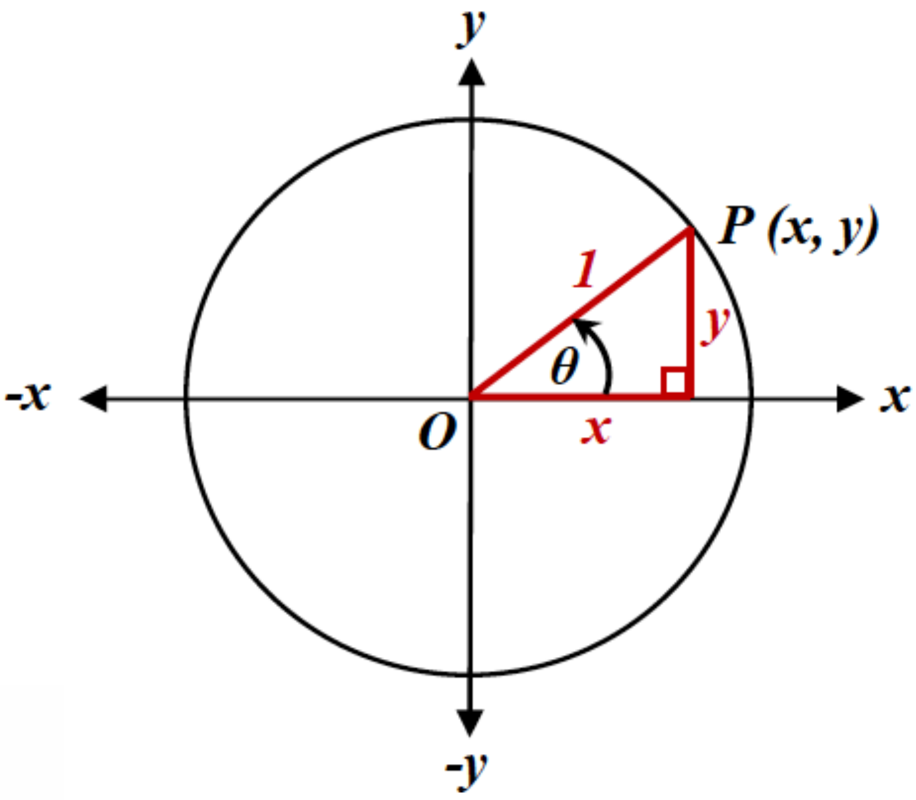


The angle is a measure of the rotation between the initial side and the terminal side. As the terminal side moves anti-clockwise around the circle, the angle increases from 0^{o} to 360^{o}. If the terminal side moves clockwise, the angle decreases.

Angles can be positive or negative, depending on the rotation. If the rotation is anti-clockwise, the angle is positive. If the rotation is clockwise, the angle is negative.

How do you define the basic trigonometric ratios using a unit circle?

Consider a unit circle shown below. Let the terminal side sweep out angle **θ** (“theta”) and let **x** and **y** be the coordinates of the point **P** where the terminal side intersects the unit circle.



We define the trig ratios as follows:

$$\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{y}{1} = y$$

$$\cos\theta = \frac{Adjacent}{Hypotenuse} = \frac{x}{1} = x$$

$$\tan\theta = \frac{Opposite}{Adjacent} = \frac{y}{x}$$

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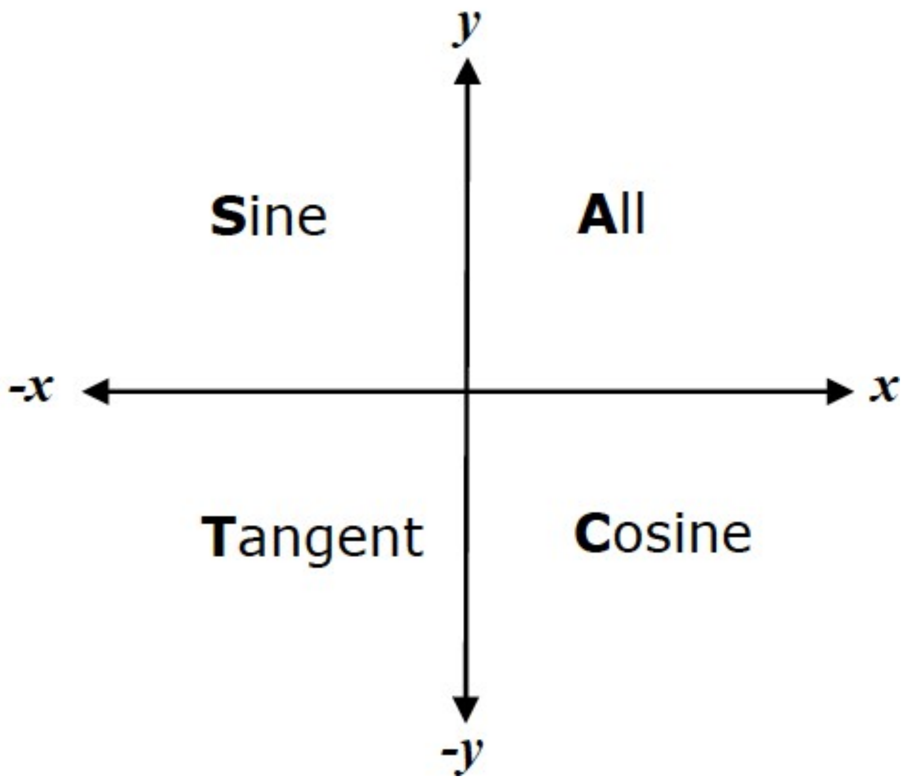
- **A**ll three trig ratios are positive in Quadrant I
- **S**ine only is positive in Quadrant II
- **T**angent only is positive in Quadrant III
- **C**osine only is positive in Quadrant IV.

This can be shown in a diagram as follows:

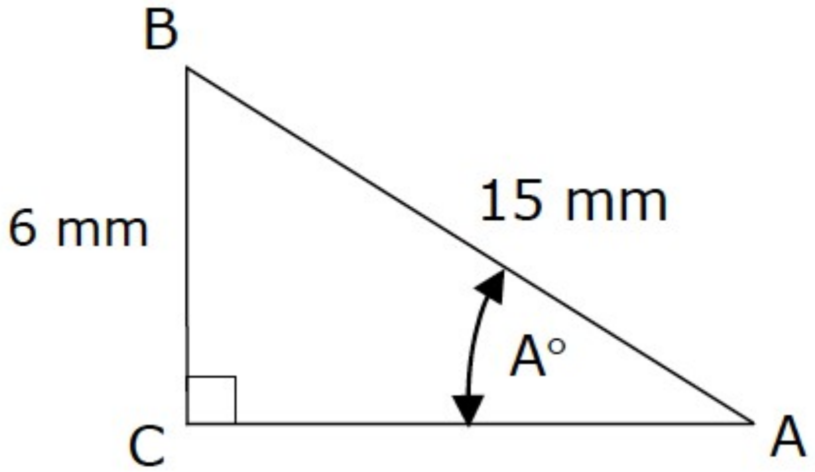


What are the inverse trigonometric ratios for sine, cosine , tangent

The inverse trig ratios allow you to find values of an unknown angle when you are given the value of one of the trig ratios. So, if you know the sine, cosine or tangent of an angle, you can find the size of the angle using the inverse ratios.

The inverse trig ratios for **sine**, **cosine** and **tangent** are **arc sine**, **arc cosine** and **arc tangent**. These are normally written as **arcsin**, **arccos** and **arctan**. However, some people and many calculators use $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ instead of **arcsin**, **arccos** and **arctan**. In this course, we'll use both notations.

Find the size of the angle labelled A in the triangle below giving your answer correct to 3 significant figures.



Solution:

Step 1: Substitute into sine ratio:

$$\sin A = \frac{6}{15} = 0.4$$

Step 2: Use the inverse sine ratio on your calculator to find the angle A:

$$A = \arcsin 0.4 = 23.58^\circ$$

So, the angle A is 23.6° (to 3 s. f.)

What are the inverse trigonometric ratios for sine, cosine and tangent?

	First Value	*Second Value*
arcsin	θ	180° - θ
arcos	θ	360° - θ
arctan	θ	θ + 180°

Find the values of the angle θ between 0° and 360° for which sin θ = +0.5264 giving your answer correct to 3 significant figures.

Solution:

Step 1: Use the inverse sine ratio on your calculator to find the angle θ:

$$\theta = \arcsin 0.5264 = 31.76^\circ$$

Step 2: Use the table to find the second value of angle θ:

$$\textit{Second value} = 180 - \theta = 180 - 31.76 = 148.24^\circ$$

So, the values of the angle θ for which sin θ = +0.5264 are 31.8° and 148.2° (to 3 s. f.)

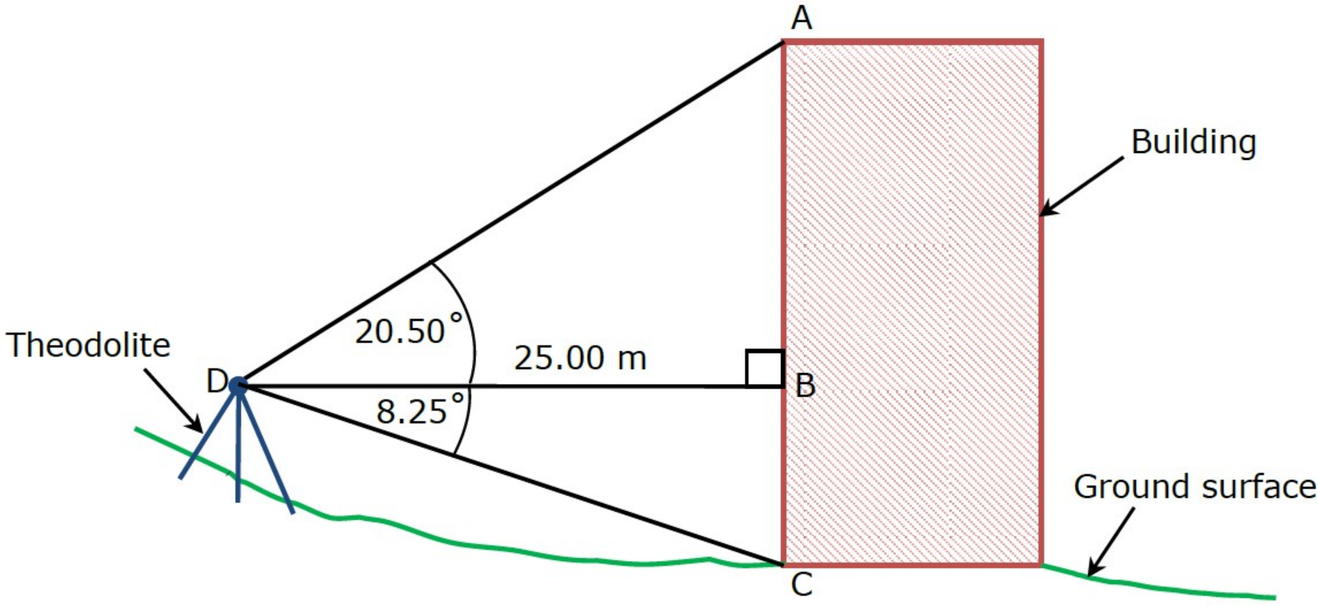
What problems are solved using trigonometric ratios?

Trigonometric ratios are used to solve a wide variety of construction problems, for example, in surveying and setting out.

Here's an example:

To determine the height AC of a building, a surveyor measures the angle of elevation ADB to be 20.50°, the angle of depression BDC to be 8.25° and the distance DB to be 25.00 m as shown below.

Calculate the height of the building AC.



Step 1: Calculate the distance AB using the tangent ratio:

$$\tan 20.50^\circ = \frac{AB}{DB}$$

$$AB = DB \times \tan 20.50^\circ$$

$$AB = 25.0 \times \tan 20.50^\circ$$

$$AB = 9.347\ m$$

Step 2: Calculate the distance BC using the tangent ratio:

$$tan\,8.25^{\circ} = \frac{BC}{DB}$$

$$BC = DB \times tan\ 8.25^{\circ}$$

$$BC = 25.0 \times tan\ 8.25^{\circ}$$

$$BC = 3.625\ m$$

Step 3: Calculate the height AC:

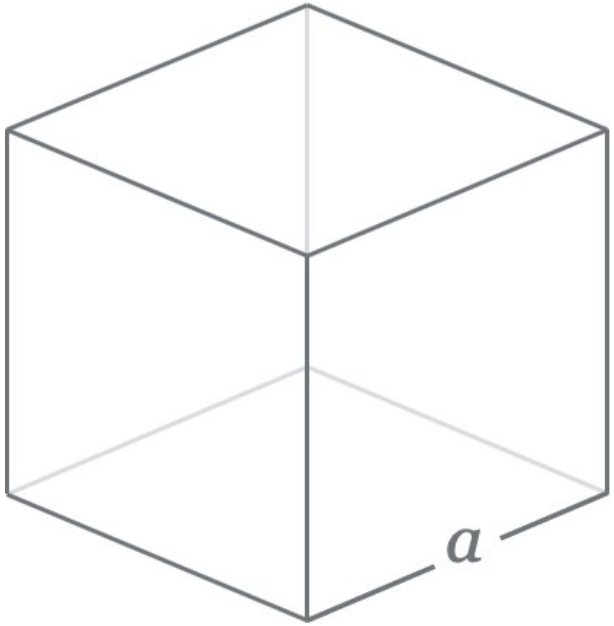
$$AC = AB + BC$$

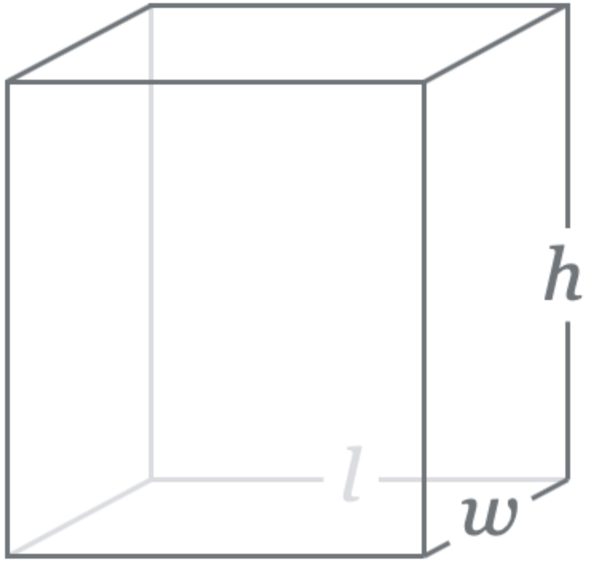
$$AC = 9.347 + 3.625$$

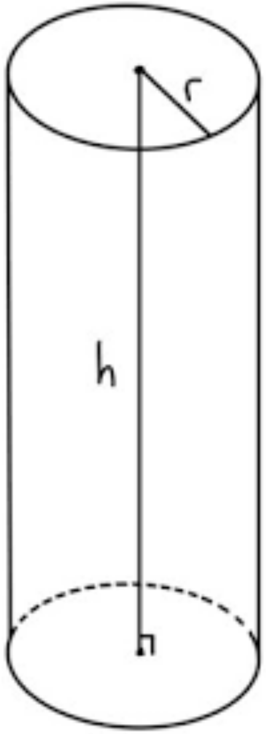
$$AC = 12.972\ m$$

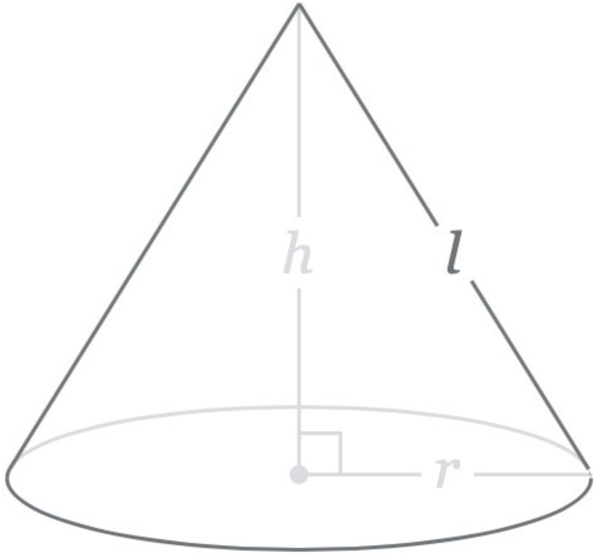
So, the height of the building = 12.972 metres.

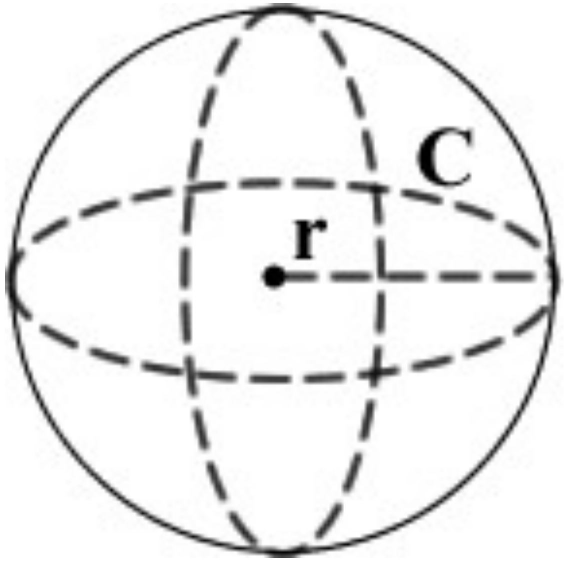
areas and volumes of regular solids











The surface area of various solid shapes are given below:

Cuboid: $2(lb+bh+hl)$, where l, b, and h are the length, breadth and height of a cuboid.

Cube: $6a^2$, a is the side of the cube.

Cylinder: $2\pi r\,(r+h)$, r is the radius of circular base and h is the height of the cylinder.

Cone: $\pi r(l+r)$, r is the radius of the circular base, l is the slant height of the cone.

Sphere: $4\pi r^2$, r is the radius of the sphere.

Volume is the capacity of any solid shape. The formulae for volumes of various shapes are:

Cuboid: l*b*h, where l, b and h are the length, breadth and height of a cuboid.

Cube: a^3, a is the side of the cube.

Cylinder: $\pi r^2 h$, r is the radius of circular base and h is the height of the cylinder.

Cone: $1/3\ \pi r^2 h$, r is the radius of the circular base, l is the slant height of the cone.

Sphere: $4/3\ \pi r^3$, r is the radius of the sphere.

irregular areas and volumes:

Areas of irregular figures

Areas of irregular plane surfaces may be approximately determined by using (a) a planimeter, (b) the trapezoidal rule, (c) the mid-ordinate rule, and (d) Simpson's rule. Such methods may be used, for example, by engineers estimating areas of indicator diagrams of steam engines, surveyors estimating areas of plots of land or naval architects estimating areas of water planes or transverse sections of ships. (a)

A planimeter is an instrument for directly measuring small areas bounded by an irregular curve.

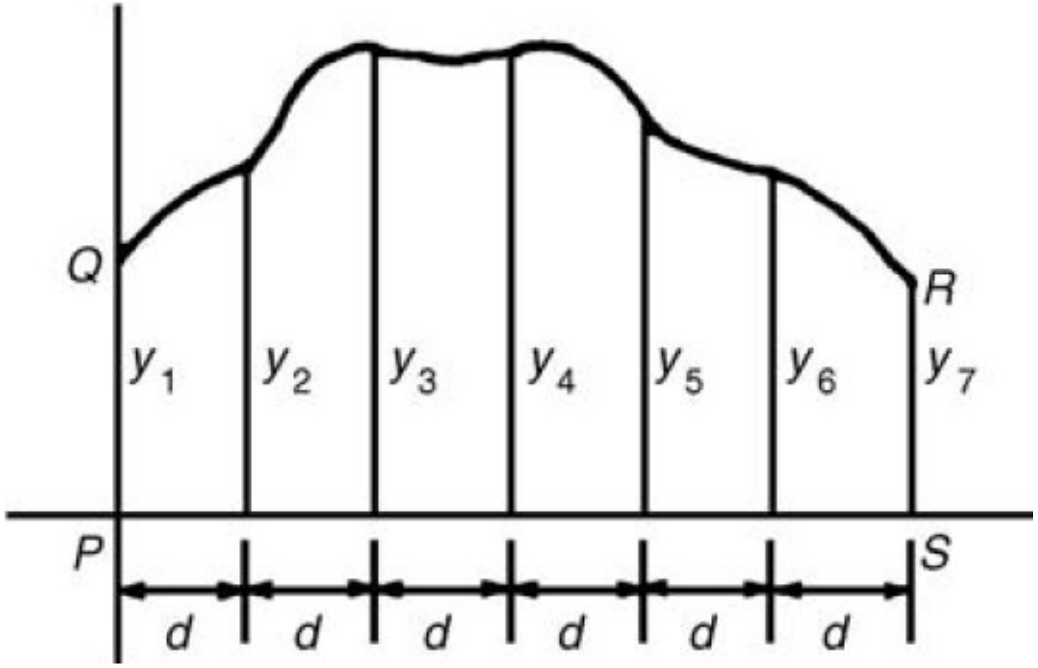
(b) Trapezoidal rule: To determine the areas PQRS in the below Fig.

(i) Divide base PS into any number of equal intervals, each of width d (the greater the number of intervals, the greater the accuracy).

(ii) Accurately measure ordinates y_1, y_2, y_3, etc.

(iii) Areas $PQRS$

$$= d\left[\frac{y_1 + y_7}{2} + y_2 + y_3 + y_4 + y_5 + y_6\right]$$



In general, the trapezoidal rule states:

$$\textbf{Area} = \left(\begin{array}{c}\textbf{width of}\\ \textbf{interval}\end{array}\right)\left[\frac{1}{2}\left(\begin{array}{l}\textbf{first} +\\ \textbf{last}\\ \textbf{ordinate}\end{array}\right) + \begin{array}{l}\textbf{sum of}\\ \textbf{remaining}\\ \textbf{ordinates}\end{array}\right]$$

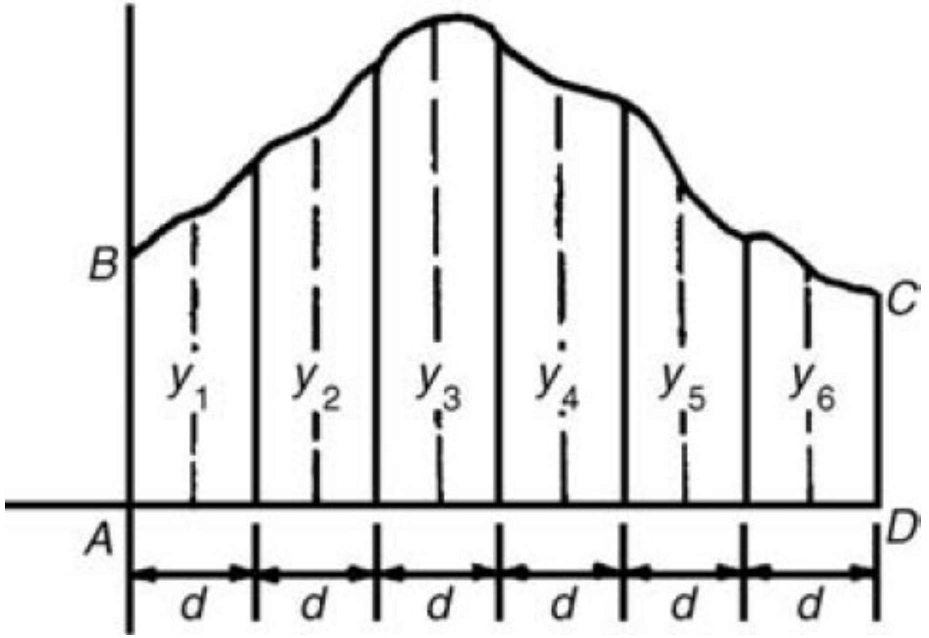
(c) Mid-ordinate rule: To determine the area ABCD of next Fig

(i) Divide base AD into any number of equal intervals, each of width d (the greater the number of intervals, the greater the accuracy).

(ii) Erect ordinates in the middle of each interval (shown by broken lines in Fig. 20.2).

(iii) Accurately measure ordinates y_1, y_2, y_3, etc.

(iv) Area $ABCD = d(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)$.



In general, the mid-ordinate rule states:

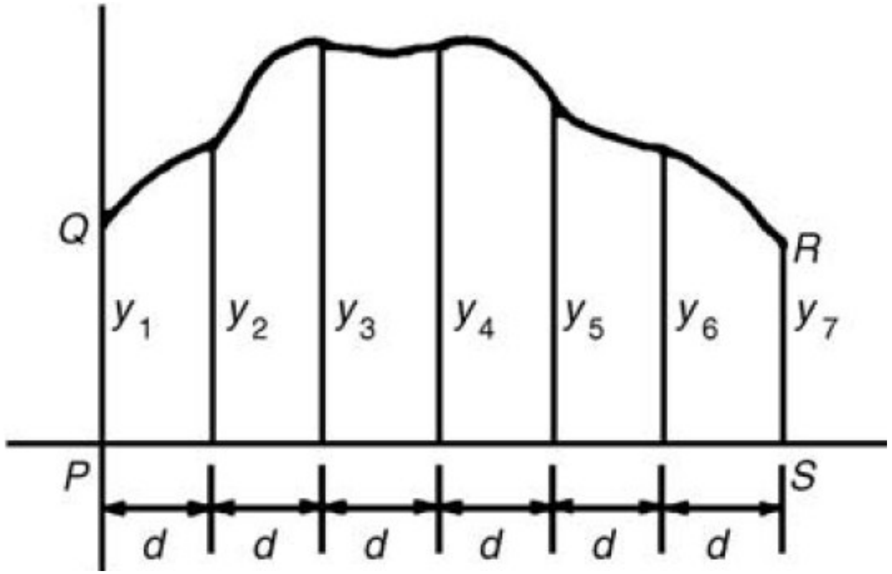
$$\textbf{Area} = \begin{pmatrix}\textbf{width of}\\ \textbf{interval}\end{pmatrix}\begin{pmatrix}\textbf{sum of}\\ \textbf{mid-ordinates}\end{pmatrix}$$

(d) Simpson's rule: To determine the area PQRS.

(i) Divide base PS into an **even** number of intervals, each of width d (the greater the number of intervals, the greater the accuracy).

(ii) Accurately measure ordinates y_1, y_2, y_3, etc.

(iii) Area $PQRS = \frac{d}{3}[(y_1 + y_7) + 4(y_2 + y_4 + y_6) + 2(y_3 + y_5)]$



In general, Simpson's rule states:

$$\textbf{Area} = \frac{1}{3}\begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left[\begin{pmatrix}\text{first + last}\\ \text{ordinate}\end{pmatrix} + 4\begin{pmatrix}\text{sum of even}\\ \text{ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{sum of remaining}\\ \text{odd ordinates}\end{pmatrix}\right]$$

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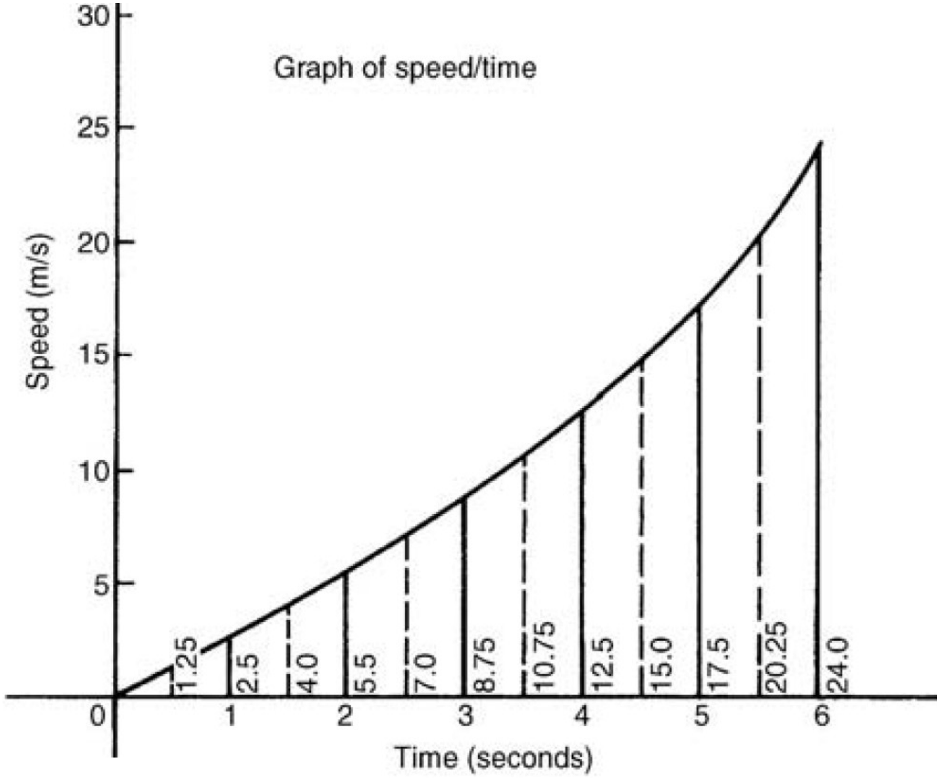
Problem 1. A car starts from rest and its speed is measured every second for 6 s:

Time $t(s)$	0	1	2	3	4	5	6
Speed v (m/s)	0	2.5	5.5	8.75	12.5	17.5	24.0

Determine the distance travelled in 6 seconds (i.e. the area under the v/t graph), by (a) the trapezoidal rule, (b) the mid-ordinate rule, and (c) Simpson's rule.



Solution:

(a) Trapezoidal rule:

The time base is divided into 6 strips each of width 1 s, and the length of the ordinates measured. Thus

$$\textbf{area} = (1)\left[\left(\frac{0+24.0}{2}\right) + 2.5 + 5.5 + 8.75 + 12.5 + 17.5\right]$$

$$= \mathbf{58.75\,m}$$

(b) Mid-ordinate rule

The time base is divided into 6 strips each of width 1 second.

Mid-ordinates are erected as shown in Fig. 20.3 by the broken lines. The length of each mid-ordinate is measured. Thus

$$\textbf{area} = (1)[1.25 + 4.0 + 7.0 + 10.75 + 15.0 + 20.25]$$

$$= \mathbf{58.25\,m}$$

(c) Simpson's rule

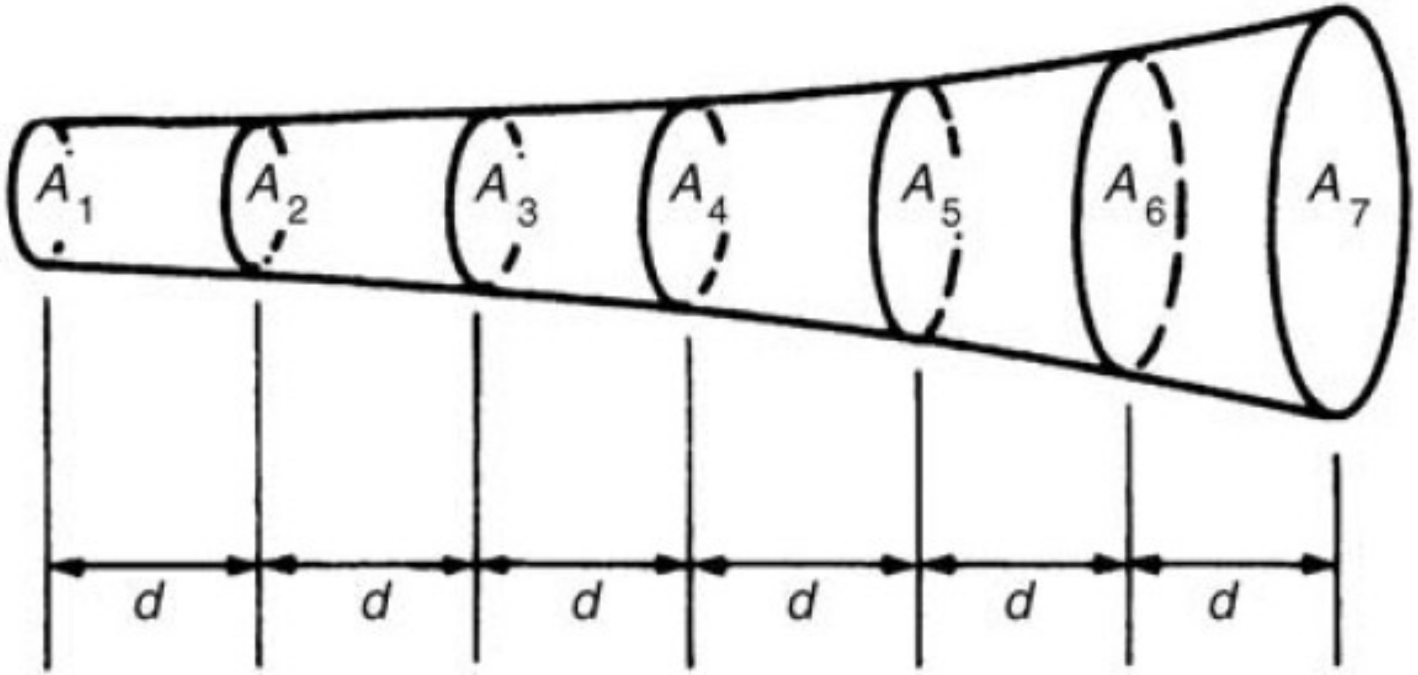
The time base is divided into 6 strips each of width 1 s, and the length of the ordinates measured. Thus

$$\textbf{area} = \tfrac{1}{3}(1)[(0 + 24.0) + 4(2.5 + 8.75 + 17.5) + 2(5.5 + 12.5)]$$

$$= \mathbf{58.33\,m}$$

Volumes of irregular solids: If the cross-sectional areas A1, A2, A3, . . . of an irregular solid bounded by two parallel planes are known at equal intervals of width d (as shown in next Fig), then by Simpson's rule:

$$\text{volume, } V = \frac{d}{3}[(A_1 + A_7) + 4(A_2 + A_4 + A_6) + 2(A_3 + A_5)]$$



Problem. A tree trunk is 12m in length and has a varying cross-section. The cross-sectional areas at intervals of 2m measured from one end are:
0.52, 0.55, 0.59, 0.63, 0.72, 0.84, 0.97 m2

Using Simpson's rule for volumes gives:

$$\text{Volume} = \frac{2}{3}[(0.52 + 0.97) + 4(0.55 + 0.63 + 0.84) + 2(0.59 + 0.72)]$$

$$= \frac{2}{3}[1.49 + 8.08 + 2.62] = \mathbf{8.13\,m^3}$$

EXAM PAPERS PRACTICE