EXAM PAPERS PRACTICE

Helping you Achieve Highest Grades in IB

IB Mathematics (Analysis and Approaches) Higher Level (HL)

Question Paper

Fully in-lined with the First Assessment Examinations in 2021 & Beyond

No Calculators Allowed

Paper: 1 (All Topics)

- **Topic 1 - Number and Algebra**
- **Topic 2 - Functions**
- **Topic 3 - Geometry and Trigonometry**
- **Topic 4 - Statistics and Probability**
- **Topic 5 - Calculus**

Marks: 723

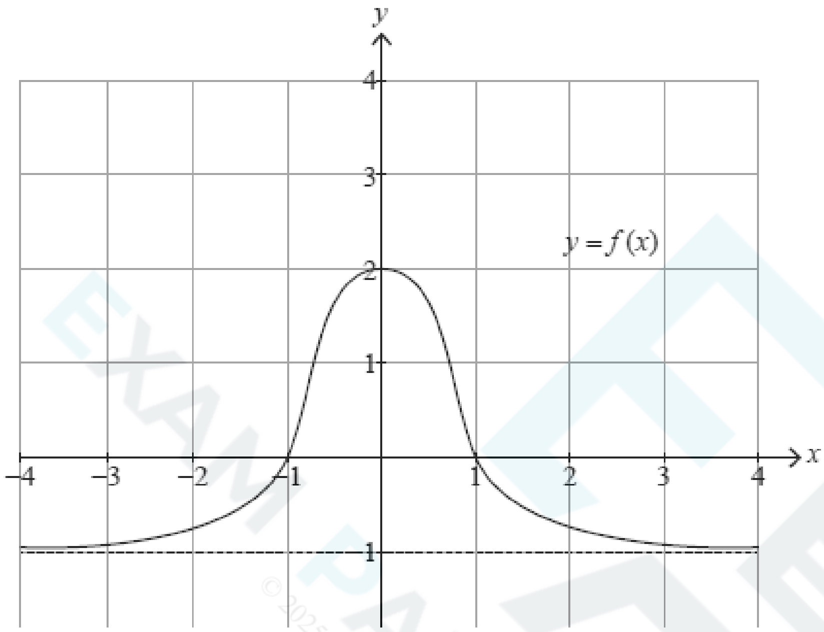
Total Marks: / 723

Suitable for HL Students sitting the 2026 exams and beyond
However, SL students may also find these resources useful

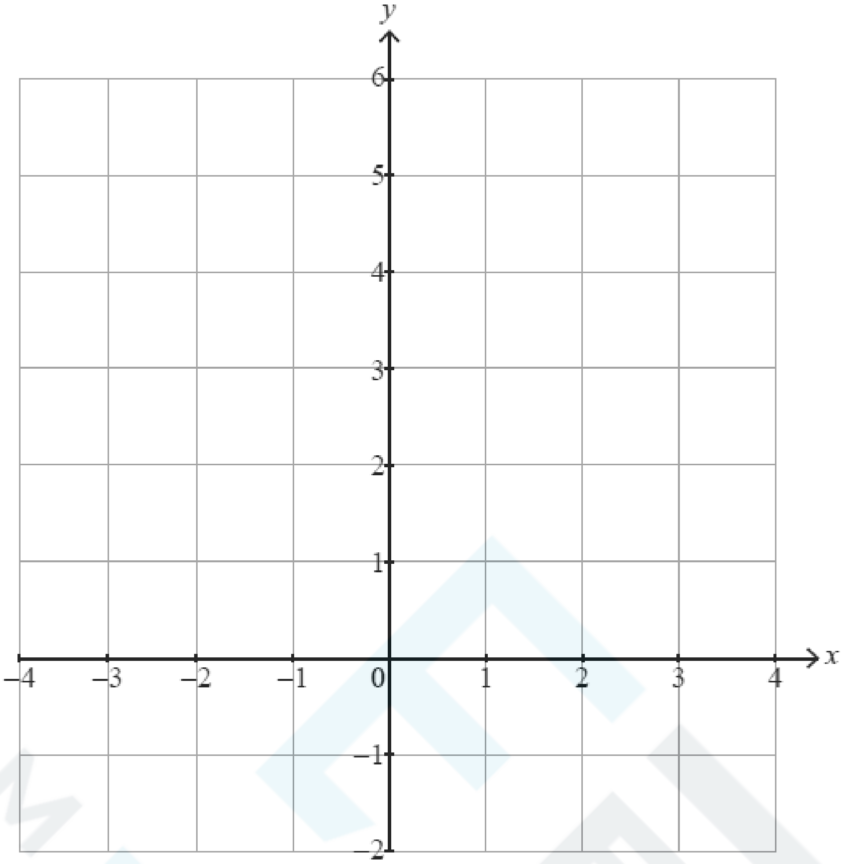
Questions

SPM.1.AHL.TZ0.4

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the x-axis at $x = -1$ and $x = 1$, and the y-axis at $y = 2$.



On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



SPM.1.AHL.TZ0.8

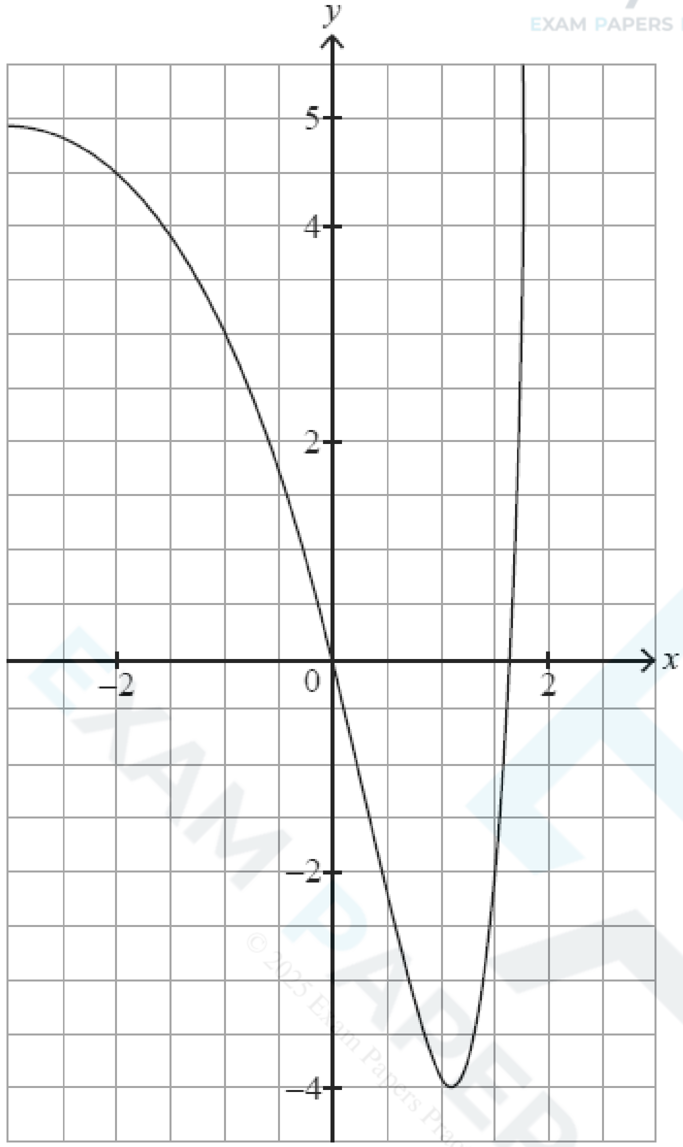
The plane Π has the Cartesian equation $2x + y + 2z = 3$

The line L has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}, \mu, p \in R$. The acute angle between the line L and the plane Π is 30°.

Find the possible values of p.

SPM.1.AHL.TZ0.9

The function f is defined by $f(x) = e^{2x} - 6e^{x} + 5, x \in R, x \leqslant a$. The graph of $y = f(x)$ is shown in the following diagram.



a. Find the largest value of a such that f has an inverse function. [3]

b. For this value of a, find an expression for $f^{-1}(x)$, stating its domain. [5]

SPM.1.AHL.TZ0.11

Let the roots of the equation $z^3 = -3 + \sqrt{3}\mathrm{i}$ be u, v and w.

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

a. Express $-3 + \sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [5]

b.

Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

[5]

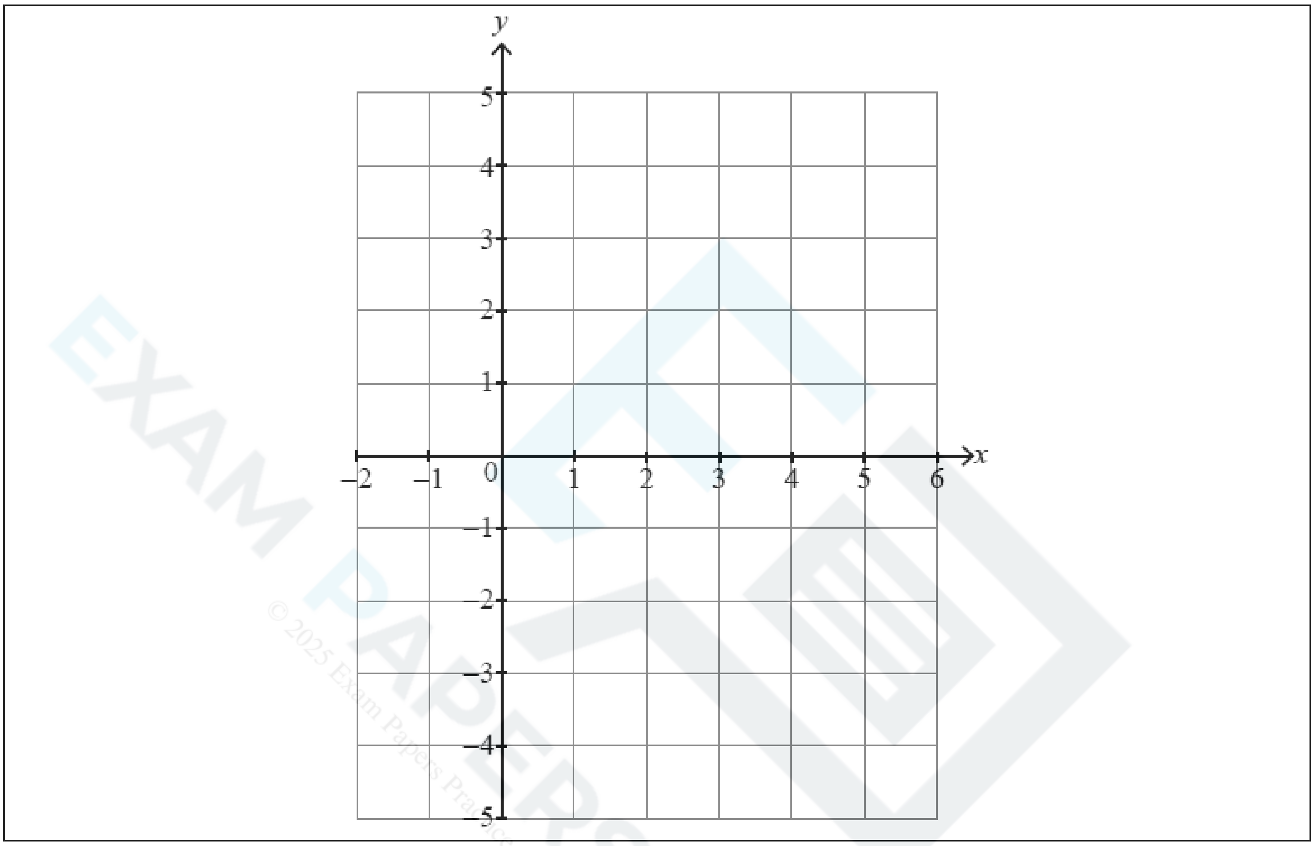
c. Find the area of triangle UVW. [4]

d. By considering the sum of the roots u, v and w, show that

$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0.$ [4]

19M.1.AHL.TZ2.H_5

Sketch the graph of $y = \frac{x-4}{2x-5}$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes.



17M.1.AHL.TZ2.H_2

The function f is defined by $f(x) = 2x^3 + 5,\ -2 \leqslant x \leqslant 2$.

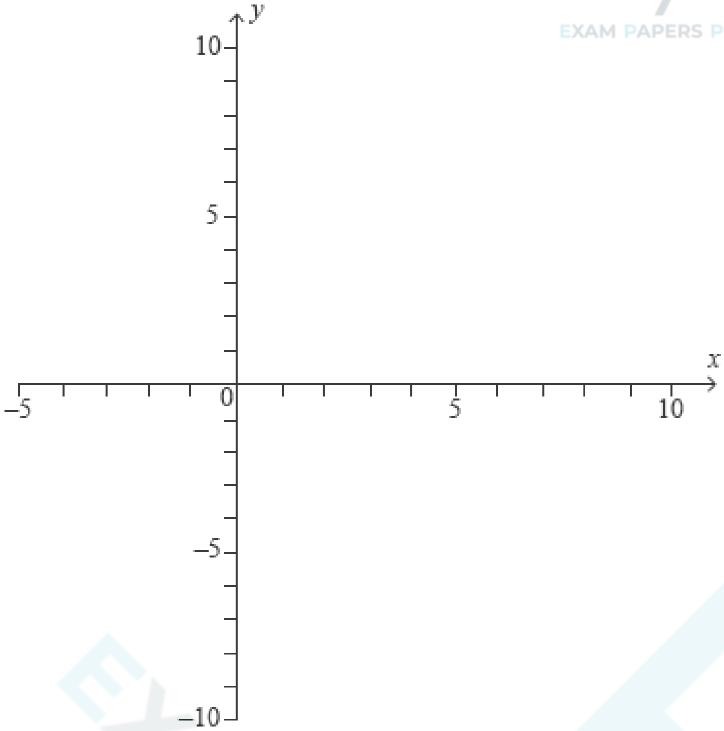
a. Write down the range of f. [2]

b. Find an expression for $f^{-1}(x)$. [2]

c. Write down the domain and range of f^{-1}. [2]

17N.1.AHL.TZ0.H_6

Sketch the graph of $y = \frac{1-3x}{x-2}$, showing clearly any asymptotes and stating the coordinates of any points of intersection with the axes.



17M.1.AHL.TZ1.H_11

Consider the function $f(x) = \frac{1}{x^2 + 3x + 2}$, $x \in R$, $x \neq -2$, $x \neq -1$.

a.i. Express $x^2 + 3x + 2$ in the form $(x + h)^2 + k$. [1]

a.ii. Factorize $x^2 + 3x + 2$. [1]

b.

Sketch the graph of $f(x)$, indicating on it the equations of the asymptotes, the coordinates of the y-intercept and the local maximum.

[5]

d. Hence find the value of p if $\int_0^1 f(x)\mathrm{d}x = \ln(p)$. [4]

e. Sketch the graph of $y = f(|x|)$. [2]

f.

Determine the area of the region enclosed between the graph of $y = f(|x|)$, the x-axis and the lines with equations $x = -1$ and $x = 1$.

[3]

17M.1.AHL.TZ2.H_9

Consider the function f defined by $f(x) = x^2 - a^2$, $x \in R$ where a is a positive constant.

The function g is defined by $g(x) = x\sqrt{f(x)}$ for $|x| > a$.

a.i.

Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes.

$y = f(x)$; [2]

a.ii.

Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes.

$y = \frac{1}{f(x)}$; [4]

a.iii.

Showing any x and y intercepts, any maximum or minimum points and any asymptotes, sketch the following curves on separate axes.

$y = \left|\frac{1}{f(x)}\right|$. [2]

b. Find $\int f(x)\cos x\mathrm{d}x$. [5]

c. By finding $g'(x)$ explain why g is an increasing function. [4]

18M.1.AHL.TZ1.H_9

Let $f(x) = \frac{2-3x^5}{2x^3}, x \in R, x \neq 0$.

a. The graph of $y = f(x)$ has a local maximum at A. Find the coordinates of A. [5]

b.i. Show that there is exactly one point of inflexion, B, on the graph of $y = f(x)$.

[5]

b.ii.

The coordinates of B can be expressed in the form B$\left(2^a, b \times 2^{-3a}\right)$ where a, $b \in Q$. Find the value of a and the value of b.

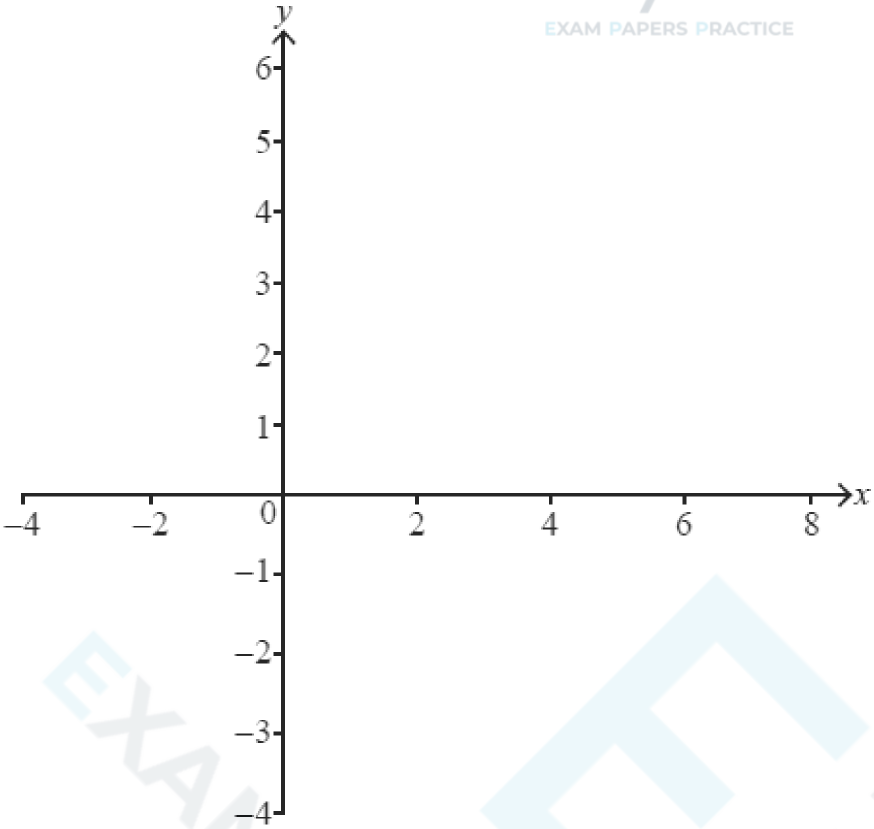
[3]

c. Sketch the graph of $y = f(x)$ showing clearly the position of the points A and B.

[4]

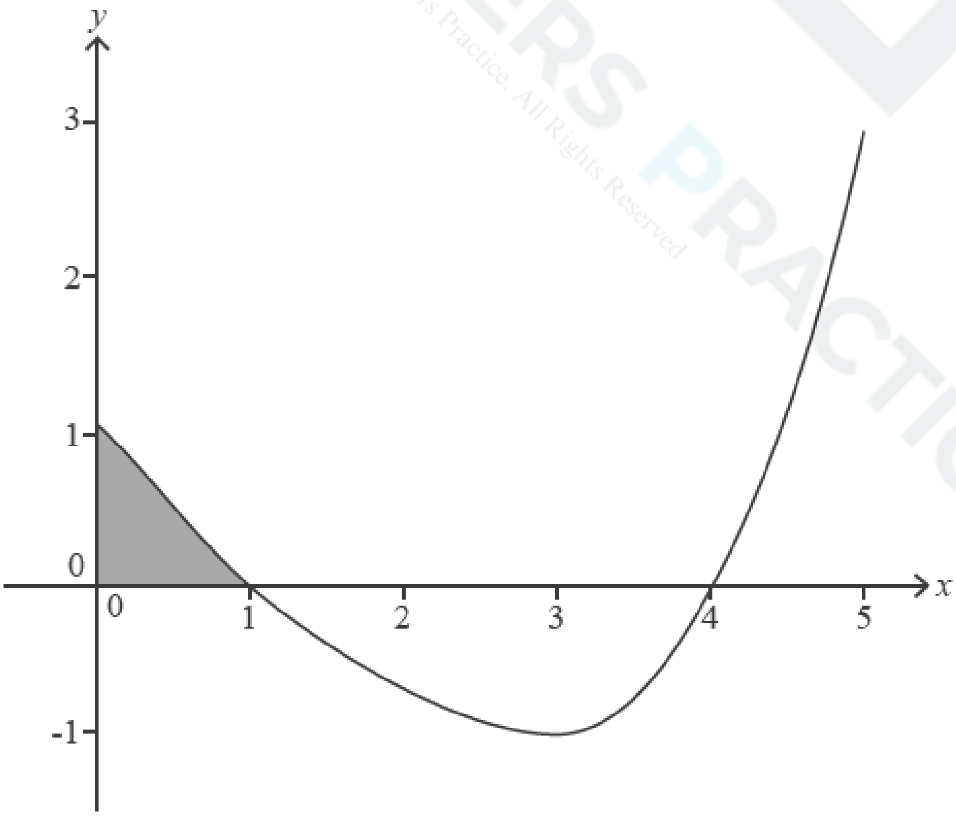
18M.1.AHL.TZ2.H_2

Sketch the graphs of $y = \frac{x}{2} + 1$ and $y = |x-2|$ on the following axes.



19M.1.AHL.TZ1.H_8

The graph of $y = f'(x)$, $0 \le x \le 5$ is shown in the following diagram. The curve intercepts the x-axis at (1, 0) and (4, 0) and has a local minimum at (3, −1).



The shaded area enclosed by the curve $y = f'(x)$, the x-axis and the y-axis is 0.5. Given that $f(0) = 3$, The area enclosed by the curve $y = f'(x)$ and the x-axis between $x = 1$ and $x = 4$ is 2.5 .

a. Write down the x-coordinate of the point of inflexion on the graph of $y = f(x)$.

[1]

b. find the value of $f(1)$. [3]

c. find the value of $f(4)$. [2]

d.

Sketch the curve $y = f(x)$, $0 \leq x \leq 5$ indicating clearly the coordinates of the maximum and minimum points and any intercepts with the coordinate axes.

[3]

19M.1.AHL.TZ2.H_3

Consider the function $f(x) = x^4 - 6x^2 - 2x + 4$, $x \in R$.

The graph of f is translated two units to the left to form the function $g(x)$.

Express $g(x)$ in the form $ax^4 + bx^3 + cx^2 + dx + e$ where $a, b, c, d, e \in Z$.

17N.1.AHL.TZ0.H_3

Consider the polynomial $q(x) = 3x^3 - 11x^2 + kx + 8$.

a. Given that $q(x)$ has a factor $(x - 4)$, find the value of k. [3]

b. Hence or otherwise, factorize $q(x)$ as a product of linear factors. [3]

18M.1.AHL.TZ1.H_1

Let $f(x) = x^4 + px^3 + qx + 5$ where p, q are constants.

The remainder when $f(x)$ is divided by $(x + 1)$ is 7, and the remainder when $f(x)$ is divided by $(x - 2)$ is 1. Find the value of p and the value of q.

17M.1.AHL.TZ1.H_12

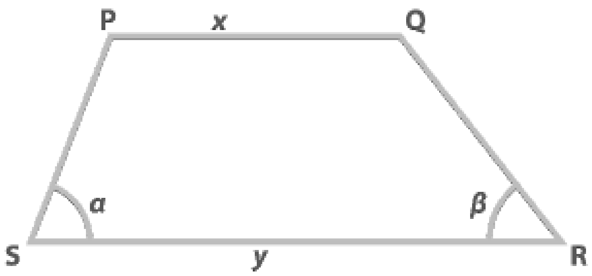
Consider the function $q(x) = x^5 - 10x^2 + 15x - 6$, $x \in R$.

e.i. Show that the graph of $y = q(x)$ is concave up for $x > 1$. [3]

e.ii. Sketch the graph of $y = q(x)$ showing clearly any intercepts with the axes. [3]

EXN.1.AHL.TZ0.7

Consider quadrilateral PQRS where PQ is parallel to SR.



In PQRS, $\text{PQ} = x$, $\text{SR} = y$, $\text{R}\hat{\text{S}}\text{P} = \alpha$ and $\text{Q}\hat{\text{R}}\text{S} = \beta$.

Find an expression for PS in terms of $x,\ \ y,\ \ \sin\ \ \beta$ and $\sin\ \ \alpha + \beta$.

EXN.1.AHL.TZ0.9

It is given that $2\ \ \cos\ \ A\ \ \sin\ \ B \equiv \sin A + B - \sin A - B$. (Do **not** prove this identity.)

Using mathematical induction and the above identity, prove that $\sum_{r=1}^{n} \cos 2r - 1\theta = \frac{\sin\ \ 2n\theta}{2\ \ \sin\ \ \theta}$ for $n \in \mathbb{Z}^{+}$.

EXN.1.AHL.TZ0.11

A function f is defined by $f x = \frac{3}{x^2 + 2},\ \ x \in \mathbb{R}$.

The region R is bounded by the curve $y = f x$, the x-axis and the lines $x = 0$ and $x = \sqrt{6}$. Let A be the area of R.

The line $x = k$ divides R into two regions of equal area.

Let m be the gradient of a tangent to the curve $y = f x$.

a.

Sketch the curve $y = f x$, clearly indicating any asymptotes with their equations and stating the coordinates of any points of intersection with the axes.

[4]

b. Show that $A = \frac{\sqrt{2}\pi}{2}$. [4]

c. Find the value of k. [4]

d. Show that $m = -\frac{6x}{x^2 + 2^2}$. [2]

e. Show that the maximum value of m is $\frac{27}{32}\sqrt{\frac{2}{3}}$. [7]

18N.1.AHL.TZ0.H_8

Consider the equation $z^4 + az^3 + bz^2 + cz + d = 0$, where $a, b, c, d \in R$ and $z \in C$.

Two of the roots of the equation are $\log_2 6$ and $i\sqrt{3}$ and the sum of all the roots is $3 + \log_2 3$.

Show that $6a + d + 12 = 0$.

16N.1.AHL.TZ0.H_5

The quadratic equation $x^2 - 2kx + (k-1) = 0$ has roots α and β such that $\alpha^2 + \beta^2 = 4$. Without solving the equation, find the possible values of the real number k.

21M.1.AHL.TZ2.2

Solve the equation $2\cos^2 x + 5\sin x = 4$, $0 \le x \le 2\pi$.

21M.1.AHL.TZ2.5

Given any two non-zero vectors, a and b, show that $|a \times b|^2 = |a|^2|b|^2 - (a \cdot b)^2$.

21M.1.AHL.TZ2.7

The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots α, β and $\alpha + \beta$.

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of k.

21M.1.AHL.TZ2.11

The acceleration, a ms^{-2}, of a particle moving in a horizontal line at time t seconds, $t \ge 0$, is given by $a = -(1 + v)$ where v ms^{-1} is the particle's velocity and $v > -1$.

At $t = 0$, the particle is at a fixed origin O and has initial velocity v_0 ms^{-1}.

Initially at O, the particle moves in the positive direction until it reaches its maximum displacement from O. The particle then returns to O.

Let s metres represent the particle's displacement from O and s_{max} its maximum displacement from O.

Let $v(T - k)$ represent the particle's velocity k seconds before it reaches s_{max}, where

$$v\left(T - k\right) = 1 + v_0 e^{-(T-k)} - 1.$$

Similarly, let $v(T + k)$ represent the particle's velocity k seconds after it reaches s_{max}.

a.

By solving an appropriate differential equation, show that the particle's velocity at time t is given by $v(t) = (1 + v_0)e^{-t} - 1$.

[6]

b.i.

Show that the time T taken for the particle to reach s_{max} satisfies the equation $e^T = 1 + v_0$.

[2]

b.ii.

By solving an appropriate differential equation and using the result from part (b) (i), find an expression for s_{max} in terms of v_0.

[5]

c. By using the result to part (b) (i), show that $vT - k = e^k - 1$. [2]

d. Deduce a similar expression for $v(T + k)$ in terms of k. [2]

e. Hence, show that $vT - k + vT + k \geq 0$. [3]

18N.1.AHL.TZ0.H_3

Consider the function $g(x) = 4\cos x + 1,\ a \leqslant x \leqslant \frac{\pi}{2}$ where $a < \frac{\pi}{2}$.

a.

For $a = -\frac{\pi}{2}$, sketch the graph of $y = g(x)$. Indicate clearly the maximum and minimum values of the function.

[3]

b. Write down the least value of a such that g has an inverse. [1]

c.i. For the value of a found in part (b), write down the domain of g^{-1}. [1]

c.ii. For the value of a found in part (b), find an expression for $g^{-1}(x)$. [2]

EXM.1.AHL.TZ0.3

Let $f(x) = \frac{4x - 5}{x^2 - 3x + 2} \quad x \neq 1, x \neq 2$.

a. Express $f(x)$ in partial fractions. [6]

b. Use part (a) to show that $f(x)$ is always decreasing. [3]

c.

Use part (a) to find the exact value of $\int_{-1}^{0} f(x)dx$, giving the answer in the form $\ln q$, $q \in Q$.

[4]

16N.1.AHL.TZ0.H_13

a.

Find the value of $\sin\frac{\pi}{4} + \sin\frac{3\pi}{4} + \sin\frac{5\pi}{4} + \sin\frac{7\pi}{4} + \sin\frac{9\pi}{4}$.

[2]

b. Show that $\frac{1-\cos 2x}{2\sin x} \equiv \sin x,\ x \neq k\pi$ where $k \in Z$. [2]

c. Use the principle of mathematical induction to prove that

$\sin x + \sin 3x + \ \ldots\ + \sin(2n-1)x = \frac{1-\cos 2nx}{2\sin x},\ n \in Z^+,\ x \neq k\pi$ where $k \in Z$. [9]

d.

Hence or otherwise solve the equation $\sin x + \sin 3x = \cos x$ in the interval $0 < x < \pi$.

[6]

18M.1.AHL.TZ1.H_3

Two unbiased tetrahedral (four-sided) dice with faces labelled 1, 2, 3, 4 are thrown and the scores recorded. Let the random variable *T* be the maximum of these two scores.

The probability distribution of *T* is given in the following table.

t	1	2	3	4
$P(T=t)$	$\frac{1}{16}$	a	b	$\frac{7}{16}$

a. Find the value of *a* and the value of *b*. [3]

b. Find the expected value of *T*. [2]

EXM.1.AHL.TZ0.4

Let $f(x) = \frac{2x+6}{x^2+6x+10}, x \in R$.

a. Show that $f(x)$ has no vertical asymptotes. [3]

b. Find the equation of the horizontal asymptote. [2]

c. Find the exact value of $\int_0^1 f(x)\,dx$, giving the answer in the form $\ln q, q \in Q$. [3]

16N.1.AHL.TZ0.H_2

The faces of a fair six-sided die are numbered 1, 2, 2, 4, 4, 6. Let X be the discrete random variable that models the score obtained when this die is rolled.

a. Complete the probability distribution table for X. [2]

x				
$P(X=x)$				

b. Find the expected value of X. [2]

EXM.1.AHL.TZ0.5

Let $f(x) = \frac{2x^2 - 5x - 12}{x+2}, x \in R, x \neq -2$.

a. Find all the intercepts of the graph of $f(x)$ with both the x and y axes. [4]

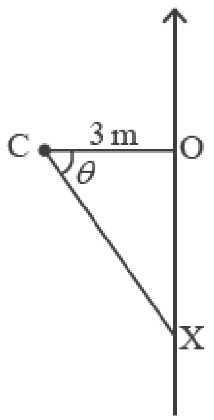
b. Write down the equation of the vertical asymptote. [1]

c. As $x \to \pm\infty$ the graph of $f(x)$ approaches an oblique straight line asymptote.

Divide $2x^2 - 5x - 12$ by $x + 2$ to find the equation of this asymptote. [4]

19M.1.AHL.TZ1.H_5

A camera at point C is 3 m from the edge of a straight section of road as shown in the following diagram. The camera detects a car travelling along the road at $t = 0$. It then rotates, always pointing at the car, until the car passes O, the point on the edge of the road closest to the camera.



A car travels along the road at a speed of 24 ms^{-1}. Let the position of the car be X and let $O\hat{C}X = \theta$.

Find $\frac{d\theta}{dt}$, the rate of rotation of the camera, in radians per second, at the instant the car passes the point O .

EXM.1.AHL.TZ0.6

Let $f(x) = \frac{x^2 - 10x + 5}{x + 1}, x \in R, x \neq -1$.

a. Find the co-ordinates of all stationary points. [4]

b. Write down the equation of the vertical asymptote. [1]

c.

With justification, state if each stationary point is a minimum, maximum or horizontal point of inflection.

[4]

19M.1.AHL.TZ1.H_6

Let X be a random variable which follows a normal distribution with mean μ. Given that $P(X < \mu - 5) = 0.2$, find

a. $P(X > \mu + 5)$. [2]

b. $P(X < \mu + 5 | X > \mu - 5)$. [5]

17M.1.AHL.TZ1.H_7

An arithmetic sequence u_1, u_2, u_3 ... has $u_1 = 1$ and common difference $d \neq 0$. Given that u_2, u_3 and u_6 are the first three terms of a geometric sequence

Given that $u_N = -15$

a. find the value of d. [4]

b. determine the value of $\sum_{r=1}^{N} u_r$. [3]

17M.1.AHL.TZ2.H_3

The 1st, 4th and 8th terms of an arithmetic sequence, with common difference d, $d \neq 0$, are the first three terms of a geometric sequence, with common ratio r. Given that the 1st term of both sequences is 9 find

a. the value of d; [4]

b. the value of r; [1]

19M.1.AHL.TZ2.H_7

Solve the simultaneous equations

$$\log_2 6x = 1 + 2\log_2 y$$

$$1 + \log_6 x = \log_6 (15y - 25).$$

18M.1.AHL.TZ1.H_5

Solve $(\ln x)^2 - (\ln 2)(\ln x) < 2(\ln 2)^2$.

16N.1.AHL.TZ0.H_7

Solve the equation $4^x + 2^{x+2} = 3$.

17N.1.AHL.TZ0.H_1

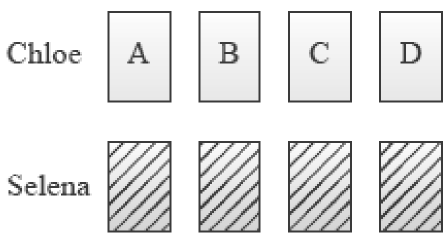
Solve the equation $\log_2(x+3) + \log_2(x-3) = 4$.

17M.1.AHL.TZ1.H_1

Find the solution of $\log_2 x - \log_2 5 = 2 + \log_2 3$.

17N.1.AHL.TZ0.H_10

Chloe and Selena play a game where each have four cards showing capital letters A, B, C and D.
Chloe lays her cards face up on the table in order A, B, C, D as shown in the following diagram.



Selena shuffles her cards and lays them face down on the table. She then turns them over one by one to see if her card matches with Chloe's card directly above.
Chloe wins if **no** matches occur; otherwise Selena wins.

Chloe and Selena repeat their game so that they play a total of 50 times.
Suppose the discrete random variable X represents the number of times Chloe wins.

a. Show that the probability that Chloe wins the game is $\frac{3}{8}$. [6]

b.i. Determine the mean of X. [3]

b.ii. Determine the variance of X. [2]

17M.1.AHL.TZ1.H_4

Three girls and four boys are seated randomly on a straight bench. Find the probability that the girls sit together and the boys sit together.

18N.1.AHL.TZ0.H_2

A team of four is to be chosen from a group of four boys and four girls.

a. Find the number of different possible teams that could be chosen. [3]

b.

Find the number of different possible teams that could be chosen, given that the team must include at least one girl and at least one boy.

[2]

18N.1.AHL.TZ0.H_7

Consider the curves C_1 and C_2 defined as follows

$$C_1 : xy = 4,\ \ x > 0$$

$$C_2 : y^2 - x^2 = 2,\ \ x > 0$$

a.

Using implicit differentiation, or otherwise, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each curve in terms of x and y.

[4]

b. Let P(a, b) be the unique point where the curves C_1 and C_2 intersect.

Show that the tangent to C_1 at P is perpendicular to the tangent to C_2 at P. [2]

16N.1.AHL.TZ0.H_9

A curve has equation $3x - 2y^2 e^{x-1} = 2$.

a. Find an expression for $\frac{dy}{dx}$ in terms of x and y. [5]

b.

Find the equations of the tangents to this curve at the points where the curve intersects the line $x = 1$.

[4]

19M.1.AHL.TZ1.H_11

Two distinct lines, l_1 and l_2, intersect at a point P. In addition to P, four distinct points are marked out on l_1 and three distinct points on l_2. A mathematician decides to join some of these eight points to form polygons.

The line l_1 has vector equation $\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\lambda \in R$ and the line l_2 has vector equation $\boldsymbol{r}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix}$, $\mu \in R$.

The point P has coordinates (4, 6, 4).

The point A has coordinates (3, 4, 3) and lies on l_1.

The point B has coordinates (−1, 0, 2) and lies on l_2.

a.i.

Find how many sets of four points can be selected which can form the vertices of a quadrilateral.

[2]

a.ii.

Find how many sets of three points can be selected which can form the vertices of a triangle.

[4]

b. Verify that P is the point of intersection of the two lines. [3]

c. Write down the value of λ corresponding to the point A. [1]

d. Write down $\overrightarrow{\mathrm{PA}}$ and $\overrightarrow{\mathrm{PB}}$. [2]

e.

Let C be the point on l_1 with coordinates (1, 0, 1) and D be the point on l_2 with parameter $\mu = -2$.

Find the area of the quadrilateral CDBA. [8]

19M.1.AHL.TZ2.H_6

The curve C is given by the equation $y = x\tan\left(\frac{\pi xy}{4}\right)$.

a. At the point (1, 1) , show that $\frac{dy}{dx} = \frac{2+\pi}{2-\pi}$. [5]

b. Hence find the equation of the normal to C at the point (1, 1). [2]

17M.1.AHL.TZ1.H_8

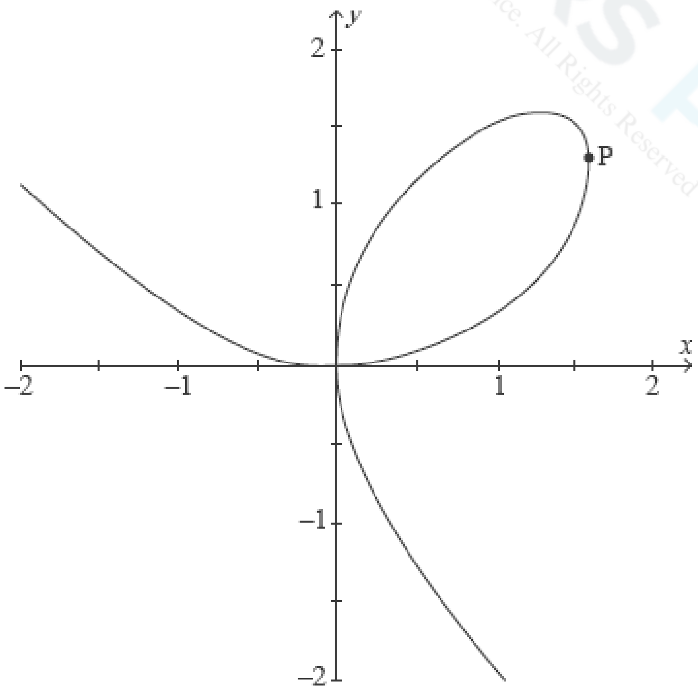
Use the method of mathematical induction to prove that $4^n + 15n - 1$ is divisible by 9 for $n \in Z^+$.

19M.1.AHL.TZ1.H_7

Find the coordinates of the points on the curve $y^3 + 3xy^2 - x^3 = 27$ at which $\frac{dy}{dx} = 0$.

17N.1.AHL.TZ0.H_7

The folium of Descartes is a curve defined by the equation $x^3 + y^3 - 3xy = 0$, shown in the following diagram.



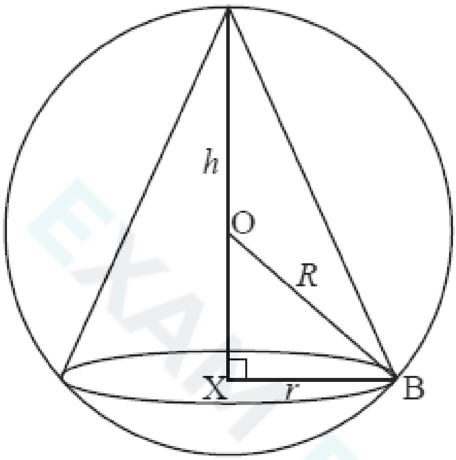
Determine the exact coordinates of the point P on the curve where the tangent line is parallel to the y-axis.

17M.1.AHL.TZ2.H_8

Prove by mathematical induction that $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\ldots+\binom{n-1}{2}=\binom{n}{3}$, where $n\in\mathbb{Z}, n\geqslant 3$.

19M.1.AHL.TZ2.H_8

A right circular cone of radius r is inscribed in a sphere with centre O and radius R as shown in the following diagram. The perpendicular height of the cone is h, X denotes the centre of its base and B a point where the cone touches the sphere.



a. Show that the volume of the cone may be expressed by $V=\frac{\pi}{3}\left(2Rh^2-h^3\right)$. [4]

b.

Given that there is one inscribed cone having a maximum volume, show that the volume of this cone is $\frac{32\pi R^3}{81}$.

[4]

18M.1.AHL.TZ1.H_6

Use the principle of mathematical induction to prove that

$1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+\ldots+n\left(\frac{1}{2}\right)^{n-1}=4-\frac{n+2}{2^{n-1}}$, where $n\in\mathbb{Z}^+$.

18N.1.AHL.TZ0.H_11

Let S be the sum of the roots found in part (a).

a.

Find the roots of $z^{24}=1$ which satisfy the condition $0<\arg(z)<\frac{\pi}{2}$, expressing your answers in the form $re^{i\theta}$, where $r, \theta\in\mathbb{R}^+$.

[5]

b.i. Show that Re S = Im S. [4]

b.ii.

By writing $\frac{\pi}{12}$ as $\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$, find the value of cos $\frac{\pi}{12}$ in the form $\frac{\sqrt{a} + \sqrt{b}}{c}$, where a, b and c are integers to be determined.

[3]

b.iii. Hence, or otherwise, show that $S = \frac{1}{2}\left(1 + \sqrt{2}\right)\left(1 + \sqrt{3}\right)\left(1 + \mathrm{i}\right)$. [4]

17M.1.AHL.TZ2.H_4

A particle moves along a straight line. Its displacement, s metres, at time t seconds is given by $s = t + \cos 2t,\ t \geqslant 0$. The first two times when the particle is at rest are denoted by t_1 and t_2, where $t_1 < t_2$.

a. Find t_1 and t_2. [5]

b. Find the displacement of the particle when $t = t_1$ [2]

16N.1.AHL.TZ0.H_12

Let ω be one of the non-real solutions of the equation $z^3 = 1$.

Consider the complex numbers $p = 1 - 3\mathrm{i}$ and $q = x + (2x + 1)\mathrm{i}$, where $x \in R$.

a. Determine the value of (i) $1 + \omega + \omega^2$; (ii) $1 + \omega^* + (\omega^*)^2$. [4]

b. Show that $(\omega - 3\omega^2)(\omega^2 - 3\omega) = 13$. [4]

c. Find the values of x that satisfy the equation $|p| = |q|$. [5]

d. Solve the inequality $\mathrm{Re}(pq) + 8 < \left(\mathrm{Im}(pq)\right)^2$. [6]

17M.1.AHL.TZ1.H_2

Consider the complex numbers $z_1 = 1 + \sqrt{3}\mathrm{i}$, $z_2 = 1 + \mathrm{i}$ and $w = \frac{z_1}{z_2}$.

a.i. By expressing z_1 and z_2 in modulus-argument form write down the modulus of w;

[3]

a.ii.

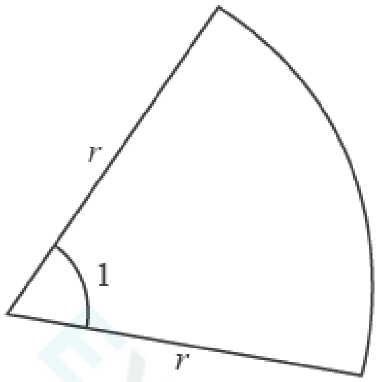
By expressing z_1 and z_2 in modulus-argument form write down the argument of w.

[1]

b. Find the smallest positive integer value of n, such that w^n is a real number. [2]

19M.1.AHL.TZ1.H_3

A sector of a circle with radius r cm , where $r > 0$, is shown on the following diagram. The sector has an angle of 1 radian at the centre.



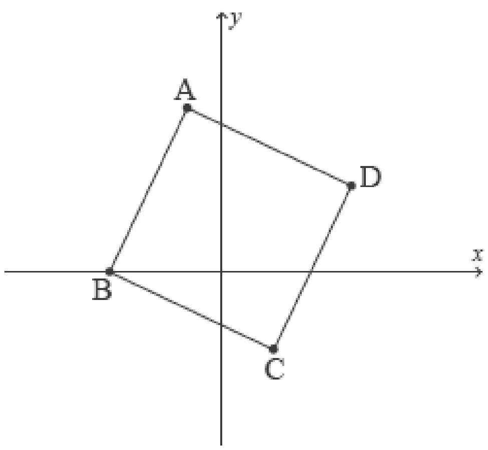
Let the area of the sector be A cm^2 and the perimeter be P cm. Given that $A = P$, find the value of r.

17M.1.AHL.TZ1.H_3

Solve the equation $\sec^2 x + 2\tan x = 0,\ 0 \leqslant x \leqslant 2\pi$.

17M.1.AHL.TZ2.H_5

In the following Argand diagram the point A represents the complex number $-1 + 4\mathrm{i}$ and the point B represents the complex number $-3 + 0\mathrm{i}$. The shape of ABCD is a square. Determine the complex numbers represented by the points C and D.



19M.1.AHL.TZ2.H_10

The random variable X has probability density function f given by

$$f(x) = \begin{cases} k(\pi - \arcsin x) & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}, \text{where } k \text{ is a positive constant.}$$

Given that $y = \left(\frac{x^2}{2}\right)\arcsin x - \left(\frac{1}{4}\right)\arcsin x + \left(\frac{x}{4}\right)\sqrt{1-x^2}$, show that

a. State the mode of X. [1]

b.i. Find $\int \arcsin x dx$. [3]

b.ii. Hence show that $k = \frac{2}{2+\pi}$. [3]

c.i. $\frac{dy}{dx} = x\arcsin x$. [4]

c.ii. $E(X) = \frac{3\pi}{4(\pi+2)}$. [5]

19N.1.AHL.TZ0.H_1

The probability distribution of a discrete random variable, X, is given by the following table, where N and p are constants.

x	1	5	10	N
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	p

a. Find the value of p. [2]

b. Given that $E(X) = 10$, find the value of N. [3]

18N.1.AHL.TZ0.H_4

Consider the following system of equations where $a \in R$.

$$2x + 4y - z = 10$$

$$x + 2y + az = 5$$

$$5x + 12y = 2a.$$

a.

Find the value of a for which the system of equations does not have a unique solution.

[2]

b. Find the solution of the system of equations when $a = 2$. [5]

19N.1.AHL.TZ0.H_10

Consider $f(x) = \frac{2x-4}{x^2-1}, -1 < x < 1.$

For the graph of $y = f(x)$,

a.i. Find $f'(x)$. [2]

a.ii. Show that, if $f'(x) = 0$, then $x = 2 - \sqrt{3}$. [3]

b.i. find the coordinates of the y-intercept. [1]

b.ii. show that there are no x-intercepts. [2]

b.iii. sketch the graph, showing clearly any asymptotic behaviour. [2]

c. Show that $\frac{3}{x+1} - \frac{1}{x-1} = \frac{2x-4}{x^2-1}$. [2]

d.

The area enclosed by the graph of $y = f(x)$ and the line $y = 4$ can be expressed as $\ln v$. Find the value of v.

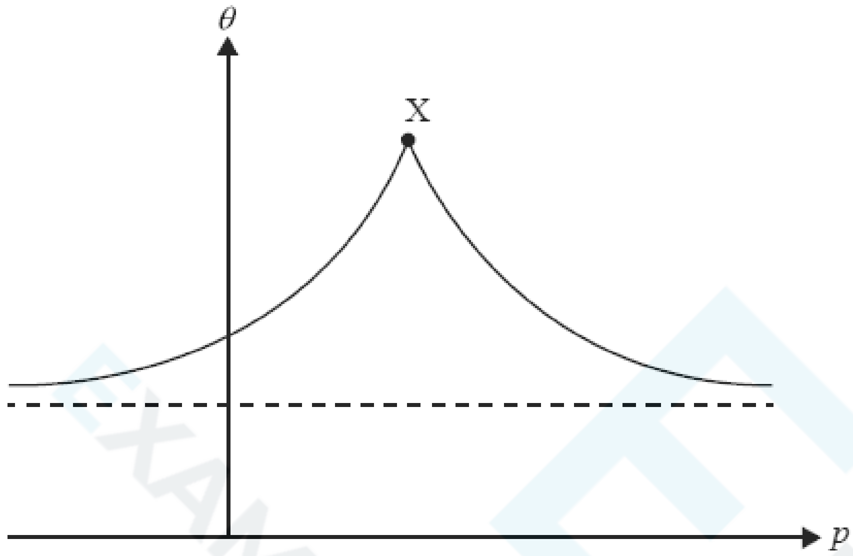
[7]

19N.1.AHL.TZ0.H_11

Points $\mathrm{A}(0, 0, 10)$, $\mathrm{B}(0, 10, 0)$, $\mathrm{C}(10, 0, 0)$, $\mathrm{V}(p, p, p)$ form the vertices of a tetrahedron.

Consider the case where the faces ABV and ACV are perpendicular.

The following diagram shows the graph of θ against p. The maximum point is shown by X.



a.i. Show that $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AV}} = -10\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix}$ and find a similar expression for $\overrightarrow{\mathrm{AC}} \times \overrightarrow{\mathrm{AV}}$. [3]

a.ii. Hence, show that, if the angle between the faces ABV and ACV is θ, then $\cos\theta = \frac{p(3p-20)}{6p^2-40p+100}$. [5]

b.i. Find the two possible coordinates of V. [3]

b.ii. Comment on the positions of V in relation to the plane ABC. [1]

c.i. At X, find the value of p and the value of θ. [3]

c.ii. Find the equation of the horizontal asymptote of the graph. [2]

19N.1.AHL.TZ0.H_5

Consider the equation $z^4 = -4$, where $z \in C$.

a. Solve the equation, giving the solutions in the form $a + \mathrm{i}b$, where $a, b \in R$. [5]

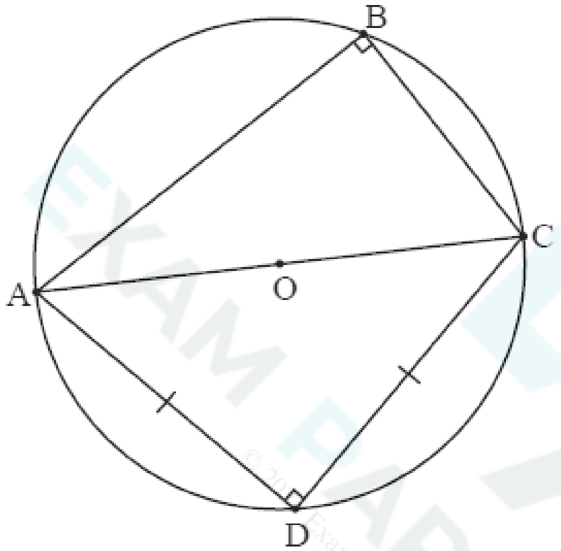
b.

The solutions form the vertices of a polygon in the complex plane. Find the area of the polygon.

[2]

19N.1.AHL.TZ0.H_9

In the following diagram, the points A, B, C and D are on the circumference of a circle with centre O and radius r. [AC] is a diameter of the circle. $BC = r$, $AD = CD$ and $A\hat{B}C = A\hat{D}C = 90^\circ$.



a. Given that $\cos 75^\circ = q$, show that $\cos 105^\circ = -q$. [1]

b. Show that $B\hat{A}D = 75^\circ$. [3]

c.i. By considering triangle ABD, show that $BD^2 = 5r^2 - 2r^2q\sqrt{6}$. [4]

c.ii. By considering triangle CBD, find another expression for BD^2 in terms of r and q.

[3]

d. Use your answers to part (c) to show that $\cos 75^\circ = \frac{1}{\sqrt{6}+\sqrt{2}}$. [3]

19N.1.AHL.TZ0.H_3

Three planes have equations:

$$2x - y + z = 5$$
$$x + 3y - z = 4 \quad \text{, where } a, b \in R.$$
$$3x - 5y + az = b$$

Find the set of values of a and b such that the three planes have no points of intersection.

19N.1.AHL.TZ0.H_4

A and B are acute angles such that $\cos A = \frac{2}{3}$ and $\sin B = \frac{1}{3}$.

Show that $\cos(2A + B) = -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$.

18M.1.AHL.TZ1.H_8

Let $a = \sin b, 0 < b < \frac{\pi}{2}$.

Find, in terms of b, the solutions of $\sin 2x = -a, 0 \leqslant x \leqslant \pi$.

18N.1.AHL.TZ0.H_1

Consider two events, A and B, such that $\mathrm{P}(A) = \mathrm{P}\left(A' \cap B\right) = 0.4$ and $\mathrm{P}(A \cap B) = 0.1$.

a. By drawing a Venn diagram, or otherwise, find $\mathrm{P}(A \cup B)$. [3]

b. Show that the events A and B are not independent. [3]

19M.1.AHL.TZ1.H_4

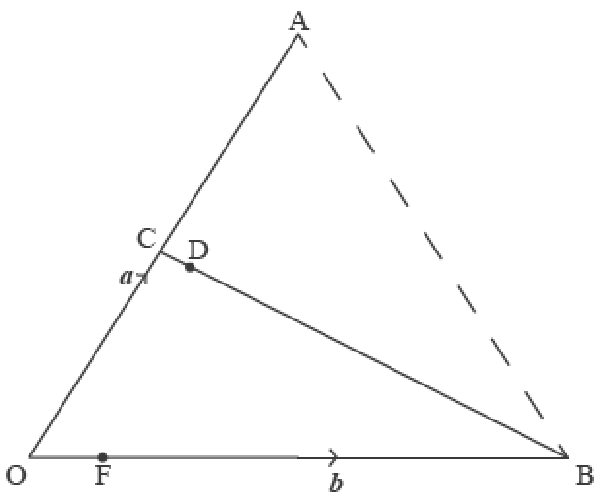
The lengths of two of the sides in a triangle are 4 cm and 5 cm. Let θ be the angle between the two given sides. The triangle has an area of $\frac{5\sqrt{15}}{2}$ cm^2.

a. Show that $\sin\theta = \frac{\sqrt{15}}{4}$. [1]

b. Find the two possible values for the length of the third side. [6]

17N.1.AHL.TZ0.H_9

In the following diagram, $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$. C is the midpoint of [OA] and $\overrightarrow{OF} = \frac{1}{6}\overrightarrow{FB}$.



It is given also that $\overrightarrow{AD} = \lambda\overrightarrow{AF}$ and $\overrightarrow{CD} = \mu\overrightarrow{CB}$, where $\lambda,\ \mu \in R$.

a.i. Find, in terms of ***a*** and ***b*** $\overrightarrow{OF}$. [1]

a.ii. Find, in terms of and $\overrightarrow{AF}$. [2]

b.i. Find an expression for $\overrightarrow{OD}$ in terms of , and λ; [2]

b.ii. Find an expression for $\overrightarrow{OD}$ in terms of , and μ. [2]

c. Show that $\mu = \frac{1}{13}$, and find the value of λ. [4]

d. Deduce an expression for $\overrightarrow{CD}$ in terms of and only. [2]

e. Given that area $\Delta OAB = k(\text{area } \Delta CAD)$, find the value of k. [5]

16N.1.AHL.TZ0.H_10

Consider two events A and A defined in the same sample space.

Given that $P(A \cup B) = \frac{4}{9}$, $P(B \mid A) = \frac{1}{3}$ and $P(B \mid A') = \frac{1}{6}$,

a. Show that $P(A \cup B) = P(A) + P(A' \cap B)$. [3]

b. (i) show that $P(A) = \frac{1}{3}$; (ii) hence find $P(B)$. [6]

18M.1.AHL.TZ2.H_9

The points A, B, C and D have position vectors ***a***, ***b***, ***c*** and ***d***, relative to the origin O.

It is given that $\vec{AB} = \vec{DC}$.

The position vectors $\vec{OA}$, $\vec{OB}$, $\vec{OC}$ and $\vec{OD}$ are given by

$$\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$$

$$\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j} + p\boldsymbol{k}$$

$$\boldsymbol{c} = q\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$$

$$\boldsymbol{d} = -\boldsymbol{i} + r\boldsymbol{j} - 2\boldsymbol{k}$$

where p , q and r are constants.

The point where the diagonals of ABCD intersect is denoted by M.

The plane Π cuts the x, y and z axes at X , Y and Z respectively.

a.i. Explain why ABCD is a parallelogram. [1]

a.ii. Using vector algebra, show that $\overrightarrow{AD} = \overrightarrow{BC}$. [3]

b. Show that $p = 1$, $q = 1$ and $r = 4$. [5]

c. Find the area of the parallelogram ABCD. [4]

d.

Find the vector equation of the straight line passing through M and normal to the plane Π containing ABCD.

[4]

e. Find the Cartesian equation of Π. [3]

f.i. Find the coordinates of X, Y and Z. [2]

f.ii. Find YZ. [2]

17M.1.AHL.TZ1.H_10

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} k\sin\left(\frac{\pi x}{6}\right), & 0 \leqslant x \leqslant 6 \\ 0, & \text{otherwise} \end{cases}.$$

a. Find the value of k. [4]

b.i. By considering the graph of f write down the mean of X; [1]

b.ii. By considering the graph of f write down the median of X; [1]

b.iii. By considering the graph of f write down the mode of X. [1]

c.i. Show that $P(0 \leqslant X \leqslant 2) = \frac{1}{4}$. [4]

c.ii. Hence state the interquartile range of X. [2]

d. Calculate $P(X \leqslant 4 \mid X \geqslant 3)$. [2]

19M.1.AHL.TZ2.H_2

Three points in three-dimensional space have coordinates A(0, 0, 2), B(0, 2, 0) and C(3, 1, 0).

a.i. Find the vector $\overrightarrow{AB}$. [1]

a.ii. Find the vector $\overrightarrow{AC}$. [1]

b. Hence or otherwise, find the area of the triangle ABC. [4]

17M.1.AHL.TZ1.H_5

ABCD is a parallelogram, where $\overrightarrow{AB} = -\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\overrightarrow{AD} = 4\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$.

a. Find the area of the parallelogram ABCD. [3]

b.

By using a suitable scalar product of two vectors, determine whether $\hat{ABC}$ is acute or obtuse.

[4]

18M.1.AHL.TZ2.H_1

The acute angle between the vectors $3\boldsymbol{i} - 4\boldsymbol{j} - 5\boldsymbol{k}$ and $5\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$ is denoted by θ.

Find $\cos \theta$.

17N.1.AHL.TZ0.H_2

The points A and B are given by $A(0,\ 3,\ -6)$ and $B(6,\ -5,\ 11)$.

The plane Π is defined by the equation $4x - 3y + 2z = 20$.

a. Find a vector equation of the line L passing through the points A and B. [3]

b. Find the coordinates of the point of intersection of the line L with the plane Π.

[3]

18N.1.AHL.TZ0.H_9

Consider a triangle OAB such that O has coordinates (0, 0, 0), A has coordinates (0, 1, 2) and B has coordinates ($2b$, 0, $b - 1$) where $b < 0$.

Let M be the midpoint of the line segment [OB].

a. Find, in terms of b, a Cartesian equation of the plane Π containing this triangle.

[5]

b.

Find, in terms of b, the equation of the line L which passes through M and is perpendicular to the plane Π.

[3]

c. Show that L does not intersect the y-axis for any negative value of b. [7]

16N.1.AHL.TZ0.H_4

Consider the vectors $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = -3\boldsymbol{j} + 2\boldsymbol{k}$.

a. Find $\boldsymbol{a} \times \boldsymbol{b}$. [2]

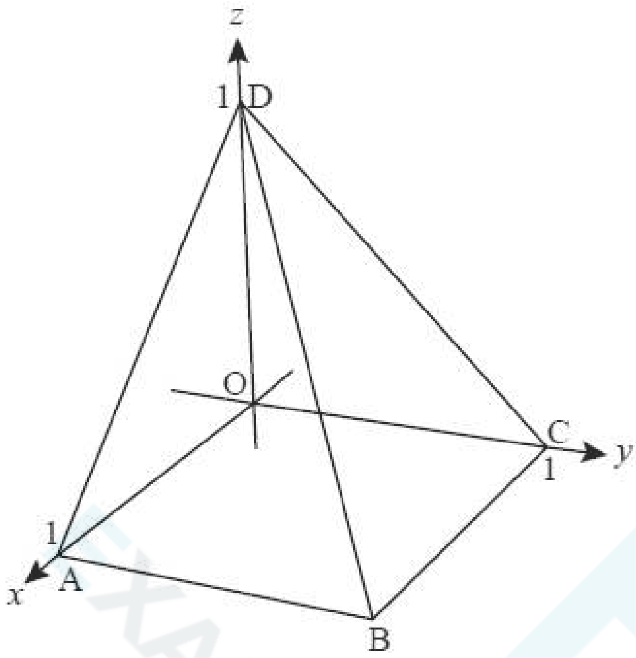
b.

Hence find the Cartesian equation of the plane containing the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and passing through the point $(1,\ 0,\ -1)$.

[3]

18M.1.AHL.TZ1.H_10

The following figure shows a square based pyramid with vertices at O(0, 0, 0), A(1, 0, 0), B(1, 1, 0), C(0, 1, 0) and D(0, 0, 1).



The Cartesian equation of the plane Π_2, passing through the points B , C and D , is $y + z = 1$.

The plane Π_3 passes through O and is normal to the line BD.

Π_3 cuts AD and BD at the points P and Q respectively.

a.

Find the Cartesian equation of the plane Π_1, passing through the points A , B and D.

[3]

b. Find the angle between the faces ABD and BCD. [4]

c. Find the Cartesian equation of Π_3. [3]

d. Show that P is the midpoint of AD. [4]

e. Find the area of the triangle OPQ. [5]

16N.1.AHL.TZ0.H_1

Find the coordinates of the point of intersection of the planes defined by the equations $x + y + z = 3$, $x - y + z = 5$ and $x + y + 2z = 6$.

18M.1.AHL.TZ2.H_3

The discrete random variable X has the following probability distribution, where p is nstant.

x	0	1	2	3	4
$\mathrm{P}(X=x)$	p	$0.5-p$	0.25	0.125	p^3

a. Find the value of p. [2]

b.i. Find μ, the expected value of X. [2]

b.ii. Find $\mathrm{P}(X > \mu)$. [2]

21M.1.AHL.TZ1.7

Consider the quartic equation $z^4 + 4z^3 + 8z^2 + 80z + 400 = 0, \quad z \in \mathbb{C}$.

Two of the roots of this equation are $a + b\mathrm{i}$ and $b + a\mathrm{i}$, where $a, \quad b \in \mathbb{Z}$.

Find the possible values of a.

21M.1.AHL.TZ1.11

Consider the line L_1 defined by the Cartesian equation $\frac{x+1}{2} = y = 3 - z$.

Consider a second line L_2 defined by the vector equation $r = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$, where $t \in \mathbb{R}$ and $a \in \mathbb{R}$.

a.i. Show that the point $(-1, \quad 0, \quad 3)$ lies on L_1. [1]

a.ii. Find a vector equation of L_1. [3]

b. Find the possible values of a when the acute angle between L_1 and L_2 is $45°$. [8]

c.

It is given that the lines L_1 and L_2 have a unique point of intersection, A, when $a \neq k$.

Find the value of k, and find the coordinates of the point A in terms of a. [7]

20N.1.AHL.TZ0.H_1

A discrete random variable X has the probability distribution given by the following table.

x	0	1	2	3
$\mathrm{P}(X=x)$	p	$\frac{1}{4}$	$\frac{1}{6}$	q

Given that $EX = \frac{19}{12}$, determine the value of p and the value of q.

20N.1.AHL.TZ0.H_10

Consider the function $fx = ax^3 + bx^2 + cx + d$, where $x \in \mathbb{R}$ and $a,\quad b,\quad c,\quad d \in \mathbb{R}$.

Consider the function $gx = \frac{1}{2}x^3 - 3x^2 + 6x - 8$, where $x \in \mathbb{R}$.

The graph of $y = g\left(x\right)$ may be obtained by transforming the graph of $y = x^3$ using a sequence of three transformations.

a.i. Write down an expression for $f'x$. [1]

a.ii. Hence, given that f^{-1} does not exist, show that $b^2 - 3ac > 0$. [3]

b.i. Show that g^{-1} exists. [2]

b.ii. $g\left(x\right)$ can be written in the form $p\left(x-2\right)^3 + q$, where $p,\quad q \quad \in \quad \mathbb{R}$.

Find the value of p and the value of q. [3]

b.iii. Hence find $g^{-1}\left(x\right)$. [3]

c. State each of the transformations in the order in which they are applied. [3]

d.

Sketch the graphs of $y = g\left(x\right)$ and $y = g^{-1}\left(x\right)$ on the same set of axes, indicating the points where each graph crosses the coordinate axes.

[5]

20N.1.AHL.TZ0.H_11

Consider the curve C defined by $y^2 = \sin\ xy \quad , \quad y \neq 0$.

a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\ \cos\ xy}{2y - x\ \cos\ xy}$. [5]

b. Prove that, when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad , \quad y = \pm 1$. [5]

c. Hence find the coordinates of all points on C, for $0 < x < 4\pi$, where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. [5]

20N.1.AHL.TZ0.H_2

Find the equation of the tangent to the curve $y = \mathrm{e}^{2x} - 3x$ at the point where $x = 0$.

20N.1.AHL.TZ0.H_4

Consider the equation $\frac{2z}{3 - z^*} = \mathrm{i}$, where $z = x + \mathrm{i}y$ and $x, \quad y \in \mathbb{R}$.

Find the value of x and the value of y.

20N.1.AHL.TZ0.H_5

The first term in an arithmetic sequence is 4 and the fifth term is $\log_2 \ 625$.

Find the common difference of the sequence, expressing your answer in the form $\log_2 \ p$, where $p \in \mathbb{Q}$.

21N.1.AHL.TZ0.7

The equation $3px^2 + 2px + 1 = p$ has two real, distinct roots.

a. Find the possible values for p. [5]

b.

Consider the case when $p = 4$. The roots of the equation can be expressed in the form $x = \frac{a \pm \sqrt{13}}{6}$, where $a \in \mathbb{Z}$. Find the value of a.

[2]

21N.1.AHL.TZ0.8

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\ln 2x}{x^2} - \frac{2y}{x}, \quad x > 0$, given that $y = 4$ at $x = \frac{1}{2}$.

Give your answer in the form $y = fx$.

22M.1.AHL.TZ2.9

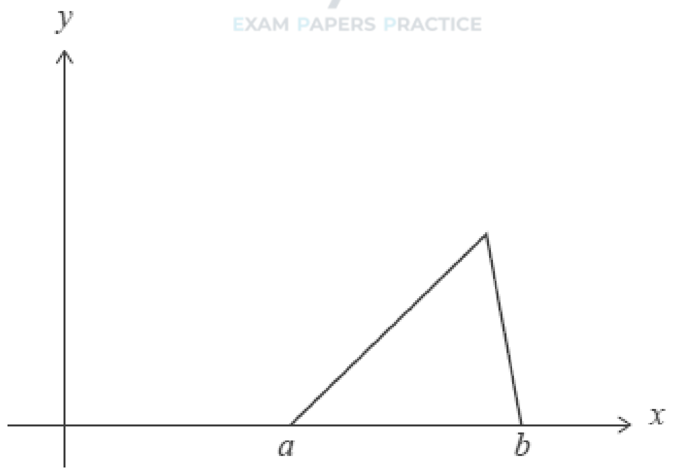
Prove by contradiction that the equation $2x^3 + 6x + 1 = 0$ has no integer roots.

22M.1.AHL.TZ2.8

A continuous random variable X has the probability density function

$$fx = \begin{cases} \frac{2}{b - ac - a}x - a, & a \le x \le c \\ \frac{2}{b - ab - c}b - x, & c < x \le b \\ 0, & \text{otherwise} \end{cases}.$$

The following diagram shows the graph of $y = fx$ for $a \le x \le b$.



Given that $c \geq \frac{a+b}{2}$, find an expression for the median of X in terms of a, b and c.

22M.1.AHL.TZ1.6

Consider the expansion of $8x^3 - \frac{1}{2x}^n$ where $n \in \mathbb{Z}^+$. Determine all possible values of n for which the expansion has a non-zero constant term.

22M.1.AHL.TZ1.8

Consider integers a and b such that $a^2 + b^2$ is exactly divisible by 4. Prove by contradiction that a and b cannot both be odd.

22M.1.AHL.TZ1.9

Consider the complex numbers $z_1 = 1 + b\text{i}$ and $z_2 = 1 - b^2 - 2b\text{i}$, where $b \in \mathbb{R}$, $b \neq 0$.

a. Find an expression for $z_1 z_2$ in terms of b. [3]

b. Hence, given that $\arg z_1 z_2 = \frac{\pi}{4}$, find the value of b. [3]

22M.1.AHL.TZ1.10

Consider the series $\ln x + p \ln x + \frac{1}{3}\ln x + \ldots$, where $x \in \mathbb{R}$, $x > 1$ and $p \in \mathbb{R}$, $p \neq 0$.

Consider the case where the series is geometric.

Now consider the case where the series is arithmetic with common difference d.

a.i. Show that $p = \pm\frac{1}{\sqrt{3}}$. [2]

a.ii. Hence or otherwise, show that the series is convergent. [1]

a.iii. Given that $p > 0$ and $S_\infty = 3 + \sqrt{3}$, find the value of x. [3]

b.i. Show that $p = \frac{2}{3}$. [3]

b.ii. Write down d in the form $k \ln x$, where $k \in \mathbb{Q}$. [1]

b.iii. The sum of the first n terms of the series is $\ln\frac{1}{x^3}$. Find the value of n. [8]

22M.1.AHL.TZ1.11

Consider the three planes

$$\Pi_1 : \quad 2x - y + z = 4$$

$$\Pi_2 : \quad x - 2y + 3z = 5$$

$$\Pi_3 : -9x + 3y - 2z = 32$$

a. Show that the three planes do not intersect. [4]

b.i. Verify that the point $\mathrm{P}(1, \ -2, \ 0)$ lies on both Π_1 and Π_2. [1]

b.ii. Find a vector equation of L, the line of intersection of Π_1 and Π_2. [4]

c. Find the distance between L and Π_3. [6]