EXAM PAPERS PRACTICE

Helping you Achieve Highest Grades in IB

IB Mathematics (Analysis and Approaches) Higher Level (HL)

Mark Scheme

Fully in-lined with the First Assessment Examinations in 2021 & Beyond

No Calculators Allowed

Paper: 1 (All Topics)

- **Topic 1 - Number and Algebra**
- **Topic 2 - Functions**
- **Topic 3 - Geometry and Trigonometry**
- **Topic 4 - Statistics and Probability**
- **Topic 5 - Calculus**

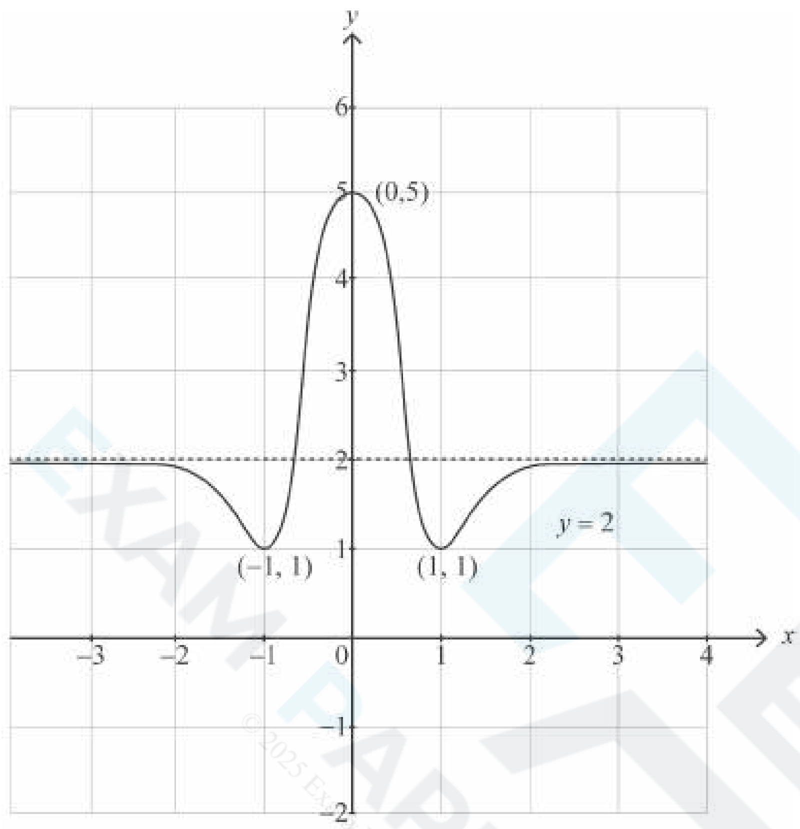
Marks: 723

Total Marks: / 732

Suitable for HL Students sitting the 2026 exams and beyond
However, SL students may also find these resources useful

Markschemes

SPM.1.AHL.TZ0.4



no y values below 1 ***A1***

horizontal asymptote at $y = 2$ with curve approaching from below as $x \to \pm\infty$ ***A1***

(±1,1) local minima ***A1***

(0,5) local maximum ***A1***

smooth curve and smooth stationary points ***A1***

[5 marks]

SPM.1.AHL.TZ0.8

recognition that the angle between the normal and the line is 60° (seen anywhere) ***R1***

attempt to use the formula for the scalar product ***M1***

$\cos 60° = \frac{\left|\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}\right|}{\sqrt{9} \times \sqrt{1 + 4 + p^2}}$ ***A1***

$\frac{1}{2} = \frac{|2p|}{3\sqrt{5 + p^2}}$ ***A1***

$3\sqrt{5+p^2} = 4|p|$

attempt to square both sides

$9(5+p^2) = 16p^2 \Rightarrow 7p^2 = 45$

$p = \pm 3\sqrt{\frac{5}{7}}$ (or equivalent)

SPM.1.AHL.TZ0.9

a.

attempt to differentiate and set equal to zero ***M1***

$f'(x) = 2e^{2x} - 6e^{x} = 2e^{x}(e^{x} - 3) = 0$ ***A1***

minimum at $x = \ln 3$

$a = \ln 3$ ***A1***

[3 marks]

b. **Note:** Interchanging x and y can be done at any stage. $y = (e^{x} - 3)^2 - 4$ ***(M1)***

$e^{x} - 3 = \pm\sqrt{y+4}$ ***A1*** as $x \leqslant \ln 3$, $x = \ln(3 - \sqrt{y+4})$ ***R1***

so $f^{-1}(x) = \ln(3 - \sqrt{x+4})$ ***A1*** domain of f^{-1} is $x \in R$, $-4 \leqslant x < 5$ ***A1*** ***[5 marks]***

SPM.1.AHL.TZ0.11

a.

attempt to find modulus ***(M1)***

$r = 2\sqrt{3}\ (= \sqrt{12})$ ***A1***

attempt to find argument in the correct quadrant ***(M1)***

$\theta = \pi + \arctan\left(-\frac{\sqrt{3}}{3}\right)$ ***A1***

$= \frac{5\pi}{6}$ ***A1***

$-3 + \sqrt{3}i = \sqrt{12}e^{\frac{5\pi i}{6}} \left(= 2\sqrt{3}e^{\frac{5\pi i}{6}}\right)$

[5 marks]

b. attempt to find a root using de Moivre's theorem ***M1*** $12^{\frac{1}{6}}e^{\frac{5\pi i}{18}}$ ***A1***

attempt to find further two roots by adding and subtracting $\frac{2\pi}{3}$ to the argument ***M1***

$12^{\frac{1}{6}}e^{-\frac{7\pi i}{18}}$ ***A1*** $12^{\frac{1}{6}}e^{\frac{17\pi i}{18}}$ ***A1*** **Note:** Ignore labels for u, v and w at this stage.

[5 marks]

c.

METHOD 1

attempting to find the total area of (congruent) triangles UOV, VOW and UOW ***M1***

Area $= 3\left(\frac{1}{2}\right)\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\sin\frac{2\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)$ and ***A1*** for $\sin\frac{2\pi}{3}$ $= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

METHOD 2 $UV^2 = \left(12^{\frac{1}{6}}\right)^2 + \left(12^{\frac{1}{6}}\right)^2 - 2\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\cos\frac{2\pi}{3}$ (or equivalent)

$UV = \sqrt{3}\left(12^{\frac{1}{6}}\right)$ (or equivalent)

attempting to find the area of UVW using Area = $\frac{1}{2} \times UV \times VW \times \sin\alpha$ for example

Area $= \frac{1}{2}\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\sin\frac{\pi}{3}$ $= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent)

d. $u + v + w = 0$

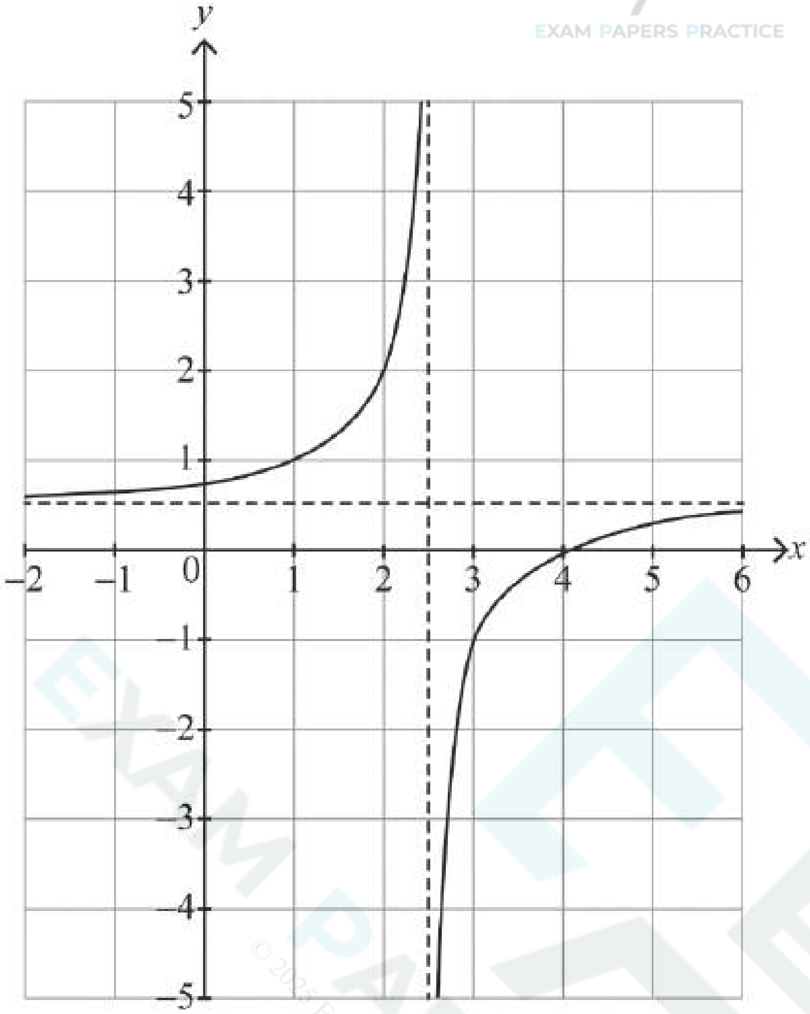
$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + i\sin\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + i\sin\frac{5\pi}{18} + \cos\frac{17\pi}{18} + i\sin\frac{17\pi}{18}\right) = 0$

consideration of real parts $12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \cos\frac{17\pi}{18}\right) = 0$

$\cos\left(-\frac{7\pi}{18}\right) = \cos\frac{17\pi}{18}$ explicitly stated $\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0$

19M.1.AHL.TZ2.H_5

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*



correct shape: two branches in correct quadrants with asymptotic behaviour

crosses at (4, 0) and $\left(0,\frac{4}{5}\right)$

asymptotes at $x = \frac{5}{2}$ and $y = \frac{1}{2}$

17M.1.AHL.TZ2.H_2

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$-11 \leqslant f(x) \leqslant 21$ ***A1A1***

Note: ***A1*** for correct end points, ***A1*** for correct inequalities.

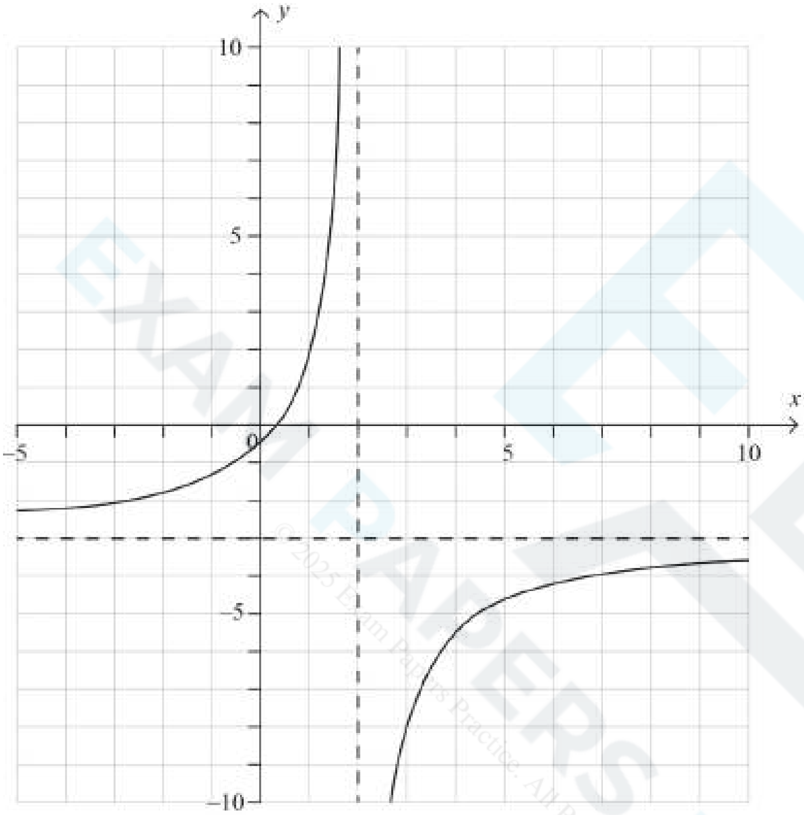
[2 marks]

b. $f^{-1}(x) = \sqrt[3]{\frac{x-5}{2}}$

c. $-11 \leqslant x \leqslant 21, \; -2 \leqslant f^{-1}(x) \leqslant 2$

17N.1.AHL.TZ0.H_6

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*



correct vertical asymptote ***A1***

shape including correct horizontal asymptote ***A1***

$\left(\frac{1}{3}, 0\right)$ ***A1***

$\left(0, -\frac{1}{2}\right)$ ***A1***

Note: Accept $x = \frac{1}{3}$ and $y = -\frac{1}{2}$ marked on the axes.

[4 marks]

17M.1.AHL.TZ1.H_11

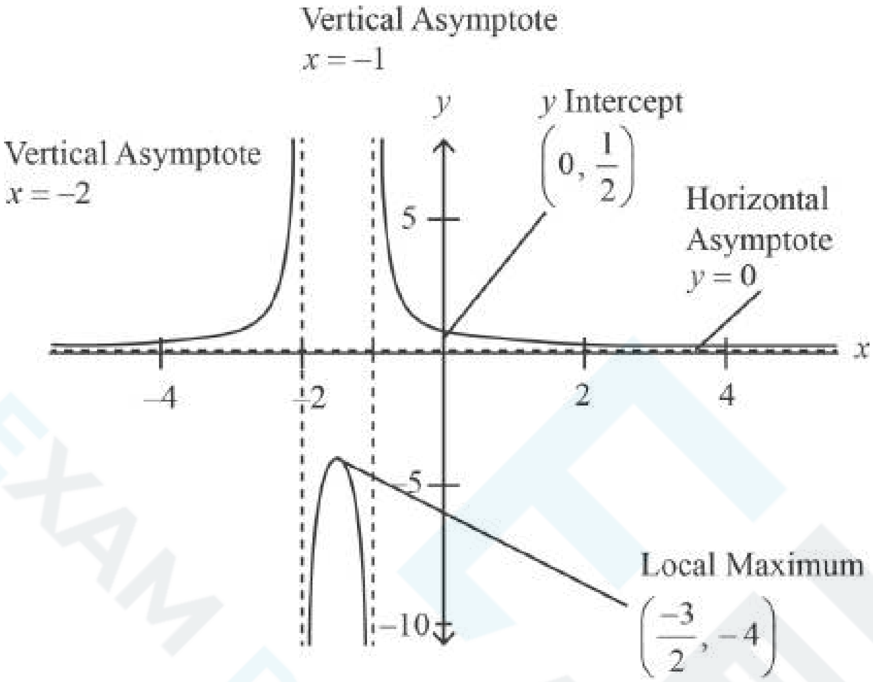
a.i.

$x^2 + 3x + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$ ***A1***

[1 mark]

a.ii. $x^2 + 3x + 2 = (x + 2)(x + 1)$ ***A1*** ***[1 mark]***

b.



A1 for the shape

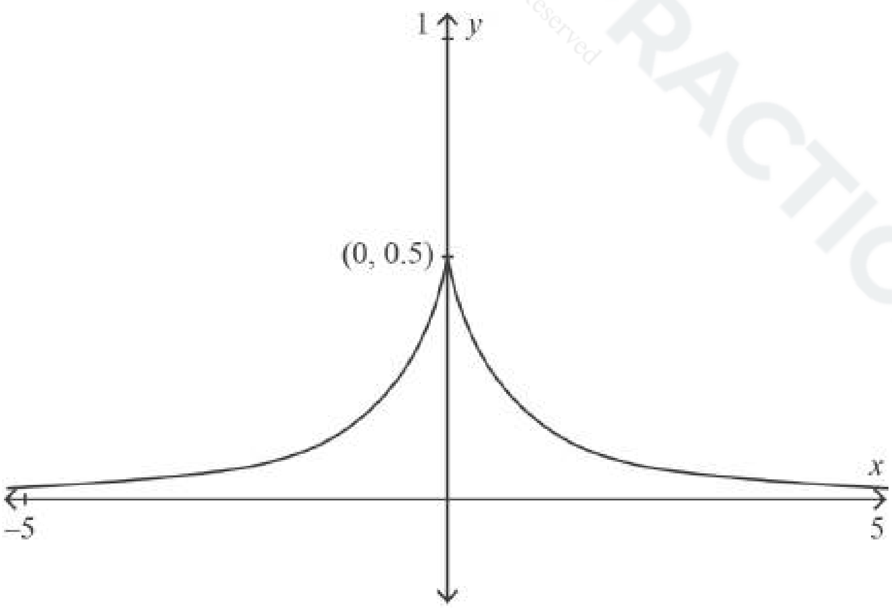
A1 for the equation $y = 0$ ***A1*** for asymptotes $x = -2$ and $x = -1$

A1 for coordinates $\left(-\frac{3}{2}, -4\right)$ ***A1*** y-intercept $\left(0, \frac{1}{2}\right)$ ***[5 marks]***

d. $\int_0^1 \frac{1}{x+1} - \frac{1}{x+2} dx$ $= \left[\ln(x+1) - \ln(x+2)\right]_0^1$ ***A1*** $= \ln 2 - \ln 3 - \ln 1 + \ln 2$ ***M1***

$= \ln\left(\frac{4}{3}\right)$ ***M1A1*** $\therefore p = \frac{4}{3}$ ***[4 marks]***

e.



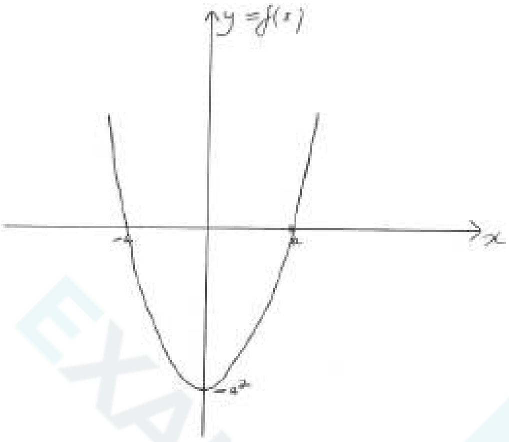
symmetry about the y-axis ***M1*** correct shape ***A1***

Note: Allow ***FT*** from part (b). ***[2 marks]***

f. $2\int_0^1 f(x)\mathrm{d}x$ $= 2\ln\left(\frac{4}{3}\right)$ Do not award from part (e).

17M.1.AHL.TZ2.H_9

a.i.

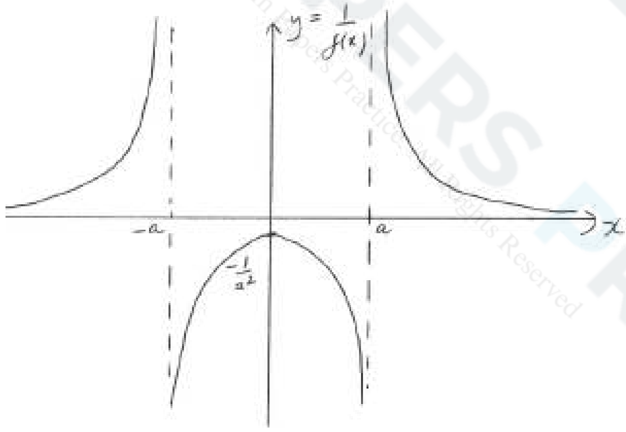


A1 for correct shape

A1 for correct x and y intercepts and minimum point

[2 marks]

a.ii.

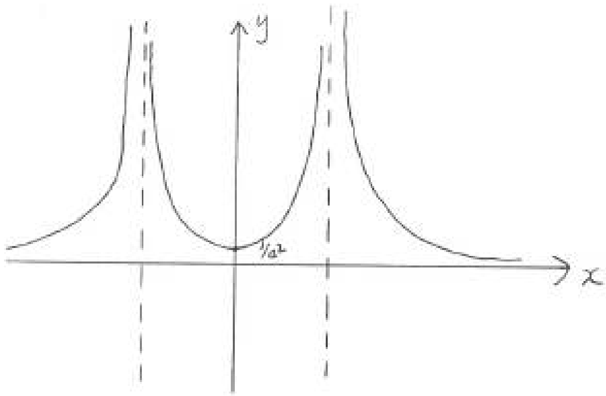


A1 for correct shape

A1 for correct vertical asymptotes ***A1*** for correct implied horizontal asymptote

A1 for correct maximum point ***[??? marks]***

a.iii.



for reflecting negative branch from (ii) in the x-axis

for correctly labelled minimum point

b. attempt at integration by parts

$\int(x^2-a^2)\cos x dx = (x^2-a^2)\sin x - \int 2x\sin x dx$

$=(x^2-a^2)\sin x - 2\left[-x\cos x + \int \cos x dx\right]$ $=(x^2-a^2)\sin x + 2x\cos - 2\sin x + c$

$\int(x^2-a^2)\cos x dx = \int x^2\cos x dx - \int a^2\cos x dx$

attempt at integration by parts $\int x^2\cos x dx = x^2\sin x - \int 2x\sin x dx$

$= x^2\sin x - 2\left[-x\cos x + \int \cos x dx\right]$ $= x^2\sin x + 2x\cos x - 2\sin x$

$-\int a^2\cos x dx = -a^2\sin x$ $\int(x^2-a^2)\cos x dx = (x^2-a^2)\sin x + 2x\cos x - 2\sin x + c$

c. $g(x) = x(x^2-a^2)^{\frac{1}{2}}$ $g'(x) = (x^2-a^2)^{\frac{1}{2}} + \frac{1}{2}x(x^2-a^2)^{-\frac{1}{2}}(2x)$

Method mark is for differentiating the product. Award for each correct term.

$g'(x) = (x^2-a^2)^{\frac{1}{2}} + x^2(x^2-a^2)^{-\frac{1}{2}}$

both parts of the expression are positive hence $g'(x)$ is positive

and therefore g is an increasing function (for $|x| > a$)

18M.1.AHL.TZ1.H_9

a.

attempt to differentiate ***(M1)***

$f'(x) = -3x^{-4} - 3x$ ***A1***

Note: Award ***M1*** for using quotient or product rule award ***A1*** if correct derivative seen even in unsimplified form, for example $f'(x) = \frac{-15x^4 \times 2x^3 - 6x^2(2-3x^5)}{(2x^3)^2}$.

$-\frac{3}{x^4} - 3x = 0$ ***M1***

$\Rightarrow x^5 = -1 \Rightarrow x = -1$ ***A1***

$\mathrm{A}\left(-1, -\frac{5}{2}\right)$ ***A1***

[5 marks]

b.i. $f''(x) = 0$ ***M1*** $f''(x) = 12x^{-5} - 3(=0)$ ***A1***

Note: Award ***A1*** for correct derivative seen even if not simplified.

$\Rightarrow x = \sqrt[5]{4}\left(= 2^{\frac{2}{5}}\right)$ ***A1*** hence (at most) one point of inflexion ***R1***

Note: This mark is independent of the two ***A1*** marks above. If they have shown or stated their equation has only one solution this mark can be awarded.

$f''(x)$ changes sign at $x = \sqrt[5]{4}\left(= 2^{\frac{2}{5}}\right)$ ***R1*** so exactly one point of inflexion

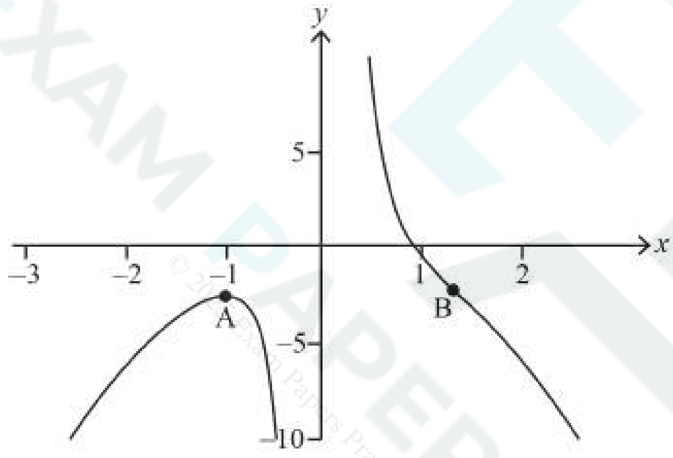
[5 marks]

b.ii. $x = \sqrt[5]{4} = 2^{\frac{2}{5}}\left(\Rightarrow a = \frac{2}{5}\right)$ ***A1***

$f\left(2^{\frac{2}{5}}\right) = \frac{2 - 3 \times 2^2}{2 \times 2^{\frac{6}{5}}} = -5 \times 2^{-\frac{6}{5}}(\Rightarrow b = -5)$ ***(M1)A1***

Award for the substitution of their value for x into $f(x)$.

c.

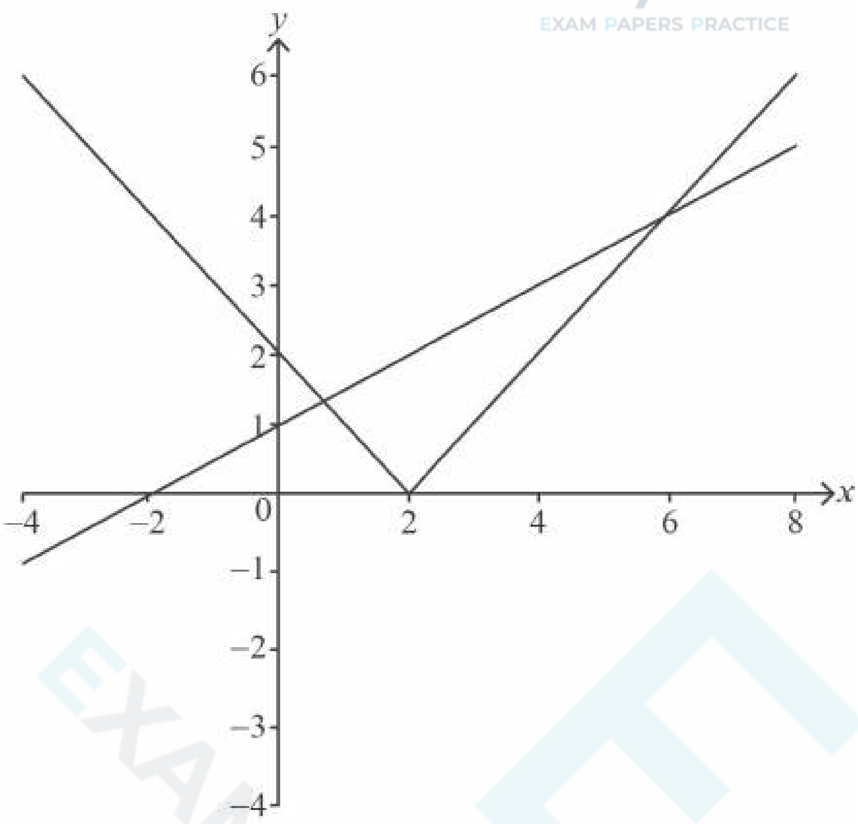


for shape for $x < 0$
for shape for $x > 0$
for maximum at A
for POI at B.

Only award last two s if A and B are placed in the correct quadrants, allowing for follow through.

18M.1.AHL.TZ2.H_2

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*



straight line graph with correct axis intercepts

modulus graph: V shape in upper half plane

modulus graph having correct vertex and *y*-intercept

19M.1.AHL.TZ1.H_8

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

3 ***A1***

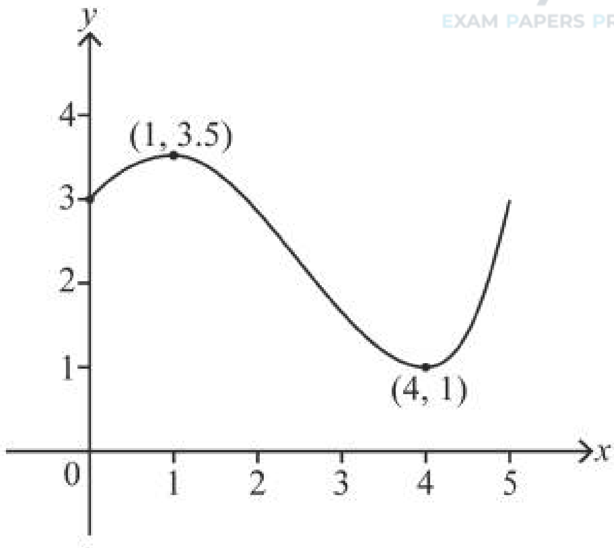
[1 mark]

b. attempt to use definite integral of $f'(x)$ ***(M1)*** $\int_0^1 f'(x)\,dx = 0.5$

$f(1) - f(0) = 0.5$ ***(A1)*** $f(1) = 0.5 + 3$ = 3.5 ***A1*** ***[3 marks]***

c. $\int_1^4 f'(x)\,dx = -2.5$ ***(A1)*** **Note: *(A1)*** is for −2.5. $f(4) - f(1) = -2.5$

$f(4) = 3.5 - 2.5$ = 1 ***A1*** ***[2 marks]*** d.



for correct shape over approximately the correct domain
for maximum and minimum (coordinates or horizontal lines from 3.5 and 1 are required),
for y-intercept at 3

19M.1.AHL.TZ2.H_3

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$g(x) = f(x+2)\left(= (x+2)^4 - 6(x+2)^2 - 2(x+2) + 4\right)$ ***M1***

attempt to expand $(x+2)^4$ ***M1***

$(x+2)^4 = x^4 + 4(2x^3) + 6\left(2^2x^2\right) + 4\left(2^3x\right) + 2^4$ ***(A1)***

$= x^4 + 8x^3 + 24x^2 + 32x + 16$ ***A1***

$g(x) = x^4 + 8x^3 + 24x^2 + 32x + 16 - 6\left(x^2 + 4x + 4\right) - 2x - 4 + 4$

$= x^4 + 8x^3 + 18x^2 + 6x - 8$ ***A1***

Note: For correct expansion of $f(x-2) = x^4 - 8x^3 + 18x^2 - 10x$ award max ***M0M1(A1)A0A1***.

[5 marks]

17N.1.AHL.TZ0.H_3

a.

$q(4) = 0$ ***(M1)***

$192 - 176 + 4k + 8 = 0\ (24 + 4k = 0)$ ***A1***

$k = -6$ ***A1***

[3 marks]

b. $3x^3 - 11x^2 - 6x + 8 = (x-4)(3x^2 + px - 2)$ equate coefficients of x^2: ***(M1)***

$-12 + p = -11 \quad p = 1$ $(x-4)(3x^2 + x - 2)$ ***(A1)*** $(x-4)(3x-2)(x+1)$ ***A1***

Note: Allow part (b) marks if any of this work is seen in part (a).

Allow equivalent methods (*eg*, synthetic division) for the marks in each part.

18M.1.AHL.TZ1.H_1

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

attempt to substitute $x = -1$ or $x = 2$ or to divide polynomials ***(M1)***

$1 - p - q + 5 = 7$, $16 + 8p + 2q + 5 = 1$ or equivalent ***A1A1***

attempt to solve their two equations ***M1***

$p = -3$, $q = 2$ ***A1***

[5 marks]

17M.1.AHL.TZ1.H_12

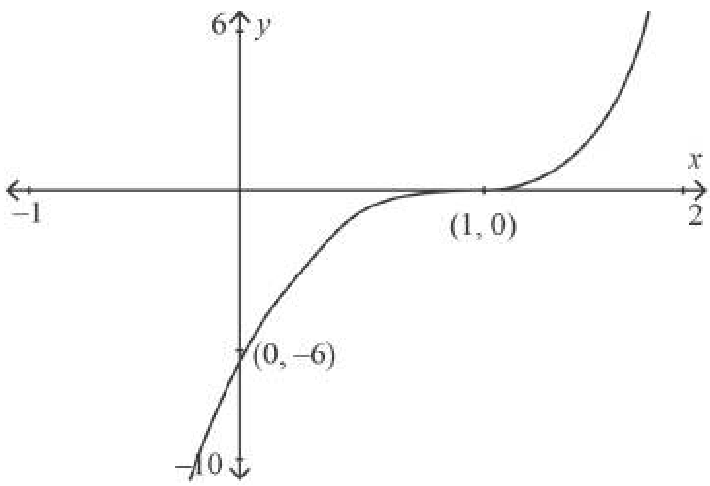
e.i.

$\frac{d^2y}{dx^2} = 20x^3 - 20$ ***M1A1***

for $x > 1$, $20x^3 - 20 > 0 \Rightarrow$ concave up ***R1AG***

[3 marks]

e.ii.



x-intercept at $(1, 0)$ ***A1***

y-intercept at $(0,\ -6)$

stationary point of inflexion at $(1,\ 0)$ with correct curvature either side

EXN.1.AHL.TZ0.7

** This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.*

METHOD 1

from vertex P, draws a line parallel to QR that meets SR at a point X **(M1)**

uses the sine rule in ΔPSX **M1**

$\frac{\text{PS}}{\sin\ \beta} = \frac{y - x}{\sin\ 180° - \alpha - \beta}$ **A1**

$\sin\ 180° - \alpha - \beta = \sin\ \alpha + \beta$ **(A1)**

$\text{PS} = \frac{y - x\ \sin\ \beta}{\sin\ \alpha + \beta}$ **A1**

METHOD 2

let the height of quadrilateral PQRS be h

$h = \text{PS}\ \sin\ \alpha$ **A1**

attempts to find a second expression for h **M1**

$h = y - x - \text{PS}\ \cos\ \alpha\ \tan\ \beta$

$\text{PS}\ \sin\ \alpha = y - x - \text{PS}\ \cos\ \alpha\ \tan\ \beta$

writes $\tan\ \beta$ as $\frac{\sin\ \beta}{\cos\ \beta}$, multiplies through by $\cos\ \beta$ and expands the RHS **M1**

$\text{PS}\ \sin\ \alpha\ \cos\ \beta = y - x\ \sin\ \beta - \text{PS}\ \cos\ \alpha\ \sin\ \beta$

$\text{PS} = \frac{y - x\ \sin\ \beta}{\sin\ \alpha\ \cos\ \beta + \cos\ \alpha\ \sin\ \beta}$ **A1**

$\text{PS} = \frac{y - x\ \sin\ \beta}{\sin\ \alpha + \beta}$ **A1**

[5 marks]

EXN.1.AHL.TZ0.9

** This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.*

let Pn be the proposition that $\sum_{r=1}^{n} \cos 2r - 1\theta = \frac{\sin \ 2n\theta}{2 \ \sin \ \theta}$ for $n \in \mathbb{Z}^+$

considering P1:

$\text{LHS} = \cos 1\theta = \cos \ \theta$ and $\text{RHS} = \frac{\sin \ 2\theta}{2 \ \sin \ \theta} = \frac{2 \ \sin \ \theta \ \cos\theta}{2 \ \sin \ \theta} = \cos\theta = \text{LHS}$

so P1 is true **R1**

assume Pk is true, i.e. $\sum_{r=1}^{k} \cos 2r - 1\theta = \frac{\sin \ 2k\theta}{2 \ \sin \ \theta} \quad k \in \mathbb{Z}^+$ **M1**

Note: Award **M0** for statements such as "let $n = k$".

Note: Subsequent marks after this **M1** are independent of this mark and can be awarded.

considering $Pk + 1$

$\sum_{r=1}^{k+1} \cos 2r - 1\theta = \sum_{r=1}^{k} \cos 2r - 1\theta + \cos 2k + 1 - 1\theta$ **M1**

$= \frac{\sin \ 2k\theta}{2 \ \sin \ \theta} + \cos 2k + 1 - 1\theta$ **A1**

$= \frac{\sin \ 2k\theta + 2 \ \cos 2k + 1\theta \ \sin \ \theta}{2 \ \sin \ \theta}$

$= \frac{\sin \ 2k\theta + \sin 2k + 1\theta + \theta - \ \sin 2k + 1\theta - \theta}{2 \ \sin \ \theta}$ **M1**

Note: Award **M1** for use of $2 \ \cos \ A \ \sin \ B = \sin A + B - \sin A - B$ with $A = 2k + 1\theta$ and $B = \theta$.

$= \frac{\sin \ 2k\theta + \sin 2k + 2\theta - \ \sin \ 2k\theta}{2 \ \sin \ \theta}$ **A1**

$= \frac{\sin \ 2k + 1\theta}{2 \ \sin \ \theta}$ **A1**

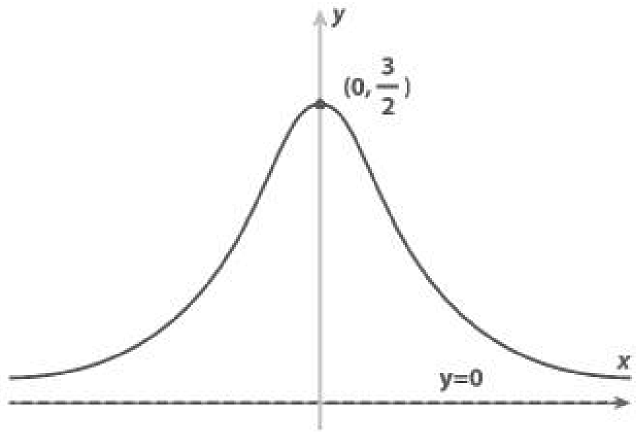
$Pk + 1$ is true whenever Pk is true, P1 is true, so Pn is true for $n \in \mathbb{Z}^+$ **R1**

Note: Award the final **R1** mark provided at least five of the previous marks have been awarded.

EXN.1.AHL.TZ0.11

a.

** This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.*



a curve symmetrical about the y-axis with correct concavity that has a local maximum point on the positive y-axis **A1**

a curve clearly showing that $y \to 0$ as $x \to \pm\infty$ **A1**

$0, \frac{3}{2}$ **A1**

horizontal asymptote $y = 0$ (x-axis) **A1**

[4 marks]

b. attempts to find $\int \frac{3}{x^2+2}\mathrm{d}x$ **(M1)** $= \frac{3}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}$ **A1**

Note: Award **M1A0** for obtaining $k \arctan\frac{x}{\sqrt{2}}$ where $k \neq \frac{3}{\sqrt{2}}$.

Note: Condone the absence of or use of incorrect limits to this stage.

$= \frac{3}{\sqrt{2}}\arctan \sqrt{3} - \arctan 0$ **(M1)** $= \frac{3}{\sqrt{2}} \times \frac{\pi}{3} = \frac{\pi}{\sqrt{2}}$ **A1** $A = \frac{\sqrt{2}\pi}{2}$ **AG**

[4 marks]

c. **METHOD 1** **EITHER** $\int_0^k \frac{3}{x^2+2}\mathrm{d}x = \frac{\sqrt{2}\pi}{4}$ $\frac{3}{\sqrt{2}}\arctan\frac{k}{\sqrt{2}} = \frac{\sqrt{2}\pi}{4}$ **(M1)** **OR**

$\int_k^{\sqrt{6}} \frac{3}{x^2+2}\mathrm{d}x = \frac{\sqrt{2}\pi}{4}$ $\frac{3}{\sqrt{2}}\arctan \sqrt{3} - \arctan\frac{k}{\sqrt{2}} = \frac{\sqrt{2}\pi}{4}$ **(M1)** $\arctan \sqrt{3} - \arctan\frac{k}{\sqrt{2}} = \frac{\pi}{6}$

THEN $\arctan\frac{k}{\sqrt{2}} = \frac{\pi}{6}$ **A1** $\frac{k}{\sqrt{2}} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ **A1** $k = \frac{\sqrt{6}}{3} = \sqrt{\frac{2}{3}}$ **A1** **METHOD 2**

$\int_0^k \frac{3}{x^2+2}\mathrm{d}x = \int_k^{\sqrt{6}} \frac{3}{x^2+2}\mathrm{d}x$ $\frac{3}{\sqrt{2}}\arctan\frac{k}{\sqrt{2}} = \frac{3}{\sqrt{2}}\arctan \sqrt{3} - \arctan\frac{k}{\sqrt{2}}$ **(M1)**

$\arctan\frac{k}{\sqrt{2}} = \frac{\pi}{6}$ **A1** $\frac{k}{\sqrt{2}} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ **A1** $k = \frac{\sqrt{6}}{3} = \sqrt{\frac{2}{3}}$ **A1** **[4 marks]**

d. attempts to find $\frac{\mathrm{d}}{\mathrm{d}x}\frac{3}{x^2+2}$ **(M1)** $= 3\text{-}12xx^2 + 2^{-2}$ **A1** so $m = -\frac{6x}{x^2+2^2}$ **AG**

e. attempts product rule or quotient rule differentiation

$\frac{\mathrm{d}\,m}{\mathrm{d}\,x} = \text{-}6x\text{-}22xx^2 + 2^{\text{-}3} + x^2 + 2^{\text{-}2}\text{-}6$ $\frac{\mathrm{d}\,m}{\mathrm{d}\,x} = \frac{x^2 + 2^2\text{-}6 \text{-} \text{-}6x22xx^2 + 2}{x^2 + 2^4}$

Award if the denominator is incorrect. Subsequent marks can be awarded.

attempts to express their $\frac{\mathrm{d}\,m}{\mathrm{d}\,x}$ as a rational fraction with a factorized numerator

$\frac{\mathrm{d}\,m}{\mathrm{d}\,x} = \frac{6x^2 + 23x^2 \text{-} 2}{x^2 + 2^4} = \frac{63x^2 \text{-} 2}{x^2 + 2^3}$ attempts to solve their $\frac{\mathrm{d}\,m}{\mathrm{d}\,x} = 0$ for x

$x = \pm\sqrt{\frac{2}{3}}$ from the curve, the maximum value of m occurs at $x = \text{-}\sqrt{\frac{2}{3}}$

(the minimum value of m occurs at $x = \sqrt{\frac{2}{3}}$)

Award for any equivalent valid reasoning.

maximum value of m is $\text{-}\frac{6\text{-}\sqrt{\frac{2}{3}}}{\text{-}\sqrt{\frac{2}{3}}^2 + 2^2}$ leading to a maximum value of $\frac{27}{32}\sqrt{\frac{2}{3}}$

18N.1.AHL.TZ0.H_8

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$-i\sqrt{3}$ is a root ***(A1)***

$3 + \log_2 3 - \log_2 6\left(= 3 + \log_2 \frac{1}{2} = 3 - 1 = 2\right)$ is a root ***(A1)***

sum of roots: $-a = 3 + \log_2 3 \Rightarrow a = -3 - \log_2 3$ ***M1***

Note: Award M1 for use of $-a$ is equal to the sum of the roots, do not award if minus is missing.

Note: If expanding the factored form of the equation, award ***M1*** for equating a to the coefficient of z^3.

product of roots: $(-1)^4 d$ $= 2\left(\log_2 6\right)\left(i\sqrt{3}\right)\left(-i\sqrt{3}\right)$ ***M1***

$= 6\log_2 6$ ***A1***

Note: Award ***M1A0*** for $d = -6\log_2 6$

$6a+d+12=-18-6\log_2 3+6\log_2 6+12$

$=-6+6\log_2 2=0$

is for a correct use of one of the log laws.

$=-6-6\log_2 3+6\log_2 3+6\log_2 2=0$

is for a correct use of one of the log laws.

16N.1.AHL.TZ0.H_5

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$\alpha+\beta=2k$ ***A1***

$\alpha\beta=k-1$ ***A1***

$(\alpha+\beta)^2=4k^2 \Rightarrow \alpha^2+\beta^2+2\underbrace{\alpha\beta}_{k-1}=4k^2$ ***(M1)***

$\alpha^2+\beta^2=4k^2-2k+2$

$\alpha^2+\beta^2=4 \Rightarrow 4k^2-2k-2=0$ ***A1***

attempt to solve quadratic ***(M1)***

$k=1,\ -\frac{1}{2}$ ***A1***

[6 marks]

21M.1.AHL.TZ2.2

attempt to use $\cos^2 x=1-\sin^2 x$ ***M1***

$2\sin^2 x-5\sin x+2=0$ ***A1***

EITHER

attempting to factorise ***M1***

$(2 \sin x - 1)(\sin x - 2)$ ***A1***

OR

attempting to use the quadratic formula ***M1***

$\sin x = \frac{5 \pm \sqrt{5^2 - 4 \times 2 \times 2}}{4} = \frac{5 \pm 3}{4}$ ***A1***

THEN

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

21M.1.AHL.TZ2.5

METHOD 1

use of $a \times b = ab\sin \theta$ on the LHS ***(M1)***

$a \times b^2 = a^2b^2 \sin^2 \theta$ ***A1***

$= a^2b^2 \ 1 - \cos^2 \theta$ ***M1***

$= a^2b^2 - a^2b^2 \cos^2 \theta$ OR $= a^2b^2 - ab \cos \theta^2$ ***A1***

$= a^2b^2 - a \cdot b^2$ ***AG***

METHOD 2

use of $a \cdot b = ab\cos \theta$ on the RHS ***(M1)***

$= a^2b^2 - a^2b^2 \cos^2 \theta$ ***A1***

$= a^2b^2 \ 1 - \cos^2 \theta$ ***M1***

$= a^2b^2 \sin^2 \theta$ OR $= ab \sin \theta^2$ ***A1***

$= a \times b^2$ ***AG***

Note: If candidates attempt this question using cartesian vectors, e.g

$a = \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix}$ $b = \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}$,

award full marks if fully developed solutions are seen.
Otherwise award no marks.

21M.1.AHL.TZ2.7

$\alpha + \beta + \alpha + \beta = k$ ***(A1)***

$\alpha + \beta = \frac{k}{2}$

$\alpha\beta\alpha + \beta = -3k$ ***(A1)***

$-\frac{k^2}{4}\frac{k}{2} = -3k$ $-\frac{k^3}{8} = -3k$ ***M1***

attempting to solve $-\frac{k^3}{8} + 3k = 0$ (or equivalent) for k ***(M1)***

$k = 2\sqrt{6}$ $= \sqrt{24}k > 0$ ***A1***

Note: Award ***A0*** for $k = \pm 2\sqrt{6}$ $\pm\sqrt{24}$.

[5 marks]

21M.1.AHL.TZ2.11

a.

$\frac{\mathrm{d}v}{\mathrm{d}t} = -1 + v$ ***(A1)***

$\int 1 \quad \mathrm{d}t = \int -\frac{1}{1+v}\mathrm{d}v$ (or equivalent / use of integrating factor) ***M1***

$t = -\ln 1 + v + C$ ***A1***

EITHER

attempt to find C with initial conditions $t = 0,\quad v = v_0$ ***M1***

$C = \ln 1 + v_0$

$t = \ln 1 + v_0 - \ln 1 + v$

$t = \ln\frac{1+v_0}{1+v} \Rightarrow e^t = \frac{1+v_0}{1+v}$ ***A1***

$e^t 1 + v = 1 + v_0$

$1 + v = 1 + v_0 e^{-t}$ ***A1***

$vt = 1 + v_0 e^{-t} - 1$ ***AG***

OR

$t - C = -\ln 1 + v \Rightarrow e^{t-C} = \frac{1}{1+v}$

Attempt to find C with initial conditions $t = 0, \quad v = v_0$ ***M1***

$e^{-C} = \frac{1}{1+v_0} \Rightarrow C = \ln 1 + v_0$

$t - \ln 1 + v_0 = -\ln 1 + v \Rightarrow t = \ln 1 + v_0 - \ln 1 + v$

$t = \ln\frac{1+v_0}{1+v} \Rightarrow e^t = \frac{1+v_0}{1+v}$ ***A1***

$e^t 1 + v = 1 + v_0$

$1 + v = 1 + v_0 e^{-t}$ ***A1***

$vt = 1 + v_0 e^{-t} - 1$ ***AG***

OR

$t - C = -\ln 1 + v \Rightarrow e^{-t+C} = 1 + v$ ***A1***

$ke^{-t} - 1 = v$

Attempt to find k with initial conditions $t = 0, \quad v = v_0$ ***M1***

$k = 1 + v_0$

$e^{-t} 1 + v_0 = 1 + v$ ***A1***

$vt = 1 + v_0 e^{-t} - 1$ ***AG***

Note: condone use of modulus within the ln function(s)

[6 marks]

b.i. recognition that when $t = T, \quad v = 0$ ***M1*** $1 + v_0 e^{-T} - 1 = 0 \Rightarrow e^{-T} = \frac{1}{1+v_0}$ ***A1***

$e^T = 1 + v_0$ ***AG***

Note: Award ***M1A0*** for substituting $v_0 = e^T - 1$ into v and showing that $v = 0$.

[6 marks]

b.ii. $s(t) = \int v(t)\,dt \quad = \int (1 + v_0)e^{-t} - 1\,dt \qquad = -(1 + v_0)e^{-t} - t + D$

$(t = 0, \quad s = 0$ so$)$ $D = 1 + v_0 \qquad s(t) = -(1 + v_0)e^{-t} - t + 1 + v_0$

at s_{max}, $\quad e^T = 1 + v_0 \Rightarrow T = \ln(1 + v_0)$ Substituting into $s(t) = -(1 + v_0)e^{-t} - t + 1 + v_0$

$s_{max} = -(1 + v_0)\frac{1}{1+v_0} - \ln(1 + v_0) + v_0 + 1 \qquad s_{max} = v_0 - \ln(1 + v_0)$

c. $\qquad v(T - k) = (1 + v_0)e^{-T}e^{k} - 1 \qquad = (1 + v_0)\frac{1}{1+v_0}e^{k} - 1$

$= e^k - 1 \qquad v(T - k) = (1 + v_0)e^{-(T-k)} - 1 \quad = e^T e^{-(T-k)} - 1$

$= e^{T-T+k} - 1 \qquad = e^k - 1$

d. $\qquad v(T + k) = (1 + v_0)e^{-T}e^{-k} - 1 \qquad = e^{-k} - 1$

$v(T + k) = (1 + v_0)e^{-(T+k)} - 1 \qquad = e^T e^{-(T+k)} - 1 \quad = e^{T-T-k} - 1 \quad = e^{-k} - 1$

e. $\qquad v(T - k) + v(T + k) = e^k + e^{-k} - 2$

attempt to express as a square $\qquad = \left(e^{\frac{k}{2}} - e^{-\frac{k}{2}}\right)^2 \quad \geq 0$

so $v(T - k) + v(T + k) \geq 0 \qquad v(T - k) + v(T + k) = e^k + e^{-k} - 2$

Attempt to solve $\frac{d}{dk}(e^k + e^{-k}) = 0 \qquad \Rightarrow k = 0$

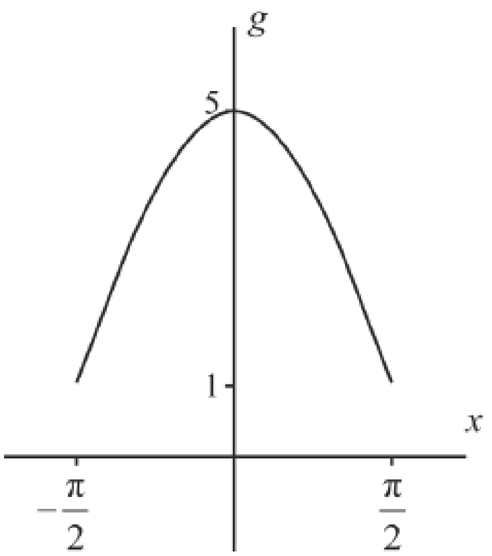
minimum value of 2, (when $k = 0$), hence $e^k + e^{-k} \geq 2$

so $v(T - k) + v(T + k) \geq 0$

18N.1.AHL.TZ0.H_3

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*



concave down and symmetrical over correct domain

indication of maximum and minimum values of the function (correct range)

b. $a = 0$ Award for $a = 0$ only if consistent with their graph.

c.i. $1 \leqslant x \leqslant 5$ Allow FT from their graph.

c.ii. $y = 4\cos x + 1$ $x = 4\cos y + 1$ $\frac{x-1}{4} = \cos y$ $\Rightarrow y = \arccos\left(\frac{x-1}{4}\right)$

$\Rightarrow g^{-1}(x) = \arccos\left(\frac{x-1}{4}\right)$

EXM.1.AHL.TZ0.3

a.

$f(x) = \frac{4x-5}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}$ ***M1A1***

$\Rightarrow 4x - 5 \equiv A(x-2) + B(x-1)$ ***M1A1***

$x = 1 \Rightarrow A = 1$ $x = 2 \Rightarrow B = 3$ ***A1A1***

$f(x) = \frac{1}{x-1} + \frac{3}{x-2}$

[6 marks]

b. $f'(x) = -(x-1)^{-2} - 3(x-2)^{-2}$ ***M1A1***

This is always negative so function is always decreasing. ***R1AG*** ***[3 marks]***

c. $\int_{-1}^{0} \frac{1}{x-1} + \frac{3}{x-2}\,dx = [\ln|x-1| + 3\ln|x-2|]_{-1}^{0}$ ***M1A1***

$= (3\ln 2) - (\ln 2 + 3\ln 3) = 2\ln 2 - 3\ln 3 = \ln\frac{4}{27}$ ***A1A1*** ***[4 marks]***

16N.1.AHL.TZ0.H_13

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$\sin\frac{\pi}{4} + \sin\frac{3\pi}{4} + \sin\frac{5\pi}{4} + \sin\frac{7\pi}{4} + \sin\frac{9\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$ ***(M1)A1***

Note: Award ***M1*** for 5 equal terms with \) + \) or $-$ signs.

[2 marks]

b. $\frac{1-\cos 2x}{2\sin x} \equiv \frac{1-(1-2\sin^2 x)}{2\sin x}$ ***M1*** $\equiv \frac{2\sin^2 x}{2\sin x}$ ***A1*** $\equiv \sin x$ ***AG*** ***[2 marks]***

c. let $\mathrm{P}(n): \sin x + \sin 3x + \ldots + \sin(2n-1)x \equiv \frac{1-\cos 2nx}{2\sin x}$ if $n = 1$

$\mathrm{P}(1): \frac{1-\cos 2x}{2\sin x} \equiv \sin x$ which is true (as proved in part (b)) ***R1***

assume $\mathrm{P}(k)$ true, $\sin x + \sin 3x + \ldots + \sin(2k-1)x \equiv \frac{1-\cos 2kx}{2\sin x}$ ***M1***

Notes: Only award ***M1*** if the words "assume" and "true" appear. Do not award ***M1*** for "let $n = k$" only. Subsequent marks are independent of this ***M1.***

consider $\mathrm{P}(k+1)$:

$\mathrm{P}(k+1): \sin x + \sin 3x + \ldots + \sin(2k-1)x + \sin(2k+1)x \equiv \frac{1-\cos 2(k+1)x}{2\sin x}$

$LHS = \sin x + \sin 3x + \ldots + \sin(2k-1)x + \sin(2k+1)x$ ***M1***

$\equiv \frac{1-\cos 2kx}{2\sin x} + \sin(2k+1)x$ ***A1*** $\equiv \frac{1-\cos 2kx + 2\sin x \sin(2k+1)x}{2\sin x}$

$\equiv \frac{1-\cos 2kx + 2\sin x\cos x\sin 2kx + 2\sin^2 x\cos 2kx}{2\sin x}$ ***M1*** $\equiv \frac{1-\left((1-2\sin^2 x)\cos 2kx - \sin 2x\sin 2kx\right)}{2\sin x}$ ***M1***

$\equiv \frac{1-(\cos 2x\cos 2kx - \sin 2x\sin 2kx)}{2\sin x}$ ***A1*** $\equiv \frac{1-\cos(2kx+2x)}{2\sin x}$ ***A1*** $\equiv \frac{1-\cos 2(k+1)x}{2\sin x}$

so if true for $n = k$, then also true for $n = k+1$

as true for $n = 1$ then true for all $n \in Z^+$ ***R1***

Note: Accept answers using transformation formula for product of sines if steps are shown clearly.

Note: Award ***R1*** only if candidate is awarded at least 5 marks in the previous steps.

[9 marks]

d. **EITHER** $\sin x + \sin 3x = \cos x \Rightarrow \frac{1-\cos 4x}{2\sin x} = \cos x$ ***M1***

$\Rightarrow 1-\cos 4x = 2\sin x\cos x,\ (\sin x \neq 0)$ ***A1*** $\Rightarrow 1-(1-2\sin^2 2x) = \sin 2x$ ***M1***

$\Rightarrow \sin 2x(2\sin 2x - 1) = 0$ ***M1*** $\Rightarrow \sin 2x = 0$ or $\sin 2x = \frac{1}{2}$ ***A1***

$2x = \pi,\ 2x = \frac{\pi}{6}$ and $2x = \frac{5\pi}{6}$ **OR** $\sin x + \sin 3x = \cos x \Rightarrow 2\sin 2x\cos x = \cos x$ ***M1A1***

$\Rightarrow (2\sin 2x - 1)\cos x = 0,\ (\sin x \neq 0)$ $\Rightarrow \sin 2x = \frac{1}{2}$ of $\cos x = 0$

$2x = \frac{\pi}{6},\ 2x = \frac{5\pi}{6}$ and $x = \frac{\pi}{2}$ $\therefore x = \frac{\pi}{2},\ x = \frac{\pi}{12}$ and $x = \frac{5\pi}{12}$

Do not award the final if extra solutions are seen.

18M.1.AHL.TZ1.H_3

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$a = \frac{3}{16}$ and $b = \frac{5}{16}$ ***(M1)A1A1***

[3 marks]

Note: Award ***M1*** for consideration of the possible outcomes when rolling the two dice.

b. $\mathrm{E}(T) = \frac{1+6+15+28}{16} = \frac{25}{8}(=3.125)$ ***(M1)A1***

Note: Allow follow through from part (a) even if probabilities do not add up to 1.

[2 marks]

EXM.1.AHL.TZ0.4

a.

$x^2 + 6x + 10 = x^2 + 6x + 9 + 1 = (x+3)^2 + 1$ ***M1A1***

So the denominator is never zero and thus there are no vertical asymptotes. (or use of discriminant is negative) ***R1***

[3 marks]

b. $x \to \pm\infty, f(x) \to 0$ so the equation of the horizontal asymptote is $y = 0$ ***M1A1***

[2 marks]

c. $\int_0^1 \frac{2x+6}{x^2+6x+10}dx = \left[\ln\left(x^2+6x+10\right)\right]_0^1 = \ln 17 - \ln 10 = \ln\frac{17}{10}$ ***M1A1A1*** ***[3 marks]***

16N.1.AHL.TZ0.H_2

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

x	1	2	4	6
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

A1A1

Note: Award ***A1*** for each correct row.

b. $\mathrm{E}(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{3} + 4 \times \frac{1}{3} + 6 \times \frac{1}{6}$ $= \frac{19}{6} \left(= 3\frac{1}{6} \right)$

If the probabilities in (a) are not values between 0 and 1 or lead to $\mathrm{E}(X) > 6$ award to correct method using the incorrect probabilities; otherwise allow marks.

EXM.1.AHL.TZ0.5

a.

$x = 0 \Rightarrow y = -6$ intercept on the y axes is (0, −6) ***A1***

$2x^2 - 5x - 12 = 0 \Rightarrow (2x+3)(x-4) = 0 \Rightarrow x = \frac{-3}{2} \text{or} 4$ ***M1***

intercepts on the x axes are $\left(\frac{-3}{2}, 0\right)$ and $(4,0)$ ***A1A1***

[4 marks]

b. $x = -2$ ***A1*** ***[1 mark]***

c. $f(x) = 2x - 9 + \frac{6}{x+2}$ ***M1A1*** So equation of asymptote is $y = 2x - 9$ ***M1A1***

[4 marks]

19M.1.AHL.TZ1.H_5

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

let OX = x

METHOD 1

$\frac{\mathrm{d}x}{\mathrm{d}t} = 24$ (or −24) ***(A1)***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} \times \frac{\mathrm{d}\theta}{\mathrm{d}x}$ ***(M1)***

$3\tan\theta = x$ ***A1***

EITHER

$3\sec^2\theta = \frac{\mathrm{d}x}{\mathrm{d}\theta}$ ***A1***

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

OR

$\theta = \arctan\left(\frac{x}{3}\right)$

$\frac{d\theta}{dx} = \frac{1}{3} \times \frac{1}{1+\frac{x^2}{9}}$ ***A1***

$\frac{d\theta}{dt} = 24 \times \frac{1}{3\left(1+\frac{x^2}{9}\right)}$

attempt to substitute for $x = 0$ into their differential equation ***M1***

THEN

$\frac{d\theta}{dt} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept -8 rad s^{-1}.

METHOD 2

$\frac{dx}{dt} = 24$ (or -24) ***(A1)***

$3\tan\theta = x$ ***A1***

attempt to differentiate implicitly with respect to t ***M1***

$3\sec^2\theta \times \frac{d\theta}{dt} = \frac{dx}{dt}$ ***A1***

$\frac{d\theta}{dt} = \frac{24}{3\sec^2\theta}$

attempt to substitute for $\theta = 0$ into their differential equation ***M1***

$\frac{d\theta}{dt} = \frac{24}{3} = 8$ (rad s^{-1}) ***A1***

Note: Accept -8 rad s^{-1}.

Note: Can be done by consideration of CX, use of Pythagoras.

METHOD 3

let the position of the car be at time t be $d - 24t$ from O ***(A1)***

$\tan\theta = \frac{d-24t}{3}\left(= \frac{d}{3} - 8t\right)$ ***M1***

Note: For $\tan\theta = \frac{24t}{3}$ award ***A0M1*** and follow through.

EITHER

attempt to differentiate implicitly with respect to t ***M1***

$\sec^2\theta\frac{d\theta}{dt} = -8$ ***A1*** attempt to substitute for $\theta = 0$ into their differential equation

$\theta = \arctan\left(\frac{d}{3} - 8t\right)$

$\frac{d\theta}{dt} = \frac{8}{1+\left(\frac{d}{3} - 8t\right)^2}$

at O, $t = \frac{d}{24}$

$\frac{d\theta}{dt} = -8$

EXM.1.AHL.TZ0.6

a.

$f'(x) = \frac{(2x-10)(x+1) - \left(x^2 - 10x + 5\right)1}{(x+1)^2}$ ***M1***

$f'(x) = 0 \Rightarrow x^2 + 2x - 15 = 0 \Rightarrow (x+5)(x-3) = 0$ ***M1***

Stationary points are $(-5, -20)$ and $(3, -4)$ ***A1A1***

[4 marks]

b. $x = -1$ ***A1*** ***[1 mark]***

c. Looking at the nature table

x		-5		-1		3	
$f'(x)$	+ve	0	-ve	undefined	-ve	0	+ve

M1A1

$(-5, -20)$ is a max and $(3, -4)$ is a min ***A1A1*** ***[4 marks]***

19M.1.AHL.TZ1.H_6

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

use of symmetry *eg* diagram ***(M1)***

$\mathrm{P}(X > \mu + 5) = 0.2$ ***A1***

[2 marks]

b. **EITHER** $\mathrm{P}\left(X < \mu + 5 \middle| X > \mu - 5\right) = \frac{\mathrm{P}(X > \mu - 5 \cap X < \mu + 5)}{\mathrm{P}(X > \mu - 5)}$ ***(M1)***

$= \frac{\mathrm{P}(\mu - 5 < X < \mu + 5)}{\mathrm{P}(X > \mu - 5)}$ ***(A1)*** $= \frac{0.6}{0.8}$ ***A1A1***

Note: ***A1*** for denominator is independent of the previous ***A*** marks. **OR**

use of diagram ***(M1)***

Only award if the region $\mu - 5 < X < \mu + 5$ is indicated and used.

$\mathrm{P}\left(X > \mu - 5\right) = 0.8$ $\mathrm{P}\left(\mu - 5 < X < \mu + 5\right) = 0.6$

Probabilities can be shown on the diagram. $= \frac{0.6}{0.8}$

$= \frac{3}{4} = (0.75)$

17M.1.AHL.TZ1.H_7

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

use of $u_n = u_1 + (n-1)d$ ***M1***

$(1+2d)^2 = (1+d)(1+5d)$ (or equivalent) ***M1A1***

$d = -2$ ***A1***

[4 marks]

b. $1 + (N-1) \times -2 = -15$ $N = 9$ ***(A1)*** $\sum_{r=1}^{9} u_r = \frac{9}{2}(2 + 8 \times -2)$ ***(M1)***

$= -63$ ***A1*** ***[3 marks]***

17M.1.AHL.TZ2.H_3

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

EITHER

the first three terms of the geometric sequence are 9, $9r$ and $9r^2$ ***(M1)***

$9 + 3d = 9r(\Rightarrow 3 + d = 3r)$ and $9 + 7d = 9r^2$ ***(A1)***

attempt to solve simultaneously ***(M1)***

$9 + 7d = 9\left(\frac{3+d}{3}\right)^2$

OR

the 1^{st}, 4^{th} and 8^{th} terms of the arithmetic sequence are

$9,\ 9 + 3d,\ 9 + 7d$ ***(M1)***

$\frac{9+7d}{9+3d} = \frac{9+3d}{9}$ ***(A1)***

attempt to solve ***(M1)***

$d = 1$

b. $r = \frac{4}{3}$

Accept answers where a candidate obtains d by finding r first. The first two marks in either method for part (a) are awarded for the same ideas and the third mark is awarded for attempting to solve an equation in r.

19M.1.AHL.TZ2.H_7

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

use of at least one "log rule" applied correctly for the first equation ***M1***

$\log_2 6x = \log_2 2 + 2\log_2 y$

$= \log_2 2 + \log_2 y^2$

$= \log_2 (2y^2)$

$\Rightarrow 6x = 2y^2$ ***A1***

use of at least one "log rule" applied correctly for the second equation ***M1***

$\log_6 (15y - 25) = 1 + \log_6 x$

$= \log_6 6 + \log_6 x$

$= \log_6 6x$

$\Rightarrow 15y - 25 = 6x$ ***A1***

attempt to eliminate x (or y) from their two equations ***M1***

$2y^2 = 15y - 25$

$2y^2 - 15y + 25 = 0$

$(2y - 5)(y - 5) = 0$

$x = \frac{25}{12}, y = \frac{5}{2}$,

or $x = \frac{25}{3}, y = 5$

x, y values do not have to be "paired" to gain either of the final two marks.

18M.1.AHL.TZ1.H_5

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$(\ln x)^2 - (\ln 2)(\ln x) - 2(\ln 2)^2 (= 0)$

EITHER

$\ln x = \frac{\ln 2 \pm \sqrt{(\ln 2)^2 + 8(\ln 2)^2}}{2}$ ***M1***

$= \frac{\ln 2 \pm 3\ln 2}{2}$ ***A1***

OR

$(\ln x - 2\ln 2)(\ln x + 2\ln 2)(= 0)$ ***M1A1***

THEN

$\ln x = 2\ln 2$ or $-\ln 2$ ***A1***

$\Rightarrow x = 4$ or $x = \frac{1}{2}$ ***(M1)A1***

Note: ***(M1)*** is for an appropriate use of a log law in either case, dependent on the previous ***M1*** being awarded, **A1** for both correct answers.

solution is $\frac{1}{2} < x < 4$ ***A1***

[6 marks]

16N.1.AHL.TZ0.H_7

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

attempt to form a quadratic in 2^x ***M1***

$(2^x)^2 + 4 \cdot 2^x - 3 = 0$ ***A1***

$2^x = \frac{-4 \pm \sqrt{16+12}}{2} \left(= -2 \pm \sqrt{7}\right)$ ***M1***

$2^x = -2 + \sqrt{7} \left(\text{as } -2 - \sqrt{7} < 0\right)$ ***R1***

$x = \log_2 \left(-2 + \sqrt{7} \right) \left(x = \frac{\ln \left(-2 + \sqrt{7} \right)}{\ln 2} \right)$

Award if final answer is $x = \log_2 \left(-2 + \sqrt{7} \right)$.

17N.1.AHL.TZ0.H_1

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$\log_2(x+3) + \log_2(x-3) = 4$

$\log_2(x^2 - 9) = 4$ ***(M1)***

$x^2 - 9 = 2^4 \ (= 16)$ ***M1A1***

$x^2 = 25$

$x = \pm 5$ ***(A1)***

$x = 5$ ***A1***

[5 marks]

17M.1.AHL.TZ1.H_1

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$\log_2 x - \log_2 5 = 2 + \log_2 3$

collecting at least two log terms ***(M1)***

eg $\log_2 \frac{x}{5} = 2 + \log_2 3$ or $\log_2 \frac{x}{15} = 2$

obtaining a correct equation without logs ***(M1)***

eg $\frac{x}{5} = 12$ **OR** $\frac{x}{15} = 2^2$ ***(A1)***

$x = 60$ ***A1***

[4 marks]

17N.1.AHL.TZ0.H_10

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

METHOD 1

number of possible "deals" $= 4! = 24$ ***A1***

consider ways of achieving "no matches" (Chloe winning):

Selena could deal B, C, D (*ie*, 3 possibilities)

as her first card ***R1***

for each of these matches, there are only 3 possible combinations for the remaining 3 cards ***R1***

so no. ways achieving no matches $= 3 \times 3 = 9$ ***M1A1***

so probability Chloe wins $= \frac{9}{23} = \frac{3}{8}$ ***A1AG***

METHOD 2

number of possible "deals" $= 4! = 24$ ***A1***

consider ways of achieving a match (Selena winning)

Selena card A can match with Chloe card A, giving 6 possibilities for this happening ***R1***

if Selena deals B as her first card, there are only 3 possible combinations for the remaining 3 cards. Similarly for dealing C and dealing D ***R1***

so no. ways achieving one match is $= 6 + 3 + 3 + 3 = 15$ ***M1A1***

so probability Chloe wins $= 1 - \frac{15}{24} = \frac{3}{8}$ ***A1AG***

METHOD 3

systematic attempt to find number of outcomes where Chloe wins (no matches)

(using tree diag. or otherwise) ***M1***

9 found ***A1***

each has probability $\frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times 1$ ***M1***

$= \frac{1}{24}$ ***A1***

their 9 multiplied by their $\frac{1}{24}$ ***M1A1***

$= \frac{3}{8}$ ***AG***

b.i. $X \sim \mathrm{B}\left(50, \frac{3}{8}\right)$ $\mu = np = 50 \times \frac{3}{8} = \frac{150}{8} \left(= \frac{75}{4}\right) (= 18.75)$

b.ii. $\sigma^2 = np(1-p) = 50 \times \frac{3}{8} \times \frac{5}{8} = \frac{750}{64} \left(= \frac{375}{32}\right) (= 11.7)$

17M.1.AHL.TZ1.H_4

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

METHOD 1

total number of arrangements 7! ***(A1)***

number of ways for girls and boys to sit together $= 3! \times 4! \times 2$ ***(M1)(A1)***

Note: Award ***M1A0*** if the 2 is missing.

probability $\frac{3! \times 4! \times 2}{7!}$ ***M1***

Note: Award ***M1*** for attempting to write as a probability.

$\frac{2 \times 3 \times 4! \times 2}{7 \times 6 \times 5 \times 4!}$

$= \frac{2}{35}$ ***A1***

Note: Award ***A0*** if not fully simplified.

METHOD 2

$\frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} + \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4}$ ***(M1)A1A1***

Note: Accept $\frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} \times 2$ or $\frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} \times 2$.

$= \frac{2}{35}$ ***(M1)A1***

Award if not fully simplified.

18N.1.AHL.TZ0.H_2

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

METHOD 1

$\binom{8}{4}$ ***(A1)***

$= \frac{8!}{4!4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 7 \times 2 \times 5$ ***(M1)***

= 70 ***A1***

METHOD 2

recognition that they need to count the teams with 0 boys, 1 boy… 4 boys ***M1***

$1 + \binom{4}{1} \times \binom{4}{3} + \binom{4}{2} \times \binom{4}{2} + \binom{4}{1} \times \binom{4}{3} + 1$

$= 1 + (4 \times 4) + (6 \times 6) + (4 \times 4) + 1$ ***(A1)***

= 70 ***A1***

[3 marks]

b. **EITHER**

recognition that the answer is the total number of teams minus the number of teams with all girls or all boys ***(M1)***

70 − 2 **OR** recognition that the answer is the total of the number of teams with 1 boy,

2 boys, 3 boys ***(M1)***

$\binom{4}{1} \times \binom{4}{3} + \binom{4}{2} \times \binom{4}{2} + \binom{4}{1} \times \binom{4}{3} = (4 \times 4) + (6 \times 6) + (4 \times 4)$ **THEN**

= 68 ***A1*** ***[2 marks]***

18N.1.AHL.TZ0.H_7

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$C_1 : y + x\frac{dy}{dx} = 0$ ***(M1)***

Note: ***M1*** is for use of both product rule and implicit differentiation.

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ ***A1***

Note: Accept $-\frac{4}{x^2}$

$C_2 : 2y\frac{dy}{dx} - 2x = 0$ ***(M1)***

$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$ ***A1***

Note: Accept $\pm\frac{x}{\sqrt{2 + x^2}}$

[4 marks]

b. substituting a and b for x and y ***M1***

product of gradients at P is $\left(-\frac{b}{a}\right)\left(\frac{a}{b}\right) = -1$ or equivalent reasoning ***R1***

Note: The ***R1*** is dependent on the previous ***M1***.

so tangents are perpendicular ***AG*** ***[2 marks]***

16N.1.AHL.TZ0.H_9

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

attempt to differentiate implicitly ***M1***

$3 - \left(4y\frac{dy}{dx} + 2y^2\right)e^{x-1} = 0$ ***A1A1A1***

Note: Award ***A1*** for correctly differentiating each term.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3 \cdot e^{1-x} - 2y^2}{4y}$ ***A1***

Note: This final answer may be expressed in a number of different ways.

[5 marks]

b. $3 - 2y^2 = 2 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm\sqrt{\frac{1}{2}}$ ***A1*** $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3 - 2 \cdot \frac{1}{2}}{\pm 4\sqrt{\frac{1}{2}}} = \pm\frac{\sqrt{2}}{2}$ ***M1***

at $\left(1, \sqrt{\frac{1}{2}}\right)$ the tangent is $y - \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}(x-1)$ and ***A1***

at $\left(1, -\sqrt{\frac{1}{2}}\right)$ the tangent is $y + \sqrt{\frac{1}{2}} = -\frac{\sqrt{2}}{2}(x-1)$

These equations simplify to $y = \pm\frac{\sqrt{2}}{2}x$.

Award if just the positive value of y is considered and just one tangent is found.

19M.1.AHL.TZ1.H_11

a.i.

appreciation that two points distinct from P need to be chosen from each line ***M1***

${}^4C_2 \times {}^3C_2$

=18 ***A1***

[2 marks]

a.ii. **EITHER** consider cases for triangles including P **or** triangles not including P ***M1***

$3 \times 4 + 4 \times {}^3C_2 + 3 \times {}^4C_2$ ***(A1)(A1)*** **Note:** Award ***A1*** for 1st term, ***A1*** for 2nd & 3rd term.

OR

consider total number of ways to select 3 points and subtract those with 3 points on the same line ***M1***

${}^8C_3 - {}^5C_3 - {}^4C_3$ ***(A1)(A1)*** **Note:** Award ***A1*** for 1st term, ***A1*** for 2nd & 3rd term.

56−10−4 **THEN** = 42 ***A1*** ***[4 marks]***

b. **METHOD 1** substitution of (4, 6, 4) into both equations ***(M1)***

$\lambda = 3$ and $\mu = 1$ ***A1A1*** (4, 6, 4) ***AG*** **METHOD 2**

attempting to solve two of the three parametric equations ***M1*** $\lambda = 3$ and $\mu = 1$ ***A1***

check both of the above give (4, 6, 4) ***M1AG***

Note: If they have shown the curve intersects for all three coordinates they only need to check (4,6,4) with one of "λ" or "μ".

[3 marks]

c. $\lambda = 2$ ***A1*** ***[1 mark]***

d. $\overrightarrow{PA} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}, \overrightarrow{PB} = \begin{pmatrix} -5 \\ -6 \\ -2 \end{pmatrix}$ ***A1A1***

Note: Award ***A1A0*** if both are given as coordinates. ***[2 marks]***

e. **METHOD 1** area triangle ABP $= \frac{1}{2}\left|\overrightarrow{PB} \times \overrightarrow{PA}\right|$ ***M1***

$\left(= \frac{1}{2}\left|\begin{pmatrix} -5 \\ -6 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}\right|\right) = \frac{1}{2}\left|\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}\right|$ ***A1*** $= \frac{\sqrt{29}}{2}$ ***A1*** **EITHER**

$\overrightarrow{PC} = 3\overrightarrow{PA}$, $\overrightarrow{PD} = 3\overrightarrow{PB}$ ***(M1)*** area triangle PCD $= 9 \times$ area triangle ABP ***(M1)A1***

$= \frac{9\sqrt{29}}{2}$ ***A1*** **OR** D has coordinates (−11, −12, −2) ***A1***

area triangle PCD $= \frac{1}{2}\left|\overrightarrow{PD} \times \overrightarrow{PC}\right| = \frac{1}{2}\left|\begin{pmatrix} -15 \\ -18 \\ -6 \end{pmatrix} \times \begin{pmatrix} -3 \\ -6 \\ -3 \end{pmatrix}\right|$ ***M1A1***

Note: ***A1*** is for the correct vectors in the correct formula. $= \frac{9\sqrt{29}}{2}$ ***A1*** **THEN**

area of CDBA $= \frac{9\sqrt{29}}{2} - \frac{\sqrt{29}}{2} = 4\sqrt{29}$ ***A1*** **METHOD 2**

D has coordinates (−11, −12, −2) ***A1*** area $= \frac{1}{2}\left|\overrightarrow{CB} \times \overrightarrow{CA}\right| + \frac{1}{2}\left|\overrightarrow{BC} \times \overrightarrow{BD}\right|$ ***M1***

Award for use of correct formula on appropriate non-overlapping triangles.

Different triangles or vectors could be used. $\overrightarrow{CB} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \overrightarrow{CA} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$

$\overrightarrow{CB} \times \overrightarrow{CA} = \begin{pmatrix} -4 \\ 6 \\ -8 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \overrightarrow{BD} = \begin{pmatrix} -10 \\ -12 \\ -4 \end{pmatrix}$ $\overrightarrow{BC} \times \overrightarrow{BD} = \begin{pmatrix} -12 \\ 18 \\ -24 \end{pmatrix}$

Other vectors which might be used are $\overrightarrow{DA} = \begin{pmatrix} 14 \\ 16 \\ 5 \end{pmatrix}, \overrightarrow{BA} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}, \overrightarrow{DC} = \begin{pmatrix} 12 \\ 12 \\ 3 \end{pmatrix}$.

Previous are all dependent on the first .

valid attempt to find a value of $\frac{1}{2}|a \times b|$ independent of triangle chosen.

area $= \frac{1}{2} \times 2 \times \sqrt{29} + \frac{1}{2} \times 6 \times \sqrt{29} \quad = 4\sqrt{29}$

accept $\frac{1}{2}\sqrt{116} + \frac{1}{2}\sqrt{1044}$ or equivalent.

19M.1.AHL.TZ2.H_6

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

attempt to differentiate implicitly ***M1***

$\frac{dy}{dx} = x\sec^2\left(\frac{\pi xy}{4}\right)\left[\left(\frac{\pi}{4}x\frac{dy}{dx} + \frac{\pi}{4}y\right)\right] + \tan\left(\frac{\pi xy}{4}\right)$ ***A1A1***

Note: Award ***A1*** for each term.

attempt to substitute $x = 1,\ y = 1$ into their equation for $\frac{dy}{dx}$ ***M1***

$\frac{dy}{dx} = \frac{\pi}{2}\frac{dy}{dx} + \frac{\pi}{2} + 1$

$\frac{dy}{dx}\left(1 - \frac{\pi}{2}\right) = \frac{\pi}{2} + 1$ ***A1***

$\frac{dy}{dx} = \frac{2+\pi}{2-\pi}$ ***AG***

[5 marks]

b. attempt to use gradient of normal $= \frac{-1}{\frac{dy}{dx}}$ ***(M1)*** $= \frac{\pi - 2}{\pi + 2}$

so equation of normal is $y - 1 = \frac{\pi-2}{\pi+2}(x - 1)$ or $y = \frac{\pi-2}{\pi+2}x + \frac{4}{\pi+2}$ ***A1*** ***[2 marks]***

17M.1.AHL.TZ1.H_8

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

let $P(n)$ be the proposition that $4^n + 15n - 1$ is divisible by 9

showing true for $n = 1$ ***A1***

ie for $n = 1,\ 4^1 + 15 \times 1 - 1 = 18$

which is divisible by 9, therefore $P(1)$ is true

assume $P(k)$ is true so $4^k + 15k - 1 = 9A,\ (A \in Z^+)$ ***M1***

Note: Only award ***M1*** if "truth assumed" or equivalent.

consider $4^{k+1} + 15(k+1) - 1$

$= 4 \times 4^k + 15k + 14$

$= 4(9A - 15k + 1) + 15k + 14$ ***M1***

$= 4 \times 9A - 45k + 18$ ***A1***

$= 9(4A - 5k + 2)$ which is divisible by 9 ***R1***

Award for either the expression or the statement above.

since $P(1)$ is true and $P(k)$ true implies $P(k+1)$ is true, therefore (by the principle of mathematical induction) $P(n)$ is true for $n \in Z^+$

Only award the final if the 2 s have been awarded.

19M.1.AHL.TZ1.H_7

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

attempt at implicit differentiation ***M1***

$3y^2\frac{dy}{dx} + 3y^2 + 6xy\frac{dy}{dx} - 3x^2 = 0$ ***A1A1***

Note: Award ***A1*** for the second & third terms, ***A1*** for the first term, fourth term & RHS equal to zero.

substitution of $\frac{dy}{dx} = 0$ ***M1***

$3y^2 - 3x^2 = 0$

$\Rightarrow y = \pm x$ ***A1***

substitute either variable into original equation ***M1***

$y = x \Rightarrow x^3 = 9 \Rightarrow x = \sqrt[3]{9}$ (or $y^3 = 9 \Rightarrow y = \sqrt[3]{9}$) ***A1***

$y = -x \Rightarrow x^3 = 27 \Rightarrow x = 3$ (or $y^3 = -27 \Rightarrow y = -3$) ***A1***

$(\sqrt[3]{9}, \sqrt[3]{9})$, $(3, -3)$ ***A1***

[9 marks]

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$x^3 + y^3 - 3xy = 0$

$3x^2 + 3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0$ ***M1A1***

Note: Differentiation wrt y is also acceptable.

$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}\left(= \frac{y - x^2}{y^2 - x}\right)$ ***(A1)***

Note: All following marks may be awarded if the denominator is correct, but the numerator incorrect.

$y^2 - x = 0$ ***M1***

EITHER

$x = y^2$

$y^6 + y^3 - 3y^3 = 0$ ***M1A1***

$y^6 - 2y^3 = 0$

$y^3(y^3 - 2) = 0$

$(y \neq 0) \therefore y = \sqrt[3]{2}$ ***A1***

$x = \left(\sqrt[3]{2}\right)^2 \left(= \sqrt[3]{4}\right)$ ***A1***

OR

$x^3 + xy - 3xy = 0$ ***M1***

$x(x^2 - 2y) = 0$

$x \neq 0 \Rightarrow y = \frac{x^2}{2}$ ***A1***

$y^2 = \frac{x^4}{4}$

$x = \frac{x^4}{4}$

$x(x^3 - 4) = 0$

$(x \neq 0) \therefore x = \sqrt[3]{4}$ ***A1***

$y = \frac{(\sqrt[3]{4})^2}{2} = \sqrt[3]{2}$

17M.1.AHL.TZ2.H_8

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{n-1}{2} = \binom{n}{3}$

show true for $n = 3$ ***(M1)***

$\text{LHS} = \binom{2}{2} = 1 \;\; \text{RHS} = \binom{3}{3} = 1$ ***A1***

hence true for $n = 3$

assume true for $n = k$: $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{k-1}{2} = \binom{k}{3}$ ***M1***

consider for $n = k + 1$: $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{k-1}{2} + \binom{k}{2}$ ***(M1)***

$= \binom{k}{3} + \binom{k}{2}$ ***A1***

$= \frac{k!}{(k-3)!3!} + \frac{k!}{(k-2)!2!} \left(= \frac{k!}{3!}\left[\frac{1}{(k-3)!} + \frac{3}{(k-2)!}\right]\right)$ or any correct expression with a visible common factor ***(A1)***

$= \frac{k!}{3!}\left[\frac{k-2+3}{(k-2)!}\right]$ or any correct expression with a common denominator ***(A1)***

$= \frac{k!}{3!}\left[\frac{k+1}{(k-2)!}\right]$

Note: At least one of the above three lines or equivalent must be seen.

$= \frac{(k+1)!}{3!(k-2)!}$ or equivalent ***A1***

$= \binom{k+1}{3}$

Result is true for $k = 3$. If result is true for k it is true for $k + 1$. Hence result is true for all $k \geqslant 3$. Hence proved by induction. ***R1***

Note: In order to award the ***R1*** at least ***[5 marks]*** must have been awarded.

19M.1.AHL.TZ2.H_8

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

attempt to use Pythagoras in triangle OXB ***M1***

$\Rightarrow r^2 = R^2 - (h - R)^2$ ***A1***

substitution of their r^2 into formula for volume of cone $V = \frac{\pi r^2 h}{3}$ ***M1***

$= \frac{\pi h}{3}\left(R^2 - (h - R)^2\right)$

$= \frac{\pi h}{3}\left(R^2 - \left(h^2 + R^2 - 2hR\right)\right)$ ***A1***

Note: This **A** mark is independent and may be seen anywhere for the correct expansion of $(h - R)^2$.

$= \frac{\pi h}{3}\left(2hR - h^2\right)$

$= \frac{\pi}{3}\left(2Rh^2 - h^3\right)$ ***AG***

[4 marks]

b. at max, $\frac{\mathrm{d}V}{\mathrm{d}h} = 0$ ***R1*** $\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi}{3}\left(4Rh - 3h^2\right)$ $\Rightarrow 4Rh = 3h^2$

$\Rightarrow h = \frac{4R}{3}$ (since $h \neq 0$) ***A1*** **EITHER** $V_{\max} = \frac{\pi}{3}\left(2Rh^2 - h^3\right)$ from part (a)

$= \frac{\pi}{3}\left(2R\left(\frac{4R}{3}\right)^2 - \left(\frac{4R}{3}\right)^3\right)$ ***A1*** $= \frac{\pi}{3}\left(2R\frac{16R^2}{9} - \left(\frac{64R^3}{27}\right)\right)$ ***A1*** **OR** $r^2 = R^2 - \left(\frac{4R}{3} - R\right)^2$

$r^2 = R^2 - \frac{R^2}{9} = \frac{8R^2}{9}$ ***A1*** $\Rightarrow V_{\max} = \frac{\pi r^2}{3}\left(\frac{4R}{3}\right)$ $= \frac{4\pi R}{9}\left(\frac{8R^2}{9}\right)$ ***A1*** **THEN** $= \frac{32\pi R^3}{81}$ ***AG***

[4 marks]

18M.1.AHL.TZ1.H_6

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

if $n = 1$

$\text{LHS} = 1; \text{RHS} = 4 - \frac{3}{2^0} = 4 - 3 = 1$ ***M1*** hence true for $n = 1$

assume true for $n = k$ ***M1***

Note: Assumption of truth must be present. Following marks are not dependent on the first two ***M1*** marks.

so $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + k\left(\frac{1}{2}\right)^{k-1} = 4 - \frac{k+2}{2^{k-1}}$

if $n = k + 1$

$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + k\left(\frac{1}{2}\right)^{k-1} + (k+1)\left(\frac{1}{2}\right)^k$

$= 4 - \frac{k+2}{2^{k-1}} + (k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

finding a common denominator for the two fractions ***M1***

$= 4 - \frac{2(k+2)}{2^k} + \frac{k+1}{2^k}$

$= 4 - \frac{2(k+2)-(k+1)}{2^k} = 4 - \frac{k+3}{2^k}\left(= 4 - \frac{(k+1)+2}{2^{(k+1)-1}}\right)$ ***A1***

hence if true for $n = k$ then also true for $n = k + 1$, as true for $n = 1$, so true (for all $n \in Z^+$)

Award the final only if the first four marks have been awarded.

18N.1.AHL.TZ0.H_11

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$(r(\cos\theta + \mathrm{i}\sin\theta))^{24} = 1(\cos 0 + \mathrm{i}\sin 0)$

use of De Moivre's theorem ***(M1)***

$r^{24} = 1 \Rightarrow r = 1$ ***(A1)***

$24\theta = 2\pi n \Rightarrow \theta = \frac{\pi n}{12}, \ (n \in Z)$ ***(A1)***

$0 < \arg(z) < \frac{\pi}{2} \Rightarrow n = 1, 2, 3, 4, 5$

$z = \mathrm{e}^{\frac{\pi \mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{2\pi \mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{3\pi \mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{4\pi \mathrm{i}}{12}}$ or $\mathrm{e}^{\frac{5\pi \mathrm{i}}{12}}$ ***A2***

Note: Award ***A1*** if additional roots are given or if three correct roots are given with no incorrect (or additional) roots.

[5 marks]

b.i. $\operatorname{Re} S = \cos\frac{\pi}{12} + \cos\frac{2\pi}{12} + \cos\frac{3\pi}{12} + \cos\frac{4\pi}{12} + \cos\frac{5\pi}{12}$

$\operatorname{Im} S = \sin\frac{\pi}{12} + \sin\frac{2\pi}{12} + \sin\frac{3\pi}{12} + \sin\frac{4\pi}{12} + \sin\frac{5\pi}{12}$ ***A1*** **Note:** Award ***A1*** for both parts correct.

but $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12}, \ \sin\frac{4\pi}{12} = \cos\frac{2\pi}{12}, \ \sin\frac{3\pi}{12} = \cos\frac{3\pi}{12}, \ \sin\frac{2\pi}{12} = \cos\frac{4\pi}{12}$ and $\sin\frac{\pi}{12} = \cos\frac{5\pi}{12}$ ***M1A1***

$\Rightarrow \operatorname{Re} S = \operatorname{Im} S$ ***AG*** **Note:** Accept a geometrical method. ***[4 marks]***

b.ii. $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$ ***M1A1*** $= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}$ $= \frac{\sqrt{6}+\sqrt{2}}{4}$ ***A1***

[3 marks]

b.iii. $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$ ***(M1)***

Note: Allow alternative methods *eg* $\cos\frac{5\pi}{12} = \sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$.

$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$ ***(A1)*** $\operatorname{Re} S = \cos\frac{\pi}{12} + \cos\frac{2\pi}{12} + \cos\frac{3\pi}{12} + \cos\frac{4\pi}{12} + \cos\frac{5\pi}{12}$

$\operatorname{Re} S = \frac{\sqrt{2}+\sqrt{6}}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} + \frac{1}{2} + \frac{\sqrt{6}-\sqrt{2}}{4}$ ***A1*** $= \frac{1}{2}\left(\sqrt{6} + 1 + \sqrt{2} + \sqrt{3}\right)$ ***A1***

$= \frac{1}{2}\left(1 + \sqrt{2}\right)\left(1 + \sqrt{3}\right)$ $S = \operatorname{Re}(S)(1 + \mathrm{i})$ since $\operatorname{Re} S = \operatorname{Im} S$, ***R1***

$S = \frac{1}{2}\left(1 + \sqrt{2}\right)\left(1 + \sqrt{3}\right)(1 + \mathrm{i})$ ***AG*** ***[4 marks]***

17M.1.AHL.TZ2.H_4

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$s = t + \cos 2t$

$\frac{\mathrm{d}s}{\mathrm{d}t} = 1 - 2\sin 2t$ ***M1A1***

$= 0$ ***M1***

$\Rightarrow \sin 2t = \frac{1}{2}$

$t_1 = \frac{\pi}{12}(s),\ t_2 = \frac{5\pi}{12}(s)$ ***A1A1***

Note: Award ***A0A0*** if answers are given in degrees.

[5 marks]

b. $s = \frac{\pi}{12} + \cos\frac{\pi}{6}\left(s = \frac{\pi}{12} + \frac{\sqrt{3}}{2}(m)\right)$ ***A1A1*** ***[2 marks]***

16N.1.AHL.TZ0.H_12

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

(i) **METHOD 1**

$1+\omega+\omega^2=\frac{1-\omega^3}{1-\omega}=0$ ***A1***

as $\omega\neq 1$ ***R1***

METHOD 2

solutions of $1-\omega^3=0$ are $\omega=1,\ \omega=\frac{-1\pm\sqrt{3}\mathrm{i}}{2}$ ***A1***

verification that the sum of these roots is 0 ***R1***

(ii) $1+\omega^*+(\omega^*)^2=0$ ***A2***

[4 marks]

b. $(\omega-3\omega^2)(\omega^2-3\omega)=-3\omega^4+10\omega^3-3\omega^2$ ***M1A1*** **EITHER**

$=-3\omega^2(\omega^2+\omega+1)+13\omega^3$ ***M1*** $=-3\omega^2\times 0+13\times 1$ ***A1*** **OR**

$=-3\omega+10-3\omega^2=-3(\omega^2+\omega+1)+13$ ***M1*** $=-3\times 0+13$ ***A1*** **OR**

substitution by $\omega=\frac{-1\pm\sqrt{3}\mathrm{i}}{2}$ in any form ***M1*** numerical values of each term seen ***A1***

THEN $=13$ ***AG*** ***[4 marks]***

c. $|p|=|q|\Rightarrow\sqrt{1^2+3^2}=\sqrt{x^2+(2x+1)^2}$ ***(M1)(A1)*** $5x^2+4x-9=0$ ***A1***

$(5x+9)(x-1)=0$ ***(M1)*** $x=1,\ x=-\frac{9}{5}$ ***A1*** ***[5 marks]***

d. $pq=(1-3\mathrm{i})\left(x+(2x+1)\mathrm{i}\right)=(7x+3)+(1-x)\mathrm{i}$ ***M1A1***

$\mathrm{Re}(pq)+8<\left(\mathrm{Im}(pq)\right)^2\Rightarrow(7x+3)+8<(1-x)^2$ ***M1*** $\Rightarrow x^2-9x-10>0$ ***A1***

$\Rightarrow(x+1)(x-10)>0$ ***M1*** $x<-1,\ x>10$ ***A1*** ***[6 marks]***

17M.1.AHL.TZ1.H_2

a.i.

$z_1=2\mathrm{cis}\left(\frac{\pi}{3}\right)$ and $z_2=\sqrt{2}\mathrm{cis}\left(\frac{\pi}{4}\right)$ ***A1A1***

Note: Award ***A1A0*** for correct moduli and arguments found, but not written in mod-arg form.

$|w| = \sqrt{2}$ ***A1***

[3 marks]

a.ii. $z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ ***A1A1***

Note: Award ***A1A0*** for correct moduli and arguments found, but not written in mod-arg form.

$\arg w = \frac{\pi}{12}$ ***A1***

Allow from incorrect answers for z_1 and z_2 in modulus-argument form.

b. $\sin\left(\frac{\pi n}{12}\right) = 0$ $\arg(w^n) = \pi$ $\frac{n\pi}{12} = \pi$

$\therefore n = 12$

19M.1.AHL.TZ1.H_3

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$A = P$

use of the correct formula for area and arc length ***(M1)***

perimeter is $r\theta + 2r$ ***(A1)***

Note: ***A1*** independent of previous ***M1***.

$\frac{1}{2}r^2(1) = r(1) + 2r$ ***A1***

$r^2 - 6r = 0$

$r = 6$ (as $r > 0$) ***A1***

Note: Do not award final ***A1*** if $r = 0$ is included.

[4 marks]

17M.1.AHL.TZ1.H_3

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

METHOD 1

use of $\sec^2 x = \tan^2 x + 1$ ***M1***

$\tan^2 x + 2\tan x + 1 = 0$

$(\tan x + 1)^2 = 0$ ***(M1)***

$\tan x = -1$ ***A1***

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$ ***A1A1***

METHOD 2

$\frac{1}{\cos^2 x} + \frac{2\sin x}{\cos x} = 0$ ***M1***

$1 + 2\sin x\cos x = 0$

$\sin 2x = -1$ ***M1A1***

$2x = \frac{3\pi}{2}, \frac{7\pi}{2}$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$ ***A1A1***

Award if extra solutions given or if solutions given in degrees (or both).

17M.1.AHL.TZ2.H_5

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

C represents the complex number $1 - 2i$ ***A2***

D represents the complex number $3 + 2i$ ***A2***

[4 marks]

19M.1.AHL.TZ2.H_10

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

mode is 0 ***A1***

[1 mark]

b.i. attempt at integration by parts ***(M1)*** $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}, dv = dx$

$= x\arcsin x - \int \frac{x dx}{\sqrt{1-x^2}}$ ***A1*** $= x\arcsin x + \sqrt{1-x^2} \,(+c)$ ***A1*** ***[3 marks]***

b.ii. $\int_0^1 (\pi - \arcsin x)\, dx = \left[\pi x - x\arcsin x - \sqrt{1-x^2}\right]_0^1$ ***A1***

$= \left(\pi - \frac{\pi}{2} - 0\right) - (0 - 0 - 1) = \frac{\pi}{2} + 1 = \frac{\pi+2}{2}$ ***A1*** $\int_0^1 k(\pi - \arcsin x)\,dx = 1$ ***(M1)***

Note: This line can be seen (or implied) anywhere.

Note: Do not allow ***FT A*** marks from bi to bii. $k\left(\frac{\pi+2}{2}\right) = 1 \quad \Rightarrow k = \frac{2}{2+\pi}$ ***AG***

[3 marks]

c.i. attempt to use product rule to differentiate ***M1***

$\frac{dy}{dx} = x\arcsin x + \frac{x^2}{2\sqrt{1-x^2}} - \frac{1}{4\sqrt{1-x^2}} - \frac{x^2}{4\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{4}$ ***A2***

Note: Award ***A2*** for all terms correct, ***A1*** for 4 correct terms.

$= x\arcsin x + \frac{2x^2}{4\sqrt{1-x^2}} - \frac{1}{4\sqrt{1-x^2}} - \frac{x^2}{4\sqrt{1-x^2}} + \frac{1-x^2}{4\sqrt{1-x^2}}$ ***A1***

Award for equivalent combination of correct terms over a common denominator.

$= x\arcsin x$

c.ii. $E(X) = k\int_0^1 x(\pi - \arcsin x)\,dx \qquad = k\int_0^1 (\pi x - x\arcsin x)\,dx$

$= k\left[\frac{\pi x^2}{2} - \frac{x^2}{2}\arcsin x + \frac{1}{4}\arcsin x - \frac{x}{4}\sqrt{1-x^2}\right]_0^1$

Award for first term, for next 3 terms. $= k\left[\left(\frac{\pi}{2} - \frac{\pi}{4} + \frac{\pi}{8}\right) - (0)\right]$

$= \left(\frac{2}{2+\pi}\right)\frac{3\pi}{8} \qquad = \frac{3\pi}{4(\pi+2)}$

19N.1.AHL.TZ0.H_1

a.

$p = 1 - \frac{1}{2} - \frac{1}{5} - \frac{1}{5}$ ***(M1)***

$= \frac{1}{10}$ ***A1***

[2 marks]

b. attempt to find $E(X)$ ***(M1)*** $\frac{1}{2} + 1 + 2 + \frac{N}{10} = 10$ ***A1*** $\Rightarrow N = 65$ ***A1***

Note: Do not allow FT in part (b) if their p is outside the range $0 < p < 1$. ***[3 marks]***

18N.1.AHL.TZ0.H_4

a.

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

an attempt at a valid method *eg* by inspection or row reduction ***(M1)***

$2 \times R_2 = R_1 \Rightarrow 2a = -1$

$\Rightarrow a = -\frac{1}{2}$ ***A1***

[2 marks]

b. using elimination or row reduction to eliminate one variable ***(M1)***

correct pair of equations in 2 variables, such as $\left.\begin{array}{c} 5x + 10y = 25 \\ 5x + 12y = 4 \end{array}\right\}$ ***A1***

Note: Award ***A1*** for $z = 0$ and one other equation in two variables.

attempting to solve for these two variables ***(M1)*** $x = 26,\ y = -10.5,\ z = 0$ ***A1A1***

Note: Award ***A1A0*** for only two correct values, and ***A0A0*** for only one.

Award marks in part (b) for equivalent steps seen in part (a).

19N.1.AHL.TZ0.H_10

a.i.

attempt to use quotient rule (or equivalent) ***(M1)***

$$f'(x) = \frac{(x^2 - 1)(2) - (2x - 4)(2x)}{(x^2 - 1)^2}$$ ***A1***

$$= \frac{-2x^2 + 8x - 2}{(x^2 - 1)^2}$$

[2 marks]

a.ii. $f'(x) = 0$ simplifying numerator (may be seen in part (i)) ***(M1)***

$\Rightarrow x^2 - 4x + 1 = 0$ or equivalent quadratic equation ***A1*** **EITHER**

use of quadratic formula $\Rightarrow x = \frac{4 \pm \sqrt{12}}{2}$ ***A1*** **OR** use of completing the square

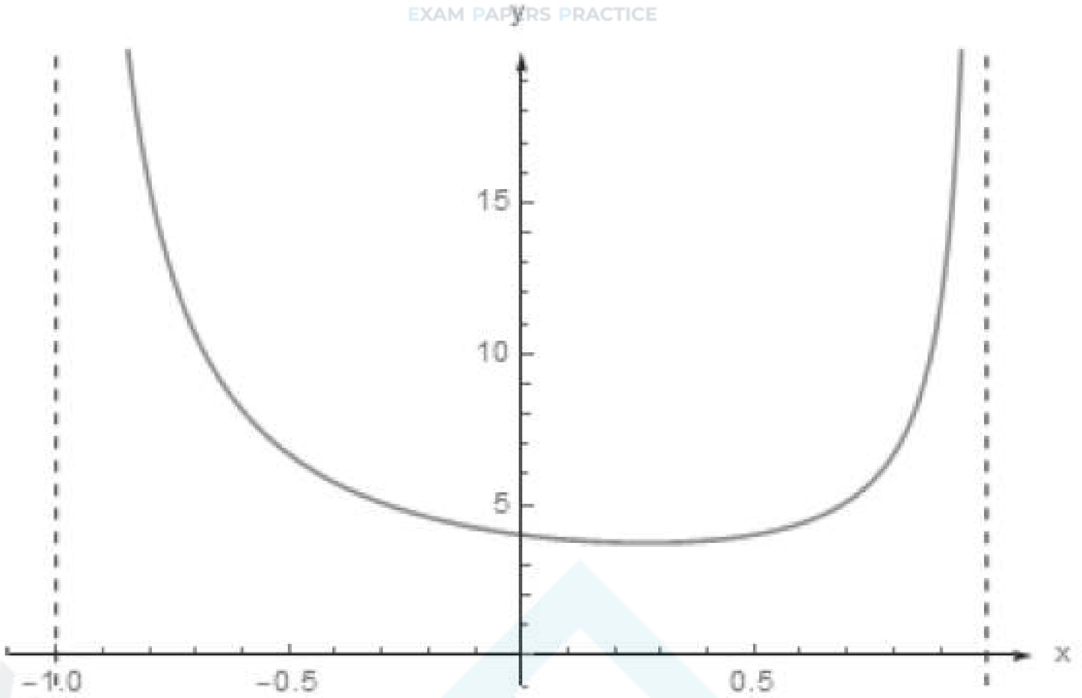
$(x - 2)^2 = 3$ ***A1*** **THEN** $x = 2 - \sqrt{3}$ (since $2 + \sqrt{3}$ is outside the domain) ***AG***

Note: Do not condone verification that $x = 2 - \sqrt{3} \Rightarrow f'(x) = 0$.

Do not award the final ***A1*** as follow through from part (i). ***[3 marks]***

b.i. (0, 4) ***A1*** ***[1 mark]***

b.ii. $2x - 4 = 0 \Rightarrow x = 2$ ***A1*** outside the domain ***R1*** ***[2 marks]*** b.iii.



award for concave up curve over correct domain with one minimum point in the first quadrant
award for approaching $x = \pm 1$ asymptotically

c. valid attempt to combine fractions (using common denominator)

$\frac{3(x-1)-(x+1)}{(x+1)(x-1)}$ $= \frac{3x-3-x-1}{x^2-1}$ $= \frac{2x-4}{x^2-1}$

d. $f(x) = 4 \Rightarrow 2x - 4 = 4x^2 - 4$ $(x = 0 \text{ or})\ x = \frac{1}{2}$

area under the curve is $\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ $= \int_0^{\frac{1}{2}} \frac{3}{x+1} - \frac{1}{x-1}\mathrm{d}x$

Ignore absence of, or incorrect limits up to this point.

$= [3\ln|x+1| - \ln|x-1|]_0^{\frac{1}{2}}$ $= 3\ln\frac{3}{2} - \ln\frac{1}{2}(-0)$ $= \ln\frac{27}{4}$

area is $2 - \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ or $\int_0^{\frac{1}{2}} 4\mathrm{d}x - \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ $= 2 - \ln\frac{27}{4}$ $= \ln\frac{4\mathrm{e}^2}{27}$

$\left(\Rightarrow v = \frac{4\mathrm{e}^2}{27}\right)$

19N.1.AHL.TZ0.H_11

a.i.

$\overrightarrow{\mathrm{AV}} = \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix}$ ***A1***

$$\overrightarrow{AB}\times\overrightarrow{AV}=\begin{pmatrix}0\\10\\-10\end{pmatrix}\times\begin{pmatrix}p\\p\\p-10\end{pmatrix}=\begin{pmatrix}10(p-10)+10p\\-10p\\-10p\end{pmatrix}$$ ***A1***

$$=\begin{pmatrix}20p-100\\-10p\\-10p\end{pmatrix}=-10\begin{pmatrix}10-2p\\p\\p\end{pmatrix}$$ ***AG***

$$\overrightarrow{AC}\times\overrightarrow{AV}=\begin{pmatrix}10\\0\\-10\end{pmatrix}\times\begin{pmatrix}p\\p\\p-10\end{pmatrix}=\begin{pmatrix}10p\\100-20p\\10p\end{pmatrix}\left(=10\begin{pmatrix}p\\10-2p\\p\end{pmatrix}\right)$$ ***A1***

[3 marks]

a.ii. attempt to find a scalar product ***M1***

$$-10\begin{pmatrix}10-2p\\p\\p\end{pmatrix}\cdot10\begin{pmatrix}p\\10-2p\\p\end{pmatrix}=100(3p^2-20p)$$

OR $-\begin{pmatrix}10-2p\\p\\p\end{pmatrix}\cdot\begin{pmatrix}p\\10-2p\\p\end{pmatrix}=3p^2-20p$ ***A1***

attempt to find magnitude of either $\overrightarrow{AB}\times\overrightarrow{AV}$ or $\overrightarrow{AC}\times\overrightarrow{AV}$ ***M1***

$$\left|-10\begin{pmatrix}10-2p\\p\\p\end{pmatrix}\right|=\left|10\begin{pmatrix}p\\10-2p\\p\end{pmatrix}\right|=10\sqrt{(10-2p)^2+2p^2}$$ ***A1***

$100(3p^2-20p)=100\left(\sqrt{(10-2p)^2+2p^2}\right)^2\cos\theta$ $\quad\cos\theta=\frac{3p^2-20p}{(10-2p)^2+2p^2}$ ***A1***

Note: Award ***A1*** for any intermediate step leading to the correct answer.

$=\frac{p(3p-20)}{6p^2-40p+100}$ ***AG*** **Note:** Do not allow FT marks from part (a)(i). ***[8 marks]***

b.i. $p(3p-20)=0\Rightarrow p=0$ or $p=\frac{20}{3}$ ***M1A1***

coordinates are (0, 0, 0) and $\left(\frac{20}{3},\frac{20}{3},\frac{20}{3}\right)$ ***A1***

Note: Do not allow column vectors for the final ***A*** mark. ***[3 marks]***

b.ii. two points are mirror images in the plane ***[1 mark]***
or opposite sides of the plane
or equidistant from the plane
or the line connecting the two Vs is perpendicular to the plane ***R1***

c.i. geometrical consideration or attempt to solve $-1=\frac{p(3p-20)}{6p^2-40p+100}$ ***(M1)***

$p=\frac{10}{3},\theta=\pi$ or $\theta=180°$ ***A1A1*** ***[3 marks]*** c.ii. $p \to \infty \Rightarrow \cos\theta \to \frac{1}{2}$ hence the asymptote has equation $\theta = \frac{\pi}{3}$

19N.1.AHL.TZ0.H_5

a.

METHOD 1

$|z| = \sqrt[4]{4}\left(= \sqrt{2}\right)$ ***(A1)***

$\arg\left(z_1\right) = \frac{\pi}{4}$ ***(A1)***

first solution is $1 + \mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/− their components) ***(M1)***

other solutions are $-1 + \mathrm{i},\ -1 - \mathrm{i},\ 1 - \mathrm{i}$ ***A1***

METHOD 2

$z^4 = -4$

$\left(a + \mathrm{i}b\right)^4 = -4$

attempt to expand and equate **both** reals and imaginaries. ***(M1)***

$a^4 + 4a^3b\mathrm{i} - 6a^2b^2 - 4ab^3\mathrm{i} + b^4 = -4$

$\left(a^4 - 6a^4 + a^4 = -4 \Rightarrow \right)a = \pm 1$ ***and*** $\left(4a^3b - 4ab^3 = 0 \Rightarrow \right)a = \pm b$ ***(A1)***

first solution is $1 + \mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/− their components) ***(M1)***

other solutions are $-1 + \mathrm{i},\ -1 - \mathrm{i},\ 1 - \mathrm{i}$ ***A1***

[5 marks]

b. complete method to find area of 'rectangle' ***(M1)*** $= 4$ ***A1*** ***[2 marks]***

19N.1.AHL.TZ0.H_9

a.

$\cos 105^\circ = \cos\left(180^\circ - 75^\circ\right) = -\cos 75^\circ$ ***R1***

$= -q$ ***AG***

Note: Accept arguments using the unit circle or graphical/diagrammatical considerations.

[1 mark]

b. $\text{AD} = \text{CD} \Rightarrow \text{C}\hat{\text{A}}\text{D} = 45^\circ$ ***A1*** valid method to find $\text{B}\hat{\text{A}}\text{C}$ ***(M1)***

for example: $\text{BC} = r \Rightarrow \text{B}\hat{\text{C}}\text{A} = 60^\circ$ $\Rightarrow \text{B}\hat{\text{A}}\text{C} = 30^\circ$ ***A1***

hence $\text{B}\hat{\text{A}}\text{D} = 45^\circ + 30^\circ = 75^\circ$ ***AG*** ***[3 marks]***

c.i. $\text{AB} = r\sqrt{3}$, $\text{AD} = (\text{CD}) = r\sqrt{2}$ ***A1A1*** applying cosine rule ***(M1)***

$\text{BD}^2 = (r\sqrt{3})^2 + (r\sqrt{2})^2 - 2(r\sqrt{3})(r\sqrt{2})\cos 75^\circ$ ***A1*** $= 3r^2 + 2r^2 - 2r^2\sqrt{6}\cos 75^\circ$

$= 5r^2 - 2r^2 q\sqrt{6}$ ***AG*** ***[4 marks]***

c.ii. $\text{B}\hat{\text{C}}\text{D} = 105^\circ$ ***(A1)*** attempt to use cosine rule on ΔBCD ***(M1)***

$\text{BD}^2 = r^2 + (r\sqrt{2})^2 - 2r(r\sqrt{2})\cos 105^\circ$ $= 3r^2 + 2r^2 q\sqrt{2}$ ***A1*** ***[3 marks]***

d. $5r^2 - 2r^2 q\sqrt{6} = 3r^2 + 2r^2 q\sqrt{2}$ ***(M1)(A1)*** $2r^2 = 2r^2 q(\sqrt{6} + \sqrt{2})$ ***A1***

Note: Award ***A1*** for any correct intermediate step seen using only two terms.

$q = \frac{1}{\sqrt{6} + \sqrt{2}}$ Do not award the final if follow through is being applied.

19N.1.AHL.TZ0.H_3

attempt to eliminate a variable (or attempt to find det A) ***M1***

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 1 & 3 & -1 & 4 \\ 3 & -5 & a & b \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 7 & -3 & 3 \\ 0 & -14 & a+3 & b-12 \end{array}\right)$$ (or $\det A = 14(a-3)$)

(or two correct equations in two variables) ***A1***

$$\rightarrow \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 7 & -3 & 3 \\ 0 & 0 & a-3 & b-6 \end{array}\right)$$ (or solving $\det A = 0$)

(or attempting to reduce to one variable, e.g. $(a-3)z = b-6$) ***M1***

$a = 3, b \neq 6$ ***A1A1***

[5 marks]

19N.1.AHL.TZ0.H_4

attempt to use $\cos(2A + B) = \cos 2A \cos B - \sin 2A \sin B$ (may be seen later) ***M1*** attempt to use any double angle formulae (seen anywhere) ***M1***

attempt to find either $\sin A$ or $\cos B$ (seen anywhere) ***M1***

$\cos A = \frac{2}{3} \Rightarrow \sin A\left(= \sqrt{1 - \frac{4}{9}}\right) = \frac{\sqrt{5}}{3}$ ***(A1)***

$\sin B = \frac{1}{3} \Rightarrow \cos B\left(= \sqrt{1 - \frac{1}{9}} = \frac{\sqrt{8}}{3}\right) = \frac{2\sqrt{2}}{3}$ ***A1***

$\cos 2A\left(= 2\cos^2 A - 1\right) = -\frac{1}{9}$ ***A1***

$\sin 2A\left(= 2\sin A\cos A\right) = \frac{4\sqrt{5}}{9}$ ***A1***

So $\cos(2A + B) = \left(-\frac{1}{9}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{4\sqrt{5}}{9}\right)\left(\frac{1}{3}\right)$

$= -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$ ***AG***

[7 marks]

18M.1.AHL.TZ1.H_8

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

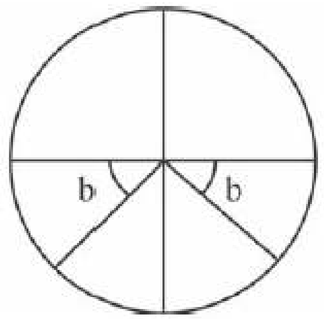
$\sin 2x = -\sin b$

EITHER

$\sin 2x = \sin(-b)$ or $\sin 2x = \sin(\pi + b)$ or $\sin 2x = \sin(2\pi - b)$ … ***(M1)(A1)***

Note: Award ***M1*** for any one of the above, ***A1*** for having final two.

OR



(M1)(A1)

Note: Award ***M1*** for one of the angles shown with b clearly labelled, ***A1*** for both angles shown. Do not award ***A1*** if an angle is shown in the second quadrant and subsequent ***A1*** marks not awarded.

THEN

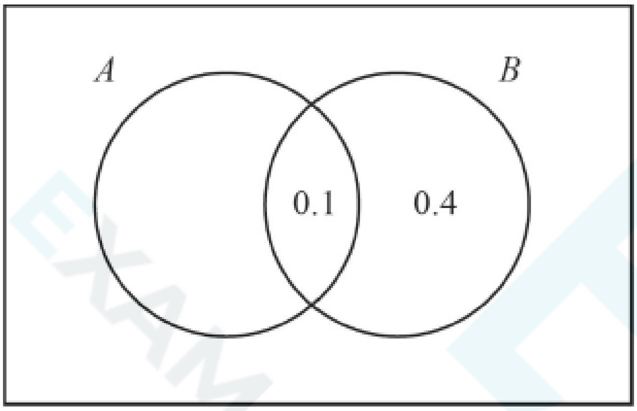
$2x = \pi + b$ or $2x = 2\pi - b$ ***(A1)(A1)***

$x = \frac{\pi}{2} + \frac{b}{2}, x = \pi - \frac{b}{2}$

18N.1.AHL.TZ0.H_1

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*



(M1)

Note: Award ***M1*** for a Venn diagram with at least one probability in the correct region.

EITHER

$\mathrm{P}\left(A \cap B'\right) = 0.3$ ***(A1)***

$\mathrm{P}(A \cup B) = 0.3 + 0.4 + 0.1 = 0.8$ ***A1***

OR

$\mathrm{P}(B) = 0.5$ ***(A1)***

$\mathrm{P}(A \cup B) = 0.5 + 0.4 - 0.1 = 0.8$ ***A1***

[3 marks]

b. **METHOD 1** $\mathrm{P}(A)\mathrm{P}(B) = 0.4 \times 0.5$ ***(M1)*** = 0.2 ***A1***

statement that their $\mathrm{P}(A)\mathrm{P}(B) \neq \mathrm{P}(A \cap B)$ ***R1***

Note: Award ***R1*** for correct reasoning from their value. $\Rightarrow A, B$ not independent ***AG***

METHOD 2 $\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{0.1}{0.5}$ ***(M1)*** = 0.2 ***A1***

statement that their $\mathrm{P}(A|B) \neq \mathrm{P}(A)$ ***R1***

Note: Award ***R1*** for correct reasoning from their value. $\Rightarrow A, B$ not independent ***AG***

Accept equivalent argument using $\mathrm{P}(B|A) = 0.25$.

19M.1.AHL.TZ1.H_4

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

EITHER

$\frac{5\sqrt{15}}{2} = \frac{1}{2} \times 4 \times 5\sin\theta$ ***A1***

OR

height of triangle is $\frac{5\sqrt{15}}{4}$ if using 4 as the base or $\sqrt{15}$ if using 5 as the base ***A1***

THEN

$\sin\theta = \frac{\sqrt{15}}{4}$ ***AG***

[1 mark]

b. let the third side be x $x^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \times \cos\theta$ ***M1***

valid attempt to find $\cos\theta$ ***(M1)***

Note: Do not accept writing $\cos\left(\arcsin\left(\frac{\sqrt{15}}{4}\right)\right)$ as a valid method. $\cos\theta = \pm\sqrt{1 - \frac{15}{16}}$

$= \frac{1}{4}, -\frac{1}{4}$ ***A1A1*** $x^2 = 16 + 25 - 2 \times 4 \times 5 \times \pm\frac{1}{4}$ $x = \sqrt{31}$ or $\sqrt{51}$ ***A1A1*** ***[6 marks]***

17N.1.AHL.TZ0.H_9

a.i.

$\overrightarrow{\mathrm{OF}} = \frac{1}{7}\boldsymbol{b}$ ***A1***

[1 mark]

a.ii. $\overrightarrow{\mathrm{AF}} = \overrightarrow{\mathrm{OF}} - \overrightarrow{\mathrm{OA}}$ ***(M1)*** $= \frac{1}{7}\boldsymbol{b} - \boldsymbol{a}$ ***A1*** ***[2 marks]***

b.i. $\overrightarrow{\mathrm{OD}} = \boldsymbol{a} + \lambda\left(\frac{1}{7}b - a\right) \left(= (1-\lambda)a + \frac{\lambda}{7}b\right)$ ***M1A1*** ***[2 marks]***

b.ii. $\overrightarrow{\mathrm{OD}} = \frac{1}{2}\boldsymbol{a} + \mu\left(-\frac{1}{2}a + b\right) \left(= \left(\frac{1}{2} - \frac{\mu}{2}\right)a + \mu b\right)$ ***M1A1*** ***[2 marks]***

c. equating coefficients: ***M1*** $\frac{\lambda}{7} = \mu,\ 1 - \lambda = \frac{1-\mu}{2}$ ***A1*** solving simultaneously: ***M1***

$\lambda = \frac{7}{13},\ \mu = \frac{1}{13}$ ***A1AG*** ***[4 marks]***

d. $\overrightarrow{CD} = \frac{1}{13}\overrightarrow{CB} \quad = \frac{1}{13}\left(b - \frac{1}{2}a\right) \left(= -\frac{1}{26}a + \frac{1}{13}b\right)$

e. $\text{area } \Delta ACD = \frac{1}{2}CD \times AC \times \sin A\hat{C}B$

$\text{area } \Delta ACB = \frac{1}{2}CB \times AC \times \sin A\hat{C}B$ ratio $\frac{\text{area } \Delta ACD}{\text{area } \Delta ACB} = \frac{CD}{CB} = \frac{1}{13}$

$k = \frac{\text{area } \Delta OAB}{\text{area } \Delta CAD} = \frac{13}{\text{area } \Delta CAB} \times \text{area } \Delta OAB \qquad = 13 \times 2 = 26$

$\text{area } \Delta OAB = \frac{1}{2}|a \times b|$ $\text{area } \Delta CAD = \frac{1}{2}\left|\overrightarrow{CA} \times \overrightarrow{CD}\right|$ or $\frac{1}{2}\left|\overrightarrow{CA} \times \overrightarrow{AD}\right|$

$= \frac{1}{2}\left|\frac{1}{2}a \times \left(-\frac{1}{26}a + \frac{1}{13}b\right)\right| \quad = \frac{1}{2}\left|\frac{1}{2}a \times \left(-\frac{1}{26}a\right) + \frac{1}{2}a \times \frac{1}{13}b\right|$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{13}|a \times b| \left(= \frac{1}{52}|a \times b|\right)$ $\text{area } \Delta OAB = k(\text{area } \Delta CAD)$ $\frac{1}{2}|a \times b| = k\frac{1}{52}|a \times b|$

$k = 26$

16N.1.AHL.TZ0.H_10

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

METHOD 1

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ***M1***

$= P(A) + P(A \cap B) + P(A' \cap B) - P(A \cap B)$ ***M1A1***

$= P(A) + P(A' \cap B)$ ***AG***

METHOD 2

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ***M1***

$= P(A) + P(B) - P(A \mid B) \times P(B)$ ***M1***

$= P(A) + \left(1 - P(A \mid B)\right) \times P(B)$

$= P(A) + P(A' \mid B) \times P(B)$ ***A1***

$= P(A) + P(A' \cap B)$ ***AG***

[3 marks]

b. (i) use $P(A \cup B) = P(A) + P(A' \cap B)$ and $P(A' \cap B) = P(B \mid A')P(A')$ ***(M1)***

$\frac{4}{9} = P(A) + \frac{1}{6}\left(1 - P(A)\right)$ ***A1*** $8 = 18P(A) + 3\left(1 - P(A)\right)$ ***M1*** $P(A) = \frac{1}{3}$ ***AG***

(ii) **METHOD 1** $P(B) = P(A \cap B) + P(A' \cap B)$ ***M1*** $= P(B \mid A)P(A) + P(B \mid A')P(A')$ ***M1***

$= \frac{1}{3} \times \frac{1}{3} + \frac{1}{6} \times \frac{2}{3} = \frac{2}{9}$ ***A1*** **METHOD 2**

$P(A \cap B) = P(B \mid A)P(A) \Rightarrow P(A \cap B) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ ***M1***

$\mathrm{P}(B) = \mathrm{P}(A \cup B) + \mathrm{P}(A \cap B) - \mathrm{P}(A)$ $\mathrm{P}(B) = \frac{4}{9} + \frac{1}{9} - \frac{1}{3} = \frac{2}{9}$

18M.1.AHL.TZ2.H_9

a.i.

a pair of opposite sides have equal length and are parallel ***R1***

hence ABCD is a parallelogram ***AG***

[1 mark]

a.ii. attempt to rewrite the given information in vector form ***M1*** ***b*** − ***a*** = ***c*** − ***d*** ***A1***

rearranging ***d*** − ***a*** = ***c*** − ***b*** ***M1*** hence $\vec{AD} = \vec{BC}$ ***AG***

Note: Candidates may correctly answer part i) by answering part ii) correctly and then deducing there
are two pairs of parallel sides.

[3 marks]

b. **EITHER** use of $\vec{AB} = \vec{DC}$ ***(M1)*** $\begin{pmatrix} 2 \\ -3 \\ p+3 \end{pmatrix} = \begin{pmatrix} q+1 \\ 1-r \\ 4 \end{pmatrix}$ ***A1A1*** **OR**

use of $\vec{AD} = \vec{BC}$ ***(M1)*** $\begin{pmatrix} -2 \\ r-2 \\ 1 \end{pmatrix} = \begin{pmatrix} q-3 \\ 2 \\ 2-p \end{pmatrix}$ ***A1A1*** **THEN**

attempt to compare coefficients of ***i***, ***j***, and ***k*** in their equation or statement to that effect ***M1***

clear demonstration that the given values satisfy their equation ***A1*** ***[5 marks]***
$p = 1$, $q = 1$, $r = 4$ ***AG***

c. attempt at computing $\vec{AB} \times \vec{AD}$ (or equivalent) ***M1*** $\begin{pmatrix} -11 \\ -10 \\ -2 \end{pmatrix}$ ***A1***

area $= \left|\vec{AB} \times \vec{AD}\right| \left(= \sqrt{225} \right)$ ***(M1)*** = 15 ***A1*** ***[4 marks]***

d. valid attempt to find $\vec{OM} = \left(\frac{1}{2}(a+c) \right)$ ***(M1)*** $\begin{pmatrix} 1 \\ \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$ ***A1*** the equation is

$= \begin{pmatrix} 1 \\ \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} + t\begin{pmatrix} 11 \\ 10 \\ 2 \end{pmatrix}$ or equivalent

: Award maximum if ' = ...' (or equivalent) is not seen.

e. attempt to obtain the equation of the plane in the form $ax + by + cz = d$

$11x + 10y + 2z = 25$ for right hand side, for left hand side.

f.i. putting two coordinates equal to zero

$X\left(\frac{25}{11}, 0, 0\right), Y\left(0, \frac{5}{2}, 0\right), Z\left(0, 0, \frac{25}{2}\right)$

f.ii. $YZ = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{25}{2}\right)^2}$ $= \sqrt{\frac{325}{2}}\left(= \frac{5\sqrt{104}}{4} = \frac{5\sqrt{26}}{2}\right)$

17M.1.AHL.TZ1.H_10

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

attempt to equate integral to 1 (may appear later) ***M1***

$k\int_0^6 \sin\left(\frac{\pi x}{6}\right)dx = 1$

correct integral ***A1***

$k\left[-\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^6 = 1$

substituting limits ***M1***

$-\frac{6}{\pi}(-1-1) = \frac{1}{k}$

$k = \frac{\pi}{12}$ ***A1***

[4 marks]

b.i. mean $= 3$ ***A1*** **Note:** Award ***A1A0A0*** for three equal answers in $(0,\ 6)$.

[1 mark]

b.ii. median $= 3$ ***A1*** **Note:** Award ***A1A0A0*** for three equal answers in $(0,\ 6)$.

[1 mark]

b.iii. mode $= 3$ ***A1*** **Note:** Award ***A1A0A0*** for three equal answers in $(0,\ 6)$.

c.i. $\frac{\pi}{12}\int_0^2 \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x$ $= \frac{\pi}{12}\left[-\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^2$

Accept without the $\frac{\pi}{12}$ at this stage if it is added later.

$\frac{\pi}{12}\left[-\frac{6}{\pi}\left(\cos\frac{\pi}{3} - 1\right)\right]$ $= \frac{1}{4}$

c.ii. from (c)(i) $Q_1 = 2$

as the graph is symmetrical about the middle value $x = 3 \Rightarrow Q_3 = 4$

so interquartile range is $4 - 2$ $= 2$

d. $P(X \leqslant 4 \mid X \geqslant 3) = \frac{P(3 \leqslant X \leqslant 4)}{P(X \geqslant 3)}$ $= \frac{\frac{1}{4}}{\frac{1}{2}}$ $= \frac{1}{2}$

19M.1.AHL.TZ2.H_2

a.i.

$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$ ***A1***

Note: Accept row vectors or equivalent.

[1 mark]

a.ii. $\overrightarrow{AC} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ ***A1*** **Note:** Accept row vectors or equivalent. ***[1 mark]***

b. **METHOD 1** attempt at vector product using $\overrightarrow{AB}$ and $\overrightarrow{AC}$. ***(M1)***

$\pm(2\boldsymbol{i} + 6\boldsymbol{j} + 6\boldsymbol{k})$ ***A1*** attempt to use area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ ***M1*** $= \frac{\sqrt{76}}{2}\left(= \sqrt{19}\right)$ ***A1***

METHOD 2 attempt to use $\overrightarrow{AB} \cdot \overrightarrow{AC} = \left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\cos\theta$ ***M1***

$\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \sqrt{0^2 + 2^2 + (-2)^2}\sqrt{3^2 + 1^2 + (-2)^2}\cos\theta$ $6 = \sqrt{8}\sqrt{14}\cos\theta$ ***A1***

$\cos\theta = \frac{6}{\sqrt{8}\sqrt{14}} = \frac{6}{\sqrt{112}}$ attempt to use area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\sin\theta$ ***M1***

$= \frac{1}{2}\sqrt{8}\sqrt{14}\sqrt{1 - \frac{36}{112}}\left(= \frac{1}{2}\sqrt{8}\sqrt{14}\sqrt{\frac{76}{112}}\right)$ $= \frac{\sqrt{76}}{2}\left(= \sqrt{19}\right)$ ***A1*** ***[4 marks]***

17M.1.AHL.TZ1.H_5

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$\overrightarrow{AB} \times \overrightarrow{AD} = -\boldsymbol{i} + 10\boldsymbol{j} - 7\boldsymbol{k}$ ***M1A1***

area $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = \sqrt{1^2 + 10^2 + 7^2}$

$= 5\sqrt{6}\left(\sqrt{150}\right)$ ***A1***

[3 marks]

b. **METHOD 1** $\overrightarrow{AB} \cdot \overrightarrow{AD} = -4 - 2 - 6$ ***M1A1*** $= -12$

considering the sign of the answer $\overrightarrow{AB} \cdot \overrightarrow{AD} < 0$, therefore angle $D\hat{A}B$ is obtuse ***M1***

(as it is a parallelogram), $A\hat{B}C$ is acute ***A1*** ***[4 marks]*** **METHOD 2**

$\overrightarrow{BA} \cdot \overrightarrow{BC} = +4 + 2 + 6$ ***M1A1*** $= 12$ considering the sign of the answer ***M1***

$\overrightarrow{BA} \cdot \overrightarrow{BC} > 0 \Rightarrow A\hat{B}C$ is acute ***A1*** ***[4 marks]***

18M.1.AHL.TZ2.H_1

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$\cos\theta = \frac{(3i - 4j - 5k) \cdot (5i - 4j + 3k)}{|3i - 4j - 5k||5i - 4j + 3k|}$ ***(M1)***

$= \frac{16}{\sqrt{50}\sqrt{50}}$ ***A1A1***

Note: ***A1*** for correct numerator and ***A1*** for correct denominator.

$= \frac{8}{25}\left(= \frac{16}{50} = 0.32\right)$ ***A1***

[4 marks]

17N.1.AHL.TZ0.H_2

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$\overrightarrow{AB} = \begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix}$ ***(A1)***

$\boldsymbol{r} = \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix} + \lambda\begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix}$ or $\boldsymbol{r} = \begin{pmatrix} 6 \\ -5 \\ 11 \end{pmatrix} + \lambda\begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix}$ ***M1A1***

Note: Award ***M1A0*** if $\boldsymbol{r}$ = is not seen (or equivalent).

[3 marks]

b. substitute line L in Π: $4(6\lambda) - 3(3 - 8\lambda) + 2(-6 + 17\lambda) = 20$ ***M1*** $82\lambda = 41$

$\lambda = \frac{1}{2}$ ***(A1)*** $= \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ \frac{5}{2} \end{pmatrix}$ so coordinate is $\left(3, -1, \frac{5}{2}\right)$ ***A1***

Accept coordinate expressed as position vector $\begin{pmatrix} 3 \\ -1 \\ \frac{5}{2} \end{pmatrix}$.

18N.1.AHL.TZ0.H_9

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

METHOD 1

$n = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2b \\ 0 \\ b-1 \end{pmatrix}$ ***(M1)***

$= \begin{pmatrix} b-1 \\ 4b \\ -2b \end{pmatrix}$ ***(M1)A1***

(0, 0, 0) on Π so $(b-1)x + 4by - 2bz = 0$ ***(M1)A1***

METHOD 2

using equation of the form $px + qy + rz = 0$ ***(M1)***

(0, 1, 2) on $\Pi \Rightarrow q + 2r = 0$

($2b$, 0, b − 1) on $\Pi \Rightarrow 2bp + r(b-1) = 0$ ***(M1)A1***

Note: Award ***(M1)A1*** for both equations seen.

solve for p, q and r ***(M1)***

$(b-1)x+4by-2bz=0$ ***A1***

[5 marks]

b. M has coordinates $\left(b, 0, \frac{b-1}{2}\right)$ ***(A1)*** $\boldsymbol{r} = \begin{pmatrix} b \\ 0 \\ \frac{b-1}{2} \end{pmatrix} + \lambda \begin{pmatrix} b-1 \\ 4b \\ -2b \end{pmatrix}$ ***M1A1***

Note: Award ***M1A0*** if ***r*** = (or equivalent) is not seen.

Note: Allow equivalent forms such as $\frac{x-b}{b-1} = \frac{y}{4b} = \frac{2z-b+1}{-4b}$. ***[3 marks]***

c. $x = z = 0$ Award for either $x = 0$ or $z = 0$ or both.

$b + \lambda(b-1) = 0$ and $\frac{b-1}{2} - 2\lambda b = 0$ attempt to eliminate λ

$\Rightarrow -\frac{b}{b-1} = \frac{b-1}{4b}$ $-4b^2 = (b-1)^2$

consideration of the signs of LHS and RHS

the LHS is negative and the RHS must be positive (or equivalent statement)

$-4b^2 = b^2 - 2b + 1$ $\Rightarrow 5b^2 - 2b + 1 = 0$ $\Delta = (-2)^2 - 4 \times 5 \times 1 = -16(<0)$

∴ no real solutions so no point of intersection

$x = z = 0$ Award for either $x = 0$ or $z = 0$ or both.

$b + \lambda(b-1) = 0$ and $\frac{b-1}{2} - 2\lambda b = 0$ attempt to eliminate b

$\Rightarrow \frac{\lambda}{1+\lambda} = \frac{1}{1-4\lambda}$ $-4\lambda^2 = 1\left(\Rightarrow \lambda^2 = -\frac{1}{4}\right)$

consideration of the signs of LHS and RHS

there are no real solutions (or equivalent statement)

so no point of intersection

16N.1.AHL.TZ0.H_4

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$\boldsymbol{a} \times \boldsymbol{b} = -12\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$ ***(M1)A1***

[2 marks]

b. **METHOD 1** $-12x - 2y - 3z = d$ ***M1*** $-12 \times 1 - 2 \times 0 - 3(-1) = d$ ***(M1)***

$\Rightarrow d = -9$ $\quad -12x - 2y - 3z = -9$ (or $12x + 2y + 3z = 9$)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -12 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -12 \\ -2 \\ -3 \end{pmatrix}$$

$-12x - 2y - 3z = -9$ (or $12x + 2y + 3z = 9$)

18M.1.AHL.TZ1.H_10

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

recognising normal to plane or attempting to find cross product of two vectors lying in the plane ***(M1)***

for example, $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ***(A1)***

$\Pi_1 : x + z = 1$ ***A1***

[3 marks]

b. **EITHER** $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1 = \sqrt{2}\sqrt{2}\cos\theta$ ***M1A1*** **OR**

$\left| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right| = \sqrt{3} = \sqrt{2}\sqrt{2}\sin\theta$ ***M1A1***

Note: ***M1*** is for an attempt to find the scalar or vector product of the two normal vectors.

$\Rightarrow \theta = 60° \left(= \frac{\pi}{3} \right)$ ***A1*** angle between faces is $20° \left(= \frac{2\pi}{3} \right)$ ***A1*** ***[4 marks]***

c. $\overrightarrow{DB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ or $\overrightarrow{BD} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ ***(A1)*** $\Pi_3 : x + y - z = k$ ***(M1)***

$\Pi_3 : x + y - z = 0$ ***A1*** ***[3 marks]***

d. **METHOD 1** line AD : ($\mathbf{r}$ =)$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ ***M1A1***

intersects Π_3 when $\lambda - (1 - \lambda) = 0$ ***M1*** so $\lambda = \frac{1}{2}$ ***A1***

hence P is the midpoint of AD ***AG*** **METHOD 2**

midpoint of AD is (0.5, 0, 0.5) ***(M1)A1*** substitute into $x + y - z = 0$ ***M1***

0.5 + 0.5 − 0.5 = 0 ***A1*** hence P is the midpoint of AD ***AG*** ***[4 marks]***

e. **METHOD 1** $\text{OP} = \frac{1}{\sqrt{2}}, \text{O}\hat{\text{P}}\text{Q} = 90°, \text{O}\hat{\text{Q}}\text{P} = 60°$ ***A1A1A1*** $\text{PQ} = \frac{1}{\sqrt{6}}$ ***A1***

area $= \frac{1}{2\sqrt{12}} = \frac{1}{4\sqrt{3}} = \frac{\sqrt{3}}{12}$ ***A1*** **METHOD 2** line BD : ($\mathbf{r}$ =)$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$\Rightarrow \lambda = \frac{2}{3}$

$\vec{OQ} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$ area $= \frac{1}{2}\left|\vec{OP} \times \vec{OQ}\right|$ $\vec{OP} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$

: This is dependent on . area $= \frac{\sqrt{3}}{12}$

16N.1.AHL.TZ0.H_1

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

METHOD 1

for eliminating one variable from two equations ***(M1)***

eg, $\begin{cases} (x + y + z = 3) \\ 2x + 2z = 8 \\ 2x + 3z = 11 \end{cases}$ ***A1A1***

for finding correctly one coordinate

eg, $\begin{cases} (x + y + z = 3) \\ (2x + 2z = 8) \\ z = 3 \end{cases}$ ***A1***

for finding correctly the other two coordinates ***A1***

$\Rightarrow \begin{cases} x = 1 \\ y = -1 \\ z = 3 \end{cases}$

the intersection point has coordinates $(1,\ -1,\ 3)$

METHOD 2

for eliminating two variables from two equations or using row reduction ***(M1)***

eg, $\begin{cases} (x + y + z = 3) \\ -2 = 2 \\ z = 3 \end{cases}$ **or** $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$ ***A1A1***

for finding correctly the other coordinates ***A1A1***

$\Rightarrow \begin{cases} x = 1 \\ y = -1 \\ (z = 3) \end{cases}$ **or** $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$

the intersection point has coordinates $(1,\ -1,\ 3)$

METHOD 3

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2$$

attempt to use Cramer's rule

$$x = \frac{\begin{vmatrix} 3 & 1 & 1 \\ 5 & -1 & 1 \\ 6 & 1 & 2 \end{vmatrix}}{-2} = \frac{-2}{-2} = 1$$

$$y = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 1 & 5 & 1 \\ 1 & 6 & 2 \end{vmatrix}}{-2} = \frac{2}{-2} = -1$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 5 \\ 1 & 1 & 6 \end{vmatrix}}{-2} = \frac{-6}{-2} = 3$$

Award only if candidate attempts to determine at least one of the variables using this method.

18M.1.AHL.TZ2.H_3

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

equating sum of probabilities to 1 ($p + 0.5 - p + 0.25 + 0.125 + p^3 = 1$) ***M1***

$p^3 = 0.125 = \frac{1}{8}$

$p = 0.5$ ***A1***

[2 marks]

b.i. $\mu = 0 \times 0.5 + 1 \times 0 + 2 \times 0.25 + 3 \times 0.125 + 4 \times 0.125$ ***M1*** $= 1.375 \left(= \frac{11}{8}\right)$ ***A1***

[2 marks]

b.ii. $P(X > \mu) = P(X = 2) + P(X = 3) + P(X = 4)$ ***(M1)*** $= 0.5$ ***A1***

Note: Do not award follow through ***A*** marks in (b)(i) from an incorrect value of p.

Note: Award ***M*** marks in both (b)(i) and (b)(ii) provided no negative probabilities, and provided a numerical value for μ has been found.

[2 marks]

21M.1.AHL.TZ1.7

METHOD 1

other two roots are $a - bi$ and $b - ai$ ***A1***

sum of roots $= -4$ and product of roots $= 400$ ***A1***

attempt to set sum of four roots equal to -4 or 4 OR
attempt to set product of four roots equal to 400 ***M1***

$a + bi + a - bi + b + ai + b - ai = -4$

$2a + 2b = -4 (\Rightarrow a + b = -2)$ ***A1***

$(a + bi)(a - bi) \quad (b + ai)(b - ai) = 400$

$a^2 + b^{2^2} = 400$ ***A1***

$a^2 + b^2 = 20$

attempt to solve simultaneous equations ***(M1)***

$a = 2$ or $a = -4$ ***A1A1***

METHOD 2

other two roots are $a - bi$ and $b - ai$ ***A1***

$z - a + biz - a - biz - b + aiz - b - ai = 0$ ***A1***

$z - a^2 + b^2 z - b^2 + a^2 = 0$

$z^2 - 2az + a^2 + b^2 z^2 - 2bz + b^2 + a^2 = 0$ ***A1***

Attempt to equate coefficient of z^3 and constant with the given quartic equation ***M1***

$-2a - 2b = 4$ and $a^2 + b^{2^2} = 400$ ***A1***

attempt to solve simultaneous equations ***(M1)***

$a = 2$ or $a = -4$ ***A1A1***

[8 marks]

21M.1.AHL.TZ1.11

a.i.

$\frac{-1+1}{2} = 0 = 3 - 3$ ***A1***

the point $(-1, \quad 0, \quad 3)$ lies on L_1. ***AG***

[1 mark]

a.ii. attempt to set equal to a parameter or rearrange cartesian form ***(M1)***

$\frac{x+1}{2} = y = 3 - z = \lambda \Rightarrow x = 2\lambda - 1, \quad y = \lambda, \quad z = 3 - \lambda$ OR $\frac{x+1}{2} = \frac{y-0}{1} = \frac{z-3}{-1}$

correct direction vector $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ or equivalent seen in vector form ***(A1)***

$r = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if $= r$ is omitted.

[3 marks]

b. attempt to use the scalar product formula ***(M1)***

$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix} = \pm\sqrt{6}\sqrt{a^2+2}\ \cos\ 45°$ ***(A1)(A1)***

Note: Award ***A1*** for LHS and ***A1*** for RHS

$2a + 2 = \frac{\pm\sqrt{6}\sqrt{a^2+2}\sqrt{2}}{2} \Rightarrow 2a + 2 = \pm\sqrt{3}\sqrt{a^2+2}$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS $4a^2 + 8a + 4 = 3a^2 + 6$ ***A1***

$a^2 + 8a - 2 = 0$ ***M1*** attempt to solve their quadratic

$a = \frac{-8 \pm \sqrt{64+8}}{2} = \frac{-8 \pm \sqrt{72}}{2} = -4 \pm 3\sqrt{2}$ ***A1*** ***[8 marks]***

c. **METHOD 1** attempt to equate the parametric forms of L_1 and L_2 ***(M1)***

$2\lambda - 1 = ta$
$\lambda = 1 + t$ ***A1***
$3 - \lambda = 2 - t$

attempt to solve equations by eliminating λ or t ***(M1)***

$2 + 2t - 1 = ta \Rightarrow 1 = ta - 2$ or $2\lambda - 1 = \lambda - 1a \Rightarrow a - 1 = \lambda a - 2$

Solutions exist unless $a - 2 = 0$ $\quad k = 2$ ***A1***

Note: This ***A1*** is independent of the following marks. $t = \frac{1}{a-2}$ or $\lambda = \frac{a-1}{a-2}$ ***A1***

A has coordinates $\left(\frac{a}{a-2}, \ 1 + \frac{1}{a-2}, \ 2 - \frac{1}{a-2}\right) = \left(\frac{a}{a-2}, \frac{a-1}{a-2}, \frac{2a-5}{a-2}\right)$ ***A2***

Note: Award ***A1*** for any two correct coordinates seen or final answer in vector form.

METHOD 2

no unique point of intersection implies direction vectors of L_1 and L_2 parallel

$k = 2$ ***A1*** **Note:** This ***A1*** is independent of the following marks.

attempt to equate the parametric forms of L_1 and L_2

$$2\lambda - 1 = ta$$
$$\lambda = 1 + t$$
$$3 - \lambda = 2 - t$$

attempt to solve equations by eliminating λ or t

$2 + 2t - 1 = ta \Rightarrow 1 = ta - 2$ or $2\lambda - 1 = \lambda - 1a \Rightarrow a - 1 = \lambda a - 2$ $\quad t = \frac{1}{a-2}$ or $\lambda = \frac{a-1}{a-2}$

A has coordinates $\frac{a}{a-2}, \quad 1 + \frac{1}{a-2}, \quad 2 - \frac{1}{a-2} \quad = \frac{a}{a-2}, \quad \frac{a-1}{a-2}, \quad \frac{2a-5}{a-2}$

Award for any two correct coordinates seen or final answer in vector form.

20N.1.AHL.TZ0.H_1

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$\mathrm{E}X = 0 \times p + 1 \times \frac{1}{4} + 2 \times \frac{1}{6} + 3q = \frac{19}{12}$ ***(M1)***

$\Rightarrow \frac{1}{4} + \frac{1}{3} + 3q = \frac{19}{12}$

$q = \frac{1}{3}$ ***A1***

$p + \frac{1}{4} + \frac{1}{6} + q = 1$ ***(M1)***

$\Rightarrow p + q = \frac{7}{12}$

$p = \frac{1}{4}$ ***A1***

[4 marks]

20N.1.AHL.TZ0.H_10

a.i.

$f'x = 3ax^2 + 2bx + c$ ***A1***

[1 mark]

a.ii. since f^{-1} does not exist, there must be two turning points ***R1***

$(\Rightarrow f'x = 0$ has more than one solution) using the discriminant $\Delta > 0$ ***M1***

$4b^2 - 12ac > 0$ ***A1*** $b^2 - 3ac > 0$ ***AG***

[4 marks]

b.i. **METHOD 1** $b^2 - 3ac = -3^2 - 3 \times \frac{1}{2} \times 6$ ***M1*** $= 9 - 9$ $= 0$ ***A1*** hence g^{-1} exists ***AG*** **METHOD 2** $g'(x) = \frac{3}{2}x^2 - 6x + 6$ ***M1*** $\Delta = (-6)^2 - 4 \times \frac{3}{2} \times 6$

$\Delta = 36 - 36 = 0 \Rightarrow$ there is (only) one point with gradient of 0 and this must be a point of inflexion (since $g(x)$ is a cubic.) ***R1***

hence g^{-1} exists ***AG***

[2 marks]

b.ii. $p = \frac{1}{2}$ ***A1*** $(x - 2)^3 = x^3 - 6x^2 + 12x - 8$ ***(M1)***

$\frac{1}{2}(x^3 - 6x^2 + 12x - 8) = \frac{1}{2}x^3 - 3x^2 + 6x - 4$ $g(x) = \frac{1}{2}(x - 2)^3 - 4 \Rightarrow q = -4$ ***A1***

[3 marks]

b.iii. $x = \frac{1}{2}(y - 2)^3 - 4$ ***(M1)***

Note: Interchanging x and y can be done at any stage.

$\sqrt[3]{2x + 4} = y - 2$ $y = \sqrt[3]{2x + 4} + 2$ $g^{-1}(x) = \sqrt[3]{2x + 4} + 2$ ***A1***

$2x + 4 = (y - 2)^3$ ***(M1)***

Note: $g^{-1}(x) = \ldots$ must be seen for the final ***A*** mark. ***[3 marks]***

c. translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, ***A1***

Note: This can be seen anywhere.

EITHER

a stretch scale factor $\frac{1}{2}$ parallel to the y-axis then a translation through $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ ***A2***

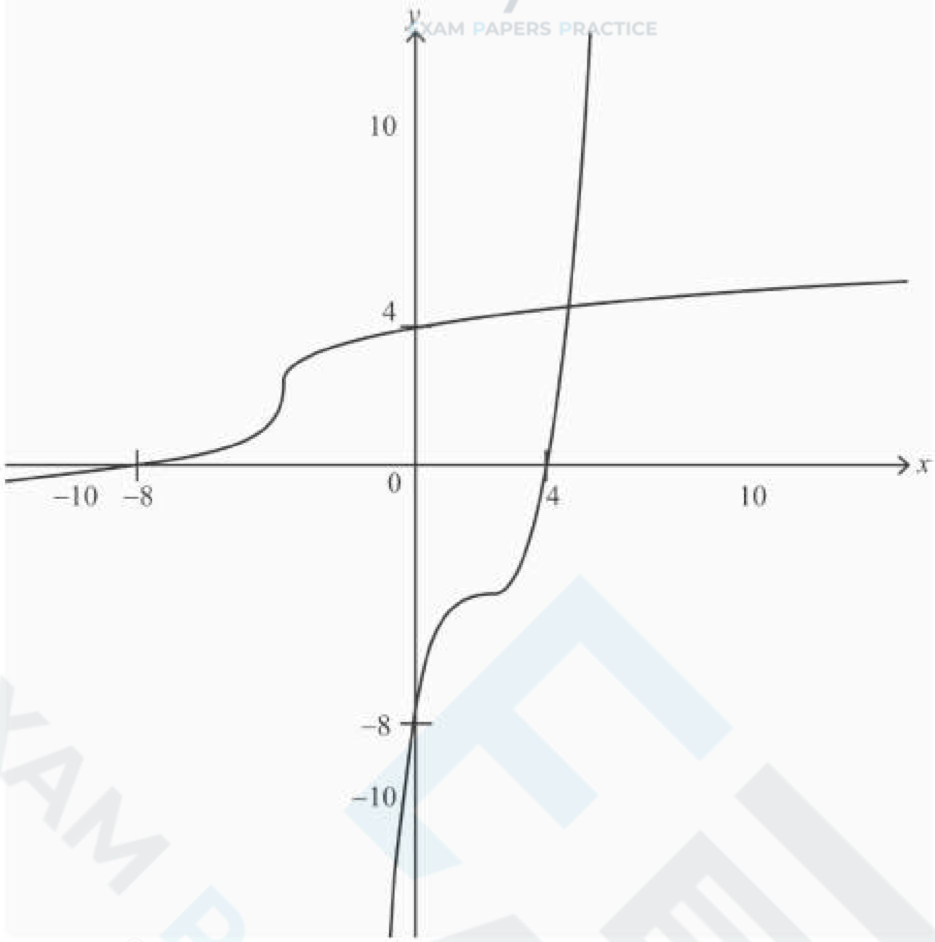
OR

a translation through $\begin{pmatrix} 0 \\ -8 \end{pmatrix}$ then a stretch scale factor $\frac{1}{2}$ parallel to the y-axis ***A2***

Note: Accept 'shift' for translation, but do not accept 'move'. Accept 'scaling' for 'stretch'.

[3 marks]

d.



Award for correct 'shape' of g (allow non-stationary point of inflexion)
Award for each correct intercept of g
Award for attempt to reflect their graph in $y = x$, for completely correct g^{-1} including intercepts

20N.1.AHL.TZ0.H_11

a.

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

attempt at implicit differentiation ***M1***

$2y\frac{\mathrm{dy}}{\mathrm{d}x} = \cos\ xyx\frac{\mathrm{dy}}{\mathrm{d}x} + y$ ***A1M1A1***

Note: Award ***A1*** for LHS, ***M1*** for attempt at chain rule, ***A1*** for RHS.

$2y\frac{\mathrm{dy}}{\mathrm{d}x} = x\frac{\mathrm{dy}}{\mathrm{d}x}\cos\ xy + y\ \cos\ xy$

$2y\frac{\mathrm{dy}}{x} - x\frac{\mathrm{dy}}{\mathrm{d}x}\cos\ xy = y\ \cos\ xy$

$\frac{dy}{dx}2y - x \quad \cos \quad xy = y \quad \cos \quad xy$ ***M1***

Note: Award ***M1*** for collecting derivatives and factorising.

$\frac{dy}{dx} = \frac{y \quad \cos \quad xy}{2y - x \quad \cos \quad xy}$ ***AG***

[5 marks]

b. setting $\frac{dy}{dx} = 0 \quad y \quad \cos \quad xy = 0$ ***(M1)*** $y \neq 0 \Rightarrow \cos \quad xy = 0$ ***A1***

$\Rightarrow \sin \quad xy = \pm\sqrt{1 - \cos^2 \quad xy} = \pm\sqrt{1 - 0} = \pm 1$ ***OR*** $xy = 2n + 1\frac{\pi}{2} \quad n \in \mathbb{Z}$ ***OR*** $xy = \frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \ldots$ ***A1***

Note: If they offer values for xy, award ***A1*** for at least two correct values in two different 'quadrants' and no incorrect values.

$\Rightarrow y^2 = 1$ ***A1*** $\Rightarrow y = \pm 1$ ***AG***

$y^2 \quad = \sin \quad xy > 0$ ***R1*** ***[5 marks]***

c.

$y = \pm 1 \Rightarrow 1 = \sin \quad \pm x \Rightarrow \sin \quad x = \pm 1$ OR $y = \pm 1 \Rightarrow 0 = \cos \quad \pm x \Rightarrow \cos \quad x = 0$

$\sin \quad x = 1 \Rightarrow \frac{\pi}{2}, \quad 1, \quad \frac{5\pi}{2}, \quad 1$ $\quad \sin \quad x = -1 \Rightarrow \frac{3\pi}{2}, \quad -1, \quad \frac{7\pi}{2}, \quad -1$

Allow 'coordinates' expressed as $x = \frac{\pi}{2}, \quad y = 1$ for example.

Each of the marks may be awarded independently and are not dependent on being awarded.

Mark only the candidate's first two attempts for each case of $\sin \quad x$.

20N.1.AHL.TZ0.H_2

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$x = 0 \Rightarrow y = 1$ ***(A1)***

appreciate the need to find $\frac{dy}{dx}$ ***(M1)***

$\frac{dy}{dx} = 2e^{2x} - 3$ ***A1***

$x = 0 \Rightarrow \frac{dy}{dx} = -1$ ***A1***

$\frac{y - 1}{x - 0} = -1 \quad y = 1 - x$ ***A1***

20N.1.AHL.TZ0.H_4

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

substituting $z = x + \mathrm{i}y$ **and** $z^* = x - \mathrm{i}y$ ***M1***

$\frac{2x + \mathrm{i}y}{3 - x - \mathrm{i}y} = \mathrm{i}$

$2x + 2\mathrm{i}y = -y + \mathrm{i}3 - x$

equate real and imaginary: ***M1***

$y = -2x$ AND $2y = 3 - x$ ***A1***

Note: If they multiply top and bottom by the conjugate, the equations $6x - 2x^2 + 2y^2 = 0$ and $6y - 4xy = 3 - x^2 + y^2$ may be seen. Allow for ***A1***.

solving simultaneously:

$x = -1, \quad y = 2 \qquad z = -1 + 2\mathrm{i}$ ***A1A1***

[5 marks]

20N.1.AHL.TZ0.H_5

** This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.*

$u_5 = 4 + 4d = \log_2 625$ ***(A1)***

$4d = \log_2 625 - 4$

attempt to write an integer (*eg* 4 or 1) in terms of $\log_2$ ***M1***

$4d = \log_2 625 - \log_2 16$

attempt to combine two logs into one ***M1***

$4d = \log_2 \frac{625}{16}$

$d = \frac{1}{4}\log_2 \frac{625}{16}$

attempt to use power rule for logs ***M1***

$d = \log_2 \frac{625}{16}^{\frac{1}{4}}$

$d = \log_2 \frac{5}{2}$

Award method marks in any order.

21N.1.AHL.TZ0.7

a.

attempt to use discriminant $b^2 - 4ac > 0$ ***M1***

$2p^2 - 43p1 - p > 0$

$16p^2 - 12p > 0$ ***(A1)***

$p4p - 3 > 0$

attempt to find critical values $p = 0, \quad p = \frac{3}{4}$ ***M1***

recognition that discriminant > 0 ***(M1)***

$p < 0$ or $p > \frac{3}{4}$ ***A1***

Note: Condone 'or' replaced with 'and', a comma, or no separator

[5 marks]

b. $p = 4 \Rightarrow 12x^2 + 8x - 3 = 0$

valid attempt to use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (or equivalent) ***M1*** $x = \frac{-8 \pm \sqrt{208}}{24}$ $x = \frac{-2 \pm \sqrt{13}}{6}$

$a = -2$ ***A1*** ***[2 marks]***

21N.1.AHL.TZ0.8

$\frac{dy}{dx} + \frac{2y}{x} = \frac{\ln 2x}{x^2}$ ***(M1)***

attempt to find integrating factor ***(M1)***

$e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$ ***(A1)***

$x^2 \frac{dy}{dx} + 2xy = \ln 2x$

$\frac{d}{dx} x^2 y = \ln 2x$

$x^2 y = \int \ln 2x \ dx$

attempt to use integration by parts **(M1)**

$x^2y = x \ln 2x - x + c$ ***A1***

$y = \frac{\ln 2x}{x} - \frac{1}{x} + \frac{c}{x^2}$

substituting $x = \frac{1}{2}$, $y = 4$ into an integrated equation involving c ***M1***

$4 = 0 - 2 + 4c$

$\Rightarrow c = \frac{3}{2}$

$y = \frac{\ln 2x}{x} - \frac{1}{x} + \frac{3}{2x^2}$

22M.1.AHL.TZ2.9

METHOD 1 (rearranging the equation)

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

Note: Award ***M1*** for equivalent statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof.

Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$.

Note: Award ***M0*** for statements such as "let's consider the equation has integer roots..." ,"let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$..."

Note: Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

attempts to rearrange their equation into a suitable form ***M1***

EITHER

$2\alpha^3 + 6\alpha = -1$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow 2\alpha^3 + 6\alpha$ is even ***R1***

$2\alpha^3 + 6\alpha = -1$ which is not even and so α cannot be an integer ***R1***

Note: Accept '$2\alpha^3 + 6\alpha = -1$ which gives a contradiction'.

OR

$1 = 2 - \alpha^3 - 3\alpha$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow -\alpha^3 - 3\alpha \in \mathbb{Z}$ ***R1***

$\Rightarrow 1$ is even which is not true and so α cannot be an integer ***R1***

Note: Accept '$\Rightarrow 1$ is even which gives a contradiction'.

OR

$\frac{1}{2} = -\alpha^3 - 3\alpha$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow -\alpha^3 - 3\alpha \in \mathbb{Z}$ ***R1***

$-\alpha^3 - 3\alpha$ is is not an integer $= \frac{1}{2}$ and so α cannot be an integer ***R1***

Note: Accept '$-\alpha^3 - 3\alpha$ is not an integer $= \frac{1}{2}$ which gives a contradiction'.

OR

$\alpha = -\frac{1}{2\alpha^2 + 3}$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow -\frac{1}{2\alpha^2 + 3} \in \mathbb{Z}$ ***R1***

$-\frac{1}{2\alpha^2 + 3}$ is not an integer and so α cannot be an integer ***R1***

Note: Accept $-\frac{1}{2\alpha^2 + 3}$ is not an integer which gives a contradiction'.

THEN

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots ***AG***

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$

Award for equivalent statements such as 'assume that α is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of x throughout the proof. Award for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$ and award subsequent marks based on this.

Award for statements such as "let's consider the equation has integer roots..." ,"let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$..."

Subsequent marks after this are independent of this and can be awarded.

let $fx = 2x^3 + 6x + 1$ (and $f\alpha = 0$)

$f'x = 6x^2 + 6 > 0$ for all $x \in \mathbb{R}$ $\Rightarrow f$ is a (strictly) increasing function

$f0 = 1$ and $f\text{-}1 = \text{-}7$

thus $fx = 0$ has only one real root between -1 and 0, which gives a contradiction

(or therefore, contradicting the assumption that $f\alpha = 0$ for some $\alpha \in \mathbb{Z}$),

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots

22M.1.AHL.TZ2.8

let m be the median

EITHER

attempts to find the area of the required triangle ***M1***

base is $m - a$ ***(A1)***

and height is $\frac{2}{b - ac - a}m - a$

area $= \frac{1}{2}m - a \times \frac{2}{b - ac - a}m - a = \frac{m - a^2}{b - ac - a}$ ***A1***

OR

attempts to integrate the correct function ***M1***

$\int_a^m \frac{2}{b - ac - a}x - a \quad \mathrm{d}\,x$

$= \frac{2}{b - ac - a}\frac{1}{2}x - a^2\Big|_a^m$ OR $\frac{2}{b - ac - a}\frac{x^2}{2} - ax\Big|_a^m$ ***A1A1***

Note: Award ***A1*** for correct integration and ***A1*** for correct limits.

THEN

sets up (their) $\int_a^m \frac{2}{b - ac - a}x - a \quad \mathrm{d}\,x$ or area $= \frac{1}{2}$ ***M1***

Note: Award ***M0A0A0M1A0A0*** if candidates conclude that $m > c$ and set up their area or sum of integrals $= \frac{1}{2}$.

$\frac{m - a^2}{b - ac - a} = \frac{1}{2}$

$m = a \pm \sqrt{\frac{b - ac - a}{2}}$

as $m > a$, rejects $m = a - \sqrt{\frac{b - ac - a}{2}}$

so $m = a + \sqrt{\frac{b - ac - a}{2}}$

22M.1.AHL.TZ1.6

EITHER

attempt to obtain the general term of the expansion

$T_{r+1} = C_r^n 8x^{3^{n-r}} - \frac{1}{2x}^{r}$ OR $T_{r+1} = C_{n-r}^n 8x^{3^r} - \frac{1}{2x}^{n-r}$ ***(M1)***

OR

recognize power of x starts at $3n$ and goes down by 4 each time ***(M1)***

THEN

recognizing the constant term when the power of x is zero (or equivalent) ***(M1)***

$r = \frac{3n}{4}$ or $n = \frac{4}{3}r$ or $3n - 4r = 0$ OR $3r - n - r = 0$ (or equivalent) ***A1***

r is a multiple of 3 $r = 3, 6, 9, \ldots$ or one correct value of n (seen anywhere) ***(A1)***

$n = 4k, \quad k \in \mathbb{Z}^+$ ***A1***

Note: Accept n is a (positive) multiple of 4 or $n = 4, 8, 12, \ldots$
Do not accept $n = 4, 8, 12$

Note: Award full marks for a correct answer using trial and error approach showing $n = 4, 8, 12, \ldots$ and for recognizing that this pattern continues.

[5 marks]

22M.1.AHL.TZ1.8

Assume that a and b are both odd. ***M1***

Note: Award ***M0*** for statements such as "let a and b be both odd".
Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

Then $a = 2m + 1$ and $b = 2n + 1$ ***A1***

$a^2 + b^2 \equiv 2m + 1^2 + 2n + 1^2$

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$ ***A1***

$= 4m^2 + m + n^2 + n + 2$ ***(A1)***

$(4m^2 + m + n^2 + n$ is always divisible by $4)$ but 2 is not divisible by 4. (or equivalent) ***R1***

$\Rightarrow a^2 + b^2$ is not divisible by 4, a contradiction. (or equivalent) ***R1***

hence a and b cannot both be odd. ***AG***

Note: Award a maximum of ***M1A0A0(A0)R1R1*** for considering identical or two consecutive odd numbers for a and b.

[6 marks]

22M.1.AHL.TZ1.9

a.

$z_1 z_2 = 1 + bi1 - b^2 - 2bi$

$= 1 - b^2 - 2i^2b^2 + i\text{-}2b + b - b^3$ ***M1***

$= 1 + b^2 + i\text{-}b - b^3$ ***A1A1***

Note: Award ***A1*** for $1 + b^2$ and A1 for $-bi - b^3i$.

[3 marks]

b. $\arg z_1 z_2 = \arctan\frac{-b - b^3}{1 + b^2} = \frac{\pi}{4}$ ***(M1)***

EITHER **OR**

$\arctan\text{-}b = \frac{\pi}{4}$ (since $1 + b^2 \neq 0$, for $b \in \mathbb{R}$) ***A1***

$-b - b^3 = 1 + b^2$ (or equivalent) $b = -1$

22M.1.AHL.TZ1.10

a.i.

EITHER

attempt to use a ratio from consecutive terms ***M1***

$\frac{p \ln x}{\ln x} = \frac{\frac{1}{3}\ln x}{p \ln x}$ OR $\frac{1}{3}\ln x = \ln xr^2$ OR $p \ln x = \ln x^{\frac{1}{3p}}$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} + \ldots$ and consider the powers of x in geometric sequence

Award ***M1*** for $\frac{p}{1} = \frac{\frac{1}{3}}{p}$.

OR

$r = p$ and $r^2 = \frac{1}{3}$ ***M1***

THEN

$p^2 = \frac{1}{3}$ OR $r = \pm\frac{1}{\sqrt{3}}$ ***A1***

$p = \pm\frac{1}{\sqrt{3}}$ ***AG***

Note: Award ***M0A0*** for $r^2 = \frac{1}{3}$ or $p^2 = \frac{1}{3}$ with no other working seen.

[2 marks]

a.ii. **EITHER** since, $p = \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}} < 1$ ***R1***

OR

since, $p = \frac{1}{\sqrt{3}}$ and $-1 < p < 1$ ***R1***

THEN

$\Rightarrow$ the geometric series converges. ***AG***

[1 mark]

Note: Accept r instead of p.
Award ***R0*** if both values of p not considered.

a.iii. $\frac{\ln x}{1 - \frac{1}{\sqrt{3}}} = 3 + \sqrt{3}$ ***(A1)***

$\ln x = 3 - \frac{3}{\sqrt{3}} + \sqrt{3} - \frac{\sqrt{3}}{\sqrt{3}}$ OR $\ln x = 3 - \sqrt{3} + \sqrt{3} - 1$ $\Rightarrow \ln x = 2$ ***A1*** $x = e^2$ ***A1***

[3 marks]

b.i. **METHOD 1** attempt to find a difference from consecutive terms or from u_2 ***M1***

correct equation ***A1***

$p \ln x - \ln x = \frac{1}{3}\ln x - p \ln x$ OR $\frac{1}{3}\ln x = \ln x + 2p \ln x - \ln x$

Note: Candidates may use $\ln x^1 + \ln x^p + \ln x^{\frac{1}{3}} + \ldots$ and consider the powers of x in arithmetic sequence.

Award ***M1A1*** for $p - 1 = \frac{1}{3} - p$ $2p \ln x = \frac{4}{3}\ln x$ $\Rightarrow 2p = \frac{4}{3}$ ***A1*** $p = \frac{2}{3}$ ***AG***

METHOD 2 attempt to use arithmetic mean $u_2 = \frac{u_1 + u_3}{2}$ ***M1***

$p \ln x = \frac{\ln x + \frac{1}{3}\ln x}{2}$ $2p \ln x = \frac{4}{3}\ln x$ $\Rightarrow 2p = \frac{4}{3}$ $p = \frac{2}{3}$

attempt to find difference using u_3

$\frac{1}{3}\ln x = \ln x + 2d$ $\Rightarrow d = -\frac{1}{3}\ln x$

$u_2 = \ln x + \frac{1}{2}\frac{1}{3}\ln x - \ln x$ OR $p \ln x - \ln x = -\frac{1}{3}\ln x$ $p \ln x = \frac{2}{3}\ln x$

$p = \frac{2}{3}$

b.ii. $d = -\frac{1}{3}\ln x$

b.iii. $S_n = \frac{n}{2}2 \ln x + n - 1 \times -\frac{1}{3}\ln x$

attempt to substitute into S_n and equate to $\ln\frac{1}{x^3}$

$\frac{n}{2}2 \ln x + n - 1 \times -\frac{1}{3}\ln x = \ln\frac{1}{x^3}$ $\ln\frac{1}{x^3} = -\ln x^3 = \ln x^{-3}$ $= -3 \ln x$

correct working with S_n (seen anywhere)

$\frac{n}{2}2 \ln x - \frac{n}{3}\ln x + \frac{1}{3}\ln x$ OR $n \ln x - \frac{nn - 1}{6}\ln x$ OR $\frac{n}{2}\ln x + \frac{4 - n}{3}\ln x$

correct equation without $\ln x$ $\frac{n}{2}\frac{7}{3} - \frac{n}{3} = -3$ OR $n - \frac{nn - 1}{6} = -3$ or equivalent

Award as above if the series $1 + p + \frac{1}{3} + \ldots$ is considered leading to $\frac{n}{2}\frac{7}{3} - \frac{n}{3} = -3$.

$n^2 - 7n - 18 = 0$

attempt to form a quadratic $= 0$

attempt to solve their quadratic $n - 9n + 2 = 0$ $n = 9$

$$\ln\frac{1}{x^3} = -\ln\ x^3 = \ln\ x^{-3} = -3\ \ln\ x$$

listing the first 7 terms of the sequence

$\ln\ x + \frac{2}{3}\ln\ x + \frac{1}{3}\ln\ x + 0 - \frac{1}{3}\ln\ x - \frac{2}{3}\ln\ x - \ln\ x + \ldots$

recognizing first 7 terms sum to 0 8^{th} term is $-\frac{4}{3}\ln\ x$

9^{th} term is $-\frac{5}{3}\ln\ x$ sum of 8^{th} and 9^{th} term $= -3\ \ln\ x$

$n = 9$

22M.1.AHL.TZ1.11

a.

METHOD 1

attempt to eliminate a variable ***M1***

obtain a pair of equations in two variables

EITHER

$-3x + z = -3$ and ***A1***

$-3x + z = 44$ ***A1***

OR

$-5x + y = -7$ and ***A1***

$-5x + y = 40$ ***A1***

OR

$3x - z = 3$ and ***A1***

$3x - z = -\frac{79}{5}$ ***A1***

THEN

the two lines are parallel ($-3 \neq 44$ or $-7 \neq 40$ or $3 \neq -\frac{79}{5}$) ***R1***

Note: There are other possible pairs of equations in two variables.
To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

METHOD 2

vector product of the two normals $= \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$ (or equivalent) ***A1***

$r = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ (or equivalent) ***A1***

Note: Award ***A0*** if "$r =$ " is missing. Subsequent marks may still be awarded.

Attempt to substitute $1 + \lambda$, $-2 + 5\lambda, 3\lambda$ in $\prod_3$ ***M1***

$-91 + \lambda + 3 - 2 + 5\lambda - 23\lambda = 32$

$-15 = 32$, a contradiction ***R1***

hence the three planes do not intersect ***AG***

METHOD 3

attempt to eliminate a variable ***M1***

$-3y + 5z = 6$ ***A1***

$-3y + 5z = 100$ ***A1***

$0 = 94$, a contradiction ***R1***

Note: Accept other equivalent alternatives. Accept other valid methods.
To obtain the final ***R1***, at least the initial ***M1*** must have been awarded.

hence the three planes do not intersect ***AG***

[4 marks]

b.i. $\prod_1 : 2 + 2 + 0 = 4$ and $\prod_2 : 1 + 4 + 0 = 5$ ***A1*** ***[1 mark]***

b.ii. **METHOD 1** attempt to find the vector product of the two normals ***M1***

$\begin{pmatrix} 2 \\ -1 \\ \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ -3 \end{pmatrix}$ ***A1*** $r = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ ***A1A1***

Award if "$r =$ " is missing.
Accept any multiple of the direction vector.
Working for (b)(ii) may be seen in part (a) Method 2. In this case penalize lack of "$r =$ " only once.

attempt to eliminate a variable from $\prod_1$ and $\prod_2$

$3x - z = 3$ OR $3y - 5z = -6$ OR $5x - y = 7$ Let $x = t$

substituting $x = t$ in $3x - z = 3$ to obtain

$z = -3 + 3t$ and $y = 5t - 7$ (for all three variables in parametric form)

$$r = \begin{pmatrix} 0 \\ -7 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$

Award if "$r =$ " is missing.
Accept any multiple of the direction vector. Accept other position vectors which satisfy both the planes $\prod_1$ and $\prod_2$.

c. the line connecting L and $\prod_3$ is given by L_1

attempt to substitute position and direction vector to form L_1

$$s = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix}$$ substitute $1 - 9t, -2 + 3t, -2t$ in $\prod_3$

$-91 - 9t + 3-2 + 3t - 2-2t = 32$ $94t = 47 \Rightarrow t = \frac{1}{2}$

attempt to find distance between $1, -2, 0$ and their point $-\frac{7}{2}, -\frac{1}{2}, -1$

$$= \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -9 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \frac{1}{2}\sqrt{-9^2 + 3^2 + -2^2} = \frac{\sqrt{94}}{2}$$

unit normal vector equation of $\prod_3$ is given by $\frac{\begin{pmatrix} -9 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\sqrt{81 + 9 + 4}} = \frac{32}{\sqrt{94}}$

let $\prod_4$ be the plane parallel to $\prod_3$ and passing through P,
then the normal vector equation of $\prod_4$ is given by

$$\begin{pmatrix} -9 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = -15$$

unit normal vector equation of $\prod_4$ is given by

$$\frac{\begin{pmatrix} -9 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\sqrt{81 + 9 + 4}} = \frac{-15}{\sqrt{94}}$$

distance between the planes is $\frac{32}{\sqrt{94}} - \frac{-15}{\sqrt{94}}$

$$= \frac{47}{\sqrt{94}} = \frac{\sqrt{94}}{2}$$