Question 1

Let G be an unweighted graph with 5 vertices. The adjacency matrix of G is shown below.

	A	B	C	D	E
A	0	1	0	0	1
B	1	0	2	0	2
C	0	2	0	1	1
D	0	0	1	2	1
E	1	2	1	1	0

(a) Draw the graph of G.

[3]

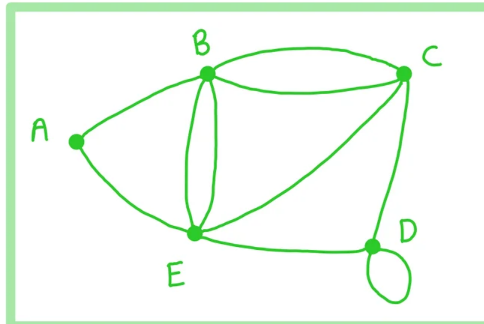
(b) (i) Define an Eulerian trail.

(ii) From which vertices in graph G must an Eulerian trail start and finish. Give a reason for your answer.

(iii) Write down an Eulerian trail in G.

[4]

a)



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D	0	0	1	2	1
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(a) Draw the graph of G.

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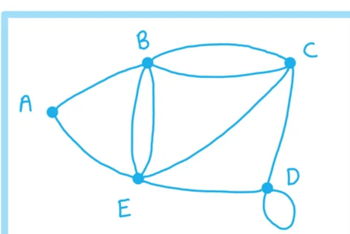


from part (a

b) (i) An Eulerian trail traverses each edge exactly once.

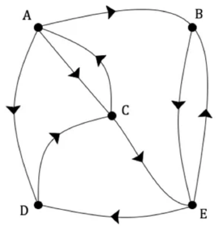
(ii) B and E are the two odd vertices. So an Eulerian trail must start at one and end at the other.

(iii) One Eulerian trail is:

$B \to C \to D \to D \to E \to C \to B \to E \to B \to A \to E$

Question 2

The graph G shown below is a strongly connected, unweighted, directed graph with 5 vertices.



(a) Explain why the graph is considered to be strongly connected.

[1]

(b) Write down the adjacency matrix $\boldsymbol{M}$ for the graph G.

[3]

(c) Find the total number of walks of length 5 that both start and finish at vertex A.

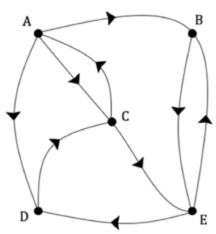
[2]

(d) Write down a possible walk of length 5 that starts and finishes at vertex A.

[1]

a) 'Strongly connected' means that a walk can be constructed between any two vertices in either direction.

The graph G shown below is a strongly connected, unweighted, directed graph with 5 vertices.



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(b) Write down the adjacency matrix $\boldsymbol{M}$ for the graph G.

[3]

(c) Find the total number of walks of length 5 that both start and finish at vertex A.

[2]

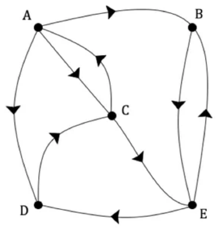
(d) Write down a possible walk of length 5 that starts and finishes at vertex A.

[1]

b)

$$\begin{array}{cc} & \text{from} \rightarrow \; A \quad B \quad C \quad D \quad E \\ M = & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{array}{l} A \\ B \\ C \\ D \\ E \end{array} \\ & \qquad\qquad\qquad\qquad\quad \uparrow \text{to} \end{array}$$

The graph G shown below is a strongly connected, unweighted, directed graph with 5 vertices.



(a) Explain why the graph is considered to be strongly connected.

[1]

(b) Write down the adjacency matrix $\boldsymbol{M}$ for the graph G.

$$M = \begin{array}{c} \text{From} \rightarrow \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \\ E \\ \uparrow \text{To} \end{array}$$

[3]

(c) Find the total number of walks of length 5 that both start and finish at vertex A.

[2]

c) Length 5, so calculate M^5 in GDC :

this is the 'from A to A' entry

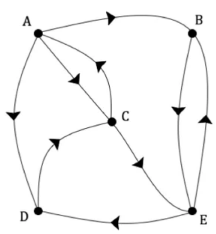
$$M^5 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}^5 = \begin{pmatrix} 4 & 0 & 1 & 2 & 2 \\ 5 & 2 & 4 & 4 & 1 \\ 3 & 2 & 6 & 1 & 2 \\ 5 & 2 & 4 & 4 & 1 \\ 6 & 1 & 5 & 2 & 4 \end{pmatrix}$$

There are 4 walks of length 5 that start and finish at A.

(d) Write down a possible walk of length 5 that starts and finishes at vertex A.

[1]

The graph G shown below is a strongly connected, unweighted, directed graph with 5 vertices.



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[1]

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[3]

(c) Find the total number of walks of length 5 that both start and finish at vertex A.

[2]

(d) Write down a possible walk of length 5 that starts and finishes at vertex A.

[1]

d) $A \rightarrow B \rightarrow E \rightarrow D \rightarrow C \rightarrow A$

The other three are :

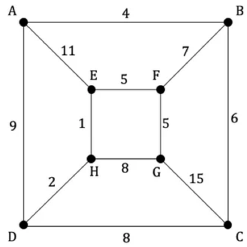
A D C A C A

A C E D C A

A C A D C A

Question 3

Consider the weighted graph G below.



(a) (i) Define a Hamiltonian cycle.

(ii) Write down a Hamiltonian cycle starting from vertex A.

[2]

(b) Use Kruskal's algorithm to find the minimum spanning tree. Show each step of the algorithm clearly.

[3]

(c) State the total weight of the minimum spanning tree.

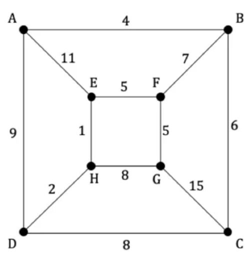
[1]

a) (i) A Hamiltonian cycle is a walk that starts and ends at the same vertex, and passes through each other vertex exactly once.

(ii) One Hamiltonian cycle is:

$A \to B \to C \to D \to H \to G \to F \to E \to A$

Consider the weighted graph G below.



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(ii) Write down a Hamiltonian cycle starting from vertex A.

[2]

(b) Use Kruskal's algorithm to find the minimum spanning tree. Show each step of the algorithm clearly.

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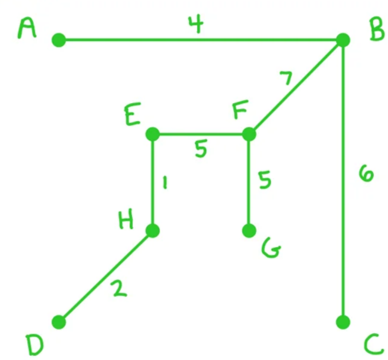
[1]

b)

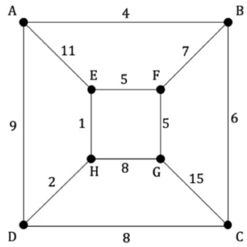


Add edges in this order:

EH, DH, AB, EF, FG, BC, BF

Note: EF and FG could trade places here.

Consider the weighted graph G below.



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(ii) Write down a Hamiltonian cycle starting from vertex A.

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[3]

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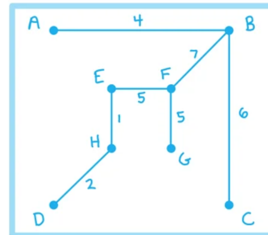
[1]



from part (b)

c) Add up weights of edges in minimum spanning tree:

$1 + 2 + 4 + 5 + 5 + 6 + 7 = \boxed{30}$

Question 4

Celeste is building a model city incorporating 6 main buildings that need to be connected to an electrical supply.

Each vertex listed in the table below represents a building and the weighting of each edge is the cost in USD of creating a link to the electrical supply between the given vertices.

	A	B	C	D	E	F
A	-	~~4~~	~~9~~	~~8~~	~~11~~	3
B	~~4~~	-	~~13~~	~~2~~	~~5~~	~~12~~
C	~~9~~	~~13~~	-	~~7~~	1	~~4~~
D	~~8~~	2	~~7~~	-	~~10~~	~~3~~
E	~~11~~	~~5~~	~~1~~	~~10~~	-	~~15~~
F	~~3~~	~~12~~	4	3	~~15~~	-

Celeste wants to find the lowest cost solution that links all 6 buildings up to the electricity supply.

(a) Starting from vertex A, use Prim's algorithm on the table to find and draw the minimum spanning tree. Show each step of the process clearly.

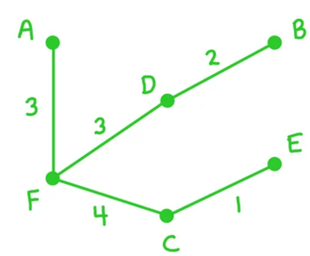
[5]

(b) State the lowest cost of connecting all of the buildings to the electricity supply.

[1]

a) Add edges in this order:

AF, DF, BD, CF, CE



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Each vertex listed in the table below represents a building and the weighting of each edge is the cost in USD of creating a link to the electrical supply between the given vertices.

	A	B	C	D	E	F
A	-	4	9	8	11	3
B	4	-	13	2	5	12
C	9	13	-	7	1	4
D	8	2	7	-	10	3
E	11	5	1	10	-	15
F	3	12	4	3	15	-

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(a) Starting from vertex A, use Prim's algorithm on the table to find and draw the minimum spanning tree. Show each step of the process clearly.

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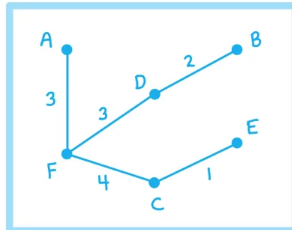


from part (a)

b) Add up weights of edges in minimum spanning tree:

$1 + 2 + 3 + 3 + 4 = \boxed{\$13 \text{ USD}}$

Question 5

5 scientists each have their own laboratory, and each laboratory is connected to other laboratories by a series of doors. Some of the doors can only be opened from one side. The table below lists which other laboratories can be accessed from the laboratory in the heading.

Anne's lab	Bilal's lab	Cariad's lab	Diego's lab	Ebele's lab
Cariad	Anne	Anne	Anne	Anne
	Cariad		Bilal	Cariad
	Diego		Ebele	Diego
	Ebele			

(a) Show the information from the table in a directed graph.

[3]

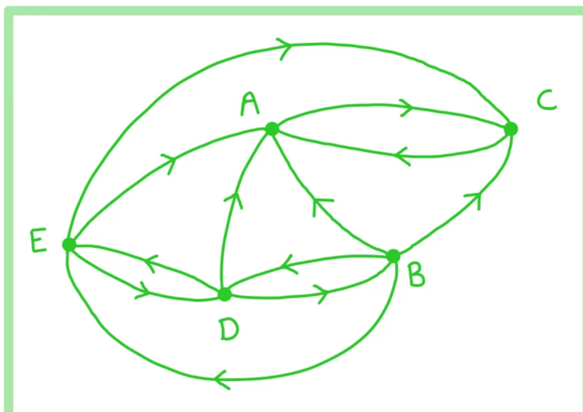
(b) Show that it is possible to move through the laboratories visiting each laboratory exactly once, and write down a viable path.

[2]

(c) Comment on any restrictions that determine the start and finish points of a path that visits each laboratory exactly once.

[2]

a)



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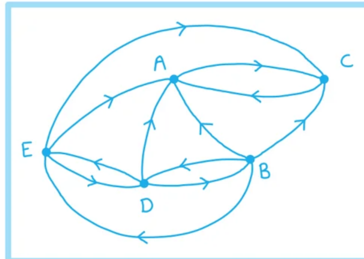
	Cariad		Bilal	Cariad
	Diego		Ebele	Diego
	Ebele			

(a) Show the information from the table in a directed graph.

[3]



from part (a)

(b) Show that it is possible to move through the laboratories visiting each laboratory exactly once, and write down a viable path.

[2]

b) One such path is :

$B \rightarrow D \rightarrow E \rightarrow A \rightarrow C$

There are others !

(c) Comment on any restrictions that determine the start and finish points of a path that visits each laboratory exactly once.

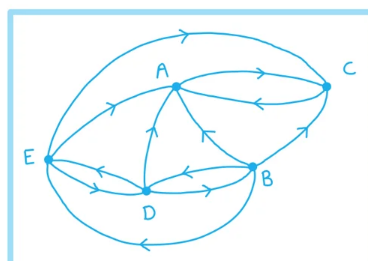
[2]

5 scientists each have their own laboratory, and each laboratory is connected to other laboratories by a series of doors. Some of the doors can only be opened from one side. The table below lists which other laboratories can be accessed from the laboratory in the heading.

Anne's lab	Bilal's lab	Cariad's lab	Diego's lab	Ebele's lab
Cariad	Anne	Anne	Anne	Anne
	Cariad		Bilal	Cariad
	Diego		Ebele	Diego
	Ebele			

(a) Show the information from the table in a directed graph.

[3]



from part (a)

(b) Show that it is possible to move through the laboratories visiting each laboratory exactly once, and write down a viable path.

[2]

(c) Comment on any restrictions that determine the start and finish points of a path that visits each laboratory exactly once.

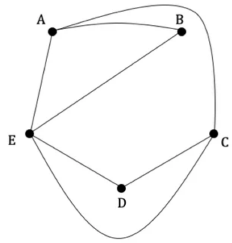
[2]

c) A and C can only access each other, so they must come last (in either order).

Either B, D, or E can be the starting point.

Question 6

Let G be the graph below.



(a) Construct the transition matrix for a random walk around G.

[3]

(b) Determine the probability that a random walk of length 3 starting at A will finish at C.

[2]

(c) (i) Find the steady state probabilities for the graph.

(ii) Hence rank the vertices in terms of importance from highest to lowest.

[2]

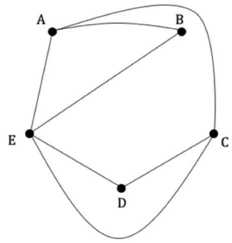
a)

$$\begin{array}{cc} & \text{from} \rightarrow \;\; A \quad\; B \quad\; C \quad\; D \quad\; E \\ T = & \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 0 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} \end{array}$$

to

Note that columns all add up to one.

Let G be the graph below.



(a) Construct the transition matrix for a random walk around G.

[3]

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(c) (i) Find the steady state probabilities for the graph.

(ii) Hence rank the vertices in terms of importance from highest to lowest.

[2]

$$\begin{array}{cc} & \text{from} \rightarrow \;\; A \quad\; B \quad\; C \quad\; D \quad\; E \\ T = & \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 0 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} \end{array}$$

to

from part (a)

b) Use GDC to calculate T^3:

To 3 s.f.,

$$T^3 = \begin{pmatrix} 0.139 & 0.285 & 0.287 & 0.146 & 0.215 \\ 0.190 & 0.0833 & 0.0972 & 0.139 & 0.174 \\ 0.287 & 0.146 & 0.139 & 0.285 & 0.215 \\ 0.0972 & 0.139 & 0.190 & 0.0833 & 0.174 \\ 0.287 & 0.347 & 0.287 & 0.347 & 0.222 \end{pmatrix}$$

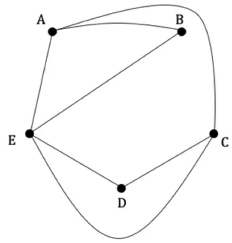
This is the 'from A to C' entry

The probability is

$$\frac{31}{108} = 0.287 \text{ (3 s.f.)}$$

Exact value from GDC

Let G be the graph below.



(a) Construct the transition matrix for a random walk around G.

[3]

(b) Determine the probability that a random walk of length 3 starting at A will finish at C.

[2]

(c) (i) Find the steady state probabilities for the graph.

(ii) Hence rank the vertices in terms of importance from highest to lowest.

[2]

$$T = \begin{array}{c} \text{From} \rightarrow \; A \quad B \quad C \quad D \quad E \\ \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 0 \end{pmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \\ E \\ \uparrow \text{To} \end{array}$$

from part (a)

c) (i) Consider high powers of T, until the entries stop changing. For example:

To 3 s.f.,

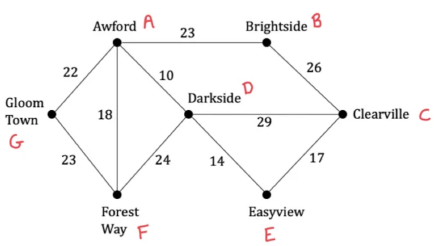
$$T^{100} = \begin{pmatrix} 0.214 & 0.214 & 0.214 & 0.214 & 0.214 \\ 0.143 & 0.143 & 0.143 & 0.143 & 0.143 \\ 0.214 & 0.214 & 0.214 & 0.214 & 0.214 \\ 0.143 & 0.143 & 0.143 & 0.143 & 0.143 \\ 0.286 & 0.286 & 0.286 & 0.286 & 0.286 \end{pmatrix}$$

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 0.214 \\ 0.143 \\ 0.214 \\ 0.143 \\ 0.286 \end{pmatrix} \quad (3 \text{ s.f.})$$

(ii) E, then A and C, then B and D.

Question 7

The graph G below shows 7 towns and the train tracks that the connect them, with the vertices representing the towns and the weighting of each edge indicating the time taken in minutes to walk along the section of track.



(a) State the degree of each vertex.

[2]

(b) Explain why G does not contain an Eulerian circuit.

[1]

The railway company in charge of maintaining the track wishes to inspect all sections of the track for defects after a storm event.

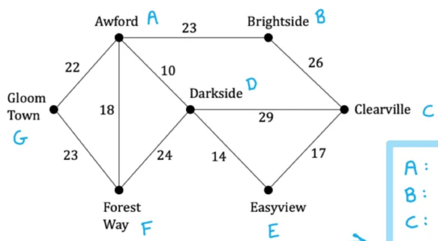
(c) Find the minimum time it would take for a railway worker who was walking to inspect all of the track.

[4]

a)

A: 4
B: 2
C: 3
D: 4
E: 2
F: 3
G: 2

The graph G below shows 7 towns and the train tracks that the connect them, with the vertices representing the towns and the weighting of each edge indicating the time taken in minutes to walk along the section of track.



(a) State the degree of each vertex.

A : 4
B : 2
C : 3
D : 4
E : 2
F : 3
G : 2

[2]

(b) Explain why G does not contain an Eulerian circuit.

[1]

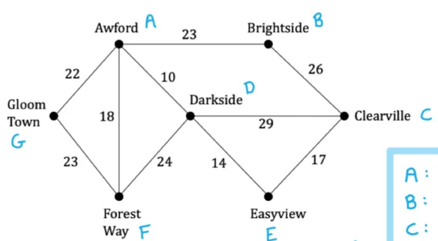
b) A graph only contains an Eulerian circuit if the degree of all vertices is even.

The railway company in charge of maintaining the track wishes to inspect all sections of the track for defects after a storm event.

(c) Find the minimum time it would take for a railway worker who was walking to inspect all of the track.

[4]

The graph G below shows 7 towns and the train tracks that the connect them, with the vertices representing the towns and the weighting of each edge indicating the time taken in minutes to walk along the section of track.



(a) State the degree of each vertex.

A : 4
B : 2
C : 3
D : 4
E : 2
F : 3
G : 2

[2]

(b) Explain why G does not contain an Eulerian circuit.

[1]

The railway company in charge of maintaining the track wishes to inspect all sections of the track for defects after a storm event.

(c) Find the minimum time it would take for a railway worker who was walking to inspect all of the track.

[4]

c) This is the 'Chinese Postman' problem. Find the sum of all edges, then add the shortest distance between the two vertices with an odd degree.

$22 + 23 + 18 + 23 + 10 + 24 + 26 + 29 + 14 + 17 = 206$

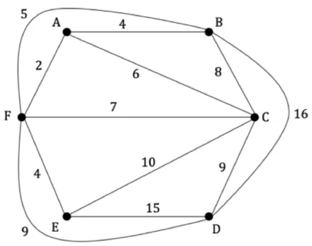
Shortest path from C to F is

$C \xrightarrow{29} D \xrightarrow{24} F \qquad 29 + 24 = 53$

$206 + 53 = \boxed{259 \text{ minutes}}$

Question 8

The graph below contains 6 vertices representing villages and their connecting bus routes. The weighting indicates the cost of each bus route in AUD.



(a) Explain why G is not a complete graph.

[1]

a) Not every vertex is connected to every other vertex. For example A is not connected to E.

(b) Complete the table of least weights below.

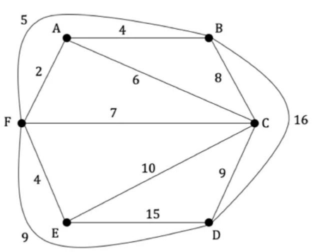
	A	B	C	D	E	F
A	-					
B	4	-				
C	6	8	-			
D		9	9	-		
E			10	15	-	
F	2	5	7	9	4	-

[4]

(c) Starting at town A, use the nearest neighbour algorithm to find an upper bound for the cost of a journey that will visit each vertex and return to A. Be sure to include the precise route of the upper bound journey.

[5]

The graph below contains 6 vertices representing villages and their connecting bus routes. The weighting indicates the cost of each bus route in AUD.



(a) Explain why G is not a complete graph.

[1]

(b) Complete the table of least weights below.

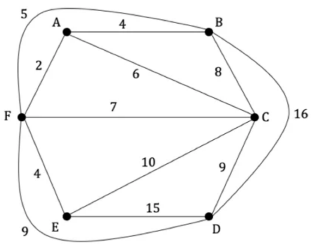
	A	B	C	D	E	F
A	-	4	6	11	6	2
B	4	-	8	9	9	5
C	6	8	-	9	10	7
D	11	9	9	-	15	9
E	6	9	10	15	-	4
F	2	5	7	9	4	-

[4]

(c) Starting at town A, use the nearest neighbour algorithm to find an upper bound for the cost of a journey that will visit each vertex and return to A. Be sure to include the precise route of the upper bound journey.

[5]

The graph below contains 6 vertices representing villages and their connecting bus routes. The weighting indicates the cost of each bus route in AUD.



(a) Explain why G is not a complete graph.

[1]

(b) Complete the table of least weights below.

	A	B	C	D	E	F
A	-	~~4~~	~~6~~	11	~~6~~	~~2~~
B	4	-	~~8~~	9	9	5
C	6	8	-	9	10	7
D	11	9	9	-	15	9
E	6	~~9~~	~~10~~	15	-	4
F	2	~~5~~	~~7~~	9	~~4~~	-

[4]

(c) Starting at town A, use the nearest neighbour algorithm to find an upper bound for the cost of a journey that will visit each vertex and return to A. Be sure to include the precise route of the upper bound journey.

[5]

c) $A \xrightarrow{2} F \xrightarrow{4} E \xrightarrow{9} B \xrightarrow{8} C \xrightarrow{9} D \xrightarrow{11} A$

$2 + 4 + 9 + 8 + 9 + 11 = 43$

$\boxed{\$43 \text{ AUD is an upper bound.}}$

Remember that the cheapest way from E to B is E → F → B, and the cheapest way from D to A is D → F → A.

The precise route is

A → F → E → F → B → C → D → F → A

Question 9

The table of least weight for a graph G of 5 vertices is shown below. Each vertex listed in the table represents a room in an art gallery and the weighting of the graph denotes the distance in metres between the rooms.

	A	B	C	D	E
A	-	15	22	18	25
B	15	-	13	17	24
C	22	13	-	30	12
D	18	17	30	-	21
E	25	24	12	21	-

(a) Draw the graph G.

[3]

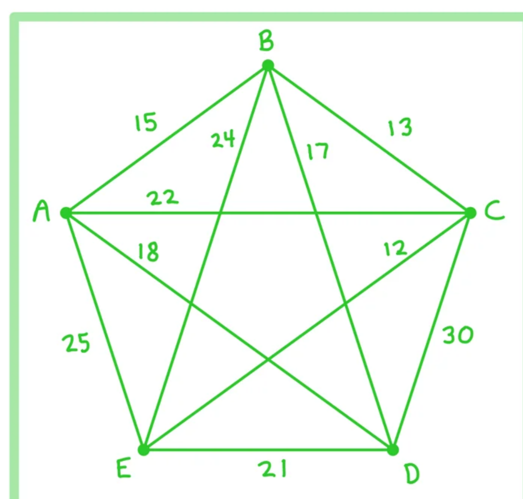
(b) By deleting vertex E and using Prim's algorithm, find a lower bound for the distance walked to visit every room in the art gallery. Show each step of the process clearly.

[4]

(c) Show that by deleting a different vertex a lower bound can be found that is higher than the one found in part (b).

[2]

a)

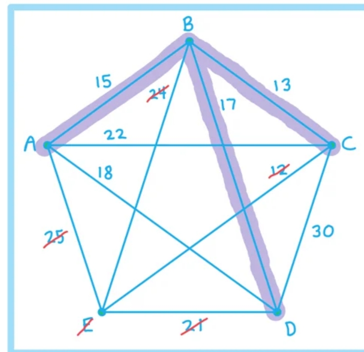


The table of least weight for a graph G of 5 vertices is shown below. Each vertex listed in the table represents a room in an art gallery and the weighting of the graph denotes the distance in metres between the rooms.

	A	B	C	D	E
A	-	15	~~22~~	~~18~~	~~25~~
B	~~15~~	-	13	17	~~24~~
C	~~22~~	~~13~~	-	~~30~~	~~12~~
D	~~18~~	~~17~~	~~30~~	-	~~21~~
E	~~25~~	~~24~~	~~12~~	~~21~~	-

(a) Draw the graph G.

[3]



from part (a)

(b) By deleting vertex E and using Prim's algorithm, find a lower bound for the distance walked to visit every room in the art gallery. Show each step of the process clearly.

[4]

b) Add edges in the order AB, BC, BD

Total weight of minimum spanning tree for ABCD is 15 + 13 + 17 = 45

Add weights of two edges of least weight connected to E (CE and DE):

45 + 12 + 21 = 78

78 metres is a lower bound

(c) Show that by deleting a different vertex a lower bound can be found that is higher than the one found in part (b).

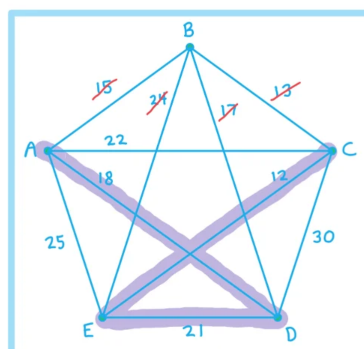
[2]

The table of least weight for a graph G of 5 vertices is shown below. Each vertex listed in the table represents a room in an art gallery and the weighting of the graph denotes the distance in metres between the rooms.

	A	B	C	D	E
A	-	~~15~~	~~22~~	18	~~25~~
B	~~15~~	-	~~13~~	~~17~~	~~24~~
C	~~22~~	~~13~~	-	~~30~~	~~12~~
D	~~18~~	~~17~~	~~30~~	-	21
E	~~25~~	~~24~~	12	~~21~~	-

(a) Draw the graph G.

[3]



from part (a)

(b) By deleting vertex E and using Prim's algorithm, find a lower bound for the distance walked to visit every room in the art gallery. Show each step of the process clearly.

78 metres is a lower bound

[4]

(c) Show that by deleting a different vertex a lower bound can be found that is higher than the one found in part (b).

[2]

c) Delete vertex B. Edges AD, DE, CE give a minimum spanning tree for ACDE, with total weight 18 + 21 + 12 = 51.

Add weights of two edges of least weight connected to B (AB and BC):

51 + 15 + 13 = 79

79 metres is a lower bound, higher than the one in (b)