Question 1

Prove that there is no $x \in \mathbb{R}$ such that $-\frac{2}{x-2} = x - 3$.

[5]

Proof by contradiction

Assume there exists a real number k, such that $\frac{-2}{k-2} = k - 3$

$-2 = (k-3)(k-2)$

$-2 = k^2 - 5k + 6$

$0 = k^2 - 5k + 8$

Solve using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ← Formula booklet

$k = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(8)}}{2(1)}$

The square root of a negative number is not a member of the set of real numbers

$\boxed{\therefore k = \frac{5 \pm \sqrt{-7}}{2} \notin \mathbb{R}}$

Question 2

Using the method of proof by contradiction, prove that $\sqrt{7}$ is irrational.

[4]

First, assume that $\sqrt{7}$ is rational

$\sqrt{7} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and have no common factors

$(\sqrt{7})^2 = \frac{a^2}{b^2}$

$7 = \frac{a^2}{b^2}$

$7b^2 = a^2$

If a^2 is a multiple of 7, then a must also be a multiple of 7

$\Rightarrow \quad a = 7k$, for $k \in \mathbb{Z}^+$

Replace $a = 7k$ into $7b^2 = a^2$

$7b^2 = (7k)^2$

$7b^2 = 49k^2$

$b^2 = 7k^2$

This means that b is also a multiple of 7 so a and b have a common factor

$\boxed{\therefore \text{ The initial assumption is false and } \sqrt{7} \text{ is irrational by proof by contradiction}}$

Question 3

Using mathematical induction, prove that $6^n - 1$ is divisible by 5 for $n \in \mathbb{Z}, n \geq 1$.

[4]

Basic step $n = 1$

$P(0)$: $6^1 - 1 = 5$

$5 \div 5 = 1$ $\therefore$ True for $n = 1$

Assume that $P(n)$ is true for some value of k where $k \geq 1$

$P(k)$: $6^k - 1 = 5A$ where $A \in \mathbb{Z}^+$

$\Rightarrow 6^k = 5A + 1$

Inductive step $n = k+1$

$P(k+1)$: $6^{k+1} - 1$

$= 6^k 6 - 1$

We have $6^k = 5A + 1$ by assumption

$= 6(5A + 1) - 1$

$= 30A + 5$

$= 5(6A + 1)$ $\therefore$ True for $n = k+1$

$\boxed{\therefore \text{ By proof of induction, } 6^n - 1 \text{ is divisible by 5 for all values of } n \geq 1}$

Question 4

The nth triangular number is given by the formula $u_n = \frac{1}{2}n(n+1)$.

(a) Write down the first five triangular numbers.

[1]

(b) Prove by exhaustion that the first five triangular numbers are all factors of 180.

[2]

(a) Substitute the values $n = 1, 2, 3, 4, 5$ into the formula

$$\boxed{\begin{aligned} u_1 &= 1 \\ u_2 &= 3 \\ u_3 &= 6 \\ u_4 &= 10 \\ u_5 &= 15 \end{aligned}}$$

The nth triangular number is given by the formula $u_n = \frac{1}{2}n(n+1)$.

(a) Write down the first five triangular numbers. [1]

$u_1 = 1$
$u_2 = 3$
$u_3 = 6$
$u_4 = 10$
$u_5 = 15$

(b) Prove by exhaustion that the first five triangular numbers are all factors of 180. [2]

(b) $180 \div 1 = 180$
$180 \div 3 = 60$
$180 \div 6 = 30$
$180 \div 10 = 18$
$180 \div 15 = 12$

$\therefore$ By proof of exhaustion the first five triangular numbers are all factors of 180

Question 5

Determine, with appropriate reasoning, whether the following statements are true or false:

(i) Given $n \in \mathbb{Z}$ and n^2 is divisible by 4, then n is divisible by 4.

(ii) Given $n \in \mathbb{Z}$ then $n^2 - 1$ is a prime number.

(iii) Given $n \in \mathbb{Z}$ and n^2 is divisible by 3, then n is divisible by 3.

(iv) Given an integer is a multiple 8 and 6 then it is a multiple of 48.

[8]

(i) Let $n^2 = 36$, $\frac{36}{4} = 9$

$n = 6$

False, 6 is not divisible by 4

(ii) Let $n = 3$, $n^2 - 1 = 8$

False, 8 is not a prime number

(iii) If $n = 3p$

then $n^2 = 9p^2 \Rightarrow n$ is divisible by 3

True, if n^2 is divisible by 3, then n is also divisible by 3

(iv) False, 24 is a multiple of 6 and 8 but is not a multiple of 48

Question 6

Prove that $x^2 - 3x + 3$ is positive for all real values of x.

[3]

Write the expression in completing the square form

$$x^2 - 3x + 3 = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 3$$

$$= \left(x - \frac{3}{2}\right)^2 + \frac{3}{4}$$

Since $\left(x - \frac{3}{2}\right)^2 \geqslant 0$ for all real values of x

then $\left(x - \frac{3}{2}\right)^2 + \frac{3}{4} \geqslant \frac{3}{4}$ for all real values of x

$\boxed{\therefore\ x^2 - 3x + 3 \text{ is positive for all real values of } x}$

Question 7

(a) Show that $(3n + 2)^2 - (n + 2)^2 = 8n^2 + 8n$, where $n \in \mathbb{Z}$.

[2]

(b) Hence, or otherwise, prove that $(3n + 2)^2 - (n + 2)^2$ is a multiple of 8.

[2]

(a) Expand and simplify LHS

$$(9n^2 + 12n + 4) - (n^2 + 4n + 2)$$

$$= 9n^2 + 12n + 4 - n^2 - 4n - 4$$

$$= 8n^2 + 8n$$

$$\boxed{(3n + 2)^2 - (n + 2)^2 = 8n^2 + 8n}$$

(a) Show that $(3n+2)^2 - (n+2)^2 = 8n^2 + 8n$, where $n \in \mathbb{Z}$. [2]

(b) Hence, or otherwise, prove that $(3n+2)^2 - (n+2)^2$ is a multiple of 8. [2]

(b) $(3n+2)^2 - (n+2)^2 = 8n^2 + 8n$

$= 8(n^2 + n)$

$\boxed{\therefore (3n+2)^2 - (n+2)^2 \text{ is a multiple of } 8}$

Question 8

Prove that $(a-b)^2 - (a+b)^2 = -4ab$. [3]

Expand and simplify LHS

$(a^2 - 2ab + b^2) - (a^2 + 2ab + b^2)$

$= a^2 - 2ab + b^2 - a^2 - 2ab - b^2$

$= -4ab$

$\boxed{\therefore (a-b)^2 - (a+b)^2 = -4ab}$

Question 9

Prove that $(4x-1)(2x+3) - (2x+1)^2 = 2(2x-1)(x+2)$. [3]

Expand and simplify LHS

$(8x^2 + 10x - 3) - (4x^2 + 4x + 1)$

$= 8x^2 + 10x - 3 - 4x^2 - 4x - 1$

$= 4x^2 + 6x - 4$

Take out a factor of 2

$= 2(2x^2 + 3x - 2)$

Factorise the expression inside the brackets

$= 2(2x-1)(x+2)$

$\boxed{\therefore (4x-1)(2x+3) - (2x+1)^2 = 2(2x-1)(x+2)}$

Question 10

Prove by mathematical induction $3^n \geq 1 + 2n$, given $n \geq 0$.

[6]

Basic step $n = 0$

$P(0)$: $3^{(0)} \geq 1 + 2(0)$

$1 \geq 1$

$\therefore$ True for $n = 0$

Assume that $P(n)$ is true for some value of k, where $k \geq 0$

$P(k)$: $3^k \geq 1 + 2k$

Inductive step $n = k+1$

$P(k+1)$: $3^{(k+1)} \geq 1 + 2(k+1)$

$3^{(k+1)} \geq 3 + 2k$

We have $3^k \geq 1 + 2k$ by assumption

Multiply both sides of the assumption by 3 to make the LHS the same as for $P(k+1)$

$3(3^k) \geq 3(1 + 2k)$

$3^{k+1} \geq 3 + 6k$

Compare the RHS of this to the RHS of $P(k+1)$

$3 + 6k \geq 3 + 2k$ since $k \geq 0$

$\Rightarrow$ $3^{(k+1)} \geq 3 + 2k$

$\therefore$ True for $n = k+1$

$\therefore$ By proof by induction, the statement holds true for all values of $n \geq 0$

Question 11

Prove by mathematical induction that if $y = \frac{1}{1-x}$ then $\frac{d^n y}{dx^n} = \frac{n!}{(1-x)^{n+1}}$.

[6]

Basic step $n = 1$

Differentiating y as usual using the chain rule

$y = (1-x)^{-1} \Rightarrow \frac{dy}{dx} = (1-x)^{-2}$

Substituting $n = 1$ into $\frac{d^n y}{dx^n}$

$$\frac{d^1 y}{dx^1} = \frac{1!}{(1-x)^{1+1}} = \frac{1}{(1-x)^2} = (1-x)^{-2}$$

$\therefore$ True for $n = 1$

Assume that $P(n)$ is true for some value of k, where $k > 0$

$$\frac{d^k y}{dx^k} = \frac{k!}{(1-x)^{k+1}}$$

Inductive step $n = k+1$

Show that the derivative of $\frac{d^k y}{dx^k}$ is equal to $\frac{d^{(k+1)} y}{dx^{(k+1)}}$

Differentiating $\frac{d^k y}{dx^k}$ using the chain rule

$$\frac{d}{dx}\left(\frac{k!}{(1-x)^{k+1}}\right) = \frac{k!\,(k+1)}{(1-x)^{(k+2)}} = \frac{(k+1)!}{(1-x)^{(k+2)}}$$

Substituting $n = k+1$ into $\frac{d^n y}{dx^n}$

$$\frac{d^{(k+1)} y}{dx^{(k+1)}} = \frac{(k+1)!}{(1-x)^{(k+1)+1}} = \frac{(k+1)!}{(1-x)^{(k+2)}}$$

$\therefore$ True for $n = k+1$

$\therefore$ By proof by induction, the statement holds true for all values of $n > 0$

Question 12

Prove by induction that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

for all values of $n, n \in \mathbb{Z}^+$.

[5]

Basic step $n = 1$

$P(1)$: LHS: $\sum_{r=1}^{1} (1)^2 = 1$

RHS: $\frac{1}{6}(1)((1)+1)(2(1)+1) = 1$

$\therefore$ True for $n = 1$

Assume that $P(n)$ is true for some value of k where $k \geqslant 1$

$P(k)$: $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$

Inductive step $n = k+1$

Substitute assumption from above

$P(k+1)$: LHS: $\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$

LHS: $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$

$= \frac{1}{6}k(2k^2+3k+1) + (k^2+2k+1)$

$= \frac{1}{6}(2k^3+3k^2+k) + \frac{1}{6}(6k^2+12k+6)$

$= \frac{1}{6}(2k^3+9k^2+13k+6)$

RHS: $\frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$

$= \frac{1}{6}(k+1)(k+2)(2k+3)$

$= \frac{1}{6}(k^2+3k+2)(2k+3)$

$= \frac{1}{6}(2k^3+9k^2+13k+6)$

$\therefore$ True for $n = k+1$

$\therefore$ By proof of induction, the statement holds true for all values of $n, n \in \mathbb{Z}^+$

Question 13

Given $z = x + yi$

(i) prove that $zz^* = |z||z^*|$,

(ii) prove that, for $x \geq 0$, $\arg(z) + \arg(z^*) = 0$.

[4]

(i) Expand and simplify LHS

$$zz^* = (x + yi)(x - yi)$$
$$= x^2 - y^2 i^2$$
$$= x^2 + y^2$$

Expand and simplify RHS

$$|z|\,|z^*| = \left(\sqrt{x^2 + y^2}\right)\left(\sqrt{x^2 + (-y)^2}\right)$$
$$= \sqrt{(x^2 + y^2)}\sqrt{(x^2 + y^2)}$$
$$= x^2 + y^2$$

$$\boxed{LHS = RHS \quad \therefore \quad zz^* = |z|\,|z^*|}$$

(ii) Simplify LHS

$$\arg(z) + \arg z^*$$
$$= \tan^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(-\frac{y}{x}\right)$$

Since $\tan(-A) = -\tan A$

$$= \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{x}\right)$$
$$= 0$$

$$\boxed{\therefore \ \arg(z) + \arg(z^*) = 0}$$