Question 1

For each of the following limits,

(i) determine whether or not l'Hôpital's rule may be used to evaluate the limit, giving a reason for your answer; and

(ii) if l'Hôpital's rule may be used, then use the rule to evaluate the limit.

(a)

$$\lim_{x\to 0}\frac{\sin x}{x^2+2x}$$

[4]

(b)

$$\lim_{x\to 0}\frac{\cos x}{x^2+2x}$$

[2]

(c)

$$\lim_{x\to\frac{\pi}{2}}\frac{\sec x}{1-\tan x}$$

[5]

a) i) $\frac{\sin 0}{0^2+0} = \frac{0}{0}$, which is one of the acceptable l'Hôpital indeterminate forms, so rule may be used.

ii) Differentiate top and bottom

$\lim_{x\to 0}\frac{\sin x}{x^2+x} = \lim_{x\to 0}\frac{\cos x}{2x+2} = \frac{1}{2}$

For each of the following limits,

(i) determine whether or not l'Hôpital's rule may be used to evaluate the limit, giving a reason for your answer; and

(ii) if l'Hôpital's rule may be used, then use the rule to evaluate the limit.

(a)

$$\lim_{x\to 0}\frac{\sin x}{x^2+2x}$$

[4]

(b)

$$\lim_{x\to 0}\frac{\cos x}{x^2+2x}$$

[2]

(c)

$$\lim_{x\to\frac{\pi}{2}}\frac{\sec x}{1-\tan x}$$

[5]

b) $\frac{\cos 0}{0^2+0} = \frac{1}{0}$, which is not one of the acceptable l'Hôpital indeterminate forms, so rule may not be used.

For each of the following limits,

(i) determine whether or not l'Hôpital's rule may be used to evaluate the limit, giving a reason for your answer; and

(ii) if l'Hôpital's rule may be used, then use the rule to evaluate the limit.

(a)

$$\lim_{x\to 0} \frac{\sin x}{x^2 + 2x}$$

[4]

(b)

$$\lim_{x\to 0} \frac{\cos x}{x^2 + 2x}$$

[2]

(c)

$$\lim_{x\to \frac{\pi}{2}} \frac{\sec x}{1 - \tan x}$$

[5]

c) i) $\frac{\sec \frac{\pi}{2}}{1 - \tan \frac{\pi}{2}} = \frac{\pm\infty}{\pm\infty}$, which is one of the acceptable l'Hôpital indeterminate forms, so rule may be used.

ii) Differentiate top and bottom and trig function identities.

$$\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{1-\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{\overbrace{\sec x \tan x}^{=\frac{d}{dx}(\sec x)}}{\underbrace{-\sec^2 x}_{=\frac{d}{dx}(1-\tan x)}} = \lim_{x\to\frac{\pi}{2}} \frac{\tan x}{-\sec x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{\sin x}{\cos x}}{-\frac{1}{\cos x}} = \lim_{x\to\frac{\pi}{2}} (-\sin x)$$

$$\boxed{\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{1-\tan x} = -1}$$

Question 2

Consider the following limit:

$$\lim_{x\to 0} \frac{-1 + \cos 2x}{x^2}$$

(a) Explain why it is appropriate to use l'Hôpital's rule to attempt to evaluate this limit.

[2]

(b) Show that employing l'Hôpital's rule once leads to an indeterminate form when you attempt to evaluate the limit.

[2]

(c) By employing l'Hôpital's rule a second time, show that the limit exists and find its value.

[2]

a) $\frac{-1 + \cos 0}{0^2} = \frac{0}{0}$, which is one of the acceptable l'Hôpital indeterminate forms, so rule may be used.

Consider the following limit:

$$\lim_{x\to 0}\frac{-1+\cos 2x}{x^2}$$

(a) Explain why it is appropriate to use l'Hôpital's rule to attempt to evaluate this limit.

[2]

(b) Show that employing l'Hôpital's rule once leads to an indeterminate form when you attempt to evaluate the limit.

[2]

(c) By employing l'Hôpital's rule a second time, show that the limit exists and find its value.

[2]

b) Differentiate top and bottom

$$\boxed{\lim_{x\to 0}\frac{1+\cos 2x}{0^2} = \lim_{x\to 0}\frac{-2\sin 2x}{2x} = \frac{0}{0},}$$

which is another indeterminate form.

Consider the following limit:

$$\lim_{x\to 0}\frac{-1+\cos 2x}{x^2}$$

(a) Explain why it is appropriate to use l'Hôpital's rule to attempt to evaluate this limit.

[2]

(b) Show that employing l'Hôpital's rule once leads to an indeterminate form when you attempt to evaluate the limit.

[2]

(c) By employing l'Hôpital's rule a second time, show that the limit exists and find its value.

[2]

c) Repeat the process from part b.

$$\lim_{x\to 0}\frac{-2\sin 2x}{2x} = \lim_{x\to 0}\frac{-4\cos 2x}{2}$$

$$\boxed{\lim_{x\to 0}(-2\cos 2x) = -2}$$

Question 3

Consider the function f defined by

$$f(x) = \frac{7-3x}{12x+5}$$

(a) Use l'Hôpital's rule to evaluate $\lim_{x\to\infty} f(x)$.

[3]

(b) Hence write down the equation(s) of any horizontal asymptotes on the graph of $y = f(x)$, giving a reason for your answer.

[2]

(c) (i) Show that $f(x)$ may be rewritten in the form

$$f(x) = \frac{\frac{7}{x}-3}{12+\frac{5}{x}}$$

(ii) Hence show that $\lim_{x\to\infty} f(x)$ may also be evaluated without the use of l'Hôpital's rule.

[4]

a) $\lim_{x\to\infty} \frac{7-3x}{12x+5} = \frac{-\infty}{+\infty}$, which is one of the acceptable l'Hôpital indeterminate forms, so rule may be used.

Differentiate top and bottom

$$\boxed{\lim_{x\to\infty} \frac{7-3x}{12x+5} = \lim_{x\to\infty} \frac{-3}{12} = -\frac{1}{4}}$$

Consider the function f defined by

$$f(x) = \frac{7-3x}{12x+5}$$

(a) Use l'Hôpital's rule to evaluate $\lim_{x\to\infty} f(x)$.

$$\boxed{\lim_{x\to\infty} \frac{7-3x}{12x+5} = -\frac{1}{4}}$$

[3]

(b) Hence write down the equation(s) of any horizontal asymptotes on the graph of $y = f(x)$, giving a reason for your answer.

[2]

(c) (i) Show that $f(x)$ may be rewritten in the form

$$f(x) = \frac{\frac{7}{x}-3}{12+\frac{5}{x}}$$

(ii) Hence show that $\lim_{x\to\infty} f(x)$ may also be evaluated without the use of l'Hôpital's rule.

[4]

b) If $\lim_{x\to\infty} f(x) = k$, then $y = k$ is a horizontal asymptote.

$$\boxed{y = -\frac{1}{4}}$$

Consider the function f defined by

$$f(x) = \frac{7 - 3x}{12x + 5}$$

(a) Use l'Hôpital's rule to evaluate $\lim_{x \to \infty} f(x)$.

[3]

(b) Hence write down the equation(s) of any horizontal asymptotes on the graph of $y = f(x)$, giving a reason for your answer.

[2]

(c) (i) Show that $f(x)$ may be rewritten in the form

$$f(x) = \frac{\frac{7}{x} - 3}{12 + \frac{5}{x}}$$

(ii) Hence show that $\lim_{x \to \infty} f(x)$ may also be evaluated without the use of l'Hôpital's rule.

[4]

c) Multiply $f(x)$ by $\left.\frac{\frac{1}{x}}{\frac{1}{x}}\right\} = 1$

i) $\boxed{f(x) = \frac{7-3x}{12x+5} \times \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{\frac{7}{x} - 3}{12 + \frac{5}{x}}}$

ii) $\boxed{\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{7-3x}{12x+5} = \lim_{x \to \infty} \frac{\frac{7}{x} - 3}{12 + \frac{5}{x}} = -\frac{3}{12} = -\frac{1}{4}}$

Question 4

(a) By substituting $-x$ into the Maclaurin series for e^x, determine the Maclaurin series for e^{-x}.

[2]

Consider the limit

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{2x}$$

(b) Use Maclaurin series to evaluate the limit.

[5]

(c) (i) Show that it would also be appropriate to use l'Hôpital's rule to attempt to evaluate the limit.

(ii) Evaluate the limit using l'Hôpital's rule, and confirm that this matches your answer in part (b).

[4]

a) Maclaurin series: $e^x = 1 + x + \frac{x^2}{2!} + \ldots$ (in formula booklet)

swap x with $-x$.

$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \ldots$

$\boxed{e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots}$

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(a) By substituting $-x$ into the Maclaurin series for e^x, determine the Maclaurin series for e^{-x}.

$$\boxed{e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots}$$

[2]

Consider the limit

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{2x}$$

(b) Use Maclaurin series to evaluate the limit.

[5]

(c) (i) Show that it would also be appropriate to use l'Hôpital's rule to attempt to evaluate the limit.

(ii) Evaluate the limit using l'Hôpital's rule, and confirm that this matches your answer in part (b).

[4]

b) Substituting the series gives :

$$\frac{e^x - e^{-x}}{2x} = \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)}{2x}$$

$$\frac{e^x - e^{-x}}{2x} = \frac{2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \ldots}{2x} = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots$$

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{2x} = \lim_{x \to 0} \left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right) = 1 + \frac{0^2}{3!} + \frac{0^4}{5!} + \ldots$$

$$\boxed{\lim_{x \to 0} \frac{e^x - e^{-x}}{2x} = 1}$$

(a) By substituting $-x$ into the Maclaurin series for e^x, determine the Maclaurin series for e^{-x}.

[2]

Consider the limit

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{2x}$$

(b) Use Maclaurin series to evaluate the limit.

$$\boxed{\lim_{x \to 0} \frac{e^x - e^{-x}}{2x} = 1}$$

[5]

(c) (i) Show that it would also be appropriate to use l'Hôpital's rule to attempt to evaluate the limit.

(ii) Evaluate the limit using l'Hôpital's rule, and confirm that this matches your answer in part (b).

[4]

c) i) $\frac{e^0 - e^{-0}}{2(0)} = \frac{0}{0}$, which is one of the acceptable l'Hôpital indeterminate forms, so rule may be used.

ii) $\lim_{x \to 0} \frac{e^x - e^{-x}}{2x} = \lim_{x \to 0} \frac{e^x + e^{-x}}{2} = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2}$

$$\boxed{\lim_{x \to 0} \frac{e^x - e^{-x}}{2x} = 1}$$

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Question 5

(a) Find the Maclaurin series for $\cos 2x$. [3]

(b) Hence evaluate the limit

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$$

[4]

a) Maclaurin series: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$ (in formula booklet)

swap x with $2x$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots$$

$$\boxed{\cos 2x = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \dots}$$

(a) Find the Maclaurin series for $\cos 2x$. [3]

$$\boxed{\cos 2x = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \dots}$$

(b) Hence evaluate the limit

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$$

[4]

b) If we sub $x = 0$, we get $\frac{1-\cos 2(0)}{(0)^2} = \frac{0}{0}$

Instead use the series expansion

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0}\frac{1-\left(1-\frac{2^2}{2!}x^2+\frac{2^4}{4!}x^4-\frac{2^6}{6!}x^6+\dots\right)}{x^2}$$

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0}\frac{\frac{2^2}{2!}x^2-\frac{2^4}{4!}x^4+\frac{2^6}{6!}x^6-\dots}{x^2}$$

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0}\frac{2^2}{2!}-\frac{2^4}{4!}x^2+\frac{2^6}{6!}x^4-\dots$$

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \frac{2^2}{2!}-\frac{2^4}{4!}(0)^2+\frac{2^6}{6!}(0)^4-\dots = 2-0+0$$

$$\boxed{\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = 2}$$

Question 6

Use an appropriate method to evaluate the limit

$$\lim_{x\to 0}\frac{\sin x - x}{x^3}$$

[5]

Method 1: Maclaurin series

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ (in formula booklet)

$$\frac{\sin x - x}{x^3} = \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) - x}{x^3}$$

$$\frac{\sin x - x}{x^3} = \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}{x^3} = \lim_{x\to 0}\left(-\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \ldots\right)$$

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = -\frac{1}{6} + \frac{0^2}{5!} - \frac{0^4}{7!}$$

$$\boxed{\lim_{x\to 0}\frac{\sin x - x}{x^3} = -\frac{1}{6}}$$

Method 2: L'Hôpital's rule

$\frac{\sin 0 - 0}{0^3} = \frac{0}{0}$, which is one of the acceptable

l'Hôpital indeterminate forms, so rule may be used.

Differentiate top and bottom

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{\cos x - 1}{3x^2}$$

$\frac{\cos 0 - 1}{3(0)^2} = \frac{0}{0}$, which is one of the acceptable

l'Hôpital indeterminate forms, so rule may be used.

Differentiate top and bottom

$$\lim_{x\to 0}\frac{\cos x - 1}{3x^2} = \lim_{x\to 0}\frac{-\sin x}{6x}$$

$\frac{-\sin 0}{6(0)} = \frac{0}{0}$, which is one of the acceptable

l'Hôpital indeterminate forms, so rule may be used.

Differentiate top and bottom

$$\lim_{x\to 0}\frac{-\sin x}{6x} = \lim_{x\to 0}\frac{-\cos x}{6} = \frac{-\cos 0}{6}$$

$$\boxed{\lim_{x\to 0}\frac{\sin x - x}{x^3} = -\frac{1}{6}}$$