Question 1

(a) Find the indefinite integral for

$$\int \sin x \; dx$$

[1]

(a) $\int \sin x \, dx = -\cos x + C$ Formula booklet

$$\boxed{\int \sin x \; dx = -\cos x + c}$$

(b) Show that the exact value of the definite integral

$$\int_1^4 \frac{1}{x} \, dx$$

is $2 \ln 2$.

[3]

(c) Find the indefinite integral for

$$\int 7e^{7x} \, dx$$

[2]

(a) Find the indefinite integral for

$$\int \sin x \; dx$$

[1]

(b) Show that the exact value of the definite integral

$$\int_1^4 \frac{1}{x} \, dx$$

is $2 \ln 2$.

(b) $\int \frac{1}{x} \, dx = \ln|x| + C$ Formula booklet

$$\int_1^4 \frac{1}{x} \, dx = \Big[\ln|x|\Big]_1^4$$

$= \ln 4 - \ln 1$ $\quad \log_a \frac{x}{y} = \log_a x - \log_a y$ Formula booklet

$= \ln 4$

$= \ln 2^2$ $\quad \log_a x^m = m \log_a x$

$$\boxed{2\ln 2}$$

[3]

(c) Find the indefinite integral for

$$\int 7e^{7x} \, dx$$

[2]

(a) Find the indefinite integral for

$$\int \sin x \; dx$$

[1]

(b) Show that the exact value of the definite integral

$$\int_1^4 \frac{1}{x} \, dx$$

is $2\ln 2$.

[3]

(c) Find the indefinite integral for

$$\int 7e^{7x} \, dx$$

[2]

(c) $\int e^x \, dx = e^x + C$ ← formula booklet

$y = g(u) \qquad u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Use the chain rule in reverse

Let $y = e^{7x} \Rightarrow y = e^u$ and $u = 7x$

$\frac{dy}{du} = e^u \qquad \frac{du}{dx} = 7$

So, $\frac{dy}{dx} = 7e^{7x} \qquad \int \frac{dy}{dx} \, dx = y + c$

(indefinite integral as antiderivative)

$$\boxed{\int 7e^{7x} \, dx = e^{7x} + c}$$

Question 2

(a) Integrate

$$\int \cos 2x \; dx$$

[2]

(b) Show that

$$\int (3x-1)^3 \, dx = \frac{1}{12}(3x-1)^4 + c$$

where c is a constant of integration.

[3]

(c) Find an expression for y given that

$$\frac{dy}{dx} = e^{5x}$$

and that $y = 1$ when $x = 0$.

[3]

(a) $f(x) = \sin x \Rightarrow f'(x) = \cos x$ ← formula booklet

$y = g(u) \qquad u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Use the chain rule in reverse

Let $y = \sin 2x \Rightarrow y = \sin u$ and $u = 2x$

$\frac{dy}{du} = \cos u \qquad \frac{du}{dx} = 2$

So, $\frac{dy}{dx} = 2\cos 2x \qquad \int \frac{dy}{dx} \, dx = y + c$

(indefinite integral as antiderivative)

Adjust and compensate for the 2 in front

$$\frac{1}{2}\int 2\cos 2x = \frac{1}{2}\sin 2x + c$$

$$\boxed{\int \cos 2x = \frac{1}{2}\sin 2x + c}$$

EXAM PAPERS PRACTICE

(a) Integrate

$$\int \cos 2x \, dx$$

[2]

(b) Show that

$$\int (3x-1)^3 \, dx = \frac{1}{12}(3x-1)^4 + c$$

where c is a constant of integration.

[3]

(c) Find an expression for y given that

$$\frac{dy}{dx} = e^{5x}$$

and that $y = 1$ when $x = 0$.

[3]

(b) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$ Formula booklet

Integration by substitution

Let $u = 3x - 1$

Find dx in terms of du

$$\frac{du}{dx} = 3 \Rightarrow du = 3\,dx \Rightarrow dx = \frac{1}{3}du$$

Rewrite integral in terms of u and du

$$\int u^3 \times \frac{1}{3} du = \frac{1}{3}\int u^3 \, du = \frac{1}{3} \times \frac{u^4}{4} + c$$

$$\boxed{\int (3x-1)^3 \, dx = \frac{1}{12}(3x-1)^4 + c}$$

Alternative method

Note that the derivative of $(3x-1)^4$ is $12(3x-1)^3$ (chain rule)

Then

$$\int (3x-1)^3 \, dx = \frac{1}{12}\int 12(3x-1)^3 \, dx = \frac{1}{12}(3x-1)^4 + c$$

(a) Integrate

$$\int \cos 2x \ \mathrm{d}x$$

[2]

(b) Show that

$$\int (3x-1)^3 \ \mathrm{d}x = \frac{1}{12}(3x-1)^4 + c$$

where c is a constant of integration.

[3]

(c) Find an expression for y given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{5x}$$

and that $y = 1$ when $x = 0$.

[3]

$$\int e^x \ \mathrm{d}x = e^x + C$$ formula booklet

$$y = g(u) \qquad u = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

(c) Use the chain rule in reverse

Let $y = e^{5x} \Rightarrow y = e^u \quad u = 5x$

$\frac{dy}{du} = e^u \quad \frac{du}{dx} = 5$

So, $\frac{dy}{dx} = 5e^{5x} \qquad \int \frac{dy}{dx} \, dx = y + c$ (indefinite integral as antiderivative)

Adjust and compensate for the 5 in front

$$\frac{1}{5}\int 5e^{5x} \, dx = \frac{1}{5}e^{5x} + c$$

When $y = 1, \ x = 0$

$(1) = \frac{1}{5}e^{5(0)} + c \Rightarrow c = \frac{4}{5}$

$$\boxed{\int e^{5x} \, dx = \frac{1}{5}e^{5x} + \frac{4}{5}}$$

Question 3

(a) Find the indefinite integral for

$$\int \left(\sqrt{x} + \frac{3}{\sqrt{x}}\right) \mathrm{d}x$$

[3]

(b) Find the indefinite integral for

$$\int \frac{x^{\frac{2}{3}} + x^{\frac{11}{6}}}{x^2} \ \mathrm{d}x$$

[3]

(a) $\int \left(\sqrt{x} + \frac{3}{\sqrt{x}}\right) dx = \int \left(x^{1/2} + 3x^{-1/2}\right) dx$

$$\int x^n \ \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$ Formula booklet

$= \frac{2}{3}x^{3/2} + 6x^{1/2} + c$

$$\boxed{\frac{2}{3}x\sqrt{x} + 6\sqrt{x} + c}$$

(a) Find the indefinite integral for

$$\int \left(\sqrt{x} + \frac{3}{\sqrt{x}}\right) \mathrm{d}x$$

[3]

(b) Find the indefinite integral for

$$\int \frac{x^{\frac{2}{3}} + x^{\frac{11}{6}}}{x^2} \mathrm{d}x$$

[3]

(b) Simplify the expression before integrating

$$\int \frac{x^{2/3} + x^{11/6}}{x^2} \, dx = \int \left(x^{-4/3} + x^{-1/6}\right) dx$$

$$\boxed{-3x^{-1/3} + \frac{6}{5}x^{5/6} + c}$$

Question 4

(a) Given that $f(x) = 2x^3 + 4x$, find $f'(x)$.

[2]

(b) Hence, or otherwise, find

$$\int \frac{3x^2 + 2}{2x^3 + 4x} \mathrm{d}x$$

[3]

(a) $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ Formula booklet

$$\boxed{f'(x) = 6x^2 + 4}$$

(a) Given that $f(x) = 2x^3 + 4x$, find $f'(x)$.

[2]

$$\boxed{f'(x) = 6x^2 + 4}$$

(b) Hence, or otherwise, find

$$\int \frac{3x^2 + 2}{2x^3 + 4x} \mathrm{d}x$$

[3]

The key here is to spot that the numerator is a multiple of the derivative of the denominator

$$3x^2 + 2 = \frac{1}{2}(6x^2 + 4)$$

(b) $\int \frac{1}{x} \mathrm{d}x = \ln|x| + C$ Formula booklet

Integration by substitution

Let $u = 2x^3 + 4x$

Find du in terms of x

$$\frac{du}{dx} = 6x^2 + 4 \Rightarrow du = (6x^2 + 4)\, dx$$

$$\Rightarrow \frac{1}{2} du = (3x^2 + 2)\, dx$$

Rewrite integral in terms of u and du

$$\int \frac{3x^2 + 2}{2x^3 + 4x} \, dx = \int \frac{1}{u} \times \frac{1}{2} \, du$$

$$= \frac{1}{2} \int \frac{1}{u} \, du$$

$$= \frac{1}{2} \ln|u| + c$$

$$\boxed{\frac{1}{2} \ln|2x^3 + 4x| + c}$$

Question 5

Consider the function $f(x) = \ln(2x^2 + 1)$.

(a) Find $f'(x)$. [3]

(b) Hence, find

$$\int \frac{x}{2x^2+1}\,dx$$ [3]

$y = \ln(u) \quad u = 2x^2 + 1$

$\frac{dy}{du} = \frac{1}{u} = \frac{1}{2x^2+1}$

$\frac{du}{dx} = 4x$

$y = g(u)$ where $u = f(x)$

$\Longrightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ } Chain rule

a) $f'(x) = \frac{1}{2x^2+1}(4x)$

$$\boxed{f'(x) = \frac{4x}{2x^2+1}}$$

$f(x) = x^n \Longrightarrow f'(x) = nx^{n-1}$ } Derivative of x^n

$f(x) = \ln x \Longrightarrow f'(x) = \frac{1}{x}$ } Derivative of $\ln x$

Consider the function $f(x) = \ln(2x^2 + 1)$.

(a) Find $f'(x)$. [3]

$$\boxed{f'(x) = \frac{4x}{2x^2+1}}$$

(b) Hence, find

$$\int \frac{x}{2x^2+1}\,dx$$ [3]

b) $\int \frac{x}{2x^2+1}\,dx = \frac{1}{4}\int \frac{4x}{2x^2+1}\,dx$

adjust

compensate

$$\boxed{= \frac{1}{4}\ln(2x^2+1) + c}$$

$\int \frac{dy}{dx}\,dx = y + c$ (indefinite integral as antiderivative)

Question 6

Let $f'(x) = x^2\cos(x^3+1)$.

Find $f(x)$ given that $f(-1) = 1$. [5]

$\rightarrow f(x) = \int f'(x)\,dx$

$y = g(u)$ where $u = f(x)$

$\Longrightarrow \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ } Chain rule

$f(x) = \sin x \Longrightarrow f'(x) = \cos x$ } Derivative of $\sin x$

Let $y = \sin(\underbrace{x^3+1}_{u = x^3+1})$. Then: ($y = \sin(u)$)

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (\cos(x^3+1))(3x^2) = 3x^2\cos(x^3+1)$$

$$f(x) = \int x^2\cos(x^3+1)\,dx = \frac{1}{3}\int 3x^2\cos(x^3+1)\,dx$$

adjust

compensate

$$\int \frac{dy}{dx}\,dx = y + c \quad \left(\text{indefinite integral as antiderivative}\right)$$

$$\Longrightarrow f(x) = \frac{1}{3}\sin(x^3+1) + c$$

But $f(-1) = 1$, so

$$\frac{1}{3}\sin((-1)^3+1) + c = 1$$

$$\frac{1}{3}\sin(0) + c = 1 \Longrightarrow c = 1 \qquad \sin(0) = 0$$

$$\boxed{f(x) = \frac{1}{3}\sin(x^3+1) + 1}$$

Question 7

(a) Show that

$$\frac{\tan x}{\sin x\cos x} = \frac{1}{\cos^2 x}$$

[2]

(b) Hence find

$$\int \frac{3\tan x}{5\sin x\cos x}\,dx$$

[3]

(a) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ ← formula booklet

LHS: $$\frac{\tan x}{\sin x\cos x} = \frac{\left(\frac{\sin x}{\cos x}\right)}{\sin x\cos x}$$

$$= \frac{\cancel{\sin x}}{\cancel{\sin x}\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

LHS = RHS

$$\boxed{\frac{\tan x}{\sin x\cos x} = \frac{1}{\cos^2 x}}$$

(a) Show that

$$\frac{\tan x}{\sin x \cos x} = \frac{1}{\cos^2 x}$$

[2]

(b) Hence find

$$\int \frac{3\tan x}{5\sin x\cos x}\,dx$$

[3]

(b) $\displaystyle\int \frac{3\tan x}{5\sin x\cos x}\,dx = \int \frac{3}{5\cos^2 x}\,dx$

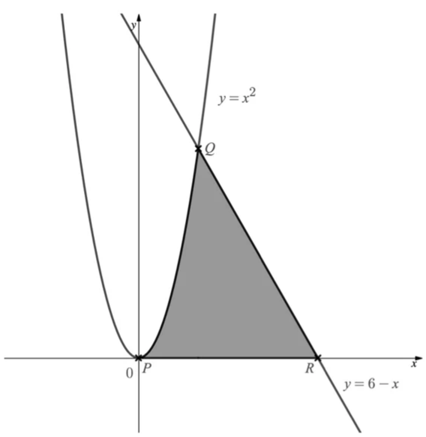
$f(x) = \tan x \Rightarrow f'(x) = \dfrac{1}{\cos^2 x}$ ← Formula booklet

$\displaystyle\int \frac{3}{5\cos^2 x}\,dx = \frac{3}{5}\int \frac{1}{\cos^2 x}\,dx$

$\boxed{\frac{3}{5}\tan x + c}$

Question 8

The diagram below shows the graphs of the line $y = 6 - x$ and the curve $y = x^2$.



Point P is the point of intersection of the curve $y = x^2$ with the x-axis. Point Q is the point of intersection of the curve $y = x^2$ with the line $y = 6 - x$ for which $x > 0$. Point R is the point of intersection of the line $y = 6 - x$ with the x-axis.

(a) Write down the x-coordinates of points P, Q and R.

[3]

(b) Calculate the area of the shaded region.

[2]

(a) P: $y = 0$ on the x-axis

$x^2 = 0 \Rightarrow x = 0$

$\boxed{x\text{-coordinate of } P = 0}$

Q: $x^2 = 6 - x$

$0 = x^2 + x - 6$

$0 = (x+3)(x-2)$

$x = \cancel{-3},\ 2$ ← $x > 0$

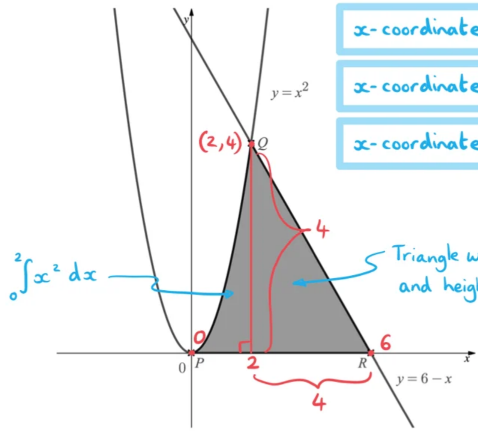
$\boxed{x\text{-coordinate of } Q = 2}$

R: $y = 0$ on the x-axis

$6 - x = 0 \Rightarrow x = 6$

$\boxed{x\text{-coordinate of } R = 6}$

The diagram below shows the graphs of the line $y = 6 - x$ and the curve $y = x^2$.



x-coordinate of $P = 0$

x-coordinate of $Q = 2$

x-coordinate of $R = 6$

Point P is the point of intersection of the curve $y = x^2$ with the x-axis. Point Q is the point of intersection of the curve $y = x^2$ with the line $y = 6 - x$ for which $x > 0$. Point R is the point of intersection of the line $y = 6 - x$ with the x-axis.

(a) Write down the x-coordinates of points P, Q and R.

[3]

(b) Calculate the area of the shaded region.

[2]

(b) Find y-coordinate of Q

$y = (2)^2 = 4 \Rightarrow Q(2,4)$

Area of the triangle

$A = \frac{1}{2}(bh)$ $\quad b$ is the base, h is the height ← Formula booklet

$A = \frac{1}{2} \times 4 \times 4 = 8$

Area between the curve $y = x^2$ and the x-axis for $0 \le x \le 2$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \ne -1$ ← Formula booklet

$\int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3 = \frac{8}{3}$

Can do on the GDC

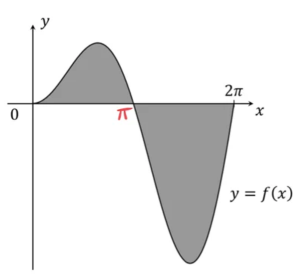
Total area $= 8 + \frac{8}{3}$

$\boxed{\text{Area of shaded region} = \frac{32}{3}\text{ units}^2}$

Question 9

The diagram below shows the graph of the function f which is defined by

$$f(x) = x\sin x, \quad 0 \le x \le 2\pi$$



The shaded region in the diagram is the region enclosed by the x-axis and the graph of $y = f(x)$.

(a) Find the area of the part of the shaded region that lies above the x-axis.

[4]

(b) Find the area of the entire shaded region.

[3]

(a) Find the limits for the part of the curve that lies above the x-axis

$f(x) = 0$ for $0 \le x \le 2\pi$

$x\sin x = 0$

$x = 0$ or $\sin x = 0$

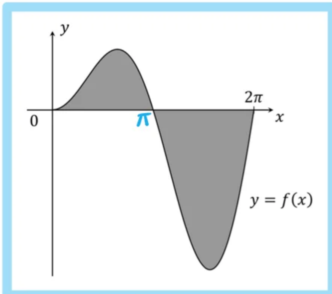
$\Rightarrow x = \pi, 2\pi$

Integrate $f(x)$ between the limits $x = 0, \pi$ using your GDC

$\int_0^{\pi} x\sin x\,dx = \boxed{\pi\text{ units}^2}$

The diagram below shows the graph of the function f which is defined by

$$f(x) = x\sin x, \quad 0 \le x \le 2\pi$$



The shaded region in the diagram is the region enclosed by the x-axis and the graph of $y = f(x)$.

(a) Find the area of the part of the shaded region that lies above the x-axis.

[4]

(b) Find the area of the entire shaded region.

[3]

(b) Integrate $f(x)$ between the limits $x = 0, 2\pi$ using your GDC

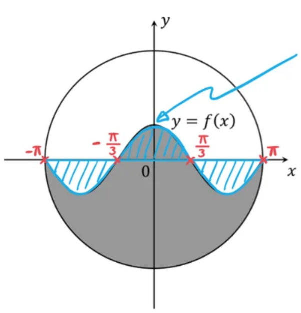
$A = \int_a^b |y| \,\mathrm{d}x$ ← Formula booklet

Remember to integrate the modulus of the function

$$\int_0^{2\pi} |x\sin x| \, dx = \boxed{4\pi \text{ units}^2}$$

Question 10

The diagram below depicts the design for a new company logo. The logo is formed by a circle centred on the origin, which is divided into two regions by the curve $y = f(x)$ where f is the function defined by $f(x) = \cos\frac{3x}{2}$, $-\pi \le x \le \pi$. The points where the circle and the curve intersect lie on the x-axis, as shown.



Each of these blue shaded parts has the same area as a result of the symmetry of the cosine curve

The shaded region in the diagram is the region inside that circle that lies below the curve $y = f(x)$.

(a) (i) Write down the radius of the circle that forms the outer border of the logo.

(ii) Hence determine the exact area of the shaded region.

[6]

(b) Find the percentage of the circular logo that is shaded.

[2]

(a) (i) The circle is centered on the origin and meets $f(x)$ at the x-axis, therefore the radius is the distance between these 2 points $(-\pi, 0)$ and $(\pi, 0)$

$$\boxed{\text{radius} = 2\pi \text{ units}}$$

(ii) Find all points of intersection of $f(x)$ with the x-axis

$$f(x) = \cos\frac{3x}{2} = 0 \quad \text{for } -\pi \le x \le x$$

$$\Rightarrow x = -\pi, -\frac{\pi}{3}, \frac{\pi}{3}, \pi$$

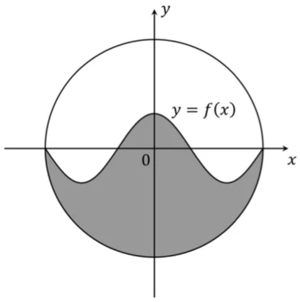
The total shaded area is a semi-circle plus one of the "humps" shaded in blue (the other two cancel each other out)

$$\text{Area} = \frac{1}{2} \times \pi \times \pi^2 - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\frac{3x}{2} \, dx$$

← solve with your GDC

$$\boxed{\frac{1}{2}\pi^3 - \frac{4}{3} \text{ units}^2}$$

The diagram below depicts the design for a new company logo. The logo is formed by a circle centred on the origin, which is divided into two regions by the curve $y = f(x)$ where f is the function defined by $f(x) = \cos\frac{3x}{2}$, $-\pi \leq x \leq \pi$. The points where the circle and the curve intersect lie on the x-axis, as shown.



The shaded region in the diagram is the region inside that circle that lies below the curve $y = f(x)$.

(a) (i) Write down the radius of the circle that forms the outer border of the logo.

(ii) Hence determine the exact area of the shaded region.

$\boxed{\frac{1}{2}\pi^3 - \frac{4}{3} \text{ units}^2}$ [6]

(b) Find the percentage of the circular logo that is shaded.

[2]

(b) Percentage shaded $= \dfrac{\frac{1}{2}\pi^3 - \frac{4}{3}}{\pi^3} \times 100$

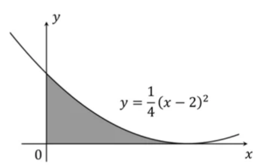
Area of whole circle

$= 45.6997954$

$\boxed{45.7\%}$

Question 11

The following diagram shows a part of the graph of the curve $y = \frac{1}{4}(x-2)^2$. The shaded region is the region enclosed by the graph and the positive x- and y-axes.



(a) (i) Find the coordinates of the points where the graph intersects the coordinate axes.

(ii) For the part of the curve that forms the boundary of the shaded region, show that $x = 2 - 2\sqrt{y}$.

[3]

(b) Find the area of the shaded region

(i) by calculating it as an area between the curve and the x-axis

(ii) by calculating it as an area between the curve and the y-axis

[6]

(c) Find the volume of the solid formed when the shaded region is rotated 2π radians about the x-axis.

[5]

(d) Find the volume of the solid formed when the shaded region is rotated 2π radians about the y-axis.

[5]

(a) (i) Find $(0, y)$, $(x, 0)$

When $x = 0 \Rightarrow y = \frac{1}{4}(0-2)^2$

$y = 1$ $\boxed{y\text{-intercept}: (0, 1)}$

When $y = 0 \Rightarrow 0 = \frac{1}{4}(x-2)^2$

$x = 2$ $\boxed{x\text{-intercept}: (2, 0)}$

(ii) Rearrange $y = \frac{1}{4}(x-2)^2$ to make x the subject

$y = \frac{1}{4}(x-2)^2$

$4y = (x-2)^2$

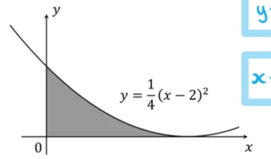
$\pm 2\sqrt{y} = x - 2 \Rightarrow x = 2 \pm 2\sqrt{y}$

The shaded region occurs when $x \leq 2$
$\therefore x = 2 - 2\sqrt{y}$

The following diagram shows a part of the graph of the curve $y = \frac{1}{4}(x-2)^2$. The shaded region is the region enclosed by the graph and the positive x- and y-axes.

y-intercept : (0, 1)

x-intercept : (2, 0)



(a) (i) Find the coordinates of the points where the graph intersects the coordinate axes.

(ii) For the part of the curve that forms the boundary of the shaded region, show that $x = 2 - 2\sqrt{y}$.

[3]

(b) Find the area of the shaded region

(i) by calculating it as an area between the curve and the x-axis

(ii) by calculating it as an area between the curve and the y-axis

[6]

(c) Find the volume of the solid formed when the shaded region is rotated 2π radians about the x-axis.

[5]

(d) Find the volume of the solid formed when the shaded region is rotated 2π radians about the y-axis.

[5]

(b) (i) $\displaystyle\int_0^2 \frac{1}{4}(x-2)^2\,dx$

Let $u = x - 2 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

Rewrite the integral with the substitutions

$$\frac{1}{4}\int_{x=0}^{x=2} (u)^2\,du$$

Convert the limits of the integral

$u = x - 2$, when $x = 2 \Rightarrow u = 0$

when $x = 0 \Rightarrow u = -2$

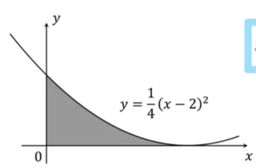
Rewrite the integral with the converted limits and evaluate

$$\frac{1}{4}\int_{u=-2}^{u=0} (u)^2\,du = \frac{1}{4}\left[\frac{1}{3}u^3\right]_{u=-2}^{u=0} = \frac{1}{4}\left[\frac{1}{3}(0)^3\right] - \frac{1}{4}\left[\frac{1}{3}(-2)^3\right]$$

$$= 0 + \frac{8}{12} \qquad \boxed{\frac{2}{3}\text{ units}^2}$$

The following diagram shows a part of the graph of the curve $y = \frac{1}{4}(x-2)^2$. The shaded region is the region enclosed by the graph and the positive x- and y-axes.

x-intercept : (2, 0)



(a) (i) Find the coordinates of the points where the graph intersects the coordinate axes.

(ii) For the part of the curve that forms the boundary of the shaded region, show that $x = 2 - 2\sqrt{y}$.

[3]

(b) Find the area of the shaded region

(i) by calculating it as an area between the curve and the x-axis

(ii) by calculating it as an area between the curve and the y-axis

[6]

(c) Find the volume of the solid formed when the shaded region is rotated 2π radians about the x-axis.

[5]

(d) Find the volume of the solid formed when the shaded region is rotated 2π radians about the y-axis.

[5]

(c) $V = \int_a^b \pi y^2\,dx$ ← Formula booklet

$$V = \int_0^2 \pi\left(\frac{1}{4}(x-2)^2\right)^2 dx = \frac{\pi}{16}\int_0^2 (x-2)^4\,dx$$

Let $u = x - 2 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

Rewrite the integral with the substitutions

$$\frac{\pi}{16}\int_{x=0}^{x=2} (u)^4\,du$$

Convert the limits of the integral

$u = x - 2$ when $x = 2 \Rightarrow u = 0$

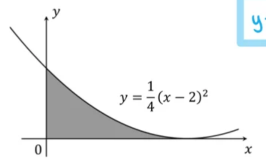
$x = 0 \Rightarrow u = -2$

Rewrite the integral with the converted limits and evaluate

$$\frac{\pi}{16}\int_{u=-2}^{u=0} (u)^4\,du = \frac{\pi}{16}\left[\frac{u^5}{5}\right]_{u=-2}^{u=0}$$

The following diagram shows a part of the graph of the curve $y = \frac{1}{4}(x-2)^2$. The shaded region is the region enclosed by the graph and the positive x- and y-axes.

y-intercept : (0, 1)



(a) (i) Find the coordinates of the points where the graph intersects the coordinate axes.

(ii) For the part of the curve that forms the boundary of the shaded region, show that $x = 2 - 2\sqrt{y}$.

[3]

(b) Find the area of the shaded region

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(c) Find the volume of the solid formed when the shaded region is rotated 2π radians about the x-axis.

[5]

(d) Find the volume of the solid formed when the shaded region is rotated 2π radians about the y-axis.

[5]

(d) $V = \int_a^b \pi x^2 \, dy$ ← Formula booklet

$$V = \int_0^1 \pi \left(2 - 2\sqrt{y}\right)^2 dy$$

$$= \pi \int_0^1 \left(2 - 2y^{1/2}\right)^2 dy = \pi \int_0^1 4 - 8y^{1/2} + 4y \, dy$$

$$= \pi \left[4y - \frac{16}{3}y^{3/2} + 2y^2\right]_0^1$$

$$= \pi \left[\left(4(1) - \frac{16}{3}(1)^{3/2} + 2(1)^2\right) - (0)\right]$$

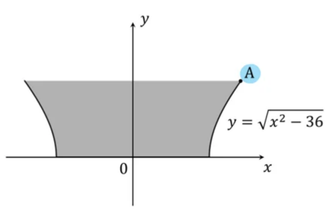
$$= \pi \left(4 - \frac{16}{3} + 2\right)$$

$$\boxed{\frac{2}{3}\pi \text{ units}^2}$$

Note: The volume when the curve is rotated around the y-axis is not necessarily the same as the volume when the curve is rotated around the x-axis

Question 12

The diagram below shows the cross-section of a bowl that a company is planning to begin producing.



As indicated on the diagram, one of the sides of the bowl in the cross-section may be described by the curve $y = \sqrt{x^2 - 36}$, where units for x and y are centimetres. The cross-section is entirely symmetrical about the y-axis. The flat circular bottom of the bowl has a diameter of 12 cm, and the vertical depth of the bowl is 6 cm. For purposes of answering this question, the thickness of the bottom and sides of the bowl may be regarded as negligible.

(a) Find the exact coordinates of the point marked A on the diagram.

[3]

(b) Show that the capacity of the bowl in cm^3 is given by

$$\pi \int_0^b (y^2 + 36) \, dy$$

where b is a constant to be determined.

[4]

(c) Hence find the capacity of the bowl.

[2]

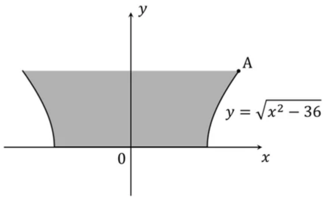
(a) Find x when $y = 6$

$6 = \sqrt{x^2 - 36}$

$x = \pm\sqrt{72} = \pm 6\sqrt{2}$

$$\boxed{A\left(6\sqrt{2}, 6\right)}$$

The diagram below shows the cross-section of a bowl that a company is planning to begin producing.



As indicated on the diagram, one of the sides of the bowl in the cross-section may be described by the curve $y = \sqrt{x^2 - 36}$, where units for x and y are centimetres. The cross-section is entirely symmetrical about the y-axis. The flat circular bottom of the bowl has a diameter of 12 cm, and the vertical depth of the bowl is 6 cm. For purposes of answering this question, the thickness of the bottom and sides of the bowl may be regarded as negligible.

(a) Find the exact coordinates of the point marked A on the diagram.

[3]

(b) Show that the capacity of the bowl in cm³ is given by

$$\pi \int_0^b (y^2 + 36)\,dy$$

where b is a constant to be determined.

[4]

(c) Hence find the capacity of the bowl.

[2]

(b) The capacity of the bowl is found by rotating the curve on one side of the y axis by 2π radians about the y-axis, $\therefore$ the upper limit of the integral will be the height of the bowl

$b = 6$

Write x^2 in terms of y

$y = \sqrt{x^2 - 36}$

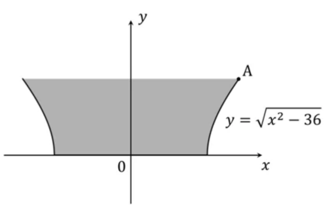
$\Rightarrow x^2 = y^2 + 36$ ← Rearrange for x^2 rather than x as it appears in the formula

$V = \int_a^b \pi x^2\,dy$ ← formula booklet

$$V = \int_0^6 \pi (y^2 + 36)\,dy$$

$$\boxed{V = \pi \int_0^6 (y^2 + 36)\,dy}$$

The diagram below shows the cross-section of a bowl that a company is planning to begin producing.



As indicated on the diagram, one of the sides of the bowl in the cross-section may be described by the curve $y = \sqrt{x^2 - 36}$, where units for x and y are centimetres. The cross-section is entirely symmetrical about the y-axis. The flat circular bottom of the bowl has a diameter of 12 cm, and the vertical depth of the bowl is 6 cm. For purposes of answering this question, the thickness of the bottom and sides of the bowl may be regarded as negligible.

(a) Find the exact coordinates of the point marked A on the diagram.

[3]

(b) Show that the capacity of the bowl in cm³ is given by

$$\pi \int_0^b (y^2 + 36)\,dy$$

where b is a constant to be determined.

$$\boxed{V = \pi \int_0^6 (y^2 + 36)\,dy}$$

[4]

(c) Hence find the capacity of the bowl.

[2]

(c) $V = \pi \left[\frac{1}{3}y^3 + 36y\right]_0^6$

$= \pi\left[\left(\frac{1}{3}(6)^3 + 36(6)\right) - \left(\frac{1}{3}(0)^3 + 36(0)\right)\right]$

$= \pi(288 - 0)$

$= 288\pi$

$= 904.778684\ldots$

$$\boxed{V = 905 \text{ cm}^3 \ (3 \text{ sf})}$$