Question 1

(a) Find the indefinite integral for

$$\int \sin x \, dx$$

[1]

(b) Find the exact value for

$$\int_1^4 \frac{1}{x} \, dx$$

[3]

(c) Find the indefinite integral for

$$\int 7e^{7x} \, dx$$

[2]

$\int \sin x \, dx = -\cos x + c$ } standard integral

a) $\boxed{-\cos x + c}$

(a) Find the indefinite integral for

$$\int \sin x \, dx$$

[1]

(b) Find the exact value for

$$\int_1^4 \frac{1}{x} \, dx$$

[3]

(c) Find the indefinite integral for

$$\int 7e^{7x} \, dx$$

[2]

$\int \frac{1}{x} \, dx = \ln|x| + c$ } standard integral

b) $\int_1^4 \frac{1}{x} \, dx = \left[\ln|x|\right]_1^4$

$= \ln|4| - \ln|1|$

$= \ln 4 - \ln 1$ $\quad \ln 1 = 0$

$\boxed{= \ln 4}$

(a) Find the indefinite integral for

$$\int \sin x \, dx$$

[1]

(b) Find the exact value for

$$\int_1^4 \frac{1}{x} \, dx$$

[3]

(c) Find the indefinite integral for

$$\int 7e^{7x} \, dx$$

[2]

c) Let $y = e^{7x}$. Then $y = e^u$ and $u = 7x$, so

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (e^u)(7) = 7e^u = 7e^{7x}$$

And $\int \frac{dy}{dx} \, dx = y + c$ (indefinite integral as antiderivative), so :

$$\boxed{\int 7e^{7x} \, dx = e^{7x} + c}$$

$f(x) = e^x \implies f'(x) = e^x$ } Derivative of e^x

$y = g(u)$ where $u = f(x)$

$\implies \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ } Chain rule

Question 2

(a) Integrate

$$\int \cos 2x \, dx$$

[2]

(b) Find the definite integral

$$\int_0^2 (3x-1)^3 \, dx$$

[4]

(c) Find an expression for y given that

$$\frac{dy}{dx} = e^{5x}$$

[2]

a) By the chain rule, $\frac{d}{dx}(\sin 2x) = 2 \cos 2x$

adjust

$$\int \cos 2x \, dx = \frac{1}{2} \int 2 \cos 2x \, dx$$

compensate

$$\boxed{= \frac{1}{2} \sin 2x + c}$$

(indefinite integral as antiderivative)

$f(x) = \sin x \implies f'(x) = \cos x$ } Derivative of $\sin x$

$y = g(u)$ where $u = f(x)$

$\implies \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ } Chain rule

(a) Integrate

$$\int \cos 2x \; dx$$

[2]

(b) Find the definite integral

$$\int_0^2 (3x-1)^3 \; dx$$

[4]

(c) Find an expression for y given that

$$\frac{dy}{dx} = e^{5x}$$

[2]

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad \Big\} \text{ Integral of } x^n \ (n \neq -1)$$

b) Let $u = 3x - 1$. Then: Substitution

$$\frac{du}{dx} = 3 \implies du = 3\,dx \implies dx = \frac{1}{3}du$$ Find du in terms of x

$$\left.\begin{aligned} x = 2 &\implies u = 3(2) - 1 = 5 \\ x = 0 &\implies u = 3(0) - 1 = -1 \end{aligned}\right\} \text{ Transform integration limits}$$

$$\int_0^2 (3x-1)^3 dx = \int_{-1}^5 u^3 \left(\tfrac{1}{3}du\right)$$

$$= \frac{1}{3}\int_{-1}^5 u^3 \, du$$

$$= \frac{1}{3}\left[\frac{1}{4}u^4\right]_{-1}^5$$

$$= \frac{1}{3}\left(\frac{1}{4}(5)^4 - \frac{1}{4}(-1)^4\right)$$

$$= \frac{1}{3}\left(\frac{625}{4} - \frac{1}{4}\right) = \frac{624}{12} = \boxed{52}$$

(a) Integrate

$$\int \cos 2x \; dx$$

[2]

ind the definite integral

$$\int_0^2 (3x-1)^3 \; dx$$

[4]

ind an expression for y given that

$$\frac{dy}{dx} = e^{5x}$$

[2]

$$f(x) = e^x \implies f'(x) = e^x \quad \Big\} \text{ Derivative of } e^x$$

$$\left.\begin{aligned} &y = g(u) \text{ where } u = f(x) \\ &\implies \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \end{aligned}\right\} \text{ Chain rule}$$

c) By the chain rule, $\frac{d}{dx}\left(e^{5x}\right) = 5e^{5x}$

$$y = \int e^{5x} dx = \underbrace{\frac{1}{5}}_{\text{compensate}} \int \overbrace{5}^{\text{adjust}} e^{5x} dx$$

$$\boxed{y = \frac{1}{5}e^{5x} + c}$$ (indefinite integral as antiderivative)

Question 3

Using a suitable substitution, show that

$$\int_1^2 \frac{x}{x+4}\,dx = 1 + 4\ln\frac{5}{6}$$

[7]

Substitution

Let $u = x + 4$. Then:

$\frac{du}{dx} = 1 \implies du = dx$ Find du in terms of x

$x = 2 \implies u = 2 + 4 = 6$
$x = 1 \implies u = 1 + 4 = 5$ } Transform integration limits

$u = x + 4 \implies x = u - 4$

$$\int_1^2 \frac{x}{x+4}\,dx = \int_5^6 \frac{u-4}{u}\,du = \int_5^6 \left(1 - \frac{4}{u}\right) du$$

$$= \left[u - 4\ln|u|\right]_5^6$$

$$= (6 - 4\ln 6) - (5 - 4\ln 5)$$

$$= 1 + 4(\ln 5 - \ln 6)$$

$$= 1 + 4\ln\frac{5}{6}$$

$\int \frac{1}{x}\,dx = \ln|x| + c$ } Standard integral

$\log_a \frac{x}{y} = \log_a x - \log_a y$ } Exponents and logarithms

Question 4

Given that $\cos 2\theta \equiv 2\cos^2\theta - 1$, find the exact value of

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

[6]

$$\cos 2\theta = 2\cos^2\theta - 1 \implies \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

$$\int_{\pi/4}^{\pi/2} \cos^2\theta\,d\theta = \int_{\pi/4}^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

By the chain rule, $\frac{d}{d\theta}(\sin 2\theta) = 2\cos 2\theta$

adjust

$$= \int_{\pi/4}^{\pi/2} \left(\frac{1}{2} + \frac{1}{4}(2\cos 2\theta)\right) d\theta$$

compensate

$$= \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{\pi/4}^{\pi/2}$$

$$= \left(\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)\right) - \left(\frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{1}{4}\sin 2\left(\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{8} + \frac{1}{4}\sin\pi - \frac{1}{4}\sin\frac{\pi}{2}$$

$$\boxed{= \frac{\pi}{8} - \frac{1}{4}}$$

$\sin\pi = 0$
$\sin\frac{\pi}{2} = 1$

$f(x) = \sin x \implies f'(x) = \cos x$ } Derivative of sin x

$y = g(u)$ where $u = f(x)$
$\implies \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ } Chain rule

Question 5

(a) Given that $f(x) = 2x^3 + 4x$, find $f'(x)$.

[2]

(b) Hence, or otherwise, find

$$\int \frac{3x^2 + 2}{2x^3 + 4x}\, dx$$

[4]

$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ } Derivative of x^n

a) $f'(x) = 2(3x^2) + 4 = \boxed{6x^2 + 4}$

(a) Given that $f(x) = 2x^3 + 4x$, find $f'(x)$. $\boxed{6x^2 + 4}$

[2]

(b) Hence, or otherwise, find

$$\int \frac{3x^2 + 2}{2x^3 + 4x}\, dx$$

[4]

The key here is to spot that the numerator is a multiple of the derivative of the denominator.

$3x^2 + 2 = \frac{1}{2}(6x^2 + 4)$

$\int \frac{1}{x}\, dx = \ln|x| + c$ } Standard integral

Substitution

b) Let $u = 2x^3 + 4x$. Then:

$\frac{du}{dx} = 6x^2 + 4$ from part (a)

$\Rightarrow du = (6x^2 + 4)\, dx$

$\Rightarrow \frac{1}{2} du = (3x^2 + 2)\, dx$

Find du in terms of x

$$\int \frac{3x^2 + 2}{2x^3 + 4x}\, dx = \int \frac{1}{u}\left(\frac{1}{2}\right) du$$

$$= \frac{1}{2}\int \frac{1}{u}\, du$$

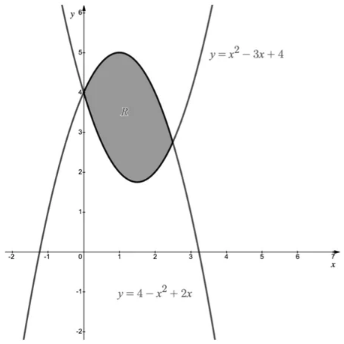
$$= \frac{1}{2}\ln|u| + c$$

$$\boxed{= \frac{1}{2}\ln|2x^3 + 4x| + c}$$

Question 6

The diagram below shows a sketch of the curves with equations

$y = x^2 - 3x + 4$ and $y = 4 - x^2 + 2x$



(a) Find the x-coordinates of the intersections of the two graphs.

[2]

(b) Show that the area of the shaded region labelled R is given by

$$\int_0^{\frac{5}{2}} (5x - 2x^2)\,dx$$

[2]

(c) Find the area of the shaded region labelled R.

[2]

a) $x^2 - 3x + 4 = 4 - x^2 + 2x$

$x^2 + x^2 - 3x - 2x + 4 - 4 = 0$

$2x^2 - 5x = 0$

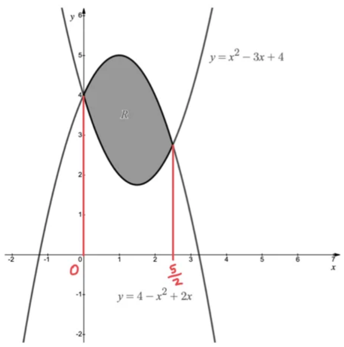
$x(2x - 5) = 0$

So $x = 0$ or $2x - 5 = 0 \Rightarrow x = \frac{5}{2}$

$\boxed{x = 0 \text{ and } x = \frac{5}{2}}$

The diagram below shows a sketch of the curves with equations

$y = x^2 - 3x + 4$ and $y = 4 - x^2 + 2x$



(a) Find the x-coordinates of the intersections of the two graphs.

$\boxed{x = 0 \text{ and } x = \frac{5}{2}}$

[2]

(b) Show that the area of the shaded region labelled R is given by

$$\int_0^{\frac{5}{2}} (5x - 2x^2)\,dx$$

[2]

(c) Find the area of the shaded region labelled R.

[2]

b) This is the area between two curves between $x = 0$ and $x = \frac{5}{2}$, so:

$$\text{Area} = \int_0^{5/2} [(\text{top curve}) - (\text{bottom curve})]\,dx$$

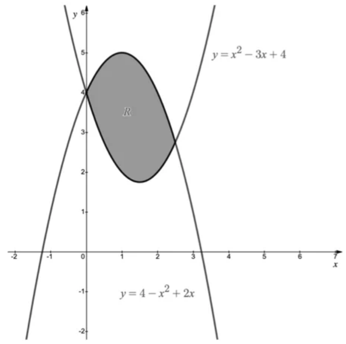
$$= \int_0^{5/2} [(4 - x^2 + 2x) - (x^2 - 3x + 4)]\,dx$$

$$= \int_0^{5/2} (4 - 4 + 2x + 3x - x^2 - x^2)\,dx$$

$$= \int_0^{5/2} (5x - 2x^2)\,dx$$

The diagram below shows a sketch of the curves with equations

$$y = x^2 - 3x + 4 \quad \text{and} \quad y = 4 - x^2 + 2x$$



(a) Find the x-coordinates of the intersections of the two graphs.

[2]

(b) Show that the area of the shaded region labelled R is given by

$$\int_0^{\frac{5}{2}} (5x - 2x^2)\, \mathrm{d}x$$

[2]

(c) Find the area of the shaded region labelled R.

[2]

c) Area of R $= \int_0^{5/2} (5x - 2x^2)\, dx$

$$\boxed{= \frac{125}{24} \text{ units}^2}$$ from GDC

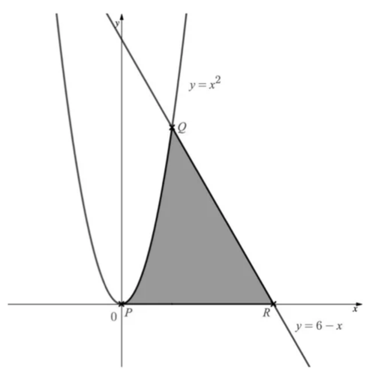
You could also work this out by hand via

$$\int_0^{5/2} (5x - 2x^2)\, dx = \left[\frac{5}{2}x^2 - \frac{2}{3}x^3\right]_0^{5/2}$$

$$= \left(\frac{125}{8} - \frac{250}{24}\right) - (0 - 0) = \frac{125}{24}$$

Question 7

The diagram below shows the graphs of the line $y = 6 - x$ and the curve $y = x^2$.



Point P is the point of intersection of the curve $y = x^2$ with the x-axis. Point Q is the point of intersection of the curve $y = x^2$ with the line $y = 6 - x$ for which $x > 0$. Point R is the point of intersection of the line $y = 6 - x$ with the x-axis.

(a) Work out the x-coordinates of points P, Q and R.

[3]

(b) Work out the area of the shaded region.

[4]

a) $x^2 = 0 \Rightarrow x = 0$ $\quad y = 0$ on the x-axis

$$\boxed{\text{The x-coordinate of } P = 0}$$

$$x^2 = 6 - x \Rightarrow x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

$$x = -3 \text{ or } x = 2$$

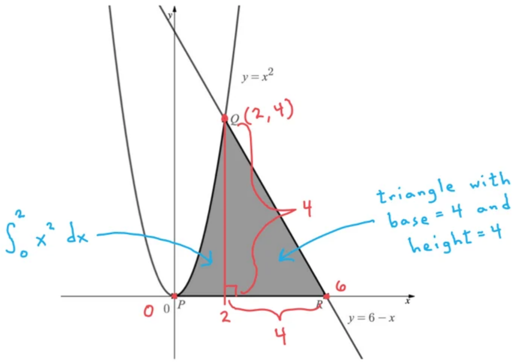
But $x > 0$, so:

$$\boxed{\text{The x-coordinate of } Q = 2}$$

$6 - x = 0 \Rightarrow x = 6$ $\quad y = 0$ on the x-axis

$$\boxed{\text{The x-coordinate of } R = 6}$$

The diagram below shows the graphs of the line $y = 6 - x$ and the curve $y = x^2$.



Point P is the point of intersection of the curve $y = x^2$ with the x-axis. Point Q is the point of intersection of the curve $y = x^2$ with the line $y = 6 - x$ for which $x > 0$. Point R is the point of intersection of the line $y = 6 - x$ with the x-axis.

(a) Work out the x-coordinates of points P, Q and R.

[3]

(b) Work out the area of the shaded region.

[4]

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ } Integral of x^n $(n \neq -1)$

b) When $x = 2$, $y = (2)^2 = 4$ (or $y = 6 - 2 = 4$) $\Rightarrow$ Q is the point $(2, 4)$

$A = \frac{1}{2}(bh)$ } Area of a triangle (b = base, h = height)

$\frac{1}{2}(4 \times 4) = \frac{1}{2}(16) = 8$ area of the triangle

$A = \int_a^b y \, dx$ } Area between a curve $y = f(x)$ and the x-axis, where $f(x) > 0$

$\int_0^2 x^2 \, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{(2)^3}{3} - \frac{(0)^3}{3} = \frac{8}{3} - 0 = \frac{8}{3}$

You could also evaluate the integral with your GDC.

Area of R $= 8 + \frac{8}{3} = \frac{32}{3}$

$\boxed{\text{Area of R} = \frac{32}{3} \text{ units}^2}$

Question 8

Consider the function $h(x)$ such that

$$\int_1^5 h(x)\,dx = 2.$$

(a) Find

$$\int_5^1 h(x)\,dx$$

[2]

(b) Find

$$\int_1^5 \frac{h(x)+1}{2}\,dx$$

[3]

(c) Find

$$\int_1^5 (h(x) + 2x)\,dx$$

[3]

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ [Property of definite integrals]

a) $\int_5^1 h(x)\,dx = -\int_1^5 h(x)\,dx = \boxed{-2}$

Consider the function $h(x)$ such that

$$\int_1^5 h(x)\,dx = 2.$$

(a) Find

$$\int_5^1 h(x)\,dx$$

[2]

(b) Find

$$\int_1^5 \frac{h(x)+1}{2}\,dx$$

[3]

(c) Find

$$\int_1^5 (h(x)+2x)\,dx$$

[3]

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad \Big\} \text{ Integral of } x^n \ (n \neq -1)$$

$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$$

[Properties of definite integrals]

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

b) $$\int_1^5 \frac{h(x)+1}{2}\,dx = \int_1^5 \tfrac{1}{2}(h(x)+1)\,dx$$

$$= \tfrac{1}{2}\int_1^5 (h(x)+1)\,dx$$

$$= \tfrac{1}{2}\left(\int_1^5 h(x)\,dx + \int_1^5 1\,dx\right)$$

And $\int_1^5 1\,dx = [x]_1^5 = 5-1 = 4$, so

$$\int_1^5 \frac{h(x)+1}{2}\,dx = \tfrac{1}{2}(2+4) = \boxed{3}$$

Consider the function $h(x)$ such that

$$\int_1^5 h(x)\,dx = 2.$$

(a) Find

$$\int_5^1 h(x)\,dx$$

[2]

(b) Find

$$\int_1^5 \frac{h(x)+1}{2}\,dx$$

[3]

(c) Find

$$\int_1^5 (h(x)+2x)\,dx$$

[3]

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad \Big\} \text{ Integral of } x^n \ (n \neq -1)$$

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

[Property of definite integrals]

c) $$\int_1^5 (h(x)+2x)\,dx = \int_1^5 h(x)\,dx + \int_1^5 2x\,dx$$

And $\int_1^5 2x\,dx = [x^2]_1^5 = 5^2 - 1^2 = 25 - 1 = 24$

So $\int_1^5 (h(x)+2x)\,dx = 2 + 24 = \boxed{26}$

Question 9

Consider the function $f(x) = \ln(2x^2 + 1)$.

(a) Find $f'(x)$. [3]

(b) Hence, find

$$\int \frac{x}{2x^2+1}\,dx$$ [3]

$y = \ln(u)$ $u = 2x^2 + 1$

$\frac{dy}{du} = \frac{1}{u} = \frac{1}{2x^2+1}$

$\frac{du}{dx} = 4x$

$y = g(u)$ where $u = f(x)$

$\Longrightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ } Chain rule

a) $f'(x) = \frac{1}{2x^2+1}(4x)$

$$\boxed{f'(x) = \frac{4x}{2x^2+1}}$$

$f(x) = x^n \Longrightarrow f'(x) = nx^{n-1}$ } Derivative of x^n

$f(x) = \ln x \Longrightarrow f'(x) = \frac{1}{x}$ } Derivative of $\ln x$

Consider the function $f(x) = \ln(2x^2 + 1)$.

(a) Find $f'(x)$. [3]

$$\boxed{f'(x) = \frac{4x}{2x^2+1}}$$

(b) Hence, find

$$\int \frac{x}{2x^2+1}\,dx$$ [3]

adjust

b) $\int \frac{x}{2x^2+1}\,dx = \frac{1}{4}\int \frac{4x}{2x^2+1}\,dx$

compensate

$$\boxed{= \frac{1}{4}\ln(2x^2+1) + c}$$

$\int \frac{dy}{dx}\,dx = y + c$ (indefinite integral as antiderivative)

Question 10

Let $f'(x) = x^2\cos(x^3+1)$.

Find $f(x)$ given that $f(-1) = 1$.

[5]

$f(x) = \int f'(x)\,dx$

$y = g(u)$ where $u = f(x)$

$\Longrightarrow \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ } Chain rule

$f(x) = \sin x \Longrightarrow f'(x) = \cos x$ } Derivative of $\sin x$

Let $y = \sin(\underbrace{x^3+1}_{u = x^3+1})$. Then: $y = \sin(u)$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (\cos(x^3+1))(3x^2) = 3x^2\cos(x^3+1)$$

$$f(x) = \int x^2\cos(x^3+1)\,dx = \frac{1}{3}\int 3x^2\cos(x^3+1)\,dx$$

adjust

compensate

$$\int \frac{dy}{dx}\,dx = y + c$$ (indefinite integral as antiderivative)

$$\Longrightarrow f(x) = \frac{1}{3}\sin(x^3+1) + c$$

But $f(-1) = 1$, so

$$\frac{1}{3}\sin((-1)^3+1) + c = 1$$

$$\frac{1}{3}\sin(0) + c = 1 \Longrightarrow c = 1 \qquad \sin(0) = 0$$

$$\boxed{f(x) = \frac{1}{3}\sin(x^3+1) + 1}$$