Question 1

Differentiate $\frac{5x^7}{\sin 2x}$ with respect to x.

[4]

Quotient Rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Let $u = 5x^7$ $\quad v = \sin 2x$

$\frac{du}{dx} = 35x^6$ $\quad \frac{dv}{dx} = 2\cos 2x$ chain rule

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{(\sin 2x)35x^6 - 5x^7(2\cos 2x)}{(\sin 2x)^2}$$

$$\boxed{\frac{d\left(\frac{5x^7}{\sin 2x}\right)}{dx} = \frac{35x^6\sin 2x - 10x^7\cos 2x}{\sin^2 2x}}$$

Question 2

Find $\frac{dy}{dx}$ for each of the following:

(a) $y = \cos(x^2 - 3x + 7) + \sin(e^x)$ [4]

(b) $y = \ln(2x^3)$ [3]

a) Chain rule 1

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Let $y = \cos u$ $\quad u = x^2 - 3x + 7$

$\frac{dy}{du} = -\sin u$ $\quad \frac{du}{dx} = 2x - 3$

$\frac{dy}{dx} = (-\sin u)(2x-3)$

sub in $u = x^2 - 3x + 7$

$\frac{dy}{dx} = (-\sin(x^2-3x+7))(2x-3)$

Chain rule 2

Let $y = \sin u$ $\quad u = e^x$

$\frac{dy}{du} = \cos u$ $\quad \frac{du}{dx} = e^x$

$\frac{dy}{dx} = (\cos u)e^x$

sub in $u = e^x$, $\frac{dy}{dx} = (\cos(e^x))e^x$

$$\boxed{\frac{dy}{dx} = -(2x-3)\sin(x^2-3x+7) + e^x\cos(e^x)}$$

Find $\frac{dy}{dx}$ for each of the following:

(a) $y = \cos(x^2 - 3x + 7) + \sin(e^x)$ [4]

(b) $y = \ln(2x^3)$ [3]

b) Method 1: Chain Rule

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$y = \ln u$ $\quad u = 2x^3$

$\frac{dy}{du} = \frac{1}{u}$ $\quad \frac{du}{dx} = 6x^2$

$\frac{dy}{dx} = \frac{1}{u}(6x^2)$

sub $u = 2x^3$

$\frac{dy}{dx} = \frac{1}{2x^3}(6x^2) = \boxed{\frac{3}{x}}$

OR

Method 2: Simplify using log laws

$y = \ln(2x^3) = \ln 2 + \ln x^3 = \ln 2 + 3\ln x$

$\frac{dy}{dx} = \boxed{\frac{3}{x}}$

Question 3

Differentiate with respect to x, simplifying your answers as far as possible:

(a) $(4\cos x - 3\sin x)e^{3x-5}$ [3]

(b) $(x^3 - 4x^2 + 7)\ln x$ [3]

a) Product rule

$\frac{dy}{dx} = u\frac{dv}{du} + v\frac{du}{dv}$

Let $u = 4\cos x - 3\sin x$ $\quad v = e^{3x-5}$

$\frac{du}{dx} = -4\sin x - 3\cos x$ $\quad \frac{dv}{dx} = 3e^{3x-5}$

$\frac{d(uv)}{dx} = (4\cos x - 3\sin x)3e^{3x-5} + e^{3x-5}(-4\sin x - 3\cos x)$

$= e^{3x-5}(12\cos x - 9\sin x - 4\sin x - 3\cos x)$

$= \boxed{e^{3x-5}(9\cos x - 13\sin x)}$

Differentiate with respect to x, simplifying your answers as far as possible:

(a) $(4\cos x - 3\sin x)e^{3x-5}$ [3]

(b) $(x^3 - 4x^2 + 7)\ln x$ [3]

b)

Let $u = x^3 - 4x^2 + 7$ $\quad v = \ln x$ $\quad \frac{dy}{dx} = u\frac{dv}{du} + v\frac{du}{dv}$

$\frac{du}{dx} = 3x^2 - 8x$ $\quad \frac{dv}{dx} = \frac{1}{x}$

$$\frac{d(uv)}{dx} = \frac{(x^3-4x^2+7)}{x} + (\ln x)(3x^2-8x)$$

$$= \boxed{x^2 - 4x + \frac{7}{x} + (\ln x)(3x^2 - 8x)}$$

Question 4

A curve has the equation $y = e^{-3x} + \ln x,\ x > 0$.

Find the gradient of the normal to the curve at the point $(1,\ e^{-3})$, giving your answer correct to 3 decimal places. [4]

differenciate using chain rule

$\frac{d(\ln x)}{dx} = \frac{1}{x}$

$m_{normal} = \frac{-1}{\frac{dy}{dx}}$

e^u $\quad u = -3x$ $\quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{d(e^u)}{du} = e^u$ $\quad \frac{du}{dx} = -3$

$\frac{d(e^{-3x})}{dx} = \frac{d(e^u)}{du} \times \frac{du}{dx}$ chain rule

$= e^u \times (-3)$

$= -3e^{-3x}$

$$\frac{dy}{dx} = -3e^{-3x} + \frac{1}{x}$$

sub $x = 1$

$$\frac{dy}{dx} = -3e^{-3(1)} + \frac{1}{1} = 1 - 3e^{-3}$$

$$m_{normal} = -\frac{1}{\frac{dy}{dx}} = \frac{-1}{1-3e^{-3}} = \boxed{-1.176} \text{ (3dp)}$$

Question 5

Find the equation of the tangent to the curve $y = e^{3x^2 + 5x - 2}$ at the point $(-2,\ 1)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[4]

$y - y_1 = m(x - x_1)$

$\downarrow$

$\frac{dy}{dx}$

Let $y = e^u$ $\quad u = 3x^2 + 5x - 2$ $\quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{du} = e^u$ $\quad \frac{du}{dx} = 6x + 5$

Chain rule

$\frac{dy}{dx} = (6x+5)e^u$

$= (6x+5)e^{3x^2+5x-2}$

At $x = -2$

$\frac{dy}{dx} = (6(-2)+5)e^{3(-2)^2+5(-2)-2}$

$= -7e^0 = -7 = m$

$y - (1) = -7(x - (-2))$

$y - 1 = -7x - 14$

$\boxed{y + 7x + 13 = 0}$

Question 6

Let $f(x) = \frac{g(x)}{h(x)}$, where $g(2) = 4$, $h(2) = -1$, $g'(2) = 0$ and $h'(2) = 2$.

Find the equation of the tangent of f at $x = 2$.

[6]

$f(2) = \frac{g(2)}{h(2)}$

$f(2) = -4$ $\quad$ point : $(2, -4)$

Quotient rule

$y = \frac{u}{v} \longrightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (in formula booklet)

$u = g(2) = 4$ $\quad v = h(2) = -1$

$\frac{du}{dx} = g'(2) = 0$ $\quad \frac{dv}{dx} = h'(2) = 2$

$f'(2) = \frac{\cancel{(-1)(0)} - (4)(2)}{(-1)^2}$

$f'(2) = -8$

Sub $(2,-4)$ and $m = -8$ into $y - y_1 = m(x - x_1)$.

$y + 4 = -8(x - 2)$

$y = -8x + 16 - 4$

$\boxed{y = -8x + 12}$

Question 7

A curve has the equation $y = x^3 - 12x + 7$.

(a) Find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]

(b) Determine the coordinates of the local minimum of the curve. [3]

a) Differenciate once

$$\frac{dy}{dx} = \boxed{3x^2 - 12}$$

... and differenciate again!

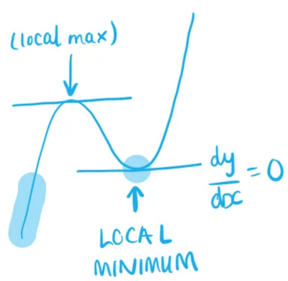
$$\frac{d^2y}{dx^2} = \boxed{6x}$$

A curve has the equation $y = x^3 - 12x + 7$.

(a) Find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]

$\frac{dy}{dx} = 3x^2 - 12$ $\quad$ $\frac{d^2y}{dx^2} = 6x$

(b) Determine the coordinates of the local minimum of the curve. [3]



b) Find x at the stationary points

$$\frac{dy}{dx} = 3x^2 - 12 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

Two values since $\frac{dy}{dx} = 0$ at local min & local max.

Classify stationary points

At $x = 2$ $\quad \frac{d^2y}{dx^2} = 6(2) = 12 > 0 \quad \therefore$ local min

At $x = -2$ $\quad \frac{d^2y}{dx^2} = 6(-2) = -12 < 0 \quad \therefore$ local max

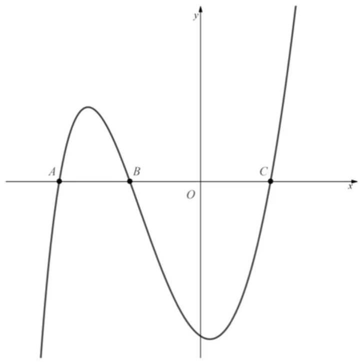
$$y = 2^3 - 12(2) + 7 = -9$$

LOCAL MINIMUM: $\boxed{(2, -9)}$

Question 8

The diagram below shows part of the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x)$ is the function defined by

$$\mathrm{f}(x) = (x^2 - 1)\ln(x + 3)\,, \quad x > -3$$



Points A, B and C are the three places where the graph intercepts the x-axis.

(a) Find $\mathrm{f}'(x)$.

[4]

(b) Show that the coordinates of point A are $(-2, 0)$.

[2]

(c) Find the equation of the tangent to the curve at point A.

[3]

a) Product rule $\quad \dfrac{dy}{dx} = u\dfrac{dv}{du} + v\dfrac{du}{dv}$

Let $u = x^2 - 1 \qquad v = \ln(x+3)$

$\dfrac{du}{dx} = 2x \qquad \dfrac{dv}{dx} = \dfrac{1}{x+3}$

$$\boxed{f'(x) = \frac{(x^2-1)}{x+3} + (\ln(x+3))2x}$$

The diagram below shows part of the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x)$ is the function defined by

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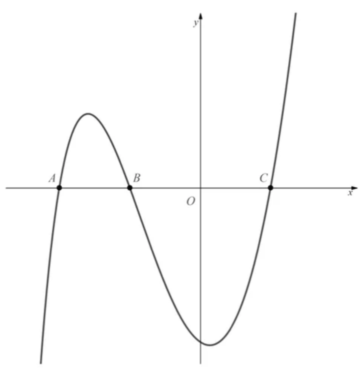


Points A, B and C are the three places where the graph intercepts the x-axis.

(a) Find $\mathrm{f}'(x)$.

[4]

(b) Show that the coordinates of point A are $(-2, 0)$.

[2]

(c) Find the equation of the tangent to the curve at point A.

[3]

b) At A, the curve intersects the x axis, so

$y = f(x) = 0$

$(x^2-1)\ln(x+3) = 0$

$(x+1)(x-1)\underbrace{\ln(x+3)}=0$

$\ln 1 = 0$

$x + 3 = 1$

$x = -1, 1 \qquad x = -2$

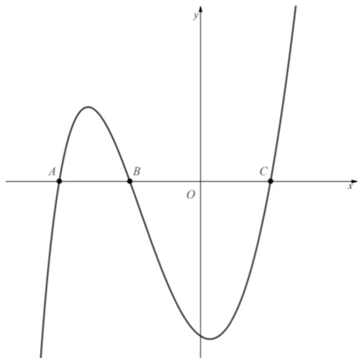
A is the most negative point of intersection,

$\therefore \boxed{A(-2, 0)}$

EXAM PAPERS PRACTICE

The diagram below shows part of the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x)$ is the function defined by

$$\mathrm{f}(x) = (x^2 - 1)\ln(x + 3), \ x > -3$$



Points A, B and C are the three places where the graph intercepts the x-axis.

(a) Find $\mathrm{f}'(x)$. [4]

$$f'(x) = \frac{x^2-1}{x+3} + 2x\ln(x+3)$$

(b) Show that the coordinates of point A are $(-2, 0)$. [2]

(c) Find the equation of the tangent to the curve at point A. [3]

c) $y - y_1 = m(x - x_1)$

When $x = -2$

$$m = f'(x) = \frac{(-2)^2-1}{(-2)+3} + 2(-2)\ln(-2+3)$$

$$= -4\ln 1 + 3 = 3$$

Sub in $x_1 = -2$, $y_1 = 0$ and $m = 3$

$$y - (0) = 3(x - (-2))$$

$$\boxed{y = 3x + 6}$$

Question 9

Let $f(x) = x^2 e^x$.

(a) Find $f'(x)$. [3]

(b) Find $f''(x)$. [3]

(c) Find the exact x coordinates of the points of inflection for the graph of f. [4]

(d) Find $\lim_{x \to -2} x^2 e^x$. [1]

a) Product rule (in formula booklet)

$$y = uv \longrightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$u = x^2$ $\qquad$ $v = e^x$

$\frac{du}{dx} = 2x$ $\qquad$ $\frac{dv}{dx} = e^x$

$$\boxed{f'(x) = x^2 e^x + 2x e^x}$$

EXAM PAPERS PRACTICE

Let $f(x) = x^2e^x$.

(a) Find $f'(x)$. $\boxed{f'(x) = x^2e^x + 2xe^x}$ [3]

(b) Find $f''(x)$. [3]

(c) Find the exact x coordinates of the points of inflection for the graph of f. [4]

(d) Find $\lim_{x \to -2} x^2e^x$. [1]

b) Product rule (in formula booklet)

$y = uv \longrightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$f'(x) = x^2e^x + 2xe^x$

$f'(x) = e^x(x^2 + 2x)$

$u = e^x \qquad v = x^2 + 2x$

$\frac{du}{dx} = e^x \qquad \frac{dv}{dx} = 2x + 2$

$f''(x) = e^x(2x + 2) + e^x(x^2 + 2x)$

$\boxed{f''(x) = e^x(x^2 + 4x + 2)}$

Let $f(x) = x^2e^x$.

(a) Find $f'(x)$. [3]

(b) Find $f''(x)$. $\boxed{f''(x) = e^x(x^2 + 4x + 2)}$ [3]

(c) Find the exact x coordinates of the points of inflection for the graph of f. [4]

(d) Find $\lim_{x \to -2} x^2e^x$. [1]

c) Points of inflection occur when $f''(x) = 0$.

$0 = e^x(x^2 + 4x + 2)$

$0 = x^2 + 4x + 2$

Quadratic formula

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (in formula booklet)

$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(2)}}{2(1)}$

$x = \frac{-4 \pm \sqrt{8}}{2}$

$x = \frac{\cancel{-4}^{-2} \pm \cancel{2}\sqrt{2}}{\cancel{2}}$

$\boxed{x = -2 \pm \sqrt{2}}$

Let $f(x) = x^2e^x$.

(a) Find $f'(x)$.

[3]

(b) Find $f''(x)$.

[3]

(c) Find the exact x coordinates of the points of inflection for the graph of f.

[4]

(d) Find $\lim_{x \to -2} x^2e^x$.

[1]

d) $\lim_{x \to -2} x^2 e^x = f(-2) = (-2)^2 e^{(-2)}$

$\lim_{x \to -2} x^2 e^x = f(-2) = \frac{4}{e^2}$

$\lim_{x \to -2} x^2 e^x = f(-2) = 0.54134\ldots$

$\boxed{\lim_{x \to -2} x^2 e^x = f(-2) \quad 0.541}$

Question 10

Let $f(x) = 2e^{2cosx}$, where $-\pi \le x \le \pi$.

(a) Find the number of points containing a horizontal tangent.

[1]

(b) Show algebraically that the gradient of the tangent at $x = \frac{\pi}{2}$ is -4.

[4]

(c) State the gradient of the tangent at $x = \frac{3\pi}{2}$.

[1]

It can be found that as the function, f, undergoes a transformation $f(kx)$, the number of stationary points found between $-\pi \le x \le \pi$ increases.

(d) Find the number of stationary points on f after a transformation of $f(2x)$ and hence, state the general rule representing the number of stationary points in terms of k where $k \in Z^+$.

[3]

a) Graph $f(x)$ on your GDC and count the number of points the gradient is 0 in the given domain.

$\boxed{3 \text{ points}}$

Let $f(x) = 2e^{2cosx}$, where $-\pi \leq x \leq \pi$.

(a) Find the number of points containing a horizontal tangent.

[1]

(b) Show algebraically that the gradient of the tangent at $x = \frac{\pi}{2}$ is -4.

[4]

(c) State the gradient of the tangent at $x = \frac{3\pi}{2}$.

[1]

It can be found that as the function, f, undergoes a transformation $f(kx)$, the number of stationary points found between $-\pi \leq x \leq \pi$ increases.

(d) Find the number of stationary points on f after a transformation of $f(2x)$ and hence, state the general rule representing the number of stationary points in terms of k where $k \in Z^+$.

[3]

b) Chain rule

$y = g(u)$, where $u = f(x)$

$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \times \frac{\cancel{du}}{dx}$ (in formula booklet)

$u = 2\cos x$ $\quad$ $y = 2e^u$

$\frac{du}{dx} = -2\sin x$ $\quad$ $\frac{dy}{du} = 2e^u$

$f'(x) = 2e^{2\cos x} \times -2\sin x$

$f'(x) = -4\sin x \, e^{2\cos x}$

Sub $x = \pi/2$ into $f'(x)$.

$f'(\pi/2) = -4\sin(\pi/2)e^{2\cos(\pi/2)}$

$f'(\pi/2) = -4(1)e^{2(0)}$

$\boxed{f'(\pi/2) = -4}$

Let $f(x) = 2e^{2cosx}$, where $-\pi \leq x \leq \pi$.

(a) Find the number of points containing a horizontal tangent.

[1]

(b) Show algebraically that the gradient of the tangent at $x = \frac{\pi}{2}$ is -4.

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(d) Find the number of stationary points on f after a transformation of $f(2x)$ and hence, state the general rule representing the number of stationary points in terms of k where $k \in Z^+$.

[3]

c) $f'(x) = -4\sin x \, e^{2\cos x}$

Sub $x = 3\pi/2$ into $f'(x)$.

$f'(3\pi/2) = -4\sin(3\pi/2)e^{2\cos(3\pi/2)}$

$f'(3\pi/2) = -4(-1)e^{2(0)}$

$\boxed{f'(3\pi/2) = 4}$

Let $f(x) = 2e^{2\cos x}$, where $-\pi \le x \le \pi$.

(a) Find the number of points containing a horizontal tangent.

$\boxed{\text{3 points}}$

[1]

(b) Show algebraically that the gradient of the tangent at $x = \frac{\pi}{2}$ is -4.

[4]

(c) State the gradient of the tangent at $x = \frac{3\pi}{2}$.

[1]

It can be found that as the function, f, undergoes a transformation $f(kx)$, the number of stationary points found between $-\pi \le x \le \pi$ increases.

(d) Find the number of stationary points on f after a transformation of $f(2x)$ and hence, state the general rule representing the number of stationary points in terms of k where $k \in Z^+$.

[3]

d) $f(x)$ has 3 stationary points in the given domain $(-\pi, 0, \pi)$.

$f(2x)$ has 5 stationary points in the given domain $(-\pi, -\pi/2, 0, \pi/2, \pi)$

$\boxed{\therefore \text{ rule} : 2k+1}$

Question 11

Let $f(x) = \sin x$ and $g(x) = \sin^2 x$, for $0 \le x \le 2\pi$.

Solve $f'(x) = g'(x)$.

[5]

Derivative of $\sin x$ (in formula booklet)

$f(x) = \sin x \longrightarrow f'(x) = \cos x$

Chain rule

$g(x) = \sin^2 x$

$g(x) = (\sin x)^2 \longrightarrow g'(x) = 2\sin x \cos x$

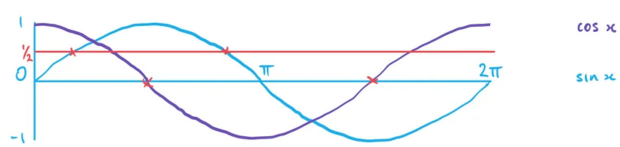
$f'(x) = g'(x)$

when $\cos x = 0$, $f'(x) = g'(x) = 0$.

$\cos x = 2\sin x \cos x$

$1 = 2\sin x$

$\sin x = 1/2$



$\boxed{\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}}$