IB Maths: AI HL

Further Differential Equations

Topic Questions

These practice questions can be used by students and teachers and is Suitable for IB Maths AI HL Topic Questions

Course	IB Maths
Section	5. Calculus
Topic	5.7 Further Differential Equations
Difficulty	Medium

Level: IB Maths

Subject: IB Maths AI HL

Board: IB Maths

Topic: Further Differential Equations

Question 1

Use the Euler method with a step size of 0.1 to find approximations for the values of x and y when $t=0.5$ for each of the following systems of coupled differential equations with the given initial conditions:

(a)

$$\frac{dx}{dt}=x^2+4ty$$

$$\frac{dy}{dt}=-3x+y-t$$

$$x=2,\ y=1 \ \ when \ \ t=0$$

[6 marks]

(b)

$$\dot{x}=-x+e^{-t}y$$

$$\dot{y}=e^{-t}x+y$$

$$x=1,\ y=-1 \ \ when \ \ t=0$$

[6 marks]

Question 2

Consider the following system of differential equations:

$$\frac{dx}{dt}=x+2y$$

$$\frac{dy}{dt}=-3x-4y$$

(a)

Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$.

[6 marks]

(b)
Hence write down the general solution of the system.

[2 marks]

When $t=0$, $x=-10$ and $y=17$.

(c)
Use the given initial condition to determine the exact solution of the system.

[3 marks]

(d)
By considering appropriate limits as $t \to \infty$, determine the long-term behaviour of the variables x and y.

[2 marks]

Question 3

The rates of change of two variables, x and y, are described by the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x + y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -5x - 2y$$

The matrix $\begin{pmatrix} 4 & 1 \\ -5 & -2 \end{pmatrix}$ has eigenvalues of 3 and -1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$. Initially $x=1$ and $y=3$.

a)
Use the above information to find the exact solution to the system of differential equations.

[5 marks]

(b)
Use the Euler method with a step size of 0.2 to find approximations for the values of x and y when $t=1$.

[6 marks]

(c)

(i)

Find the percentage error of the approximations from part (b) compared with the exact values of x and y when $t = 1$.

(ii)

Comment on the accuracy of the approximations in part (b), and explain how they could be improved.

[5 marks]

Question 4

For each of the general solutions to a system of coupled differential equations given below,

(i)

sketch the phase portrait for the system

(ii)

state whether the point $(0, 0)$ is a stable equilibrium point or an unstable equilibrium point.

(a)

$$\boldsymbol{x} = Ae^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{2t}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

[4 marks]

(b)

$$\boldsymbol{x} = Ae^{-2t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + Be^{-3t}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

[4 marks]

(c)

$$\boldsymbol{x} = Ae^{t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + Be^{-t}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

[4 marks]

Question 5

The behaviour of two variables, x and y, is modelled by the following system of differential equations:

$$\frac{dx}{dt} = 3x - 5y \qquad \frac{dy}{dt} = x - y$$

where $x = 1$ and $y = 1$ when $t = 0$.

The matrix $\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$ has eigenvalues of $1 + i$ and $1 - i$.

(a)

(i)

Find the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at the point $(0, 1)$.

(ii)

Hence sketch the phase portrait of the system with the given initial condition.

[4 marks]

It is suggested that the variables might better be described by the system

$$\frac{dx}{dt} = -3x - 5y \qquad \frac{dy}{dt} = x + y$$

with the same initial conditions.

(b) Calculate the eigen values of the matrix $\begin{pmatrix} -3 & -5 \\ 1 & 1 \end{pmatrix}$

[3 marks]

(c)

Hence describe how your phase portrait from part (a)(ii) would change to represent this new system of differential equations.

[2 marks]

Question 6

Scientists have been tracking levels, x and y, of two atmospheric pollutants, and recording the levels of each relative to historical baseline figures (so a positive value indicates an amount higher than the baseline and a negative value indicates an amount less than the baseline). Based on known interactions of the pollutants with each other and with other substances in the atmosphere, the scientists propose modelling the situation with the following system of differential equations:

$$\frac{dx}{dt} = x - 2y$$

$$\frac{dy}{dt} = x - y$$

a) Find the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at the points $(1, 0)$ and $(0, 1)$.

[2 marks]

(b) Find the eigenvalues of the matrix $\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$.

[3 marks]

At the start of the study both pollutants are above baseline levels, with $x = 5$ and $y = 3$.

(c)
Use the above information to sketch a phase portrait showing the long-term behaviour of x and y.

[4 marks]

Question 7

Two types of bacteria, X and Y, are being grown on a culture plate in a research lab. From past studies of the two bacteria and their interactions, the researchers believe that the growth of the two populations may be represented by the following differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -5x + 4y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -8x + 7y$$

for populations of x thousand and y thousand bacteria of types X and Y respectively. Initially the plate contains 20000 bacteria of type X and 21000 of type Y.

a)

Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -5 & 4 \\ -8 & 7 \end{pmatrix}$.

[6 marks]

(b) Find the values of $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ when $t = 0$.

[2 marks]

(c)

Sketch a possible trajectory for the growth of the two populations of bacteria, being sure to indicate any asymptotic behaviour.

[4 marks]