Question 1

Use the Euler method with a step size of 0.1 to find approximations for the values of x and y when $t = 0.5$ for each of the following systems of coupled differential equations with the given initial conditions:

(a)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 + 4ty$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -3x + y - t$$

$$x = 2,\ y = 1 \ \text{ when } \ t = 0$$

[6]

(b)

$$\dot{x} = -x + \mathrm{e}^{-t}y$$

$$\dot{y} = \mathrm{e}^{-t}x + y$$

$$x = 1,\ y = -1 \ \text{ when } \ t = 0$$

[6]

a)

Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$	h is a constant (step length)
Euler's method for coupled systems	$x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$ $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$ $t_{n+1} = t_n + h$	h is a constant (step length)

$t_{n+1} = t_n + 0.1$,

$x_{n+1} = x_n + 0.1(x_n^2 + 4t_n y_n)$, $\quad y_{n+1} = y_n + 0.1(-3x_n + y_n - t_n)$

Use recursion feature in GDC.

n	t_n	x_n	y_n
0	0	2	1
1	0.1	2.4	0.5
2	0.2	2.996	-0.18
3	0.3	3.8792	-1.116
4	0.4	5.25	-2.422
5	0.5	7.6187	-4.279

$\boxed{\text{at } t = 0.5,\ x = 7.62,\ y = -4.28 \ \ (3sf)}$

Use the Euler method with a step size of 0.1 to find approximations for the values of x and y when $t = 0.5$ for each of the following systems of coupled differential equations with the given initial conditions:

(a)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 + 4ty$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -3x + y - t$$

$$x = 2,\ y = 1 \ \text{ when } \ t = 0$$

[6]

(b)

$$\dot{x} = -x + \mathrm{e}^{-t}y$$

$$\dot{y} = \mathrm{e}^{-t}x + y$$

$$x = 1,\ y = -1 \ \text{ when } \ t = 0$$

[6]

b)

Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$	h is a constant (step length)
Euler's method for coupled systems	$x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$ $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$ $t_{n+1} = t_n + h$	h is a constant (step length)

$t_{n+1} = t_n + 0.1$,

$x_{n+1} = x_n + 0.1(-x_n + e^{-t_n} y_n)$, $\quad y_{n+1} = y_n + 0.1(e^{-t_n} x_n + y_n)$

Use recursion feature in GDC.

n	t_n	x_n	y_n
0	0	1	-1
1	0.1	0.8	-1
2	0.2	0.6295	-1.027
3	0.3	0.4824	-1.078
4	0.4	0.3542	-1.15
5	0.5	0.2416	-1.242

$\boxed{\text{at } t = 0.5,\ x = 0.242,\ y = -1.24 \ \ (3sf)}$

Question 2

Consider the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = -3x - 4y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$.

[6]

(b) Hence write down the general solution of the system.

[2]

When $t = 0$, $x = -10$ and $y = 17$.

(c) Use the given initial condition to determine the exact solution of the system.

[3]

(d) By considering appropriate limits as $t \to \infty$, determine the long-term behaviour of the variables x and y.

[2]

a) $|T - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ -3 & -4-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0$

$\boxed{\lambda = -1, -2}$

$\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (-1)\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x + 2y \\ -3x - 4y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$

$x = -y \rightarrow \boxed{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

$\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (-2)\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x + 2y \\ -3x - 4y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$

$3x = -2y \rightarrow \boxed{\begin{pmatrix} 2 \\ -3 \end{pmatrix}}$

Consider the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = -3x - 4y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$.

$\boxed{\lambda = -1, -2}$ $\boxed{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ $\boxed{\begin{pmatrix} 2 \\ -3 \end{pmatrix}}$

[6]

(b) Hence write down the general solution of the system.

[2]

When $t = 0$, $x = -10$ and $y = 17$.

(c) Use the given initial condition to determine the exact solution of the system.

[3]

(d) By considering appropriate limits as $t \to \infty$, determine the long-term behaviour of the variables x and y.

[2]

b)

Exact solution for coupled linear differential equations	$x = Ae^{\lambda_1 t}p_1 + Be^{\lambda_2 t}p_2$

$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}}$

Consider the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = -3x - 4y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$.

[6]

(b) Hence write down the general solution of the system.

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}}$$

[2]

When $t = 0$, $x = -10$ and $y = 17$.

(c) Use the given initial condition to determine the exact solution of the system.

[3]

(d) By considering appropriate limits as $t \to \infty$, determine the long-term behaviour of the variables x and y.

[2]

c) At $t = 0$ both exponentials equal 1, so

$$\begin{pmatrix} A + 2B \\ -A - 3B \end{pmatrix} = \begin{pmatrix} -10 \\ 17 \end{pmatrix} \longrightarrow A = 4, \ B = -7$$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = 4e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 7e^{-2t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}}$$

Consider the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = -3x - 4y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$.

[6]

(b) Hence write down the general solution of the system.

[2]

When $t = 0$, $x = -10$ and $y = 17$.

(c) Use the given initial condition to determine the exact solution of the system.

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = 4e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 7e^{-2t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}}$$

[3]

(d) By considering appropriate limits as $t \to \infty$, determine the long-term behaviour of the variables x and y.

[2]

d) $\lim_{x\to\infty} e^{-t} = \lim_{x\to\infty} e^{-2t} = 0$, so

$$\lim_{x\to\infty} \begin{pmatrix} x \\ y \end{pmatrix} = \lim_{x\to\infty}\left(4e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 7e^{-2t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\boxed{\therefore\ x \text{ and } y \text{ will converge to zero.}}$

Question 3

The rates of change of two variables, x and y, are described by the following system of differential equations:

$$\frac{dx}{dt} = 4x + y$$

$$\frac{dy}{dt} = -5x - 2y$$

The matrix $\begin{pmatrix} 4 & 1 \\ -5 & -2 \end{pmatrix}$ has eigenvalues of 3 and -1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$. Initially $x = 1$ and $y = 3$.

(a) Use the above information to find the exact solution to the system of differential equations.

[5]

(b) Use the Euler method with a step size of 0.2 to find approximations for the values of x and y when $t = 1$.

[6]

(c) (i) Find the percentage error of the approximations from part (b) compared with the exact values of x and y when $t = 1$.

(ii) Comment on the accuracy of the approximations in part (b), and explain how they could be improved.

[5]

a)

Exact solution for coupled linear differential equations	$\boldsymbol{x} = Ae^{\lambda_1 t}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{p}_2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

At $t = 0$ both exponentials equal 1, so

$$\begin{pmatrix} A + B \\ -A - 5B \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \longrightarrow A = 2, \ B = -1$$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = 2e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - e^{-t}\begin{pmatrix} 1 \\ -5 \end{pmatrix}}$$

The rates of change of two variables, x and y, are described by the following system of differential equations:

$$\frac{dx}{dt} = 4x + y$$

$$\frac{dy}{dt} = -5x - 2y$$

The matrix $\begin{pmatrix} 4 & 1 \\ -5 & -2 \end{pmatrix}$ has eigenvalues of 3 and -1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$. Initially $x = 1$ and $y = 3$.

(a) Use the above information to find the exact solution to the system of differential equations.

[5]

(b) Use the Euler method with a step size of 0.2 to find approximations for the values of x and y when $t = 1$.

[6]

(c) (i) Find the percentage error of the approximations from part (b) compared with the exact values of x and y when $t = 1$.

(ii) Comment on the accuracy of the approximations in part (b), and explain how they could be improved.

[5]

b)

Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$	h is a constant (step length)
Euler's method for coupled systems	$x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$ $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$ $t_{n+1} = t_n + h$	h is a constant (step length)

$t_{n+1} = t_n + 0.2$,

$x_{n+1} = x_n + 0.2(4x_n + y_n)$, $\quad y_{n+1} = y_n + 0.2(-5x_n - 2y_n)$

Use recursion feature in GDC.

n	t_n	x_n	y_n
0	0	1	3
1	0.2	2.4	0.8
2	0.4	4.48	-1.92
3	0.6	7.68	-5.632
4	0.8	12.697	-11.05
5	1.0	20.643	-19.33

$$\boxed{\text{at } t = 1, \ x = 20.6, \ y = -19.3 \quad (3sf)}$$

The rates of change of two variables, x and y, are described by the following system of differential equations:

$$\frac{dx}{dt} = 4x + y$$

$$\frac{dy}{dt} = -5x - 2y$$

The matrix $\begin{pmatrix} 4 & 1 \\ -5 & -2 \end{pmatrix}$ has eigenvalues of 3 and -1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$. Initially $x = 1$ and $y = 3$.

(a) Use the above information to find the exact solution to the system of differential equations.

[5]

(b) Use the Euler method with a step size of 0.2 to find approximations for the values of x and y when $t = 1$.

[6]

(c) (i) Find the percentage error of the approximations from part (b) compared with the exact values of x and y when $t = 1$.

(ii) Comment on the accuracy of the approximations in part (b), and explain how they could be improved.

[5]

c) i) Exact values are

$$\begin{pmatrix} x \\ y \end{pmatrix} = 2e^3\begin{pmatrix} 1 \\ -1 \end{pmatrix} - e^{-1}\begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 2e^3 - e^{-1} \\ -2e^3 + 5e^{-1} \end{pmatrix} = \begin{pmatrix} 39.8031\ldots \\ -38.3316\ldots \end{pmatrix}$$

Percentage error	$\varepsilon = \left\lvert \frac{v_A - v_E}{v_E} \right\rvert \times 100\%$	v_E is the exact value and v_A is the approximate value of v

$$\varepsilon_x = \left\lvert \frac{(2e^3 - e^{-1}) - 20.6438\ldots}{(2e^3 - e^{-1})} \right\rvert \times 100\% = 48.1352\ldots$$

$$\varepsilon_y = \left\lvert \frac{(-2e^3 + 5e^{-1}) - (-19.3331\ldots)}{(-2e^3 + 5e^{-1})} \right\rvert \times 100\% = 49.5635\ldots$$

$\boxed{\varepsilon_x = 48.1\%, \quad \varepsilon_y = 49.6\% \quad (3sf)}$

ii) The approximations are inaccurate. They could be improved by decreasing the step size.

Question 4

For each of the general solutions to a system of coupled differential equations given below,

(i) sketch the phase portrait for the system

(ii) state whether the point $(0, 0)$ is a stable equilibrium point or an unstable equilibrium point.

(a)
$$\mathbf{x} = Ae^{t}\begin{pmatrix}1\\1\end{pmatrix} + Be^{2t}\begin{pmatrix}-1\\3\end{pmatrix}$$
[4]

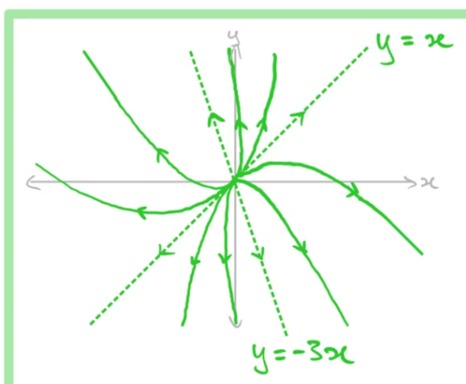
(b)
$$\mathbf{x} = Ae^{-2t}\begin{pmatrix}4\\1\end{pmatrix} + Be^{-3t}\begin{pmatrix}1\\5\end{pmatrix}$$
[4]

(c)
$$\mathbf{x} = Ae^{t}\begin{pmatrix}-1\\1\end{pmatrix} + Be^{-t}\begin{pmatrix}4\\3\end{pmatrix}$$
[4]

a) Your sketch needs to reflect these things:

- Both eigenvalues are positive, so all solutions will move away from the origin.
- Straight lines through the origin parallel to the eigenvectors (so here $y = x$ and $y = -3x$), these correspond to the solutions when either A or B is zero.
- Between the eigenvector lines, the paths will curve towards the eigenvector line with the greater eigenvalue (so here, as t increases, e^{2t} will dominate e^{t}).

i)



ii) Unstable (even moving a little bit away from the origin, x and y will start flowing outwards.

For each of the general solutions to a system of coupled differential equations given below,

(i) sketch the phase portrait for the system

(ii) state whether the point $(0, 0)$ is a stable equilibrium point or an unstable equilibrium point.

(a)

$$\boldsymbol{x} = Ae^{t}\begin{pmatrix}1\\1\end{pmatrix} + Be^{2t}\begin{pmatrix}-1\\3\end{pmatrix}$$

[4]

(b)

$$\boldsymbol{x} = Ae^{-2t}\begin{pmatrix}4\\1\end{pmatrix} + Be^{-3t}\begin{pmatrix}1\\5\end{pmatrix}$$

[4]

(c)

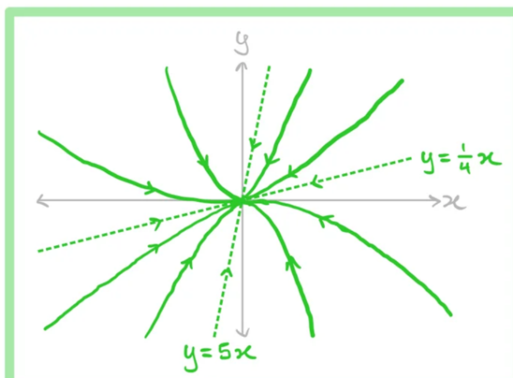
$$\boldsymbol{x} = Ae^{t}\begin{pmatrix}-1\\1\end{pmatrix} + Be^{-t}\begin{pmatrix}4\\3\end{pmatrix}$$

[4]

b) Your sketch needs to reflect these things:

- Both eigenvalues are negative, so all solutions will move towards the origin.
- Straight lines through the origin parallel to the eigenvectors (so here $y = \frac{1}{4}x$ and $y = 5x$), these correspond to the solutions when either A or B is zero.
- Between the eigenvector lines, the paths will curve towards the eigenvector line with the more negative eigenvalue (so here, as t increases in the negative direction, e^{-3t} will dominate e^{-2t}).

i)



ii) Stable (moving a little bit away from the origin, x and y will start flowing inward).

For each of the general solutions to a system of coupled differential equations given below,

(i) sketch the phase portrait for the system

(ii) state whether the point $(0, 0)$ is a stable equilibrium point or an unstable equilibrium point.

(a)

$$\boldsymbol{x} = Ae^{t}\begin{pmatrix}1\\1\end{pmatrix} + Be^{2t}\begin{pmatrix}-1\\3\end{pmatrix}$$

[4]

(b)

$$\boldsymbol{x} = Ae^{-2t}\begin{pmatrix}4\\1\end{pmatrix} + Be^{-3t}\begin{pmatrix}1\\5\end{pmatrix}$$

[4]

(c)

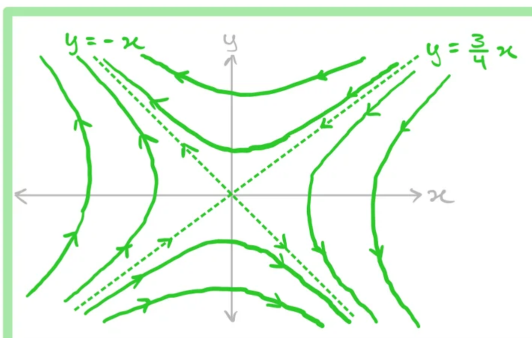
$$\boldsymbol{x} = Ae^{t}\begin{pmatrix}-1\\1\end{pmatrix} + Be^{-t}\begin{pmatrix}4\\3\end{pmatrix}$$

[4]

c) Your sketch needs to reflect these things:

- One eigenvalue is positive and one is negative, so the origin is a saddle point.
- Straight lines through the origin parallel to the eigenvectors (so here $y = -x$ and $y = \frac{3}{4}x$), these correspond to the solutions when either A or B is zero. Direction is away from the origin along $y = -x$ and towards the origin along $y = \frac{3}{4}x$
- The other paths will loop around between the eigenvector lines. For the directions, the solutions will move along the curves towards the origin when those curves are closer to $y = \frac{3}{4}x$, then swoop around and move away from the origin when those curves are closer to $y = -x$.

i)



ii) Unstable (once you move away from the origin, x and y will get caught up in one of the flows and end up moving off one way or the other in the direction of $y = -x$.

Question 5

The behaviour of two variables, x and y, is modelled by the following system of differential equations:

$$\frac{dx}{dt} = 3x - 5y \qquad \frac{dy}{dt} = x - y$$

where $x = 1$ and $y = 1$ when $t = 0$.

The matrix $\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$ has eigenvalues of $1 + i$ and $1 - i$.

(a) (i) Find the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at the point $(0, 1)$.

(ii) Hence sketch the phase portrait of the system with the given initial condition.

[4]

It is suggested that the variables might better be described by the system

$$\frac{dx}{dt} = -3x - 5y \qquad \frac{dy}{dt} = x + y$$

with the same initial conditions.

(b) Calculate the eigenvalues of the matrix $\begin{pmatrix} -3 & -5 \\ 1 & 1 \end{pmatrix}$

[3]

(c) Hence describe how your phase portrait from part (a)(ii) would change to represent this new system of differential equations.

[2]

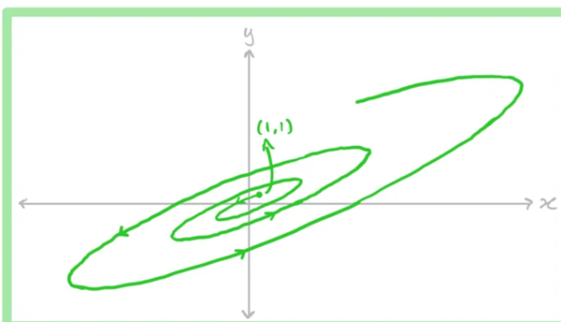
a) At $(0, 1)$

$\frac{dx}{dt} = 3(0) - 5(1) = -5 \qquad \frac{dy}{dt} = (0) - (1) = -1$

i) $\boxed{\dot{x} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}}$

ii) Your sketch needs to reflect these things:

- Complex eigenvalue with real parts means it's spiralling away from the origin.
- Negative $\frac{dx}{dt}$ at y-axis means it goes anti-clockwise.
- Initial point $(1,1)$ needs to be marked, with spiral spiralling outwards from there.
- $\dot{x} \begin{pmatrix} -5 \\ -1 \end{pmatrix}$ means it's going left and down at the y-axis, this gives the general 'tilt' of the spiral.



The behaviour of two variables, x and y, is modelled by the following system of differential equations:

$$\frac{dx}{dt} = 3x - 5y \qquad \frac{dy}{dt} = x - y$$

where $x = 1$ and $y = 1$ when $t = 0$.

The matrix $\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$ has eigenvalues of $1 + i$ and $1 - i$.

(a) (i) Find the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at the point $(0, 1)$.

(ii) Hence sketch the phase portrait of the system with the given initial condition.

[4]

It is suggested that the variables might better be described by the system

$$\frac{dx}{dt} = -3x - 5y \qquad \frac{dy}{dt} = x + y$$

with the same initial conditions.

(b) Calculate the eigenvalues of the matrix $\begin{pmatrix} -3 & -5 \\ 1 & 1 \end{pmatrix}$

[3]

(c) Hence describe how your phase portrait from part (a)(ii) would change to represent this new system of differential equations.

[2]

b) $|T - \lambda I| = 0$

$$\begin{vmatrix} -3-\lambda & -5 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0$$

$\boxed{\lambda = -1 \pm i}$

The behaviour of two variables, x and y, is modelled by the following system of differential equations:

$$\frac{dx}{dt} = 3x - 5y \qquad \frac{dy}{dt} = x - y$$

where $x = 1$ and $y = 1$ when $t = 0$.

The matrix $\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}$ has eigenvalues of $1 + i$ and $1 - i$.

(a) (i) Find the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at the point $(0, 1)$.

(ii) Hence sketch the phase portrait of the system with the given initial condition.

[4]

It is suggested that the variables might better be described by the system

$$\frac{dx}{dt} = -3x - 5y \qquad \frac{dy}{dt} = x + y$$

with the same initial conditions.

(b) Calculate the eigenvalues of the matrix $\begin{pmatrix} -3 & -5 \\ 1 & 1 \end{pmatrix}$

[3]

(c) Hence describe how your phase portrait from part (a)(ii) would change to represent this new system of differential equations.

[2]

c) The spiral will spiral towards the origin because the real parts of the eigenvalues are now both negative.

Question 6

Scientists have been tracking levels, x and y, of two atmospheric pollutants, and recording the levels of each relative to historical baseline figures (so a positive value indicates an amount higher than the baseline and a negative value indicates an amount less than the baseline). Based on known interactions of the pollutants with each other and with other substances in the atmosphere, the scientists propose modelling the situation with the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - 2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - y$$

(a) Find the values of $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ at the points $(1, 0)$ and $(0, 1)$.

[2]

(b) Find the eigenvalues of the matrix $\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$.

[3]

At the start of the study both pollutants are above baseline levels, with $x = 5$ and $y = 3$.

(c) Use the above information to sketch a phase portrait showing the long-term behaviour of x and y.

[4]

a) At $(1,0)$

$\frac{dx}{dt} = (1) - 2(0) = 1$ $\quad \frac{dy}{dt} = (1) - (0) = 1$

$$\boxed{\dot{x} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

At $(0,1)$

$\frac{dx}{dt} = (0) - 2(1) = -2$ $\quad \frac{dy}{dt} = (0) - (1) = -1$

$$\boxed{\dot{x} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}}$$

Scientists have been tracking levels, x and y, of two atmospheric pollutants, and recording the levels of each relative to historical baseline figures (so a positive value indicates an amount higher than the baseline and a negative value indicates an amount less than the baseline). Based on known interactions of the pollutants with each other and with other substances in the atmosphere, the scientists propose modelling the situation with the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - 2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - y$$

(a) Find the values of $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ at the points $(1, 0)$ and $(0, 1)$.

[2]

(b) Find the eigenvalues of the matrix $\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$.

[3]

At the start of the study both pollutants are above baseline levels, with $x = 5$ and $y = 3$.

(c) Use the above information to sketch a phase portrait showing the long-term behaviour of x and y.

[4]

b) $|T - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\boxed{\lambda = \pm i}$$

Scientists have been tracking levels, x and y, of two atmospheric pollutants, and recording the levels of each relative to historical baseline figures (so a positive value indicates an amount higher than the baseline and a negative value indicates an amount less than the baseline). Based on known interactions of the pollutants with each other and with other substances in the atmosphere, the scientists propose modelling the situation with the following system of differential equations:

$$\frac{dx}{dt} = x - 2y$$

$$\frac{dy}{dt} = x - y$$

(a) Find the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at the points $(1, 0)$ and $(0, 1)$.

[2]

(b) Find the eigenvalues of the matrix $\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$.

[3]

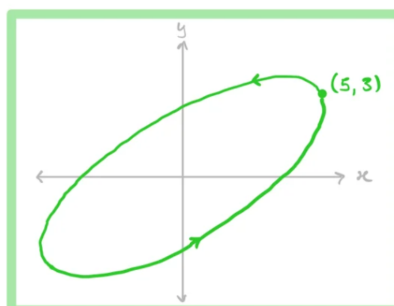
At the start of the study both pollutants are above baseline levels, with $x = 5$ and $y = 3$.

(c) Use the above information to sketch a phase portrait showing the long-term behaviour of x and y.

[4]

c) Your sketch needs to reflect these things:

- Purely imaginary eigenvalues means that all possible paths will be circles or ellipses around the origin.
- The derivatives show thall all possible paths will be going up and to the right' at the positive x-axis and 'down and to the left at the positive y-axis. So, the paths will go anti-clockwise.
- The initial point $(5, 3)$ should be marked.
- The derivative vectors don't make right angles with the axes – so it's an ellipse.



Question 7

Two types of bacteria, X and Y, are being grown on a culture plate in a research lab. From past studies of the two bacteria and their interactions, the researchers believe that the growth of the two populations may be represented by the following differential equations

$$\frac{dx}{dt} = -5x + 4y \qquad \frac{dy}{dt} = -8x + 7y$$

for populations of x thousand and y thousand bacteria of types X and Y respectively. Initially the plate contains 20000 bacteria of type X and 21000 of type Y.

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -5 & 4 \\ -8 & 7 \end{pmatrix}$.

[6]

(b) Find the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ when $t = 0$.

[2]

(c) Sketch a possible trajectory for the growth of the two populations of bacteria, being sure to indicate any asymptotic behaviour.

[4]

a) $|T - \lambda I| = 0$

$\begin{vmatrix} -5-\lambda & 4 \\ -8 & 7-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$

$\boxed{\lambda = 3, -1}$

$\begin{pmatrix} -5 & 4 \\ -8 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (3)\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -5x + 4y \\ -8x + 7y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$

$2x = y \rightarrow \boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

$\begin{pmatrix} -5 & 4 \\ -8 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (-1)\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -5x + 4y \\ -8x + 7y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$

$x = y \rightarrow \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

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[4]

b) At $t = 0 \rightarrow x = 20,\ y = 21$

$\boxed{\frac{dx}{dt} = -5(20) + 4(21) = -16 \qquad \frac{dy}{dt} = -8(20) + 7(21) = -20}$

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[2]

(c) Sketch a possible trajectory for the growth of the two populations of bacteria, being sure to indicate any asymptotic behaviour.

[4]

c) Note $x, y \geq 0$

Your sketch needs to reflect these things:

- Starts at $(20, 21)$ and heads off 'down and to the left.'
- Eigenvector lines $y = 2x$ (and $y = x$).
- Trajectory will head towards and ultimately have an asymptote at the eigenvector line corresponding to the positive eigenvalue (in this case $y = 2x$) - so it will need to loop around from its starting direction and head off in the direction of $y = 2x$.