Question 1

Consider $w = \frac{z_1}{z_2}$, where $z_1 = 2 + 2\sqrt{3}i$ and $z_2 = 2 + 2i$.

(a) Express w in the form $w = a + bi$. [3]

(b) Write the complex numbers z_1 and z_2 in the form $re^{i\theta}$, $r \geq 0$, $-\pi < \theta < \pi$. [4]

(c) Express w in the form $re^{i\theta}$, $r \geq 0$, $-\pi < \theta < \pi$. [3]

(a) $\omega = \frac{2+2\sqrt{3}i}{2+2i} \times \frac{2-2i}{2-2i}$

Multiply by conjugate of denominator to cancel i

$= \frac{4+4\sqrt{3}i-4i+4\sqrt{3}}{4+4i-4i+4}$

$= \frac{4(1+\sqrt{3}) + 4(\sqrt{3}-1)i}{8}$

$\boxed{w = \frac{(1+\sqrt{3})}{2} + \frac{(\sqrt{3}-1)}{2}i}$

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(c) Express w in the form $re^{i\theta}$, $r \geq 0$, $-\pi < \theta < \pi$. [3]

(b) Find the modulus and argument for z_1 and z_2

$|z_1| = \sqrt{2^2 + (2\sqrt{3})^2}$

$= \sqrt{4+12}$

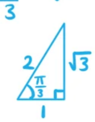
$= 4$

$\theta_1 = \tan^{-1}\left(\frac{\cancel{2}\sqrt{3}}{\cancel{2}}\right)$

$= \frac{\pi}{3}$

Trig exact values

$\tan\frac{\pi}{3} = \sqrt{3}$



$\boxed{z_1 = 4e^{i\left(\frac{\pi}{3}\right)}}$

$|z_2| = \sqrt{2^2 + 2^2}$

$= 2\sqrt{2}$

$\theta_2 = \tan^{-1}\left(\frac{\cancel{2}}{\cancel{2}}\right)$

$= \frac{\pi}{4}$

Trig exact values

$\tan\frac{\pi}{4} = 1$

$\boxed{z_2 = 2\sqrt{2}e^{i\left(\frac{\pi}{4}\right)}}$

Consider $w = \frac{z_1}{z_2}$, where $z_1 = 2 + 2\sqrt{3}i$ and $z_2 = 2 + 2i$.

(a) Express w in the form $w = a + bi$.

[3]

(b) Write the complex numbers z_1 and z_2 in the form $re^{i\theta}$, $r \geq 0$, $-\pi < \theta < \pi$.

$\boxed{z_1 = 4e^{i\left(\frac{\pi}{3}\right)}}$

[4]

(c) Express w in the form $re^{i\theta}$, $r \geq 0$, $-\pi < \theta < \pi$.

$\boxed{z_2 = 2\sqrt{2}\, e^{i\left(\frac{\pi}{4}\right)}}$

[3]

(c) $w = \dfrac{4e^{i\frac{\pi}{3}}}{2\sqrt{2}\, e^{i\frac{\pi}{4}}}$

$= \dfrac{2}{\sqrt{2}} e^{i\left(\frac{\pi}{3} - \frac{\pi}{4}\right)}$

$= \dfrac{\cancel{2}\sqrt{2}}{\cancel{2}} e^{i\left(\frac{\pi}{12}\right)}$

$\boxed{w = \sqrt{2}\, e^{i\left(\frac{\pi}{12}\right)}}$

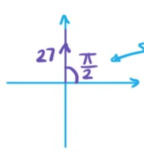
Question 2

Solve the equation $z^3 = 27i$, giving your answers in the form $a + bi$.

[6]

Write the number $27i$ in polar form

Sketching an Argand diagram can help



Use $\theta = \frac{\pi}{2} + 2k\pi$ for multiple solutions

$z^3 = 27e^{i\left(\frac{\pi}{2} + 2k\pi\right)}$

$z = 3e^{i\left(\frac{\pi}{2} + 2k\pi\right) \times \frac{1}{3}}$

$z = 3e^{i\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)}$

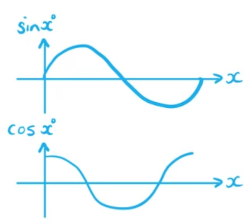
$z = 3e^{i\left(\frac{\pi(1+4k)}{6}\right)}$

Remember exact trig values and the sine and cosine graphs

Exact trig values:

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\qquad$ $\sin\frac{\pi}{6} = \frac{1}{2}$

$\cos\frac{\pi}{2} = 0$ $\qquad$ $\sin\frac{\pi}{2} = 1$



As it is a cubic, there should be 3 solutions, find z for $k = 0, 1, 2$

$k = 0: \quad z = 3e^{i\left(\frac{\pi}{6}\right)} = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$\boxed{z = \frac{3\sqrt{3}}{2} + \frac{3}{2}i}$

$k = 1: \quad z = 3e^{i\left(\frac{5\pi}{6}\right)} = 3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$\boxed{z = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i}$

$k = 2: \quad z = 3e^{i\left(\frac{3\pi}{2}\right)} = 3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 3(0 - i)$

$\boxed{z = -3i}$

Question 3

Let $z_1 = 6\text{cis}\left(\frac{\pi}{6}\right)$ and $z_2 = 3\sqrt{2}e^{i\left(\frac{\pi}{4}\right)}$.

(a) Giving your answers in the form $r\text{cis}\theta$, find

(i) $z_1 z_2$

(ii) $\frac{z_1}{z_2}$.

[4]

(b) Write z_1 and z_2 in the form $a + bi$.

[2]

(c) Find $z_1 + z_2$, giving your answer in the form $a + bi$.

[2]

It is given that z_1^* and z_2^* are the complex conjugates of z_1 and z_2 respectively.

(d) Find $z_1^* + z_2^*$, giving your answer in the form $a + bi$.

[2]

(a) (i) $z_1 z_2 = 6e^{i\left(\frac{\pi}{6}\right)} \times 3\sqrt{2}e^{i\left(\frac{\pi}{4}\right)}$

$= 18\sqrt{2}\,e^{i\left(\frac{\pi}{6}+\frac{\pi}{4}\right)}$

$= 18\sqrt{2}\,e^{i\left(\frac{5\pi}{12}\right)}$

$$\boxed{z_1 z_2 = 18\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)}$$

(ii) $\frac{z_1}{z_2} = \frac{6e^{i\left(\frac{\pi}{6}\right)}}{3\sqrt{2}e^{i\left(\frac{\pi}{4}\right)}}$

$= \frac{2}{\sqrt{2}}\,e^{i\left(\frac{\pi}{6}-\frac{\pi}{4}\right)}$

$= \sqrt{2}\,e^{i\left(-\frac{\pi}{12}\right)}$

$$\boxed{\frac{z_1}{z_2} = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{12}\right)}$$

Let $z_1 = 6\text{cis}\left(\frac{\pi}{6}\right)$ and $z_2 = 3\sqrt{2}e^{i\left(\frac{\pi}{4}\right)}$.

(a) Giving your answers in the form $r\text{cis}\theta$, find

(i) $z_1 z_2$

(ii) $\frac{z_1}{z_2}$. [4]

(b) Write z_1 and z_2 in the form $a + bi$. [2]

(c) Find $z_1 + z_2$, giving your answer in the form $a + bi$. [2]

It is given that z_1^* and z_2^* are the complex conjugates of z_1 and z_2 respectively.

(d) Find $z_1^* + z_2^*$, giving your answer in the form $a + bi$. [2]

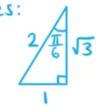
(b) $z_1 = x + yi$

Express $\tan\theta$ in terms of x and y

$$\tan\frac{\pi}{6} = \left(\frac{y}{x}\right)$$

Exact trig values:

$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$



$$\frac{1}{\sqrt{3}} = \frac{y}{x}$$

$\frac{y}{x}$ is a multiple of $\frac{1}{\sqrt{3}}$

$$\Rightarrow \quad y = \frac{x}{\sqrt{3}}$$

Find the modulus, 6 in terms of x and y

$$6 = \sqrt{x^2 + y^2}$$

Solve simultaneous equations

$$6 = \sqrt{x^2 + \left(\frac{x}{\sqrt{3}}\right)^2}$$

$$36 = \frac{4}{3}x^2$$

$$x = 3\sqrt{3} \quad \Rightarrow \quad y = 3$$

$$\boxed{z_1 = 3\sqrt{3} + 3i}$$

$z_2 = x + yi$

Express $\tan\theta$ in terms of x and y

$$\tan\frac{\pi}{4} = \left(\frac{y}{x}\right)$$

Exact trig values:

$\tan\left(\frac{\pi}{4}\right) = 1$

$$\Rightarrow y = x$$

Find the modulus, $3\sqrt{2}$ in terms of x and y

$$3\sqrt{2} = \sqrt{x^2 + y^2}$$

Solve simultaneous equations

$$3\sqrt{2} = \sqrt{x^2 + x^2}$$

$$18 = 2x^2$$

$$x = 3 \quad \Rightarrow \quad y = 3$$

$$\boxed{z_2 = 3 + 3i}$$

Let $z_1 = 6\text{cis}\left(\frac{\pi}{6}\right)$ and $z_2 = 3\sqrt{2}e^{i\left(\frac{\pi}{4}\right)}$.

(a) Giving your answers in the form $r\text{cis}\theta$, find

(i) $z_1 z_2$

(ii) $\frac{z_1}{z_2}$.

[4]

(b) Write z_1 and z_2 in the form $a + bi$.

[2]

(c) Find $z_1 + z_2$, giving your answer in the form $a + bi$.

[2]

It is given that z_1^* and z_2^* are the complex conjugates of z_1 and z_2 respectively.

(d) Find $z_1^* + z_2^*$, giving your answer in the form $a + bi$.

[2]

Exact trig values:

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\quad \sin\frac{\pi}{6} = \frac{1}{2}$

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

(c) $z_1 + z_2 = 6\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) + 3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= 6\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) + 3\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= 3\sqrt{3} + 3i + 3 + 3i$

$\boxed{z_1 + z_2 = 3(1+\sqrt{3}) + 6i}$

Question 3d

Let $z_1 = 6\text{cis}\left(\frac{\pi}{6}\right)$ and $z_2 = 3\sqrt{2}e^{i\left(\frac{\pi}{4}\right)}$.

(a) Giving your answers in the form $r\text{cis}\theta$, find

(i) $z_1 z_2$

(ii) $\frac{z_1}{z_2}$.

[4]

(b) Write z_1 and z_2 in the form $a + bi$.

$\boxed{z_1 = 3\sqrt{3} + 3i}$ $\quad \boxed{z_2 = 3 + 3i}$

[2]

(c) Find $z_1 + z_2$, giving your answer in the form $a + bi$.

[2]

It is given that z_1^* and z_2^* are the complex conjugates of z_1 and z_2 respectively.

(d) Find $z_1^* + z_2^*$, giving your answer in the form $a + bi$.

[2]

(d) $z_1^* = 3\sqrt{3} - 3i \qquad z_2^* = 3 - 3i$

$z_1^* + z_2^* = (3\sqrt{3} - 3i) + (3 - 3i)$

$= 3\sqrt{3} + 3 - 3i - 3i$

$\boxed{z_1^* + z_2^* = 3(\sqrt{3}+1) - 6i}$

Question 4a

Let $z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = 2 + 2i$.

(a) Express

(i) z_1 in the form $a + bi$

(ii) z_2 in the form $r\text{cis}\theta$.

[2]

(b) Find $w_1 = z_1 + z_2$, giving your answer in the form $a + bi$.

[2]

(c) Find $w_2 = z_1 z_2$, giving your answer in the form $r\text{cis}\theta$.

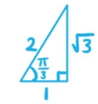
[3]

(d) Sketch w_1 and w_2 on a single Argand diagram.

[2]

(a) (i) $z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ ← Trig exact values

$\cos\frac{\pi}{3} = \frac{1}{2}$



$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$\boxed{z_1 = 1 + \sqrt{3}i}$

(ii) $|z_2| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$\theta_2 = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$ ← Trig exact values

$\tan\left(\frac{\pi}{4}\right) = 1$

$\boxed{z_2 = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)}$

Let $z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = 2 + 2i$.

(a) Express

(i) z_1 in the form $a + bi$

$\boxed{z_1 = 1 + \sqrt{3}i}$

(ii) z_2 in the form $r\text{cis}\theta$.

[2]

(b) Find $w_1 = z_1 + z_2$, giving your answer in the form $a + bi$.

[2]

(c) Find $w_2 = z_1 z_2$, giving your answer in the form $r\text{cis}\theta$.

[3]

(d) Sketch w_1 and w_2 on a single Argand diagram.

[2]

(b) $w_1 = (1 + \sqrt{3}i) + (2 + 2i)$

$\boxed{w_1 = 3 + (2 + \sqrt{3})i}$

Let $z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = 2 + 2i$.

(a) Express

(i) z_1 in the form $a + bi$

(ii) z_2 in the form $r\text{cis}\theta$.

$$\boxed{z_2 = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)}$$

[2]

(b) Find $w_1 = z_1 + z_2$, giving your answer in the form $a + bi$.

[2]

(c) Find $w_2 = z_1 z_2$, giving your answer in the form $r\text{cis}\theta$.

[3]

(d) Sketch w_1 and w_2 on a single Argand diagram.

[2]

(c) $w_2 = \left(2\text{cis}\left(\frac{\pi}{3}\right)\right)\left(2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)$

$= 2e^{i\left(\frac{\pi}{3}\right)} \times 2\sqrt{2}e^{i\left(\frac{\pi}{4}\right)}$

$= 4\sqrt{2}e^{i\left(\frac{\pi}{3}+\frac{\pi}{4}\right)}$

$$\boxed{w_2 = 4\sqrt{2}\,\text{cis}\left(\frac{7\pi}{12}\right)}$$

Let $z_1 = 2\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = 2 + 2i$.

(a) Express

(i) z_1 in the form $a + bi$

(ii) z_2 in the form $r\text{cis}\theta$.

[2]

(b) Find $w_1 = z_1 + z_2$, giving your answer in the form $a + bi$.

$$\boxed{w_1 = 3 + \left(2+\sqrt{3}\right)i}$$

[2]

(c) Find $w_2 = z_1 z_2$, giving your answer in the form $r\text{cis}\theta$.

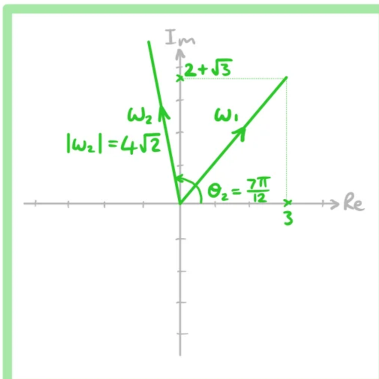
$$\boxed{w_2 = 4\sqrt{2}\,\text{cis}\left(\frac{7\pi}{12}\right)}$$

[3]

(d) Sketch w_1 and w_2 on a single Argand diagram.

[2]

(d)



Question 5

It is given that that $z_1 = 2e^{i\left(\frac{\pi}{3}\right)}$ and $z_2 = 3\text{cis}\left(\frac{n\pi}{12}\right)$, $n \in \mathbb{Z}^+$.

(a) Find the value of $z_1 z_2$ for $n = 3$. [3]

(b) Find the least value of n such that $z_1 z_2 \in \mathbb{R}^+$. [3]

(a)

$$z_1 z_2 = 2e^{i\left(\frac{\pi}{3}\right)} \times 3e^{i\left(\frac{3\pi}{12}\right)}$$

$$= 6e^{i\left(\frac{\pi}{3} + \frac{3\pi}{12}\right)}$$

$$= 6e^{i\left(\frac{7\pi}{12}\right)}$$

$$\boxed{z_1 z_2 = 6\,\text{cis}\left(\frac{7\pi}{12}\right)}$$

It is given that that $z_1 = 2e^{i\left(\frac{\pi}{3}\right)}$ and $z_2 = 3\text{cis}\left(\frac{n\pi}{12}\right)$, $n \in \mathbb{Z}^+$.

(a) Find the value of $z_1 z_2$ for $n = 3$. [3]

(b) Find the least value of n such that $z_1 z_2 \in \mathbb{R}^+$. [3]

(b) $z_1 z_2 = 6e^{i\left(\frac{\pi}{3} + \frac{n\pi}{12}\right)}$

$$= 6e^{i\left(\frac{(4+n)\pi}{12}\right)}$$

For $z_1 z_2$ to be $\mathbb{R}^+$, it must lie on the positive, real axis and θ must therefore be a multiple of 2π

The smallest value for n will be when $\theta = 2\pi$

$$\frac{(4+n)\cancel{\pi i}}{12} = 2\cancel{\pi i}$$

$$4+n = 24$$

$$\boxed{n = 20}$$

Question 6

Consider the complex number $w = \frac{z_1}{z_2}$ where $z_1 = 3 - \sqrt{3}i$ and $z_2 = 2\text{cis}\left(\frac{2\pi}{3}\right)$.

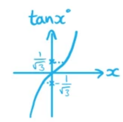
(a) Express w in the form $r\text{cis}\theta$. [5]

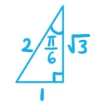
(a) Rewrite z_1 in the form of $r\text{cis}\theta$

$|z_1| = \sqrt{3^2 + (-\sqrt{3})^2} = 2\sqrt{3}$

$\theta_1 = \tan^{-1}\left(\frac{-\sqrt{3}}{3}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ Trig exact values

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$





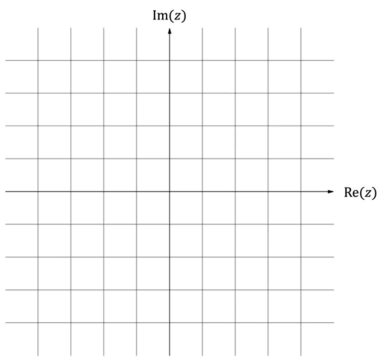
$= -\frac{\pi}{6}$

$w = \frac{2\sqrt{3}e^{i\left(-\frac{\pi}{6}\right)}}{2e^{i\left(\frac{2\pi}{3}\right)}}$

$= \frac{2\sqrt{3}}{2}e^{i\left(-\frac{\pi}{6} - \frac{2\pi}{3}\right)}$

$\boxed{w = \sqrt{3}\,\text{cis}\left(-\frac{5\pi}{6}\right)}$

(b) Sketch z_1, z_2 and w on the Argand diagram below.



[3]

(c) Find the smallest positive integer value of n such that w^n is a real number.

[2]

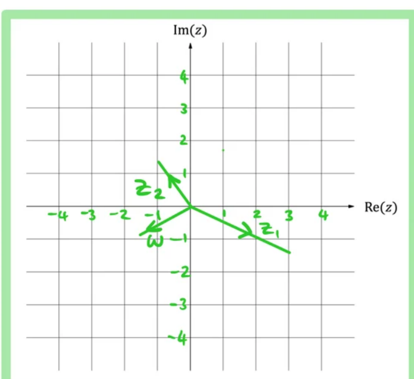
Consider the complex number $w = \frac{z_1}{z_2}$ where $z_1 = 3 - \sqrt{3}i$ and $z_2 = 2\text{cis}\left(\frac{2\pi}{3}\right)$.

(a) Express w in the form $r\text{cis}\theta$.

[5]

(b) Sketch z_1, z_2 and w on the Argand diagram below.

(b)



[3]

(c) Find the smallest positive integer value of n such that w^n is a real number.

[2]

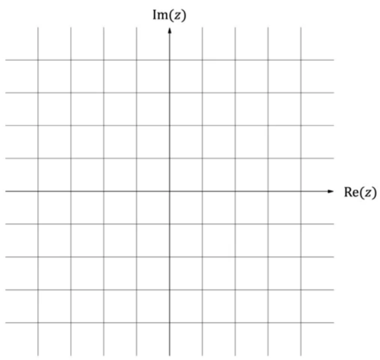
Consider the complex number $w = \frac{z_1}{z_2}$ where $z_1 = 3 - \sqrt{3}i$ and $z_2 = 2\text{cis}\left(\frac{2\pi}{3}\right)$.

(a) Express w in the form $r\text{cis}\theta$.

$\boxed{w = \sqrt{3}\ \text{cis}\left(-\frac{5\pi}{6}\right)}$

[5]

(b) Sketch z_1, z_2 and w on the Argand diagram below.



[3]

(c) Find the smallest positive integer value of n such that w^n is a real number.

[2]

(c) $w = \sqrt{3}\, e^{i\left(-\frac{5\pi}{6}\right)}$

$w^n = \sqrt{3}^n e^{i\left(-\frac{5\pi n}{6}\right)}$

For w^n to be $\mathbb{R}$, it must lie on the real axis and θ must therefore be a multiple of π

The smallest value for n will be when $\theta = -k\pi$

$-\frac{5\cancel{\pi}n}{6} = -k\cancel{\pi}$

For k to be an integer value

$\boxed{n = 6}$

Question 7

Consider the complex number $z = -1 + \sqrt{3}\text{i}$.

(a) Express z in the form $r\text{cis}\,\theta$, where $r > 0$ and $-\pi < \theta \leq \pi$.

[4]

(b) Find the three roots of the equation $z^3 = -1 + \sqrt{3}\text{i}$, expressing your answers in the form $r\text{cis}\,\theta$, where $r > 0$ and $-\pi < \theta \leq \pi$.

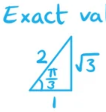
[4]

(a) $r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \frac{2\pi}{3}$

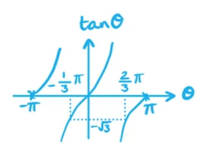
$\boxed{z = 2\,\text{cis}\left(\frac{2\pi}{3}\right)}$

Exact values:



$\tan\frac{\pi}{3} = \sqrt{3}$

$\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$



Consider the complex number $z = -1 + \sqrt{3}\mathrm{i}$.

(a) Express z in the form $r\operatorname{cis}\theta$, where $r > 0$ and $-\pi < \theta \leq \pi$.

$\boxed{z = 2\operatorname{cis}\left(\frac{2\pi}{3}\right)}$

[4]

(b) Find the three roots of the equation $z^3 = -1 + \sqrt{3}\mathrm{i}$, expressing your answers in the form $r\operatorname{cis}\theta$, where $r > 0$ and $-\pi < \theta \leq \pi$.

[4]

De Moivre's theorem	$[r(\cos\theta + \mathrm{i}\sin\theta)]^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta) = r^n \mathrm{e}^{\mathrm{i}n\theta} = r^n \operatorname{cis} n\theta$

Formula booklet

(b) If $z^3 = -1 + \sqrt{3}i = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$

Then $z = (-1 + \sqrt{3}i)^{1/3}$

$$= 2^{1/3}\left(\cos\left(\frac{2\pi}{3} + 2\pi k\right) + i\sin\left(\frac{2\pi}{3} + 2\pi k\right)\right)^{1/3}$$

$$= \sqrt[3]{2}\left(\cos\left(\frac{\frac{2\pi}{3} + 2\pi k}{3}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 2\pi k}{3}\right)\right)$$

$k = 0, 1, 2 \ldots, n-1$

$k = 0$

$$z_1 = \sqrt[3]{2}\left(\cos\left(\frac{\frac{2\pi}{3} + 0}{3}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 0}{3}\right)\right)$$

$$= \sqrt[3]{2}\left(\cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right)\right)$$

$\boxed{z_1 = \sqrt[3]{2}\operatorname{cis}\left(\frac{2\pi}{9}\right)}$

$k = 1$

$$z_2 = \sqrt[3]{2}\left(\cos\left(\frac{\frac{2\pi}{3} + 2\pi}{3}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 2\pi}{3}\right)\right)$$

$$= \sqrt[3]{2}\left(\cos\left(\frac{8\pi}{9}\right) + i\sin\left(\frac{8\pi}{9}\right)\right)$$

$\boxed{z_2 = \sqrt[3]{2}\operatorname{cis}\left(\frac{8\pi}{9}\right)}$

$k = 2$

$$z_3 = \sqrt[3]{2}\left(\cos\left(\frac{\frac{2\pi}{3} + 4\pi}{3}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 4\pi}{3}\right)\right)$$

$$= \sqrt[3]{2}\left(\cos\left(\frac{14\pi}{9}\right) + i\sin\left(\frac{14\pi}{9}\right)\right)$$

$\boxed{z_3 = \sqrt[3]{2}\operatorname{cis}\left(\frac{14\pi}{9}\right)}$

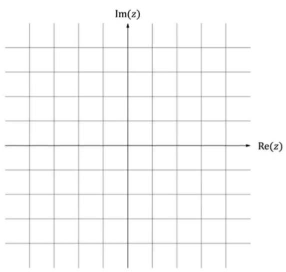
Question 8

Consider the equation $z^4 - 1 = 15$, where $z \in \mathbb{C}$.

(a) Find the four distinct roots of the equation, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

[4]

(b) Represent the roots found in part (a) on the Argand diagram below.



[2]

(c) Find the area of the polygon whose vertices are represented by the four roots on the Argand diagram.

[3]

(a) $z^4 - 1 = 15 \Rightarrow z^4 = 16$

Define a new complex number, w

Let $z^2 = w \Rightarrow w^2 = 16$

$w = \pm 4$

Substitute $w = \pm 4$ back into $z^2 = w$

$z^2 = 4$ $\quad$ $z^2 = -4$

$z = \pm 2$ $\quad$ $z = \pm 2i$

$\boxed{z_1 = 2 \quad z_2 = -2 \quad z_3 = 2i \quad z_4 = -2i}$

Consider the equation $z^4 - 1 = 15$, where $z \in \mathbb{C}$.

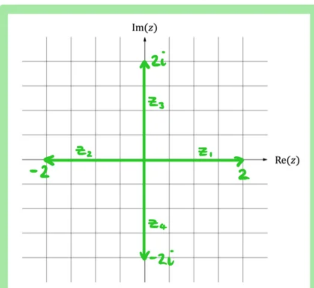
(a) Find the four distinct roots of the equation, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

$\boxed{z_1 = 2 \quad z_2 = -2 \quad z_3 = 2i \quad z_4 = -2i}$

[4]

(b) Represent the roots found in part (a) on the Argand diagram below.

(b)



[2]

(c) Find the area of the polygon whose vertices are represented by the four roots on the Argand diagram.

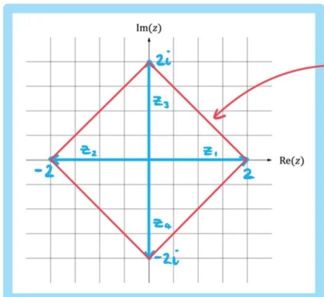
[3]

Consider the equation $z^4 - 1 = 15$, where $z \in \mathbb{C}$.

(a) Find the four distinct roots of the equation, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

[4]

(b) Represent the roots found in part (a) on the Argand diagram below.



$l = 2\sqrt{2}$

[2]

(c) Find the area of the polygon whose vertices are represented by the four roots on the Argand diagram.

[3]

(c) The shape formed is a square

Length, $l = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

Area, $A = (2\sqrt{2})^2$

$\boxed{A = 8 \text{ units}^2}$

Question 9

Consider the complex numbers $w = 3\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)$ and $z = 3 - \sqrt{3}\mathrm{i}$.

(a) Write w and z in the form $r\,\mathrm{cis}\,\theta$, where $r > 0$ and $-\pi < \theta \le \pi$.

[4]

(b) Find the modulus and argument of zw.

[2]

(c) Write down the value of zw.

[2]

(a) $w = 3\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

spot the negative!

$-\sin\theta = \sin(-\theta)$

$\cos\theta = \cos(-\theta)$

$\boxed{w = 3\,\mathrm{cis}\left(-\frac{\pi}{3}\right)}$

$|z| = \sqrt{3^2 + (-\sqrt{3})^2}$

$= 2\sqrt{3}$

$\theta = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

$\theta = -\frac{\pi}{6}$

Exact values:

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

$\boxed{z = 2\sqrt{3}\,\mathrm{cis}\left(-\frac{\pi}{6}\right)}$

Consider the complex numbers $w = 3\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)$ and $z = 3 - \sqrt{3}\mathrm{i}$.

(a) Write w and z in the form $r\operatorname{cis}\theta$, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

$\boxed{w = 3\operatorname{cis}\left(-\frac{\pi}{3}\right)}$ $\boxed{z = 2\sqrt{3}\operatorname{cis}\left(-\frac{\pi}{6}\right)}$

(b) Find the modulus and argument of zw. [2]

(c) Write down the value of zw. [2]

(b) $|zw| = 2\sqrt{3} \times 3$

$\boxed{|zw| = 6\sqrt{3}}$

$\arg(zw) = -\frac{\pi}{6} + -\frac{\pi}{3}$

$\boxed{\arg(zw) = -\frac{\pi}{2}}$

Consider the complex numbers $w = 3\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)$ and $z = 3 - \sqrt{3}\mathrm{i}$.

(a) Write w and z in the form $r\operatorname{cis}\theta$, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

(b) Find the modulus and argument of zw. [2]

$\boxed{|zw| = 6\sqrt{3}}$ $\boxed{\arg(zw) = -\frac{\pi}{2}}$

(c) Write down the value of zw. [2]

(c) $zw = 6\sqrt{3}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

$zw = 6\sqrt{3}(0 - i)$

$\boxed{zw = 6\sqrt{3}i}$

Question 10

Let $z = 12 + 16i$, where $a, b \in \mathbb{R}$.

(a) Verify that $4 + 2i$ and $-4 - 2i$ are the second roots of z.

[2]

(b) Hence, or otherwise, find two distinct roots of the equation $w^2 + 4w + (1 - 4i) = 0$, where $w \in \mathbb{C}$. Give your answer in the form $a + bi$, where $a, b \in \mathbb{R}$.

[4]

(a) $(4 + 2i)^2 = 16 + 8i + 8i - 4$ to

$\boxed{12 + 16i}$

$(-4 - 2i)^2 = 16 + 8i + 8i - 4$

$\boxed{12 + 16i}$

Let $z = 12 + 16i$, where $a, b \in \mathbb{R}$.

(a) Verify that $4 + 2i$ and $-4 - 2i$ are the second roots of z.

[2]

(b) Hence, or otherwise, find two distinct roots of the equation $w^2 + 4w + (1 - 4i) = 0$, where $w \in \mathbb{C}$. Give your answer in the form $a + bi$, where $a, b \in \mathbb{R}$.

[4]

(b) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Formula booklet

$$\omega = \frac{-4 \pm \sqrt{4^2 - 4(1)(1 - 4i)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{12 + 16i}}{2}$$

$$= \frac{-4 \pm (4 + 2i)}{2}$$

$+$ $-$

$\omega = \dfrac{0 + 2i}{2}$ $\qquad$ $\omega = \dfrac{-8 - 2i}{2}$

$\boxed{\omega = i}$ $\qquad$ $\boxed{\omega = -4 - i}$

Question 11

The complex numbers $\omega_1 = 3$ and $\omega_2 = 2 - 2i$ are roots of the cubic equation $\omega^3 + p\omega^2 + q\omega + r = 0$, where $p, q, r \in \mathbb{R}$.

(a) Write down the third root, ω_3, of the equation.

[1]

(b) Find the values of p, q and r.

[4]

(c) Express ω_1, ω_2 and ω_3 in the form $r\,\text{cis}\theta$.

[4]

(a) The third root will be the complex conjugate of ω_2

$$\boxed{\omega_3 = 2 + 2i}$$

The complex numbers $\omega_1 = 3$ and $\omega_2 = 2 - 2i$ are roots of the cubic equation $\omega^3 + p\omega^2 + q\omega + r = 0$, where $p, q, r \in \mathbb{R}$.

(a) Write down the third root, ω_3, of the equation.

$$\boxed{\omega_3 = 2 + 2i}$$

[1]

(b) Find the values of p, q and r.

[4]

(c) Express ω_1, ω_2 and ω_3 in the form $r\,\text{cis}\theta$.

[4]

(b) If ω_1, ω_2 and ω_3 are the roots, then

$$\omega^3 + p\omega^2 + q\omega + r = (\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3)$$

$$\omega^3 + (-\omega_1 - \omega_2 - \omega_3)\omega^2 + (\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3)\omega - \omega_1\omega_2\omega_3$$

Substitute the values in for ω_1, ω_2 and ω_3 to find the coefficient of each term and equate it to the corresponding coefficient or constant p, q or r

ω^2 term:

$$p = -\omega_1 - \omega_2 - \omega_3$$

$$p = -(3) - (2 - 2i) - (2 + 2i)$$

$$\boxed{p = -7}$$

ω term:

$$q = (\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3)$$

$$q = (3)(2 - 2i) + (3)(2 + 2i) + (2 - 2i)(2 + 2i)$$

$$q = 6 - 6i + 6 + 6i + 4 - 4i + 4i + 4$$

$$\boxed{q = 20}$$

constant:

$$r = -\omega_1\omega_2\omega_3$$

$$r = -(3)(2 - 2i)(2 + 2i) = -3(4 + 4)$$

$$\boxed{r = -24}$$

The complex numbers $\omega_1 = 3$ and $\omega_2 = 2 - 2i$ are roots of the cubic equation $\omega^3 + p\omega^2 + q\omega + r = 0$, where $p, q, r \in \mathbb{R}$.

(a) Write down the third root, ω_3, of the equation. [1]

$\boxed{\omega_3 = 2 + 2i}$

(b) Find the values of p, q and r. [4]

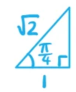
(c) Express ω_1, ω_2 and ω_3 in the form $r\,\text{cis}\theta$. [4]

(c) $\boxed{\omega_1 = 3\,\text{cis}\,0}$

$|\omega_2| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$

$\theta_2 = \tan^{-1}\left(\frac{-2}{2}\right) = -\frac{\pi}{4}$ ← Trig exact values

$\tan\left(\frac{\pi}{4}\right) = 1$



$\boxed{\omega_2 = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)}$

$|\omega_3| = |\omega_2| = 2\sqrt{2}$

$\theta_3 = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

$\boxed{\omega_2 = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)}$