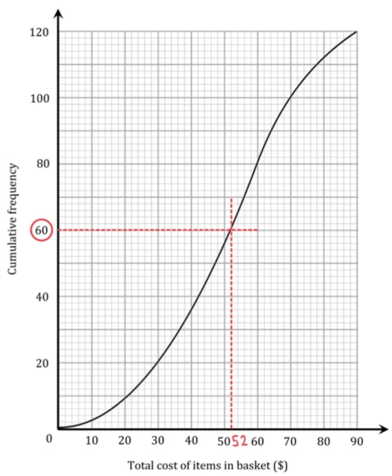
Question 1

A supermarket manager wishes to gather information about the spending habits of the store's customers. During each of his lunchbreaks on Monday through Friday of a given week, he chooses 24 customers at random and notes the total cost, in dollars ($), of the items in their baskets when they check out at the tills. The results of his survey are represented by the following cumulative frequency graph.



(a) Find the median total cost of the items these customers had in their baskets.

[2]

(b) Find the interquartile range of the total cost of the items these customers had in their baskets.

[3]

(c) Given that two thirds of customers had a total of more than $$p$ of goods in their baskets, find the value of p.

[3]

a) The median can be found using the cumulative frequency (c.f.) graph.

Median c.f. $= \frac{\text{Total}}{2}$

Median c.f. $= \frac{120}{2}$

Median c.f. $= 60$

Horizontal line at c.f. $= 60$

$\boxed{\therefore \text{ Median} = \$52}$

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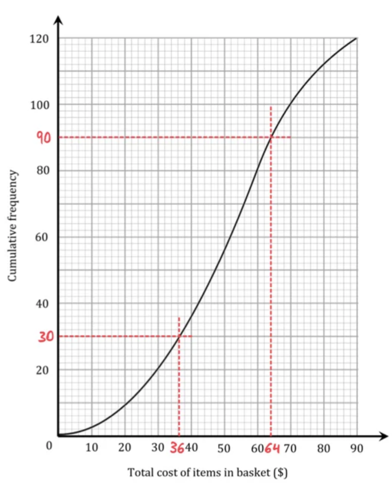

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b) Interquartile range

$IQR = Q_3 - Q_1$ (in formula booklet)

Horizontal lines at Q_3 and Q_1 c.f.

Q_3 c.f. $= \frac{3}{4} \times 120$

Q_3 c.f. $= 90$ $\therefore Q_3 = 64$

Q_1 c.f. $= \frac{1}{4} \times 120$

Q_1 c.f. $= 30$ $\therefore Q_1 = 36$

$IQR = 64 - 36$

$\boxed{IQR = \$28}$

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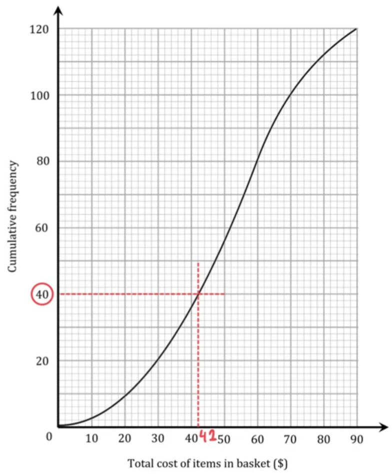

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c) $\frac{2}{3}$ of customers' baskets > $p

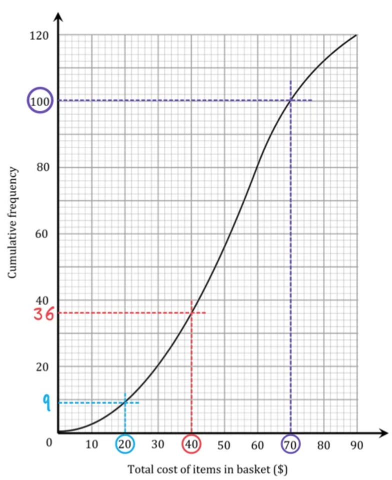
$\therefore$ $\frac{1}{3}$ of customers' baskets < $p

$c.f. = 120 - \left(\frac{2}{3} \times 120\right)$

$c.f. = 40$

Horizontal line at c.f. = 40.

$\boxed{\therefore p = \$42}$



The same survey information is represented by the following table:

Total cost ($m) of goods in basket	$0 < m \leq 20$	$20 < m \leq 40$	$40 < m \leq 70$	$70 < m \leq 90$
Frequency	9	q	r	20

(d) Find the value of q and the value of r.

[4]

In an average week, the manager estimates that the store has a total of 3600 customers.

(e) Use the results of the manager's survey to estimate the number of customers in a week who have goods totalling more than $50 in their baskets.

[3]

(f) (i) Explain why the manager's survey sample might not provide an accurate representation of the spending habits of *all* the shop's customers.

(ii) Suggest a sampling method that might obtain a more representative sample.

[2]

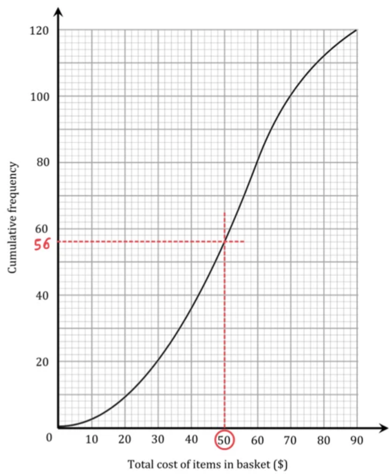
d) Find q and r by drawing vertical lines using the given intervals.

$q = 36 - 9$

$\boxed{q = 27 \text{ customers}}$

$r = 100 - 36$

$\boxed{r = 64 \text{ customers}}$



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e) $50 corresponds to cf = 56.

$120 - 56 = 64$

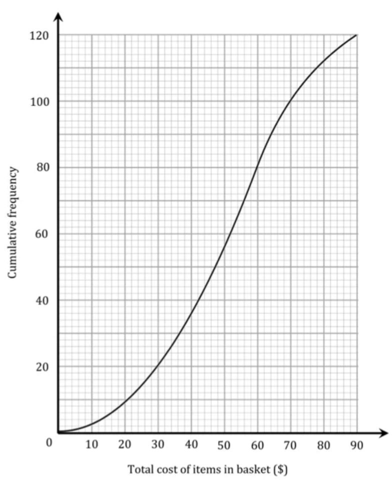
$\therefore \frac{64}{120}$ spent over $50.

estimate $= \frac{64}{120} \times 3600$

estimate $= 64 \times \frac{\cancel{3600}^{30}}{\cancel{120}}$

estimate $= 64 \times 30$

$\boxed{\text{estimate} = 1920 \text{ customers}}$



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[2]

f) i) Only sampling customers at lunchtimes on weekdays is unlikely to provide a good representation of all customers.

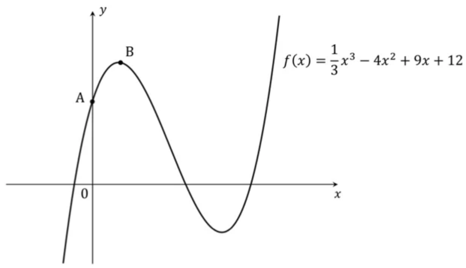
ii) There are many possible answers.

A form of quota sampling, with a set number of customers randomly chosen throughout all opening hours on all days of the week.

Question 2

The diagram below shows a part of the graph of the function

$$f(x) = \frac{1}{3}x^3 - 4x^2 + 9x + 12$$



(a) Point A is the point of intersection between the graph and the y-axis. Write down the coordinates of point A.

[1]

(b) Find $f'(x)$.

[2]

(c) Using the graph, explain why the equation $f'(x) = 0$ must have exactly two distinct real solutions.

[3]

a) y-intercepts occur when $x = 0$

$f(0) = \frac{1}{3}(0)^3 - 4(0)^2 + 9(0) + 12$

$f(0) = 12$

$\boxed{A(0, 12)}$

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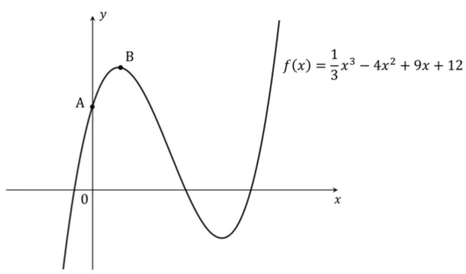

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[3]

b) Derivative of x^n formula (in formula booklet)

$f(x) = y = x^n \longrightarrow f'(x) = \frac{dy}{dx} = nx^{n-1}$

$f'(x) = \cancel{(3)} \cdot \frac{1}{\cancel{3}} x^2 - (2) \cdot 4x + 9$

$\boxed{f'(x) = x^2 - 8x + 9}$

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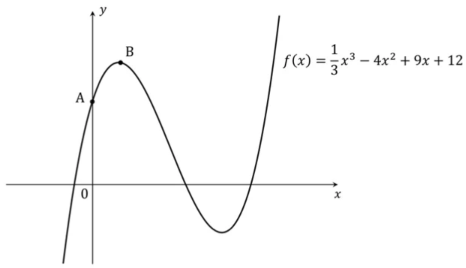

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(c) Using the graph, explain why the equation $f'(x) = 0$ must have exactly two distinct real solutions.

[3]

c) $f'(x) = 0$ when the graph of f has a local maximum or a local minimum.

The graph of f shows a local maximum and a local minimum.
$\therefore$ There are two distinct real solutions.

Alternatively

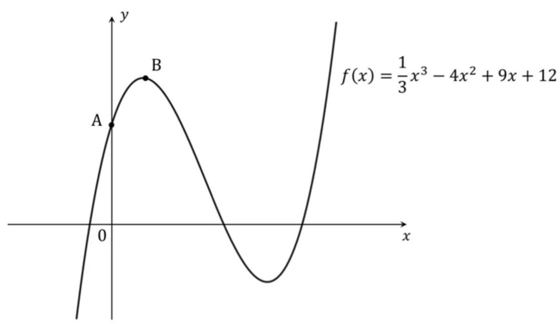
$f'(x)$ is a quadratic and when $\Delta > 0$ the equation $f'(x) = 0$ has two solutions.

Discriminant

$\Delta = b^2 - 4ac$ (in formula booklet)

$\Delta = (8)^2 - 4(1)(9) > 0$

$\boxed{\Delta = 28 > 0 \ \therefore \text{two distinct real solutions.}}$



Point B is the point on the graph with x-coordinate $\frac{8-\sqrt{26}}{2}$.

(d) Given that $\left(\frac{8-\sqrt{26}}{2}\right)^2 = \frac{45-8\sqrt{26}}{2}$, find the gradient of the tangent line to the graph at point B.

[2]

Points C and D are the points on the graph at which the tangent lines are perpendicular to the tangent line at point B.

(e) By first determining the gradient of the tangents at points C and D, find the x-coordinates of points C and D.

[5]

(f) Given that point C lies between points A and B on the graph, find the equation of the tangent line to the graph at point C. Give your answer in the form $y = mx + c$.

[4]

d) Sub $x = \frac{8-\sqrt{26}}{2}$ into $f'(x)$.

$f'\left(\frac{8-\sqrt{26}}{2}\right) = \left(\frac{8-\sqrt{26}}{2}\right)^2 - \cancel{8}^{4}\left(\frac{8-\sqrt{26}}{\cancel{2}}\right) + 9$

$f'\left(\frac{8-\sqrt{26}}{2}\right) = \frac{45-8\sqrt{26}}{2} - 4(8-\sqrt{26}) + 9$

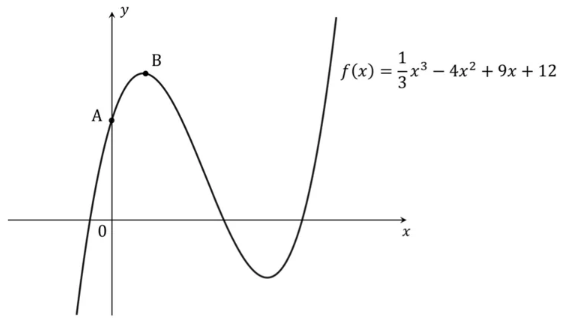
$f'\left(\frac{8-\sqrt{26}}{2}\right) = \frac{45}{2} - \frac{\cancel{8}^{4}\sqrt{26}}{\cancel{2}} - 4(8-\sqrt{26}) + 9$

$f'\left(\frac{8-\sqrt{26}}{2}\right) = \frac{45}{2} \cancel{-4\sqrt{26}} - 32 \cancel{+4\sqrt{26}} + 9$

$f'\left(\frac{8-\sqrt{26}}{2}\right) = \frac{45}{2} - 32 + 9$

$f'\left(\frac{8-\sqrt{26}}{2}\right) = -\frac{1}{2}$

$\boxed{\therefore \text{gradient at point } B \text{ is } -\tfrac{1}{2}.}$



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$\therefore$ gradient at point B is $-\frac{1}{2}$.

[2]

Points C and D are the points on the graph at which the tangent lines are perpendicular to the tangent line at point B.

(e) By first determining the gradient of the tangents at points C and D, find the x-coordinates of points C and D.

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[4]

e) Perpendicular gradients

$m_\perp = -\frac{1}{m}$

$m_\perp = -\frac{1}{\left(-\frac{1}{2}\right)}$ $\quad \therefore m_\perp = 2$

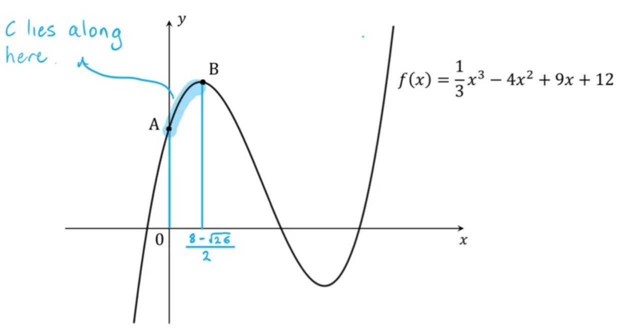
Set $f'(x) = 2$ and find x.

$2 = x^2 - 8x + 9$

$x^2 - 8x + 7 = 0$

$(x-7)(x-1) = 0$

$\therefore$ x-coordinates of points C and D are $x = 1$ and $x = 7$.



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$\boxed{m_\perp = 2}$

[5]

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[4]

f) x-coordinate of C is $x = 1$ or $x = 7$ and is in the interval $0 < x < \frac{8-\sqrt{26}}{2}$.

$\therefore$ x-coordinate of C is $x = 1$.

Find $f(1)$

$f(1) = \frac{1}{3}(1)^3 - 4(1)^2 + 9(1) + 12$

$f(1) = \frac{52}{3}$ $\quad \therefore C\left(1, \frac{52}{3}\right)$

Sub $C\left(1, \frac{52}{3}\right)$ and $m = 2$ into $y - y_1 = m(x - x_1)$.

$y - \frac{52}{3} = 2(x-1)$

$y = 2x - 2 + \frac{52}{3}$

$\boxed{y = 2x + \frac{46}{3}}$

Question 3

After escaping from a research station, a small population of rabbits has become established on an island in the Southern Ocean. Scientists have begun to study this rabbit population, and have determined that the number of rabbits, P, at a time t months after the beginning of the study can be modelled by the function

$$P(t) = \frac{3000}{1 + 99e^{-kt}}$$

Where k is a positive constant.

(a) Determine the number of rabbits on the island at the beginning of the study.

[2]

(b) (i) Explain what happens to the values of e^{-kt} as t becomes large.

(ii) Hence determine the maximum number of rabbits that the model predicts the island can support. Be sure to show clear mathematical reasoning to support your answer.

[4]

(c) Show that

$$P'(t) = \frac{3000 \times 99ke^{-kt}}{(1 + 99e^{-kt})^2}$$

[3]

(d) (i) Use the result from part (c) to show that $P(t)$ is an increasing function for all values of $t \geq 0$.

(ii) Explain why this does not contradict the result of (b)(ii).

[4]

a) $t = 0$ at beginning of the study.

Find $P(0)$.

$$P(0) = \frac{3000}{1 + 99\underbrace{e^{-k(0)}}_{=1}}$$

$$P(0) = \frac{3000}{1 + 99}$$

$$\boxed{P(0) = 30}$$

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[4]

b) i) Because $k > 0$.

$$e^{-kt} = \frac{1}{e^{kt}} \qquad (t \to \infty,\ e^{-kt} \to 0)$$

As t becomes large, e^{-kt} tends towards zero.

ii) $$\lim_{t \to \infty} P(t) = \frac{3000}{1 + 99(0)}$$

$$\lim_{t \to \infty} P(t) = \frac{3000}{1}$$

Hence the max population predicted by the model is 3000.

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[4]

c) Method 1 : Chain rule

$y = g(u)$, where $u = f(x)$

$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \times \frac{\cancel{du}}{dx}$ (in formula booklet)

$P(t) = \frac{3000}{1 + 99e^{-kt}} = 3000(1 + 99e^{-kt})^{-1}$

$P'(t) = -3000(1 + 99e^{-kt})^{-2}(-99ke^{-kt})$

Method 2 : Quotient rule

$y = \frac{u}{v} \longrightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (in formula booklet)

$u = 3000 \quad v = 1 + 99e^{-kt}$

$P'(t) = \frac{\cancel{(1 + 99e^{-kt})(0)} - (3000)(-99ke^{-kt})}{(1 + 99e^{-kt})^2}$

$$\boxed{P'(t) = \frac{3000 \times 99ke^{-kt}}{(1 + 99e^{-kt})^2}}$$

After escaping from a research station, a small population of rabbits has become established on an island in the Southern Ocean. Scientists have begun to study this rabbit population, and have determined that the number of rabbits, P, at a time t months after the beginning of the study can be modelled by the function

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[4]

d) i) $e^x > 0$ for all (real) values of x.

$\therefore P'(t) = \frac{3000 \times 99ke^{-kt}}{(1 + 99e^{-kt})^2} = \frac{\text{positive number}}{\text{positive number}} > 0$

$P'(t) > 0$ for all values of t,
$\therefore P(t)$ is an increasing function.

ii) The graph of P has a horizontal asymptote at $y = 3000$.
Hence $P(t)$ is always increasing but never exceeds 3000.

The model predicts that the population of rabbits will double in the first two months after the beginning of the study.

(e) (i) Use this information to show that $k = \frac{1}{2}\ln\left(\frac{99}{49}\right)$.

(ii) Hence find the exact rate of change of the rabbit population at the beginning of the study, as predicted by the model.

[7]

e) i) $P(0) = 30 \quad \therefore P(2) = 60$

$60 = \dfrac{3000}{1 + 99e^{-k(2)}}$

$1 + 99e^{-2k} = 50$

$99e^{-2k} = 49$

$e^{2k} = \dfrac{99}{49}$

$2k = \ln \dfrac{99}{49}$

$\boxed{k = \frac{1}{2}\ln\left(\frac{99}{49}\right)}$

ii) Find $P'(0)$.

$P'(0) = \dfrac{3000 \times 99ke^{-k(0)}}{(1 + 99e^{-k(0)})^2}$

$P'(0) = \dfrac{\cancel{3000}^{3} \times 99k}{\cancel{(100)^2}_{10}}$

$P'(0) = \dfrac{3}{10} \times 99k$

Sub in $k = \frac{1}{2}\ln\left(\frac{99}{49}\right)$

$P'(0) = \dfrac{3}{10} \times 99\left(\frac{1}{2}\ln\left(\frac{99}{49}\right)\right)$

$P'(0) = \dfrac{297}{10}\left(\frac{1}{2}\ln\left(\frac{99}{49}\right)\right)$

$P'(0) = 29.7 \times \frac{1}{2}\ln\left(\frac{99}{49}\right)$

$\boxed{14.85\ln\left(\frac{99}{49}\right) \text{ or } 29.7\ln\frac{\sqrt{99}}{7} \text{ rabbits/month}}$

Question 4

The Strike A Light! matchstick company produces matchsticks with a length, X mm, that is normally distributed with mean 45 and variance σ^2.

The probability that X is greater than 45.37 is 0.1714.

(a) Find $P(44.63 < X < 45.37)$.

[2]

(b) (i) Find σ, the standard deviation of X.

(ii) Hence, find the probability that a randomly selected matchstick has a length less than 44.5 mm.

[5]

Andrew has a box of Strike A Light! matches with fifteen matchsticks remaining in it. Those matchsticks may be assumed to be a random sample. Let Y represent the number of matchsticks in Andrew's box with lengths less than 44.5 mm.

(c) Find E(Y).

[3]

(d) Find the probability that exactly one of the matchsticks in Andrew's box has a length less than 44.5 mm.

[2]

A Strike A Light! matchstick is selected at random and is found to have a length greater than 44.5 mm.

(e) Find the probability that the length of the matchstick is between 44.63 mm and 45.37 mm.

[3]

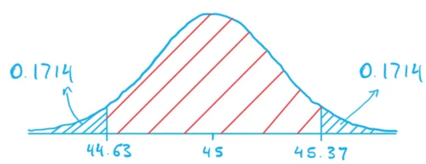
a) $X \sim N(45, \sigma^2)$

$45.37 - 45 = 0.37$

$45 - 44.63 = 0.37$

45.37 and 44.63 are equidistant from 45.

$P(X > 45.37) = P(X < 44.63) = 0.1714$



$P(44.63 < X < 45.37) = 1 - 2(0.1714)$

$P(44.63 < X < 45.37) = 0.6572$

$\boxed{P(44.63 < X < 45.37) = 0.657 \text{ (3sf)}}$

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[3]

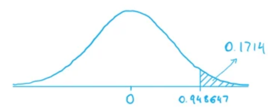
b) i) Standardised normal variable

$Z \sim N(0, 1) \longrightarrow Z = \frac{x - \mu}{\sigma}$ (in formula booklet)

Inverse normal function on your GDC.

Area = 0.1714 $\mu = 0$ $\sigma = 1$

$P(Z > 0.948647) = 0.1714$



$0.948647 = \frac{45.37 - 45}{\sigma}$

$\therefore \sigma = 0.390029...$

$\boxed{\sigma = 0.390 \text{ (3sf)}}$

ii) $X \sim N(45, 0.390^2)$

$P(X < 44.5) = 0.09992...$

$\boxed{P(X < 44.5) = 0.0999 \text{ (3sf)}}$

The Strike A Light! matchstick company produces matchsticks with a length, X mm, that is normally distributed with mean 45 and variance σ^2.

The probability that X is greater than 45.37 is 0.1714.

(a) Find $P(44.63 < X < 45.37)$.

[2]

(b) (i) Find σ, the standard deviation of X.

(ii) Hence, find the probability that a randomly selected matchstick has a length less than 44.5 mm.

[5]

Andrew has a box of Strike A Light! matches with fifteen matchsticks remaining in it. Those matchsticks may be assumed to be a random sample. Let Y represent the number of matchsticks in Andrew's box with lengths less than 44.5 mm.

(c) Find $E(Y)$.

[3]

(d) Find the probability that exactly one of the matchsticks in Andrew's box has a length less than 44.5 mm.

[2]

A Strike A Light! matchstick is selected at random and is found to have a length greater than 44.5 mm.

(e) Find the probability that the length of the matchstick is between 44.63 mm and 45.37 mm.

[3]

c) Binomial distribution (in formula booklet)

$E(X) = np \qquad Var(X) = np(1-p)$

$Y \sim B(15, 0.0999)$

$E(Y) = 15 \times 0.09992$
$= 1.4989...$

$\boxed{E(Y) = 1.50 \text{ (3sf)}}$

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A Strike A Light! matchstick is selected at random and is found to have a length greater than 44.5 mm.

(e) Find the probability that the length of the matchstick is between 44.63 mm and 45.37 mm.

[3]

d) $Y \sim B(15,\ 0.0999)$

$P(Y=1) = 0.343287\ldots$

$\boxed{P(Y=1) = 0.343 \text{ (3sf)}}$

The Strike A Light! matchstick company produces matchsticks with a length, X mm, that is normally distributed with mean 45 and variance σ^2.

The probability that X is greater than 45.37 is 0.1714.

(a) Find $P(44.63 < X < 45.37)$.

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(e) Find the probability that the length of the matchstick is between 44.63 mm and 45.37 mm.

[3]

e) Conditional probability

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ (in formula booklet)

$P(A \cap B)$

$P(44.63 < X < 45.37) = 0.6572$ (part (a))

$P(B)$

$P(X > 44.5) = 1 - P(X < 44.5)$

$P(X > 44.5) = 0.90007\ldots$

Apply formula.

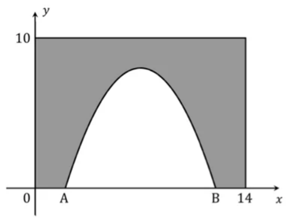
$P(A \mid B) = \dfrac{0.6572}{0.90007}$

$P(A \mid B) = 0.73016\ldots$

$\boxed{P(A \mid B) = 0.730 \text{ (3sf)}}$

Question 5

K. C. Jones & Company produces tunnels for model railroad layouts. Each tunnel has the form of a right prism, and the cross-section of one of the tunnels the company produces is shown in the diagram below. The upper and right-hand borders of the shaded area are parallel to the x-axis and y-axis respectively, and all units are in centimetres.



The shape of the opening of the tunnel may be modelled by the function

$$f(x) = -k(x^2 - 14x + 24)$$

where k is a positive constant.

Points A and B are the points where the tunnel opening meets the x-axis in the diagram.

(a) Find the coordinates of points A and B.

[3]

The maximum height of the tunnel opening above the x-axis is 8 cm.

(b) Use this information to determine the value of k.

[3]

(c) By setting up and solving an appropriate definite integral, show that the area of the tunnel opening is $\frac{160}{3}$ cm^2. You must use calculus and show the steps of your working.

[4]

The material from which the tunnel is made has a density of 1060 kg/m^3.

(d) Given that the mass of the tunnel is 2067 g, find the length of the tunnel.

[5]

a) $f(x) = 0$ at A and B

$0 = -k(x^2 - 14x + 24)$

$0 = x^2 - 14x + 24$

$0 = (x-2)(x-12)$

$x = 2 \qquad x = 12$

$\boxed{A\,(2, 0) \text{ and } B\,(0, 12)}$

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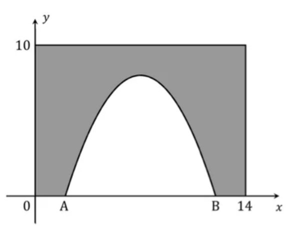

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(d) Given that the mass of the tunnel is 2067 g, find the length of the tunnel.

[5]

b) Vertex $= (7, 8)$, by symmetry

$f(7) = 8$

$8 = -k((7)^2 - 14(7) + 24)$

$8 = 25k$

$\boxed{k = \frac{8}{25}}$

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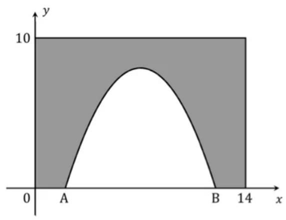

The shape of the opening of the tunnel may be modelled by the function

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where k is a positive constant.

Points A and B are the points where the tunnel opening meets the x-axis in the diagram.

(a) Find the coordinates of points A and B.

[3]

The maximum height of the tunnel opening above the x-axis is 8 cm.

(b) Use this information to determine the value of k.

[3]

(c) By setting up and solving an appropriate definite integral, show that the area of the tunnel opening is $\frac{160}{3}$ cm^2. You must use calculus and show the steps of your working.

[4]

The material from which the tunnel is made has a density of 1060 kg/m^3.

(d) Given that the mass of the tunnel is 2067 g, find the length of the tunnel.

[5]

c) Integral of x^n (in formula booklet)

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$

$\int f(x)\,dx = \int_2^{12} -\frac{8}{25}(x^2 - 14x + 24)\,dx$

$= -\frac{8}{25}\int_2^{12}(x^2 - 14x + 24)\,dx$

$= -\frac{8}{25}\left[\frac{1}{3}x^3 - 7x^2 + 24x\right]_2^{12}$

$= -\frac{8}{25}\left[\left(\frac{1}{3}(12)^3 - 7(12)^2 + 24(12)\right) - \left(\frac{1}{3}(2)^3 - 7(2)^2 + 24(2)\right)\right]$

$= -\frac{8}{\cancel{25}}\left(-\frac{\cancel{500}^{\,20}}{3}\right)$

$= \frac{-8 \times -20}{3}$

$\boxed{= \frac{160}{3} \text{ cm}^2}$

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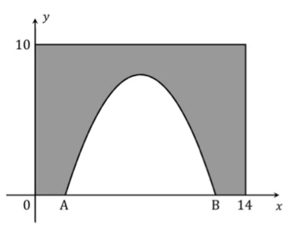

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[3]

(c) By setting up and solving an appropriate definite integral, show that the area of the tunnel opening is $\frac{160}{3}$ cm^2. You must use calculus and show the steps of your working.

[4]

The material from which the tunnel is made has a density of 1060 kg/m^3.

(d) Given that the mass of the tunnel is 2067 g, find the length of the tunnel.

[5]

d) Shaded area $= (10 \times 14) - \frac{160}{3}$

Shaded area $= \frac{260}{3}$ cm^2

Convert to g/cm^3

1060 kg/m^3 = 1.06 g/cm^3

mass = density × volume

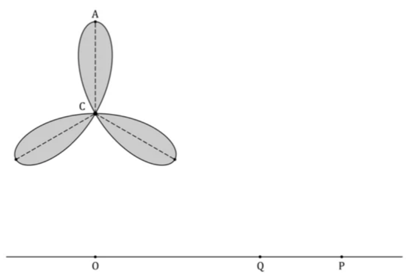
Let l be the length.

$2067 = 1.06 \times \frac{260}{3} l$

$\boxed{l = 22.5 \text{ cm}}$

Question 6

Badon Iron Works is building a new ship called the Gargantuan, which will be a full-sized replica of the original RMS Titanic. Eleanor is an engineer at the company, and is involved with construction and testing of the ship's screws (commonly known as 'propellers'). The diagram below depicts one of the ship's screws mounted in the testing facility.



Point C is the centre of the screw, which is fixed in place so that the screw is able to rotate about it. Point A is the marked tip of one of the three identical blades of the screw. Point O is the point on the horizontal floor of the testing facility that lies directly below point C. Points O, A, C, P and Q lie at all times in the same plane.

The height, h m, of point A above the testing facility floor once the screw begins to rotate may be modelled by the function

$$h(t) = 5.59 + 3.6\cos(k\pi t)$$

where t is the time in seconds since the screw began rotating, and k is a constant.

(a) Use the above information to determine:

(i) The distance of point A from point C.

(ii) The height of point C above point O.

[2]

a) $-1 \le \cos(k\pi t) \le 1$

$h(t)_{max} = 5.59 + 3.6$
$= 9.19\text{ m}$

$h(t)_{min} = 5.59 - 3.6$
$= 1.99\text{ m}$

i) $AC = \frac{1}{2}(h(t)_{max} - h(t)_{min})$

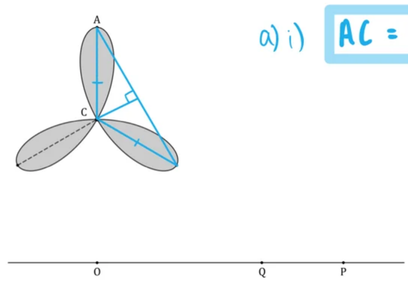
$AC = \frac{1}{2}(9.19 - 1.99)$

$\boxed{AC = 3.6\text{ m}}$

ii) $OC = AC + h(t)_{min}$

$OC = 3.6 + 1.99$

$\boxed{OC = 5.59\text{ m}}$



a) i) $\boxed{AC = 3.6\text{ m}}$

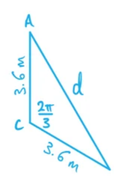
(b) Given that the tips of the three blades of the screw are located at equal distances from each other around the circumference of a circle with centre C, determine the exact distance of point A from the tip of one of the other blades of the screw.

[3]

b) Method 1: Cosine rule

$c^2 = a^2 + b^2 - 2ab\cos C$ (in formula booklet)

$a = b = 3.6 \qquad C = \frac{2\pi}{3}$

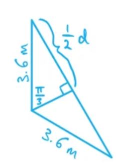


$d = \sqrt{(3.6)^2 + (3.6)^2 - 2(3.6)(3.6)\cos\left(\frac{2\pi}{3}\right)}$

$\boxed{d = \frac{18\sqrt{3}}{5}\text{ m or } 6.24\text{ m (3sf)}}$

Method 2: Isosceles triangle + SOHCAHTOA

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = \frac{\frac{1}{2}d}{3.6}$



$d = \cancel{2} \times 3.6 \times \frac{\sqrt{3}}{\cancel{2}}$

$d = 3.6 \times \sqrt{3}$

$\boxed{d = \frac{18\sqrt{3}}{5}\text{ m or } 6.24\text{ m (3sf)}}$

When it is rotating, the screw makes 75 complete revolutions every minute.

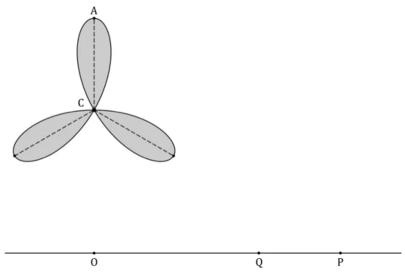
(c) Given that the argument of the cosine in the equation for $h(t)$ is measured in radians, use this information to determine the value of the constant k.

[3]

Paul, a mathematician, has been hired as a consultant on the Gargantuan project. Because of his height of 1.96 m, Eleanor is concerned about whether he will be able to walk safely beneath the screw while it is rotating.

(d) Determine whether Eleanor is right to be concerned, giving a mathematical reason for your answer.

[2]



(b) Given that the tips of the three blades of the screw are located at equal distances from each other around the circumference of a circle with centre C, determine the exact distance of point A from the tip of one of the other blades of the screw.

[3]

When it is rotating, the screw makes 75 complete revolutions every minute.

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(d) Determine whether Eleanor is right to be concerned, giving a mathematical reason for your answer.

[2]

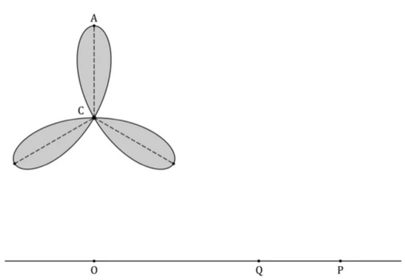
c) 75 rpm = 0.8 s per revolution

$k\pi(0) = 0$

$k\cancel{\pi}(0.8) = 2\cancel{\pi}$

$k = \frac{2}{0.8}$

$\boxed{k = 2.5}$



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(d) Determine whether Eleanor is right to be concerned, giving a mathematical reason for your answer.

[2]

d) $h(t)_{min} = 1.99\text{ m}$

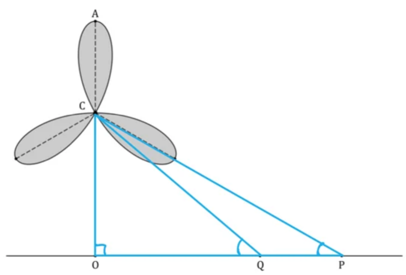
Paul's height = 1.96 m

Paul is safe if his height < $h(t)_{min}$.

$1.96 < 1.99$

$\boxed{\therefore \text{Paul is safe.}}$

N.B the argument can be made that 0.03 m is not enough clearance for Paul to be safe.



The screw has been locked in place so that point A is at its highest possible position above the floor. Paul is standing at point P, which is at a distance of 9.69 m from point O. He walks towards point O until he arrives at point Q, which is located such that

$$\tan O\hat{Q}C = \frac{3}{2}\tan O\hat{P}C$$

(e) Determine the distance of point Q from point P.

[3]

(f) Given that point A remains fixed at its highest possible position above the floor, determine the area of triangle APQ.

[4]

e) $\tan O\hat{Q}C = \frac{3}{2}\tan O\hat{P}C$ $\left(\tan\theta = \frac{opp}{adj}\right)$

$\frac{OC}{OQ} = \frac{3}{2}\left(\frac{OC}{OP}\right)$

$\frac{OQ}{OC} = \frac{2\,OP}{3\,OC}$

$OQ = \frac{2\,OP}{3\,\cancel{OC}} \times \cancel{OC}$ $(OP = 9.69\text{ m})$

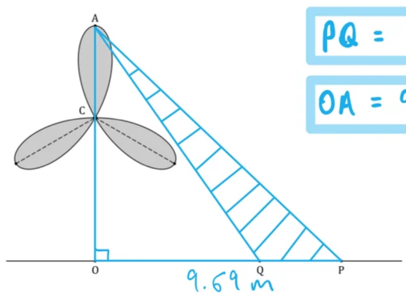
$OQ = \frac{2}{3}(9.69)$

$OQ = 6.46\text{ m}$

$PQ = OP - OQ$

$PQ = 9.69 - 6.46$

$\boxed{PQ = 3.23\text{ m}}$



$\boxed{PQ = 3.23\text{ m}}$

$\boxed{OA = 9.19\text{ m}}$

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(e) Determine the distance of point Q from point P.

[3]

(f) Given that point A remains fixed at its highest possible position above the floor, determine the area of triangle APQ.

[4]

f) Method 1

$A_{APQ} = A_{AOP} - A_{AOQ}$

$A_{APQ} = \frac{1}{2}(9.19)(9.69) - \frac{1}{2}(9.19)(6.46)$

$A_{APQ} = 14.84185$

$\boxed{A_{APQ} = 14.8\text{ m}^2 \ (3sf)}$

Method 2

$A_{APQ} = \frac{1}{2}(AP)(PQ)\sin A\hat{P}Q$ $\left(\sin A\hat{P}Q = \left(\frac{OA}{AP}\right)\right)$

$A_{APQ} = \frac{1}{2}\left(\sqrt{9.19^2 + 9.69^2}\right)(3.23)\left(\frac{9.19}{\sqrt{9.19^2 + 9.69^2}}\right)$

$A_{APQ} = 14.84185$

$\boxed{A_{APQ} = 14.8\text{ m}^2 \ (3sf)}$