Question 1

The function f is defined by $f(x) = \frac{2x-1}{x^2+3x-4}$, for $x \in \mathbb{R}$, $x \neq m$, $x \neq n$.

(a) Find the values of m and n.

[2]

(b) Find an expression for $f'(x)$.

[3]

The graph of $y = f(x)$ has exactly one point of inflection.

(c) Find the x-coordinate of the point of inflection.

[2]

(d) Sketch the graph of $y = f(x)$ for $-6 \leq x \leq 6$, showing the coordinates of any axis intercepts and local maxima and local minima, and giving the equations of any asymptotes.

[4]

The function g is defined by $g(x) = \frac{x^2+3x-4}{2x-1}$, for $x \in \mathbb{R}$, $x \neq \frac{1}{2}$.

(e) Find the equation of the oblique asymptote of the graph of $y = g(x)$.

[3]

(f) By considering the graph of $y = f(x) - g(x)$, or otherwise, solve $g(x) < f(x)$ for $x \in \mathbb{R}$.

[4]

(a) $f(x)$ is undefined when the denominator $= 0$.

Solve $x^2 + 3x - 4 = 0$

$x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$x = 1, -4$

$\boxed{m = -4, \quad n = 1}$

* $m = 1$ and $n = -4$ is also correct.

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[4]

(b) formula book:

Quotient rule	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Let $u = 2x - 1$ and $v = x^2 + 3x - 4$

$u' = 2$, $v' = 2x + 3$

Substitute into quotient rule:

$$f'(x) = \frac{(x^2+3x-4)(2) - (2x-1)(2x+3)}{(x^2+3x-4)^2}$$

$$= \frac{2(x^2+3x-4) - (4x^2+4x-3)}{(x^2+3x-4)^2}$$

$$= \frac{-2x^2+2x-5}{(x^2+3x-4)^2}$$

$$\boxed{f'(x) = \frac{-2x^2+2x-5}{(x^2+3x-4)^2}}$$

The question does not ask for $f'(x)$ to be simplified, so giving any correct unsimplified expression for $f'(x)$ is fine.

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(a) Find the values of m and n. [2]

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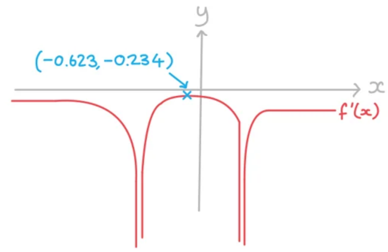
The function g is defined by $g(x) = \frac{x^2+3x-4}{2x-1}$, for $x \in \mathbb{R}$, $x \neq \frac{1}{2}$.

(e) Find the equation of the oblique asymptote of the graph of $y = g(x)$. [3]

(f) By considering the graph of $y = f(x) - g(x)$, or otherwise, solve $g(x) < f(x)$ for $x \in \mathbb{R}$. [4]

(c) Use GDC to graph $f'(x)$ and find the x-coordinate of the local maximum.



$\boxed{x = -0.623}$

The function f is defined by $f(x) = \frac{2x-1}{x^2+3x-4}$, for $x \in \mathbb{R}$, $x \neq m$, $x \neq n$.

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(b) Find an expression for $f'(x)$. [3]

The graph of $y = f(x)$ has exactly one point of inflection.

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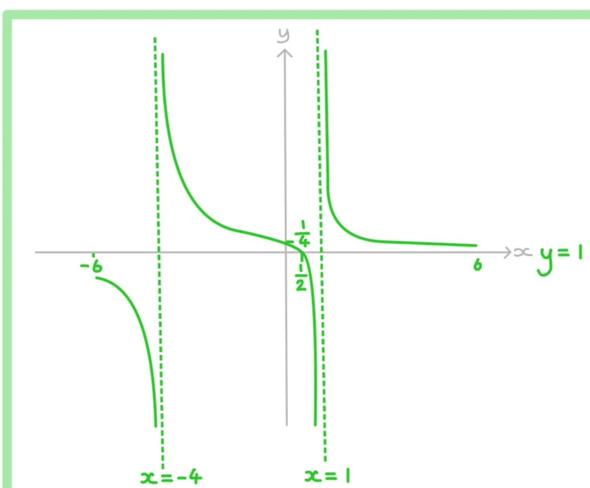
(d) Sketch the graph of $y = f(x)$ for $-6 \leq x \leq 6$, showing the coordinates of any axis intercepts and local maxima and local minima, and giving the equations of any asymptotes. [4]

The function g is defined by $g(x) = \frac{x^2+3x-4}{2x-1}$, for $x \in \mathbb{R}$, $x \neq \frac{1}{2}$.

(e) Find the equation of the oblique asymptote of the graph of $y = g(x)$. [3]

(f) By considering the graph of $y = f(x) - g(x)$, or otherwise, solve $g(x) < f(x)$ for $x \in \mathbb{R}$. [4]

(d) Use your GDC to sketch the graph.
Draw and label asymptotes at $x = -4$ and $x = 1$.
Label the x axis $y = 0$ as this is also an asymptote.
Label the axes at the intercepts $(0, \frac{1}{4})$ and $(\frac{1}{2}, 0)$
Draw the curves carefully.



* There are no local maxima or minima.

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(e) Find the equation of the oblique asymptote of the graph of $y = g(x)$.

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(f) By considering the graph of $y = f(x) - g(x)$, or otherwise, solve $g(x) < f(x)$ for $x \in \mathbb{R}$.

[4]

(e) Numerator is higher order than denominator so find the oblique asymptote by writing the numerator in the form

$(2x-1)(px+q)+r$

Use polynomial division:

$$\begin{array}{r|l} & \frac{1}{2}x + \frac{7}{4} \\ 2x-1 & x^2+3x-4 \\ & x^2 - \frac{1}{2}x \\ \hline & \frac{7}{2}x - 4 \\ & \frac{7}{2}x - \frac{7}{4} \\ \hline & -\frac{9}{4} \end{array}$$

So $g(x) = \dfrac{(2x-1)\left(\frac{1}{2}x+\frac{7}{4}\right)-\frac{9}{4}}{2x-1} = \frac{1}{2}x + \frac{7}{4} - \dfrac{\frac{9}{4}}{2x-1}$

As $x \to \pm\infty$, $\dfrac{\frac{9}{4}}{2x-1} \to 0$,

$\therefore$ oblique asymptote is at $y = \frac{1}{2}x + \frac{7}{4}$

$$\boxed{y = \frac{1}{2}x + \frac{7}{4}}$$

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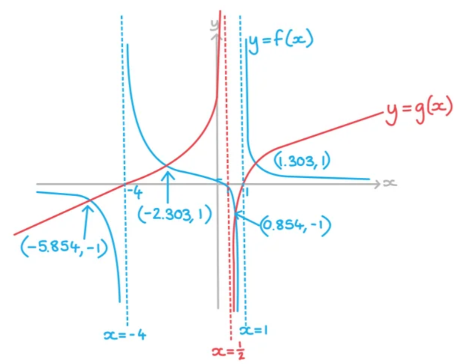
(f) By considering the graph of $y = f(x) - g(x)$, or otherwise, solve $g(x) < f(x)$ for $x \in \mathbb{R}$.

[4]

(f) Use your GDC to graph $g(x)$ on the same set of axes as $f(x)$.



Find the points of intersection of the two graphs.

Consider the parts of the graph where $g(x) < f(x)$

$x < -5.854$, $-4 < x < -2.303$, $\frac{1}{2} < x < 0.854$, $1 < x < 1.303$

$$\boxed{x < -5.85,\ -4 < x < -2.30,\ 0.5 < x < 0.85,\ 1 < x < 1.30}$$

Question 2

The function f has a derivative given by $f'(x) = \frac{1}{3x(k-x)}$, $x \in \mathbb{R}$, $x \neq 0$, where k is a positive constant.

(a) The expression for $f'(x)$ can be written in the form $\frac{a}{3x} + \frac{b}{k-x}$ where $p, q \in \mathbb{R}$. Find a and b in terms of k.

[3]

(b) Hence find an expression for $f(x)$.

[3]

R is the population of rabbits on an island. The rate of change of the population can be modelled by the differential equation $\frac{dR}{dt} = \frac{3R(k-R)}{4k}$, where t is the time measured in years, $t \geq 0$, and k is the maximum population that the island can support.

The initial population of the rabbits is 20.

(c) By solving the differential equation, show that $R = \frac{20ke^{\frac{3}{4}t}}{k-20+20e^{\frac{3}{4}t}}$.

[7]

After two years, the population of rabbits has risen to 70.

(d) Find k.

[3]

(e) Find the value of t at which the population of rabbits is growing at its fastest rate.

[2]

(a)

$$\frac{1}{3x(k-x)} \equiv \frac{a}{3x} + \frac{b}{k-x}$$

Multiply through by the denominator to eliminate fractions

$$1 \equiv a(k-x) + b(3x)$$

Choose values of x to substitute into the identity that will eliminate each constant

Let $x = 0$: (eliminates b) $\quad 1 = a(k-0) + b(3 \times 0)$

$$1 = ak$$

$$a = \frac{1}{k}$$

Let $x = k$: $\quad 1 = a(k-k) + b(3 \times 1)$

$$1 = 3b$$

$$b = \frac{1}{3k}$$

$$\boxed{a = \frac{1}{k}, \quad b = \frac{1}{3k}}$$

The function f has a derivative given by $f'(x) = \frac{1}{3x(k-x)}$, $x \in \mathbb{R}$, $x \neq 0$, where k is a positive constant.

(a) The expression for $f'(x)$ can be written in the form $\frac{a}{3x} + \frac{b}{k-x}$ where $p, q \in \mathbb{R}$. Find a and b in terms of k.

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(b) Integrate $f'(x)$, use the form found in part (a).

$$\int f'(x)\,dx = \int \frac{a}{3x} + \frac{b}{k-x}\,dx$$

Substitute a and b from part (a)

$$\int f'(x)\,dx = \int \frac{1}{3kx} + \frac{1}{3k(k-x)}\,dx$$

factorise, $\frac{1}{3k}$ is a factor.

$$\int f'(x)\,dx = \frac{1}{3k}\int\left(\frac{1}{x} + \frac{1}{k-x}\right)dx$$

Formula book

Standard integrals	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

$$= \frac{1}{3k}\left[\ln|x| + (-1)\ln|k-x|\right] + c$$

constant of integration

$$= \frac{1}{3k}\left[\ln|x| - \ln|k-x|\right] + c$$

$$\boxed{f(x) = \frac{1}{3k}\left(\ln\left|\frac{x}{k-x}\right|\right) + c}$$

The function f has a derivative given by $f'(x)=\frac{1}{3x(k-x)}$, $x\in\mathbb{R}$, $x\neq 0$, where k is a positive constant.

(a) The expression for $f'(x)$ can be written in the form $\frac{a}{3x}+\frac{b}{k-x}$ where $p,q\in\mathbb{R}$. Find a and b in terms of k.

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(c) $$\int \frac{dR}{\cancel{dt}}\,\cancel{dt}=\int\frac{3R(k-R)}{4k}\,dt$$

Separate the variables to bring R to one side

$$\int\frac{1}{3R(k-R)}\,dR=\int\frac{1}{4k}\,dt$$

Recognise that this is the same expression as $f'(x)$ in part (a), so substitute answer from (b).

$$\frac{1}{3k}\left[\ln\left|\frac{R}{k-R}\right|\right]=\frac{1}{4k}t+c$$ new constant of integration needed.

When $t=0$ (initial population), $R=20$, so

$$\frac{1}{3k}\left[\ln\left(\frac{20}{k-20}\right)\right]=\frac{1}{4k}(0)+c\quad\therefore\quad c=\frac{1}{3k}\ln\left(\frac{20}{k-20}\right)$$

Substitute c and rearrange for R.

$$\frac{1}{3k}\ln\left|\frac{R}{k-R}\right|=\frac{1}{4k}t+\frac{1}{3k}\ln\left(\frac{20}{k-20}\right)$$

$$\ln\left|\frac{R}{k-R}\right|=\frac{3}{4}t+\ln\left(\frac{20}{k-20}\right)$$

$$\frac{3}{4}t=\ln\left|\frac{R}{k-R}\right|-\ln\left(\frac{20}{k-20}\right)$$

$$\frac{3}{4}t=\ln\left|\frac{R}{k-R}\div\frac{20}{k-20}\right|$$

$$\frac{3}{4}t=\ln\left|\frac{R(k-20)}{20(k-R)}\right|$$

$$\frac{R(k-20)}{20(k-R)}=e^{\frac{3}{4}t}$$

$$R(k-20)=20e^{\frac{3}{4}t}(k-R)$$

$$Rk-20R=20ke^{\frac{3}{4}t}-20Re^{\frac{3}{4}t}$$

$$Rk-20R+20Re^{\frac{3}{4}t}=20ke^{\frac{3}{4}t}$$

$$R\left(k-20+20e^{\frac{3}{4}t}\right)=20ke^{\frac{3}{4}t}$$

$$\boxed{R=\frac{20ke^{\frac{3}{4}t}}{k-20+20e^{\frac{3}{4}t}}}$$

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The initial population of the rabbits is 20.

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[7]

After two years, the population of rabbits has risen to 70.

(d) Find k.

[3]

(e) Find the value of t at which the population of rabbits is growing at its fastest rate.

[2]

(d) When $t = 2$, $R = 70$.

Substitute in and solve for k.

$$70 = \frac{20ke^{\frac{3}{4}(2)}}{k - 20 + 20e^{\frac{3}{4}(2)}}$$

Solve using GDC, solver.

$k = 248.264...$

$\boxed{k = 248 \ (3\text{s.f.})}$

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After two years, the population of rabbits has risen to 70.

(d) Find k.

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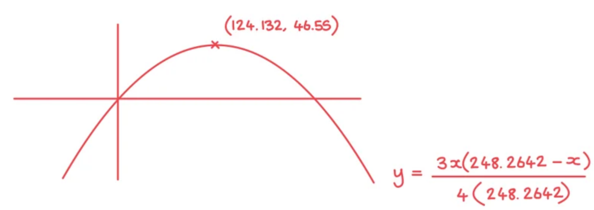
(e) Find the value of t at which the population of rabbits is growing at its fastest rate.

[2]

(e) Use your GDC to find when $\frac{dR}{dt}$ is at a maximum.

Graph $\frac{dR}{dt} = \frac{3R(248.2642 - R)}{4(248.2642)}$ and find the maximum point.



$\frac{dR}{dt}$ is at its maximum when $R = 124.132$

Substitute in and solve for t.

$$124.32 = \frac{20(248.2642)e^{\frac{3}{4}(t)}}{248.2642 - 20 + 20e^{\frac{3}{4}(t)}}$$

$\boxed{t = 3.25 \text{ years}}$

Question 3

A particle is moving in a vertical line and its acceleration, in ms^{-2}, at time t seconds, $t \geq 0$, is given by $a = -\frac{1-v}{2}$, where v is the velocity in meters per second and $v < 1$.

The particle starts at a fixed origin O with initial velocity v_o ms^{-1}.

(a) By solving a suitable differential equation, show that the particle's velocity at time t is given by $v(t) = 1 - e^{-\frac{t}{2}}(1 - v_0)$.

[6]

The particle moves down in the negative direction, until its displacement relative to the origin reaches a minimum. Then the particle changes direction and starts moving up, in a positive direction.

(b) (i) If the initial velocity of the particle is -3 ms^{-1}, find the time at which the minimum displacement of the particle from the origin occurs, giving your answer in exact form.

(ii) If T is the time in seconds when the displacement reaches its smallest value, show that $T = 2\ln(1 - v_0)$.

[4]

(c) (i) Find a general expression for the displacement, in terms of t and v_0.

(ii) Combine this general expression with the result from part (b)(ii) to find an expression for the minimum displacement of the particle in terms of v_o.

[5]

Let $v(T - k)$ represent the particle's velocity k seconds before the minimum displacement and $v(T + k)$ the particle's velocity k seconds after the minimum displacement.

(d) (i) Show that $v(T - k) = 1 - e^{\frac{k}{2}}$.

(ii) Given that $v(T + k) = 1 - e^{-\frac{k}{2}}$, show that $v(T - k) + v(T + k) \geq 0$.

[5]

(a) $$a = \frac{dv}{dt} = -\frac{1-v}{2}$$

Separation of variables

$$\frac{1}{1-v}\,dv = -\frac{1}{2}\,dt$$

Integrate

$$\int \frac{1}{1-v}\,dv = \int -\frac{1}{2}\,dt$$

$$\ln|1-v| = -\frac{t}{2} + c$$

Find c, when $t = 0$, $v = v_0$

$$\ln|1 - v_0| = c$$

Substitute $\ln|1 - v_0|$ for c

$$\ln|1-v| = -\frac{t}{2} + \ln|1 - v_0|$$

$$\ln|1-v| - \ln|1 - v_0| = -\frac{t}{2}$$

$$\ln\frac{|1-v|}{|1-v_0|} = -\frac{t}{2}$$

$$\frac{1-v}{1-v_0} = e^{-\frac{t}{2}}$$

$$1 - v = e^{-\frac{t}{2}}(1 - v_0)$$

$$\boxed{v(t) = 1 - (1 - v_0)e^{-\frac{t}{2}}}$$

EXAM PAPERS PRACTICE

A particle is moving in a vertical line and its acceleration, in ms^{-2}, at time t seconds, $t \geq 0$, is given by $a = -\frac{1-v}{2}$, where v is the velocity in meters per second and $v < 1$.

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(d) (i) Show that $v(T-k) = 1 - e^{\frac{k}{2}}$.

(ii) Given that $v(T+k) = 1 - e^{-\frac{k}{2}}$, show that $v(T-k) + v(T+k) \geq 0$.

[5]

(b) The displacement will be at a minimum when $v = 0$

(i) Substitute $v_0 = -3$, $v(t) = 0$

$$0 = 1 - e^{-\frac{t}{2}}(1-(-3))$$
$$0 = 1 - 4e^{-\frac{t}{2}}$$
$$4e^{-\frac{t}{2}} = 1$$
$$e^{-\frac{t}{2}} = \frac{1}{4}$$
$$\frac{1}{e^{\frac{t}{2}}} = \frac{1}{4}$$
$$e^{\frac{t}{2}} = 4$$
$$\frac{t}{2} = \ln 4$$
$$\boxed{t = 2\ln 4}$$

(ii) Substitute $t = T$, $v(t) = 0$ and find an expression for T in terms of v_0

$$0 = 1 - e^{-\frac{T}{2}}(1 - v_0)$$
$$(1 - v_0)e^{-\frac{T}{2}} = 1$$
$$e^{-\frac{T}{2}} = \frac{1}{1 - v_0}$$
$$\frac{1}{e^{\frac{T}{2}}} = \frac{1}{1 - v_0}$$
$$e^{\frac{T}{2}} = 1 - v_0$$
$$\frac{T}{2} = \ln(1 - v_0)$$
$$\boxed{T = 2\ln(1 - v_0)}$$

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$\boxed{T = 2\ln(1 - v_0)}$

[4]

(c) (i) Find a general expression for the displacement, in terms of t and v_0.

(ii) Combine this general expression with the result from part (b)(ii) to find an expression for the minimum displacement of the particle in terms of v_o.

[5]

Let $v(T - k)$ represent the particle's velocity k seconds before the minimum displacement and $v(T + k)$ the particle's velocity k seconds after the minimum displacement.

(d) (i) Show that $v(T - k) = 1 - e^{\frac{k}{2}}$.

(ii) Given that $v(T + k) = 1 - e^{-\frac{k}{2}}$, show that $v(T - k) + v(T + k) \geq 0$.

[5]

(c) (i) Integrate $v(t)$ to find $s(t)$

$$s(t) = \int v(t)\, dt = \int \left(1 - e^{-\frac{t}{2}}(1 - v_0)\right) dt$$

$$s(t) = t + 2e^{-\frac{t}{2}}(1 - v_0) + c$$

Substitute $t = 0$, $s(0) = 0$ and solve to find c

$$0 = 0 + 2e^{-\frac{0}{2}}(1 - v_0) + c$$

$$0 = 2(1 - v_0) + c$$

$$c = -2(1 - v_0)$$

$$\boxed{s(t) = t + 2e^{-\frac{t}{2}}(1 - v_0) - 2(1 - v_0)}$$

(ii) Substitute the expression for T into $s(t)$

$$s(T) = 2\ln(1 - v_0) + 2e^{-\frac{\cancel{2}\ln(1-v_0)}{\cancel{2}}}(1 - v_0) - 2(1 - v_0)$$

$$= 2\ln(1 - v_0) + \frac{2}{e^{\ln(1-v_0)}}(1 - v_0) - 2(1 - v_0)$$

$$= 2\ln(1 - v_0) + \frac{2}{\cancel{(1 - v_0)}}\cancel{(1 - v_0)} - 2(1 - v_0)$$

$$= 2\ln(1 - v_0) + 2 - 2(1 - v_0)$$

$$\boxed{s(T) = 2(\ln(1 - v_0) - (1 - v_0) + 1)}$$

A particle is moving in a vertical line and its acceleration, in ms^{-2}, at time t seconds, $t \geq 0$, is given by $a = -\frac{1-v}{2}$, where v is the velocity in meters per second and $v < 1$.

The particle starts at a fixed origin O with initial velocity v_o ms^{-1}.

(a) By solving a suitable differential equation, show that the particle's velocity at time t is given by $v(t) = 1 - e^{-\frac{t}{2}}(1 - v_0)$.

[6]

The particle moves down in the negative direction, until its displacement relative to the origin reaches a minimum. Then the particle changes direction and starts moving up, in a positive direction.

(b) (i) If the initial velocity of the particle is -3 ms^{-1}, find the time at which the minimum displacement of the particle from the origin occurs, giving your answer in exact form.

(ii) If T is the time in seconds when the displacement reaches its smallest value, show that $T = 2\ln(1 - v_0)$.

[4]

(c) (i) Find a general expression for the displacement, in terms of t and v_0.

(ii) Combine this general expression with the result from part (b)(ii) to find an expression for the minimum displacement of the particle in terms of v_o.

[5]

Let $v(T - k)$ represent the particle's velocity k seconds before the minimum displacement and $v(T + k)$ the particle's velocity k seconds after the minimum displacement.

(d) (i) Show that $v(T - k) = 1 - e^{\frac{k}{2}}$.

(ii) Given that $v(T + k) = 1 - e^{-\frac{k}{2}}$, show that $v(T - k) + v(T + k) \geq 0$.

[5]

(d) (i) Substitute $t = T - k$ into the original expression for $v(t)$

$$v(T-k) = 1 - e^{-\frac{T-k}{2}}(1 - v_0)$$

$$= 1 - e^{-\frac{2\ln(1-v_0)-k}{2}}(1 - v_0)$$

$$= 1 - e^{-\ln(1-v_0)}e^{\frac{k}{2}}(1 - v_0)$$

$$= 1 - \frac{1}{e^{\ln(1-v_0)}}e^{\frac{k}{2}}(1 - v_0)$$

$$= 1 - \frac{1}{(1-v_0)}e^{\frac{k}{2}}(1 - v_0)$$

$$\boxed{v(T-k) = 1 - e^{\frac{k}{2}}}$$

(ii) Substitute $v(T-k)$ and $v(T+k)$ into the LHS of the inequality to get an expression for the sum of the velocities in terms of k

$$\left(1 - e^{\frac{k}{2}}\right) + \left(1 - e^{-\frac{k}{2}}\right) \geq 0$$

$$2 - e^{\frac{k}{2}} - e^{-\frac{k}{2}} \geq 0$$

Differentiate LHS with respect to k to get an expression for the acceleration in terms of k

$$\frac{d}{dk}\left(2 - e^{\frac{k}{2}} - e^{-\frac{k}{2}}\right) = -\frac{1}{2}e^{\frac{k}{2}} + \frac{1}{2}e^{-\frac{k}{2}}$$

Equate the derivative to zero to find the value of k for which the sum of the velocities k seconds before and after T is at a minimum

$$0 = -\frac{1}{2}e^{\frac{k}{2}} + \frac{1}{2}e^{-\frac{k}{2}}$$

$$e^{\frac{k}{2}} = e^{-\frac{k}{2}}$$

$$k = -k \Rightarrow k = 0$$

Substitute $k = 0$ into expression for the sum of the velocities to find the minimum value

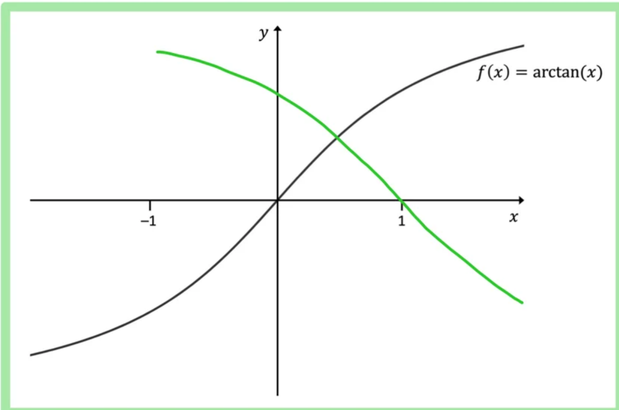
$$2 - e^0 - e^0 = 0$$

$$\boxed{\begin{array}{c}\text{The smallest value for } v(T-k) + v(T+k) \text{ is } 0 \\ \therefore\ v(T-k) + v(T+k) \geq 0\end{array}}$$

Question 4

The diagram below shows the graph of $f(x) = \arctan(x)$, $x \in \mathbb{R}$. The graph has rotational symmetry of order 2 about the origin.

(ii)



(a) A different function, g, is described by $g(x) = -\arctan(x-1)$, $x \in \mathbb{R}$.

(i) Describe the sequence of transformations that transforms $f(x)$ to $g(x)$.

(ii) Sketch the graph of $g(x)$ on the axes above.

(iii) Using your answers to parts (i) and (ii) to help you, describe the relationship between $\int_0^1 \arctan(x)\,dx$ and $\int_0^1 -\arctan(x-1)\,dx$.

[5]

(b) (i) Prove that $\arctan p - \arctan q \equiv \arctan\left(\frac{p-q}{1+pq}\right)$.

(ii) Show that $\arctan\left(\frac{1}{x^2-x+1}\right)$ can be written as $\arctan(x) - \arctan(x-1)$.

[6]

(c) Using the results from parts (a) and (b), evaluate $\int_0^1 \arctan\left(\frac{1}{x^2-x+1}\right)dx$, leaving your answer in exact form.

[7]

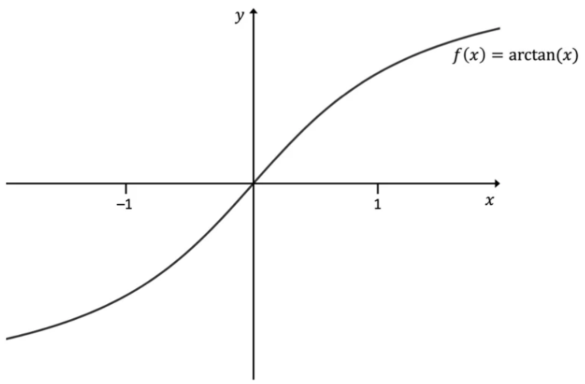
(a) (i)

$\boxed{\text{Reflection in the } x\text{-axis} \quad \text{Translation } \begin{pmatrix}1\\0\end{pmatrix}}$

(iii) $f(x)$ and $g(x)$ intersect at the midpoint of the integral interval and as one is a reflection of the other in the x-axis the area under the curves must be equal

$$\boxed{\int_0^1 \arctan x\,dx = \int_0^1 -\arctan(x-1)\,dx}$$

The diagram below shows the graph of $f(x) = \arctan(x),\ x \in \mathbb{R}$. The graph has rotational symmetry of order 2 about the origin.



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[6]

(c) Using the results from parts (a) and (b), evaluate $\int_0^1 \arctan\left(\frac{1}{x^2-x+1}\right)dx$, leaving your answer in exact form.

[7]

(b) (i) Let $a = \arctan p \Rightarrow \tan a = p$

$b = \arctan q \Rightarrow \tan b = q$

$\Rightarrow \arctan p - \arctan q = a - b$

Take the tan of $a-b$ and use the compound angle formula to simplify

Compound angle identities	$\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$
	$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
	$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

Take arctan of both sides

$$a - b = \arctan\left(\frac{\tan a - \tan b}{1 + \tan a \tan b}\right)$$

Substitute p and q back into the equation using the relationships between a, b, p and q listed above

$$\boxed{\arctan p - \arctan q = \arctan\left(\frac{p-q}{1+pq}\right)}$$

(ii) Write the denominator in the form $1 - pq$, and find p and q in terms of x

$$x^2 - x + 1 = 1 + pq$$
$$1 + x(x-1) = 1 + pq$$
$$\Rightarrow p = x,\quad q = (x-1)$$

Check that the expressions for p and q result in the numerators being equivalent

$$p - q = x - (x-1)$$
$$p - q = 1$$

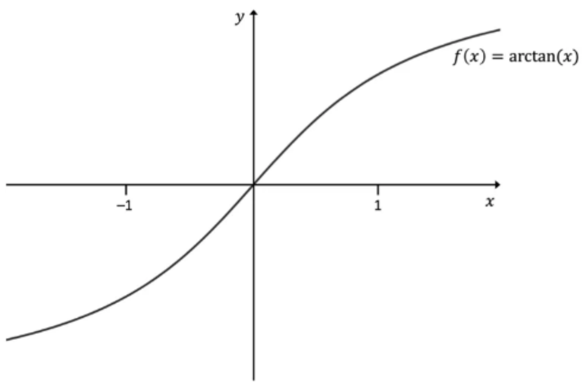
$\frac{1}{x^2 - x + 1}$ can be written in the form $\frac{p-q}{1+pq}$

$$\frac{1}{x^2-x+1} = \left(\frac{x-(x-1)}{1+x(x-1)}\right)$$

The arctan of the expression can therefore be re-written using b(i)

$$\boxed{\arctan\left(\frac{1}{1+x(x-1)}\right) = \arctan(x) - \arctan(1-x)}$$

The diagram below shows the graph of $f(x) = \arctan(x),\ x \in \mathbb{R}$. The graph has rotational symmetry of order 2 about the origin.



(a) A different function, g, is described by $g(x) = -\arctan(x-1),\ x \in \mathbb{R}$.

(i) Describe the sequence of transformations that transforms $f(x)$ to $g(x)$.

(ii) Sketch the graph of $g(x)$ on the axes above.

(iii) Using your answers to parts (i) and (ii) to help you, describe the relationship between $\int_0^1 \arctan(x)\,dx$ and $\int_0^1 -\arctan(x-1)\,dx$.

$\boxed{\int_0^1 \arctan x \, dx = \int_0^1 -\arctan(x-1)\,dx}$

[5]

(b) (i) Prove that $\arctan p - \arctan q \equiv \arctan\left(\frac{p-q}{1+pq}\right)$.

(ii) Show that $\arctan\left(\frac{1}{x^2-x+1}\right)$ can be written as $\arctan(x) - \arctan(x-1)$.

[6]

(c) Using the results from parts (a) and (b), evaluate $\int_0^1 \arctan\left(\frac{1}{x^2-x+1}\right)dx$, leaving your answer in exact form.

[7]

(c) $\int_0^1 \frac{1}{x^2-x+1}\,dx = \int_0^1 (\arctan(x) - \arctan(x-1))\,dx$

Using part (a) (iii)

$\int_0^1 (\arctan(x) - \arctan(x-1))\,dx = \int_0^1 (\arctan(x) + \arctan(x))\,dx$

$= 2\int_0^1 \arctan(x)\,dx$

Integration by parts

Integration by parts	$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ or $\int u\,dv = uv - \int v\,du$

$\arctan x$	$f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$

$u = \arctan(x) \Rightarrow \frac{du}{dx} = \frac{1}{1+x^2}$

$\frac{dv}{dx} = 1 \Rightarrow v = x$

$2\int_0^1 \arctan(x)\,dx = 2\left[x\arctan(x) - \int_0^1 \frac{x}{1+x^2}\,dx\right]_0^1$

Integrate this part by substitution

$u = 1 + x^2 \Rightarrow \frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\int_0^1 \frac{x}{1+x^2}\,dx = \int_0^1 \frac{x}{u}\,dx$

$= \int_0^1 \frac{1}{2} \times \frac{1}{u}\,du$

$= \frac{1}{2}\int_0^1 \frac{1}{u}\,du$

$= \left[\frac{1}{2}\ln|u|\right]_0^1$

$= \left[\frac{1}{2}\ln|1+x^2|\right]_0^1$

Look at the whole integrated expression and substitute in the bounds to evaluate it

$$2\int_0^1 \arctan(x)\,dx = 2\left[x\arctan(x) - \int_0^1 \frac{x}{1+x^2}\,dx\right]_0^1$$

$$= 2\left[x\arctan(x) - \frac{1}{2}\ln|1+x^2|\right]_0^1$$

$$= 2\Big(\big((1)\arctan(1) - \tfrac{1}{2}\ln|1+(1)^2|\big) - \big((0)\arctan(0) - \tfrac{1}{2}\ln|1+(0)^2|\big)\Big)$$

$$= 2\Big(\big(\tfrac{\pi}{4} - \tfrac{1}{2}\ln 2\big) - (0-0)\Big)$$

$$\boxed{\int_0^1 \arctan\frac{1}{x^2 - x + 1}\,dx = \frac{\pi}{2} - \ln 2}$$

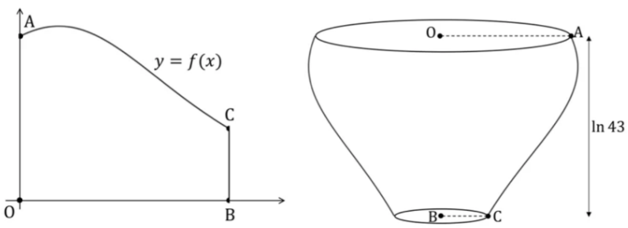
Question 5

Paola is modelling a small vase from her house for her maths project. To model the edge of the vase in cross-section, she decides to use a function f of the form

$$f(x) = \frac{qe^{\frac{x}{2}}}{2 + e^x}$$

where $x \in \mathbb{R}$, $x \geq 0$ and $q \in \mathbb{R}^+$.

The function and the vase are represented in the diagrams below.



The vertical height of the vase, OB, is measured along the x-axis. The radius of the vase's opening is OA, and its base radius is BC.

To model the vase, she will rotate by 2π radians about the x-axis the region enclosed by the graph of $y = f(x)$, the x-axis, the y-axis, and the line $x = \ln 43$.

(a) Show that the volume of the solid of revolution thus formed is $\frac{14q^2\pi}{45}$ units^3.

[6]

Volume of revolution about the x or y-axes	$V = \int_a^b \pi y^2 \, dx$ or $V = \int_a^b \pi x^2 \, dy$

(a) Substitute $f(x)$ and the limits into the volume of revolution formula

$$V = \int_0^{\ln 43} \pi \left(\frac{qe^{\frac{x}{2}}}{2+e^x}\right)^2 dx$$

$$= \int_0^{\ln 43} q^2\pi \left(\frac{e^{\frac{x}{2}}}{2+e^x}\right)^2 dx$$

$$= q^2\pi \int_0^{\ln 43} \left(\frac{e^{\frac{x}{2}}}{2+e^x}\right)^2 dx$$

$$= q^2\pi \int_0^{\ln 43} \frac{e^x}{(2+e^x)^2} dx$$

Integrate by substitution

$$u = 2 + e^x$$

$$\frac{du}{dx} = e^x$$

$$du = e^x \, dx$$

Substitute into the integral

$$V = q^2\pi \int_0^{\ln 43} \frac{1}{u^2} du$$

$$= q^2\pi \int_0^{\ln 43} u^{-2} du$$

Integrate

$$= q^2\pi \left[-u^{-1}\right]_0^{\ln 43}$$

Substitute $2 + e^x$ for u

$$= q^2\pi \left[-\frac{1}{2+e^x}\right]_0^{\ln 43}$$

Evaluate

$$= q^2\pi \left(\left(-\frac{1}{2+e^{\ln 43}}\right) + \left(\frac{1}{2+e^0}\right)\right)$$

$$= q^2\pi \left(-\left(\frac{1}{2+43}\right) + \left(\frac{1}{2+1}\right)\right)$$

$$= q^2\pi \left(\frac{1}{3} - \frac{1}{45}\right)$$

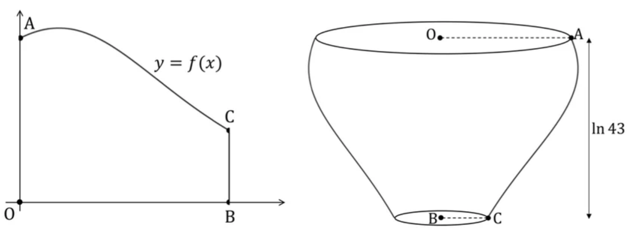
$$\boxed{V = \frac{14q^2\pi}{45}}$$

Paola is modelling a small vase from her house for her maths project. To model the edge of the vase in cross-section, she decides to use a function f of the form

$$f(x) = \frac{qe^{\frac{x}{2}}}{2 + e^x}$$

where $x \in \mathbb{R}, x \geq 0$ and $q \in \mathbb{R}^+$.

The function and the vase are represented in the diagrams below.



The vertical height of the vase, OB, is measured along the x-axis. The radius of the vase's opening is OA, and its base radius is BC.

To model the vase, she will rotate by 2π radians about the x-axis the region enclosed by the graph of $y = f(x)$, the x-axis, the y-axis, and the line $x = \ln 43$.

(a) Show that the volume of the solid of revolution thus formed is $\frac{14q^2\pi}{45}$ units^3.

[6]

The volume of the actual vase is 100 cm^3.

(b) Use this information to find the value of q.

[2]

(b) Substitute 100 for V in the answer to part (a)

$$100 = \frac{14q^2\pi}{45}$$

$$q = \sqrt{\frac{100 \times 45}{14\pi}}$$

Only take the + root as $q \in \mathbb{R}^+$

$$q = 10 \cdot 115032\ldots$$

$$\boxed{q = 10 \cdot 1}$$

Paola is modelling a small vase from her house for her maths project. To model the edge of the vase in cross-section, she decides to use a function f of the form

$$f(x) = \frac{qe^{\frac{x}{2}}}{2 + e^x}$$

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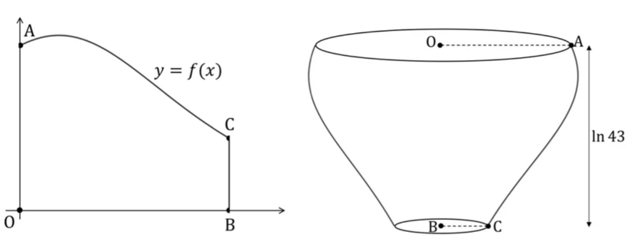


The vertical height of the vase, OB, is measured along the x-axis. The radius of the vase's opening is OA, and its base radius is BC.

To model the vase, she will rotate by 2π radians about the x-axis the region enclosed by the graph of $y = f(x)$, the x-axis, the y-axis, and the line $x = \ln 43$.

(c) Find the cross-sectional radius of the vase

(i) at its base,

(ii) at its widest point.

$$\boxed{q = 10 \cdot 115032\ldots}$$

[4]

(c) (i) Find the radius at the base by substituting $x = \ln 43$ into $f(x)$

$$BC = f(\ln 43) = \frac{10 \cdot 115032\ldots e^{\frac{\ln 43}{2}}}{2 + e^{\ln 43}}$$

$$= \frac{10 \cdot 115032\ldots (e^{\ln 43})^{\frac{1}{2}}}{2 + e^{\ln 43}}$$

$$= \frac{10 \cdot 115032\ldots \times 43^{\frac{1}{2}}}{2 + 43}$$

$$= 1 \cdot 473971\ldots$$

$$\boxed{BC = 1 \cdot 47 \text{ cm}}$$

(ii) First find the coordinates of the maximum point in $f(x)$ by graphing $f(x)$ on your GDC and analysing the curve

$f(x)_{max}$ is at $(0 \cdot 693147\ldots, 3 \cdot 576203\ldots)$

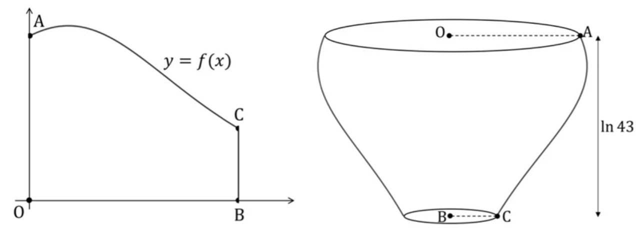
$$\boxed{\text{Max radius} = 3 \cdot 58}$$

Paola is modelling a small vase from her house for her maths project. To model the edge of the vase in cross-section, she decides to use a function f of the form

$$f(x)=\frac{qe^{\frac{x}{2}}}{2+e^{x}}$$

where $x \in \mathbb{R}, x \geq 0$ and $q \in \mathbb{R}^{+}$.

The function and the vase are represented in the diagrams below.



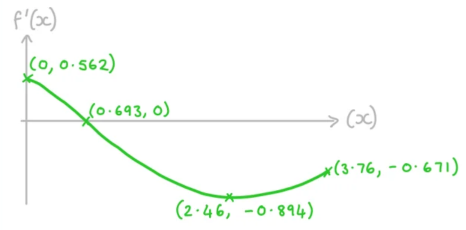
The vertical height of the vase, OB, is measured along the x-axis. The radius of the vase's opening is OA, and its base radius is BC.

To model the vase, she will rotate by 2π radians about the x-axis the region enclosed by the graph of $y=f(x)$, the x-axis, the y-axis, and the line $x=\ln 43$.
Paola wants to investigate how the cross-sectional radius of the vase changes.

(d) Sketch a graph of the derivative of f, and use it to find the value of x at which the cross-sectional radius of the vase is decreasing most rapidly.

[4]

(d) Graph $f(x)$ on your GDC, analyse the gradient along the curve and use to sketch $f'(x)$



Cross-sectional area is decreasing most rapidly at $x = 2.46$