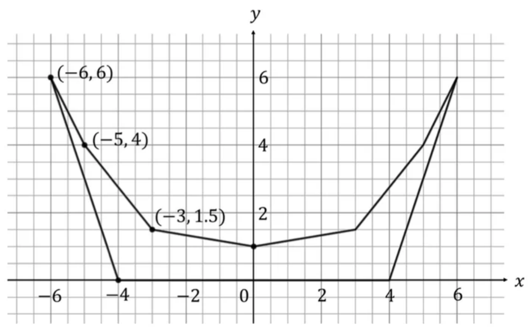
Question 1

Paul finds an unusually shaped bowl when excavating his garden. It appears to be made out of bronze, and Paul decides to model the shape in order to work out its volume.

By uploading a photograph of the object onto some graphing software, Paul identifies that the cross-section of the bowl goes through the points $(-4,0)$, $(-6,6)$, $(-5,4)$, $(-3,1.5)$ and $(0,1)$. The cross-section is symmetrical about the y-axis as shown in the diagram. All of the units are in centimetres.



He models the section from $(-4,0)$ to $(-6,6)$ as a straight line.

(a) Find the equation of the line passing through these two points.

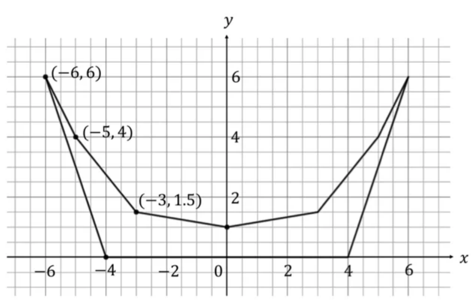
[2]

Equations of a straight line	$y = mx + c$; $ax + by + d = 0$; $y - y_1 = m(x - x_1)$
Gradient formula	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

a) gradient $= \dfrac{6-0}{-6-(-4)} = -3$

$\Rightarrow \quad y - 0 = -3(x - (-4))$ $\quad m = -3$, $(x_1, y_1) = (-4, 0)$

$$\boxed{y = -3x - 12}$$



Paul models the section of the bowl that passes through the points $(-6,6)$, $(-5,4)$, $(-3,1.5)$ and $(0,1)$ with a quadratic curve.

(b) (i) Find the equation of the least squares quadratic curve for these four points.

(ii) By considering the gradient of this curve when $x = -0.5$, explain why it may not be a good model.

[3]

b) (i) Use quadratic least squares regression calculator on GDC

$y = ax^2 + bx + c$

$a = 0.22348484 \quad b = 0.51136363$

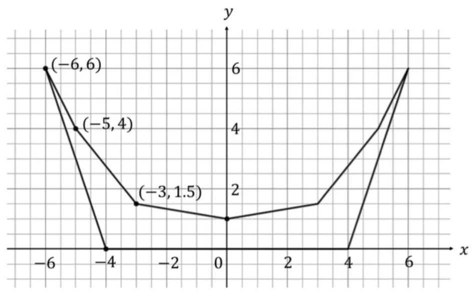
$c = 1.00378787$

$$\boxed{y = 0.223x^2 + 0.511x + 1.00 \text{ (3 s.f.)}}$$

(ii) Use GDC to find gradient

At $x = -0.5$, $\dfrac{dy}{dx} = 0.288$ (3 s.f.).

$$\boxed{\text{The gradient is positive, but the diagram clearly shows a negative gradient when } x = -0.5.}$$



Paul thinks that a quadratic with a minimum at (0, 1) and passing through the point (−6, 6) is a better option.

(c) Find the equation of the new model.

[4]

Believing this to be a better model for the bowl, Paul finds the volume of revolution about the y-axis to estimate the volume of the bowl.

(d) Re-arrange the answers to parts (a) and (c) to make x a function of y.

[3]

(e) (i) Write down an expression for Paul's estimate of the volume as the difference of two integrals.

(ii) Hence find the value of Paul's estimate.

[5]

Axis of symmetry of the graph of a quadratic function	$f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry is $x = -\frac{b}{2a}$

c) We need to find a, b, c for $y = ax^2 + bx + c$.

$(-6, 6) \Rightarrow y = 6$ when $x = -6$

$6 = a(-6)^2 + b(-6) + c \Rightarrow 36a - 6b + c = 6$

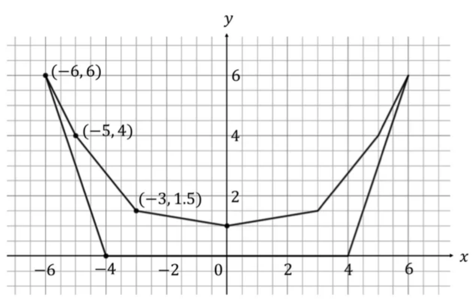
$(0, 1) \Rightarrow y = 1$ when $x = 0$

$1 = a(0)^2 + b(0) + c \Rightarrow c = 1 \Rightarrow 36a - 6b = 5$

Min. at $(0, 1) \Rightarrow$ axis of symmetry is $x = 0$

$-\frac{b}{2a} = 0 \Rightarrow b = 0 \Rightarrow 36a = 5 \Rightarrow a = \frac{5}{36}$

$$\boxed{y = \frac{5}{36}x^2 + 1}$$



Paul thinks that a quadratic with a minimum at (0, 1) and passing through the point (−6, 6) is a better option.

(c) Find the equation of the new model.

[4]

Believing this to be a better model for the bowl, Paul finds the volume of revolution about the y-axis to estimate the volume of the bowl.

(d) Re-arrange the answers to parts (a) and (c) to make x a function of y.

[3]

(e) (i) Write down an expression for Paul's estimate of the volume as the difference of two integrals.

(ii) Hence find the value of Paul's estimate.

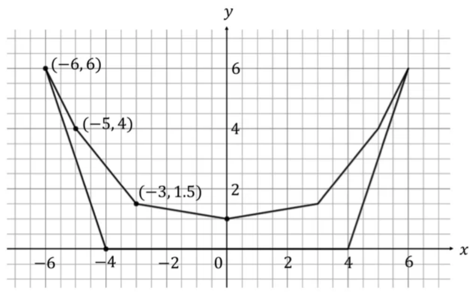
[5]

$\boxed{y = -3x - 12}$ from part (a)

$\boxed{y = \frac{5}{36}x^2 + 1}$ from part (c)

d) $y = -3x - 12 \Rightarrow \boxed{x = -\frac{y + 12}{3}}$

$y = \frac{5}{36}x^2 + 1 \Rightarrow \boxed{x^2 = \frac{36}{5}(y - 1)}$



Paul thinks that a quadratic with a minimum at $(0, 1)$ and passing through the point $(-6, 6)$ is a better option.

(c) Find the equation of the new model.

[4]

Believing this to be a better model for the bowl, Paul finds the volume of revolution about the y-axis to estimate the volume of the bowl.

(d) Re-arrange the answers to parts (a) and (c) to make x a function of y.

$\boxed{x = -\frac{y+12}{3}}$ $\boxed{x^2 = \frac{36}{5}(y-1)}$

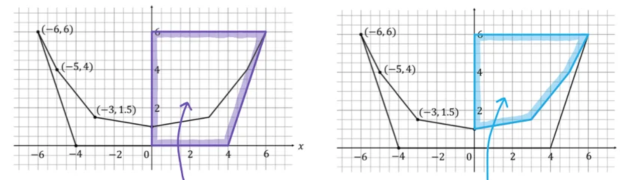
[3]

(e) (i) Write down an expression for Paul's estimate of the volume as the difference of two integrals.

(ii) Hence find the value of Paul's estimate.

[5]

Volume of revolution about x or y-axes	$V = \int_a^b \pi y^2\,dx$ or $V = \int_a^b \pi x^2\,dy$



$\int_{y=0}^{y=6} \pi\left(\frac{y+12}{3}\right)^2 dy$ $\int_{y=1}^{y=6} \pi\left(\frac{36}{5}(y-1)\right) dy$ (approximately)

e) (i) $x = -\frac{y+12}{3} \Rightarrow x^2 = \left(-\frac{y+12}{3}\right)^2 = \left(\frac{y+12}{3}\right)^2$

$$\boxed{V = \int_0^6 \pi\left(\frac{y+12}{3}\right)^2 dy - \int_1^6 \pi\left(\frac{36}{5}(y-1)\right) dy}$$

(ii) Use GDC to evaluate integrals

$V = 62\pi = 194.778744\ldots$

$$\boxed{V = 195 \text{ cm}^3 \ (3 \text{ s.f.})}$$

Question 2

A boat is moving such that its position vector when viewed from above at time t seconds can be modelled by

$$\boldsymbol{r} = \begin{pmatrix} 10 - a\sin\left(\frac{\pi t}{600}\right) \\ b\left(1 - \cos\left(\frac{\pi t}{600}\right)\right) \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O, where the non-zero constants a and b can be determined. All distances are given in metres.

The boat leaves its mooring point, at time $t = 0$ seconds and 5 minutes later is at the point with coordinates $(-20, 40)$.

(a) Find

(i) the values of a and b,

(ii) the displacement of the boat from its mooring point.

[4]

(b) Find the velocity vector of the boat at time t seconds.

[2]

After setting off, the boat reaches a point P where it is moving parallel to the x-axis.

(c) Find OP.

[6]

(d) Find the time that the boat returns to its mooring point and the acceleration of the boat at this moment.

[3]

(a)(i) When $t = 5(60) = 300$, $r = \begin{pmatrix} -20 \\ 40 \end{pmatrix}$

Do not assume the mooring point is at the origin.

$\therefore \ 10 - a\sin\left(\frac{300\pi}{600}\right) = -20$

$a = \frac{30}{\sin\left(\frac{\pi}{2}\right)} = 30$

$b\left(1 - \cos\left(\frac{300\pi}{600}\right)\right) = 40$

$b = \frac{40}{1 - \cos\left(\frac{\pi}{2}\right)} = 40$

$$\boxed{a = 30, \ b = 40}$$

(ii) At $t = 0$, $r = \begin{pmatrix} 10 - 30\sin(0) \\ 40(1 - \cos(0)) \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$

Displacement $= \sqrt{(-20-10)^2 + 40^2} = 50$

$$\boxed{50 \text{ metres}}$$

A boat is moving such that its position vector when viewed from above at time t seconds can be modelled by

$$\boldsymbol{r} = \begin{pmatrix} 10 - a\sin\left(\frac{\pi t}{600}\right) \\ b\left(1 - \cos\left(\frac{\pi t}{600}\right)\right) \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O, where the non-zero constants a and b can be determined. All distances are given in metres.

The boat leaves its mooring point, at time $t = 0$ seconds and 5 minutes later is at the point with coordinates $(-20,\ 40)$.

(a) Find

(i) the values of a and b,

(ii) the displacement of the boat from its mooring point.

[4]

(b) Find the velocity vector of the boat at time t seconds.

[2]

After setting off, the boat reaches a point P where it is moving parallel to the x-axis.

(c) Find OP.

[6]

(d) Find the time that the boat returns to its mooring point and the acceleration of the boat at this moment.

[3]

(b) $\underline{v} = \frac{d\underline{r}}{dt}$ formula book

Derivative of $\sin x$	$f(x) = \sin x \Rightarrow f'(x) = \cos x$
Derivative of $\cos x$	$f(x) = \cos x \Rightarrow f'(x) = -\sin x$

Chain rule	$y = g(u) \quad u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\underline{v} = \begin{pmatrix} -30\left(\frac{\pi}{600}\cos\left(\frac{\pi t}{600}\right)\right) \\ -40\left(\frac{-\pi}{600}\sin\left(\frac{\pi t}{600}\right)\right) \end{pmatrix} = \begin{pmatrix} -\frac{\pi}{20}\cos\left(\frac{\pi t}{600}\right) \\ \frac{\pi}{15}\sin\left(\frac{\pi t}{600}\right) \end{pmatrix}$$

$$\boxed{\underline{v} = \begin{pmatrix} -\frac{\pi}{20}\cos\left(\frac{\pi t}{600}\right) \\ \frac{\pi}{15}\sin\left(\frac{\pi t}{600}\right) \end{pmatrix}}$$

A boat is moving such that its position vector when viewed from above at time t seconds can be modelled by

$$\boldsymbol{r} = \begin{pmatrix} 10 - a\sin\left(\frac{\pi t}{600}\right) \\ b\left(1 - \cos\left(\frac{\pi t}{600}\right)\right) \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O, where the non-zero constants a and b can be determined. All distances are given in metres.

The boat leaves its mooring point, at time $t = 0$ seconds and 5 minutes later is at the point with coordinates $(-20,\ 40)$.

(a) Find

(i) the values of a and b,

(ii) the displacement of the boat from its mooring point.

[4]

(b) Find the velocity vector of the boat at time t seconds.

$$\boxed{\underline{v} = \begin{pmatrix} -\frac{\pi}{20}\cos\left(\frac{\pi t}{600}\right) \\ \frac{\pi}{15}\sin\left(\frac{\pi t}{600}\right) \end{pmatrix}}$$

[2]

After setting off, the boat reaches a point P where it is moving parallel to the x-axis.

(c) Find OP.

[6]

(d) Find the time that the boat returns to its mooring point and the acceleration of the boat at this moment.

[3]

(c) Parallel to the x-axis → only the x-coordinates are changing.
Velocity in the y-direction will be zero.

$$\frac{\pi}{15}\sin\left(\frac{\pi t}{600}\right) = 0 \Rightarrow \sin\left(\frac{\pi t}{600}\right) = 0$$

$$\frac{\pi t}{600} = \sin^{-1} 0$$

$$\frac{\cancel{\pi} t}{600} = n\cancel{\pi}$$

$t = 600n$ when $n = 1$, $t = 600$

Find the position of the boat when $t = 600$ s.

$$\underline{r} = \begin{pmatrix} 10 - 30\sin\pi \\ 40(1 - \cos\pi) \end{pmatrix} = \begin{pmatrix} 10 \\ 80 \end{pmatrix}$$

$$\text{Displacement} = \sqrt{(10-0)^2 + (80-0)^2} = \sqrt{6500}$$
$$= 10\sqrt{65}$$

$\boxed{80.6 \text{ metres}}$

A boat is moving such that its position vector when viewed from above at time t seconds can be modelled by

$$\mathbf{r} = \begin{pmatrix} 10 - a\sin\left(\frac{\pi t}{600}\right) \\ b\left(1 - \cos\left(\frac{\pi t}{600}\right)\right) \end{pmatrix}$$

with respect to a rectangular coordinate system from a point O, where the non-zero constants a and b can be determined. All distances are given in metres.

The boat leaves its mooring point, at time $t = 0$ seconds and 5 minutes later is at the point with coordinates $(-20,\ 40)$.

(a) Find

(i) the values of a and b,

(ii) the displacement of the boat from its mooring point.

[4]

(b) Find the velocity vector of the boat at time t seconds.

[2]

After setting off, the boat reaches a point P where it is moving parallel to the x-axis.

(c) Find OP.

[6]

(d) Find the time that the boat returns to its mooring point and the acceleration of the boat at this moment.

[3]

(d) The functions of sin and cos both repeat every 2π radians.

The boat will be back at its mooring point when

$$\frac{\pi t}{600} = 2\pi$$

$$t = 1200 \text{ seconds} = 20 \text{ minutes}$$

$$\underline{a} = \frac{d\underline{v}}{dt}$$

$$\underline{a} = \begin{pmatrix} -\frac{\pi}{20}\left(\frac{-\pi}{600}\sin\left(\frac{\pi t}{600}\right)\right) \\ \frac{\pi}{15}\left(\frac{\pi}{600}\cos\left(\frac{\pi t}{600}\right)\right) \end{pmatrix} = \begin{pmatrix} \frac{\pi^2}{12000}\sin\left(\frac{\pi t}{600}\right) \\ \frac{\pi^2}{9000}\cos\left(\frac{\pi t}{600}\right) \end{pmatrix}$$

when $t = 1200$, $\underline{a} = \begin{pmatrix} \frac{\pi^2}{12000}\sin(2\pi) \\ \frac{\pi^2}{9000}\cos(2\pi) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\pi^2}{9000} \end{pmatrix}$

$$\boxed{t = 20\text{ mins},\quad \underline{a} = \frac{\pi^2}{9000}\text{ ms}^{-2}}$$

Question 3

On a particular island, a particular species of bird was initially recorded as having a population of 80 at the start of a programme of observations. Over time, the scientists conducting the programme determined that the growth rate of the bird population could be modelled by the following differential equation

$$\frac{dx}{dt} = \frac{7}{5}x$$

where x is the size of the bird population, and t is the length of time in years since the start of the programme.

(a) Find the population of the bird species two years after the start of the programme.

[5]

a) Use separation of variables to solve the differential equation:

$$\frac{dx}{dt} = \frac{7}{5}x \implies \frac{1}{x}\frac{dx}{dt} = \frac{7}{5}$$

$$\implies \int \frac{1}{x}\,dx = \int \frac{7}{5}\,dt$$

$$\implies \ln x = \frac{7}{5}t + c$$

Don't forget constant of integration!

$\int \frac{1}{x}dx = \ln|x| + c$. But x is the size of a population and can't be negative, so we don't need the modulus sign here.

$$\implies x = Ae^{\frac{7}{5}t}$$

$e^{\frac{7}{5}t + c} = (e^{\frac{7}{5}t})(e^c) = Ae^{\frac{7}{5}t}$, with $A = e^c$

When $t = 0$ ('initially'), $x = 80$. So:

$$80 = Ae^0 \implies A = 80 \implies x = 80e^{\frac{7}{5}t}$$

So when $t = 2$,

$$x = 80e^{\frac{7}{5}(2)} = 80e^{\frac{14}{5}} = 1315.57174\ldots$$

$$\boxed{1320 \text{ birds (3 s.f.)}}$$

When the population of the bird species reaches 2000, a new reptile species is introduced to the island in order to control the bird population. Initially 280 reptiles are introduced to the island. Based on their research the scientists believe that the interaction between the two species after the introduction of the reptiles can be modelled by the system of coupled differential equations

$$\frac{dx}{dt} = (3 - 0.012y)x$$

$$\frac{dy}{dt} = (0.0007x - 1)y$$

Where x and y represent the size of the bird and reptile populations respectively.

(b) Using the Euler method with a step size of 0.5, find an estimate for

(i) the bird population 2 years after the reptiles were introduced

(ii) the reptile population 2 years after the reptiles were introduced.

[6]

(c) Explain how the approximation in part (b) could be improved.

[1]

(d) Show that the origin is an equilibrium point for the system, and determine the coordinates of the other equilibrium point.

[3]

Euler's method for coupled systems	$x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$ $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$ $t_{n+1} = t_n + h$	h is a constant (step length)

b) $\frac{2}{0.5} = 4$, so will need to use 4 steps

$x_{n+1} = x_n + 0.5((3 - 0.012y_n)x_n)$

$y_{n+1} = y_n + 0.5((0.0007x_n - 1)y_n)$

$t_{n+1} = t_n + 0.5$

n	t_n	x_n	y_n
0	0	2000	280
1	0.5	1640	336
2	1	793.76	360.86
3	1.5	265.76	280.68
4	2	216.83	166.45

(i) $\boxed{\text{217 birds}}$ (ii) $\boxed{\text{166 reptiles}}$

When the population of the bird species reaches 2000, a new reptile species is introduced to the island in order to control the bird population. Initially 280 reptiles are introduced to the island. Based on their research the scientists believe that the interaction between the two species after the introduction of the reptiles can be modelled by the system of coupled differential equations

$$\frac{dx}{dt} = (3 - 0.012y)x$$

$$\frac{dy}{dt} = (0.0007x - 1)y$$

Where x and y represent the size of the bird and reptile populations respectively.

(b) Using the Euler method with a step size of 0.5, find an estimate for

(i) the bird population 2 years after the reptiles were introduced

(ii) the reptile population 2 years after the reptiles were introduced.

[6]

(c) Explain how the approximation in part (b) could be improved.

[1]

(d) Show that the origin is an equilibrium point for the system, and determine the coordinates of the other equilibrium point.

[3]

c) $\boxed{\text{Decrease the step size.}}$

When the population of the bird species reaches 2000, a new reptile species is introduced to the island in order to control the bird population. Initially 280 reptiles are introduced to the island. Based on their research the scientists believe that the interaction between the two species after the introduction of the reptiles can be modelled by the system of coupled differential equations

$$\frac{dx}{dt} = (3 - 0.012y)x$$

$$\frac{dy}{dt} = (0.0007x - 1)y$$

Where x and y represent the size of the bird and reptile populations respectively.

(b) Using the Euler method with a step size of 0.5, find an estimate for

(i) the bird population 2 years after the reptiles were introduced

(ii) the reptile population 2 years after the reptiles were introduced. [6]

(c) Explain how the approximation in part (b) could be improved. [1]

(d) Show that the origin is an equilibrium point for the system, and determine the coordinates of the other equilibrium point. [3]

d) An equilibrium point is where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

At $(0,0)$, $\frac{dx}{dt} = (3 - 0.012(0))(0) = 0$ and $\frac{dy}{dt} = (0.0007(0) - 1)(0) = 0$, so the origin is an equilibrium point.

Also $\frac{dx}{dt} = 0$ when

$3 - 0.012y = 0 \Rightarrow y = 250$

and $\frac{dy}{dt} = 0$ when

$0.0007x - 1 = 0 \Rightarrow x = \frac{10000}{7} = 1428.571428\ldots$

The other equilibrium point is $\left(\frac{10000}{7}, 250\right)$, or approximately $(1429, 250)$.

Question 4

A new car costs \$20 000 and its value depreciates to \$14 792 after 2 years.

(a) Calculate

(i) the annual rate of depreciation of the car

(ii) the value of the car after 5 years. Give your answer correct to 2 decimal places. [3]

(b) Find the number of years and months it will take for the car's value to be approximately \$4000. [3]

a) i) Depreciation formula

$FV = PV\left(1 - \frac{r}{100}\right)^n$ (not in formula booklet)

$FV = 14792$ $PV = 20\,000$ $n = 2$

Sub FV, PV and n into formula and solve for r using your GDC.

$14792 = 20\,000\left(1 - \frac{r}{100}\right)^2$

$r = 14\%$

ii) $PV = 20\,000$ $r = 14\%$ $n = 5$

Sub PV, r and n into formula.

$FV = 20\,000\left(1 - \frac{14}{100}\right)^5$

$FV \approx \$9408.54$ (2dp)

In a game, enemies appear independently and randomly at an average rate of 2.5 enemies every minute.

(a) Find the probability that exactly 3 enemies will appear during one particular minute. [1]

(b) Find the probability that exactly 10 enemies will appear in a five-minute period. [2]

(c) Find the probability that at least 3 enemies will appear in a 90-second period. [2]

(d) The probability that at least one enemy appears in k seconds is 0.999. Find the value of k correct to 3 significant figures. [2]

(e) A 10-minute interval is divided into ten 1-minute periods (first minute, second minute, third minute, etc.). Find the probability that there will be exactly two of those 1-minute periods in which no enemies appear. [4]

Poisson distribution	$X \sim \mathrm{Po}(m)$
Mean	$\mathrm{E}(X) = m$
Variance	$\mathrm{Var}(X) = m$

b) Scale the mean m to the different time interval:

$m = 5 \times 2.5 = 12.5$

$X \sim Po(12.5)$

$P(X = 10) = 0.09564363$ from GDC

$\boxed{0.0956 \text{ (3 s.f.)}}$

In a game, enemies appear independently and randomly at an average rate of 2.5 enemies every minute.

(a) Find the probability that exactly 3 enemies will appear during one particular minute. [1]

(b) Find the probability that exactly 10 enemies will appear in a five-minute period. [2]

(c) Find the probability that at least 3 enemies will appear in a 90-second period. [2]

(d) The probability that at least one enemy appears in k seconds is 0.999. Find the value of k correct to 3 significant figures. [2]

(e) A 10-minute interval is divided into ten 1-minute periods (first minute, second minute, third minute, etc.). Find the probability that there will be exactly two of those 1-minute periods in which no enemies appear. [4]

Poisson distribution	$X \sim \mathrm{Po}(m)$
Mean	$\mathrm{E}(X) = m$
Variance	$\mathrm{Var}(X) = m$

c) Scale the mean m to the different time interval:

$m = \frac{90}{60} \times 2.5 = 3.75$

$X \sim Po(3.75)$

$P(X \geq 3) = 0.72293155$ from GDC

$\boxed{0.723 \text{ (3 s.f.)}}$

In a game, enemies appear independently and randomly at an average rate of 2.5 enemies every minute.

(a) Find the probability that exactly 3 enemies will appear during one particular minute.

[1]

(b) Find the probability that exactly 10 enemies will appear in a five-minute period.

[2]

(c) Find the probability that at least 3 enemies will appear in a 90-second period.

[2]

(d) The probability that at least one enemy appears in k seconds is 0.999. Find the value of k correct to 3 significant figures.

[2]

(e) A 10-minute interval is divided into ten 1-minute periods (first minute, second minute, third minute, etc.). Find the probability that there will be exactly two of those 1-minute periods in which no enemies appear.

[4]

Poisson distribution	$X \sim \mathrm{Po}(m)$
Mean	$\mathrm{E}(X) = m$
Variance	$\mathrm{Var}(X) = m$

d) $P(X \geq 1) = 0.999 \Rightarrow P(X=0) = 1 - 0.999 = 0.001$

Can use trial and error to find associated Poisson mean or use equation solver on GDC.

Solve N(Poisson PD(0, x) = 0.001) = 6.907755279

So $m = \frac{2.5}{60} k = 6.907755279$

rate per second

number of seconds

$\Rightarrow k = \frac{60}{2.5} \times 6.907755279 = 165.7861267$

$\boxed{166 \text{ seconds (3 s.f.)}}$

In a game, enemies appear independently and randomly at an average rate of 2.5 enemies every minute.

(a) Find the probability that exactly 3 enemies will appear during one particular minute.

[1]

(b) Find the probability that exactly 10 enemies will appear in a five-minute period.

[2]

(c) Find the probability that at least 3 enemies will appear in a 90-second period.

[2]

(d) The probability that at least one enemy appears in k seconds is 0.999. Find the value of k correct to 3 significant figures.

[2]

(e) A 10-minute interval is divided into ten 1-minute periods (first minute, second minute, third minute, etc.). Find the probability that there will be exactly two of those 1-minute periods in which no enemies appear.

[4]

Poisson distribution	$X \sim \mathrm{Po}(m)$
Mean	$\mathrm{E}(X) = m$
Variance	$\mathrm{Var}(X) = m$

Binomial distribution $X \sim \mathrm{B}(n, p)$	
Mean	$\mathrm{E}(X) = np$
Variance	$\mathrm{Var}(X) = np(1-p)$

e) For any one-minute period, $X \sim Po(2.5)$ and

$P(X=0) = 0.08208499$ from GDC

If N is the number of those periods in which no enemies appear, then $N \sim B(10, 0.08208499)$.

$P(N=2) = 0.15281281$ from GDC

$\boxed{0.153 \text{ (3 s.f.)}}$

On the next level of the game, there is a boss enemy and a number of additional henchmen to fight against.

The number of times that the boss enemy appears in a one-minute period can be modelled by a Poisson distribution with a mean of 1.1.

The number of times that an individual henchman appears in a one-minute period can be modelled by a Poisson distribution with a mean of 0.6.

It may be assumed that the boss enemy and the henchmen each appear randomly and independently of one another. This means the sum will also be Poisson

Each time that the boss enemy or any particular henchman appears, it is counted as one 'enemy appearance'.

(f) Determine the least number of henchmen required in order that the probability of 40 or more 'enemy appearances' occurring in a 3-minute period is greater than 0.38. You may assume that neither the boss enemy nor any of the henchmen are able to be totally eliminated from the game during this 3-minute period.

[4]

f) For the boss, the mean for 3 minutes is $3 \times 1.1 = 3.3$.

For one henchman, the mean for 3 minutes is $3 \times 0.6 = 1.8$

So with n henchmen, the total mean is $3.3 + 1.8n$

If A is the total number of enemy appearances, then $A \sim Po(3.3 + 1.8n)$.

Can use trial and error to find the required Poisson mean or use equation solver on GDC.

$\text{Solve N}(\text{Poisson CD}(\underbrace{40, 1 \times 10^{99}}_{40 \text{ or more}}, x) = 0.38) = \underbrace{37.77549904}_{\text{The total mean needs to be bigger than this}}$

$\Rightarrow 3.3 + 1.8n > 37.77549904$

$\Rightarrow n > 19.153055\ldots$ But remember, n has to be an integer!

$\boxed{\text{minimum of 20 henchmen}}$

Question 5

James throws a ball to his friend Mia. The height, h, in metres, of the ball above the ground is modelled by the function

$$h(t) = -1.05t^2 + 3.84t + 1.97, \qquad t \geq 0$$

where t is the time, in seconds, from the moment that James releases the ball.

(a) Write down the height of the ball when James releases it.

$t = 0$

[1]

a) This is the height when $t = 0$

$h(0) = -1.05(0)^2 + 3.84(0) + 1.97 = 1.97$

$\boxed{1.97 \text{ m}}$

After 4 seconds the ball is at a height of q metres above the ground.

(b) Find the value of q.

[2]

(c) Find $h'(t)$

[2]

(d) Find the maximum height reached by the ball and write down the corresponding time t.

[3]

The highest Mia can reach her hands up to catch the ball is 1.92 m above the ground, and she is able to catch the ball at that height or at any height less than that.

(e) Find the total time that the ball would be at a height at which Mia could catch it, if the ball were allowed to move freely until it hit the ground.

[3]

James throws a throws ball to his friend Mia. The height, h, in metres, of the ball above the ground is modelled by the function

$$h(t) = -1.05t^2 + 3.84t + 1.97, \quad t \geq 0$$

where t is the time, in seconds, from the moment that James releases the ball.

(a) Write down the height of the ball when James releases it.

[1]

$t = 4$

After 4 seconds the ball is at a height of q metres above the ground.

(b) Find the value of q.

[2]

(c) Find $h'(t)$

[2]

(d) Find the maximum height reached by the ball and write down the corresponding time t.

[3]

b) $h(4) = -1.05(4)^2 + 3.84(4) + 1.97$

$= -16.8 + 15.36 + 1.97 = 0.53$

$\boxed{q = 0.53}$

James throws a throws ball to his friend Mia. The height, h, in metres, of the ball above the ground is modelled by the function

$$h(t) = -1.05t^2 + 3.84t + 1.97, \quad t \geq 0$$

where t is the time, in seconds, from the moment that James releases the ball.

(a) Write down the height of the ball when James releases it.

[1]

After 4 seconds the ball is at a height of q metres above the ground.

(b) Find the value of q.

[2]

(c) Find $h'(t)$

[2]

(d) Find the maximum height reached by the ball and write down the corresponding time t.

[3]

$f(x) = x^n \implies f'(x) = nx^{n-1}$ } Derivative of x^n

c) $h'(t) = -1.05(2t) + 3.84(1) + 0$

$\boxed{h'(t) = -2.1t + 3.84}$

James throws a throws ball to his friend Mia. The height, h, in metres, of the ball above the ground is modelled by the function

$$h(t) = -1.05t^2 + 3.84t + 1.97, \qquad t \geq 0$$

where t is the time, in seconds, from the moment that James releases the ball.

(a) Write down the height of the ball when James releases it.

[1]

After 4 seconds the ball is at a height of q metres above the ground.

(b) Find the value of q.

[2]

(c) Find $h'(t)$ $\boxed{h'(t) = -2.1t + 3.84}$

[2]

(d) Find the maximum height reached by the ball and write down the corresponding time t.

[3]

d) $h(t)$ has a maximum when $h'(t) = 0$

$-2.1t + 3.84 = 0$

$2.1t = 3.84$

exact answer

$t = \frac{3.84}{2.1} = \frac{64}{35} = 1.828571\ldots$

$\boxed{t = \frac{64}{35} \text{ seconds} = 1.83 \text{ seconds (3 s.f.)}}$

$h\left(\frac{64}{35}\right) = -1.05\left(\frac{64}{35}\right)^2 + 3.84\left(\frac{64}{35}\right) + 1.97$

$= 5.480857\ldots$

$\boxed{\text{maximum height} = 5.48 \text{ m (3 s.f.)}}$

* You could also find these values by graphing $h(t)$ on your GDC and finding the max.

James then drives a remote-controlled car in a straight horizontal line from a starting position right in front of his feet. The velocity of the remote-controlled car in ms^{-1} is given by the equation

$$v(t) = \frac{5}{4}t^3 - \frac{19}{2}t^2 + 18t - 2$$

(e) Find an expression for the horizontal displacement of the remote-controlled car from its starting position at time t seconds.

[4]

(f) Find the total horizontal distance that the remote-controlled car has travelled in the first 5 seconds.

[3]

e) $v(t) = \frac{ds}{dt} \implies s(t) = \int v(t)\,dt$

$s(t) = \int \left(\frac{5}{4}t^3 - \frac{19}{2}t^2 + 18t - 2\right) dt$

$= \frac{5}{16}t^4 - \frac{19}{6}t^3 + 9t^2 - 2t + c$ constant of integration

But $s = 0$ when $t = 0$, so

$\frac{5}{16}(0)^4 - \frac{19}{6}(0)^3 + 9(0)^2 - 2(0) + c = 0$

$\implies c = 0$

$\boxed{s(t) = \frac{5}{16}t^4 - \frac{19}{6}t^3 + 9t^2 - 2t}$

James then drives a remote-controlled car in a straight horizontal line from a starting position right in front of his feet. The velocity of the remote-controlled car in ms^{-1} is given by the equation

$$v(t) = \frac{5}{4}t^3 - \frac{19}{2}t^2 + 18t - 2$$

(e) Find an expression for the horizontal displacement of the remote-controlled car from its starting position at time t seconds.

[4]

(f) Find the total horizontal distance that the remote-controlled car has travelled in the first 5 seconds.

Distance, not displacement!

[3]

f) $\text{distance} = \int_{t_1}^{t_2} |v(t)|\, dt$ } Distance travelled from t_1 to t_2

from GDC

$$\int_0^5 \left| \frac{5}{4}t^3 - \frac{19}{2}t^2 + 18t - 2 \right| dt = 18.718574\ldots$$

$\boxed{18.7 \text{ metres } (3 \text{ s.f.})}$

Question 6

Consider the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = -3x - 4y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$.

[6]

(b) Hence write down the general solution of the system.

[2]

When $t = 0$, $x = 2$ and $y = 4$.

(c) Use the given initial condition to determine the exact solution of the system.

[3]

(d) (i) Find the value of $\frac{dy}{dx}$ when $t = 0$.

(ii) Find the values of x, y and $\frac{dy}{dx}$ when $t = \ln\frac{9}{7}$.

[3]

(e) Hence sketch the solution trajectory of the system for $t \geq 0$.

[3]

a) Use characteric equation to find the eigenvalues:

$$\begin{vmatrix} 1-\lambda & 2 \\ -3 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda) - (2)(-3) = 0$$

$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$ Solve with GDC or by factorising

$\Rightarrow \lambda = -1, -2$

Then use $(A - \lambda I)\underline{x} = 0$ to find the eigenvectors:

For $\lambda = -1$,

$$\begin{pmatrix} 1-(-1) & 2 \\ -3 & -4-(-1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 2y \\ -3x - 3y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow y = -x \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector

Any multiple of this is also an eigenvector, for example $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

For $\lambda = -2$,

$$\begin{pmatrix} 1-(-2) & 2 \\ -3 & -4-(-2) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 2y \\ -3x - 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow y = -\frac{3}{2}x \Rightarrow \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ is an eigenvector

Any multiple of this is also an eigenvector, for example $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

The eigenvalues are -1 with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and -2 with eigenvector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Consider the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = -3x - 4y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$.

$\boxed{-1 \text{ with } \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ and } -2 \text{ with } \begin{pmatrix} 2 \\ -3 \end{pmatrix}}$

[6]

(b) Hence write down the general solution of the system.

[2]

When $t = 0$, $x = 2$ and $y = 4$.

(c) Use the given initial condition to determine the exact solution of the system.

[3]

(d) (i) Find the value of $\frac{dy}{dx}$ when $t = 0$.

(ii) Find the values of x, y and $\frac{dy}{dx}$ when $t = \ln\frac{9}{7}$.

[3]

(e) Hence sketch the solution trajectory of the system for $t \geq 0$.

[3]

Exact solution for coupled linear differential equations	$\boldsymbol{x} = Ae^{\lambda_1 t}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{p}_2$

b) $\boxed{\underline{x} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}}$

Consider the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = -3x - 4y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$.

[6]

(b) Hence write down the general solution of the system.

$\boxed{\underline{x} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}}$

[2]

When $t = 0$, $x = 2$ and $y = 4$.

(c) Use the given initial condition to determine the exact solution of the system.

[3]

(d) (i) Find the value of $\frac{dy}{dx}$ when $t = 0$.

(ii) Find the values of x, y and $\frac{dy}{dx}$ when $t = \ln\frac{9}{7}$.

[3]

(e) Hence sketch the solution trajectory of the system for $t \geq 0$.

[3]

c) I.e., when $t = 0$, $\underline{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$:

$Ae^0\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^0\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$\Longrightarrow \begin{pmatrix} A + 2B \\ -A - 3B \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ Simultaneous equations (solve by hand or with GDC)

$\Longrightarrow A = 14, \ B = -6$

$\boxed{\underline{x} = 14e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 6e^{-2t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}}$

Consider the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = -3x - 4y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$. [6]

(b) Hence write down the general solution of the system. [2]

When $t = 0$, $x = 2$ and $y = 4$.

(c) Use the given initial condition to determine the exact solution of the system. [3]

$$\boxed{\underline{x} = 14e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 6e^{-2t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}}$$

(d) (i) Find the value of $\frac{dy}{dx}$ when $t = 0$.

(ii) Find the values of x, y and $\frac{dy}{dx}$ when $t = \ln\frac{9}{7}$. [3]

(e) Hence sketch the solution trajectory of the system for $t \geq 0$. [3]

d) (i) $\frac{dy}{dt} = -3(2) - 4(4) = -22$

$\frac{dx}{dt} = (2) + 2(4) = 10$

So the initial trajectory is 'down and to the right'

$$\Longrightarrow \boxed{\frac{dy}{dx} = \frac{-22}{10} = -\frac{11}{5}} \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

(ii) $e^{-\ln\frac{9}{7}} = \frac{7}{9}$ and $e^{-2\ln\frac{9}{7}} = \frac{49}{81}$, so at $t = \ln\frac{9}{7}$

$$\underline{x} = 14\left(\frac{7}{9}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 6\left(\frac{49}{81}\right)\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 98/27 \\ 0 \end{pmatrix}$$

$$\boxed{x = \frac{98}{27} \qquad y = 0}$$

$\frac{dy}{dt} = -3\left(\frac{98}{27}\right) - 4(0) = -\frac{98}{9}$

$\frac{dx}{dt} = \left(\frac{98}{27}\right) + 2(0) = \frac{98}{27}$

So on the positive x-axis the trajectory is also 'down and to the right'

$$\Longrightarrow \boxed{\frac{dy}{dx} = \frac{-98/9}{98/27} = -3} \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Consider the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = -3x - 4y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$. [6]

(b) Hence write down the general solution of the system. [2]

When $t = 0$, $x = 2$ and $y = 4$.

(c) Use the given initial condition to determine the exact solution of the system. [3]

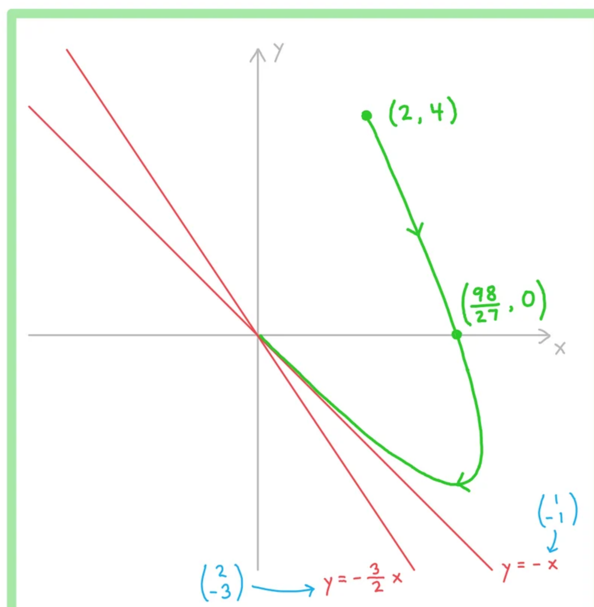
$$\boxed{\underline{x} = 14e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 6e^{-2t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}}$$

(d) (i) Find the value of $\frac{dy}{dx}$ when $t = 0$. $\boxed{\frac{dy}{dx} = -\frac{11}{5}}$

(ii) Find the values of x, y and $\frac{dy}{dx}$ when $t = \ln\frac{9}{7}$. $\boxed{x = \frac{98}{27} \quad y = 0 \quad \frac{dy}{dx} = -3}$ [3]

(e) Hence sketch the solution trajectory of the system for $t \geq 0$. [3]

e)



Include the points you know, and use the answers from part (d) to get the gradients and directions at those points.

Because both eigenvalues are real and negative, the trajectory will converge on the origin as t increases.