IB Maths: AI HL

Eigenvalues & Eigenvectors

Topic Questions

These practice questions can be used by students and teachers and is Suitable for IB Maths AI HL Topic Questions

Course	IB Maths
Section	1. Number & Algebra
Topic	1.8 Eigenvalues & Eigenvectors
Difficulty	Medium

Level: IB Maths

Subject: IB Maths AI HL

Board: IB Maths

Topic: Eigenvalues & Eigenvectors

Question 1

Consider the 2×2 matrix $\boldsymbol{A}$ defined by

$$\boldsymbol{A} = \begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix}$$

(a)

(i)

Find the characteristic polynomial of $\boldsymbol{A}$.

(ii)

By solving an appropriate equation with the characteristic polynomial, find the eigenvalues λ_1 and λ_2 of $\boldsymbol{A}$.

[3 marks]

Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be the eigenvectors of $\boldsymbol{A}$ corresponding to λ_1 and λ_2 respectively.

(b)

By solving the eigenvector equations $\boldsymbol{A}\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$ and $\boldsymbol{A}\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2$, find eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

[4 marks]

(c) Show that the answers to part (b) could alternatively have been found by solving the equations $(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\,\boldsymbol{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\,\boldsymbol{x}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where $\boldsymbol{I}$ is the 2×2 identity matrix.

[3 marks]

Question 2

Find the eigenvalues and corresponding eigenvectors for the matrix $\boldsymbol{A}$ defined as

$$\boldsymbol{A} = \begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix}$$

[1 mark]

Question 3

Consider the matrix $\boldsymbol{B}$ defined as

$$\boldsymbol{B} = \begin{pmatrix} 4 & -6 \\ 1 & -2 \end{pmatrix}$$

Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{B}$.

[6 marks]

Question 4

Find the eigenvalues for each of the following matrices:

(a)

$$\boldsymbol{C} = \begin{pmatrix} -2 & 13 \\ -1 & 2 \end{pmatrix}$$

[3 marks]

(b)

$$\boldsymbol{D} = \begin{pmatrix} 6 & -1 \\ 17 & -2 \end{pmatrix}$$

[3 marks]

Question 5

Consider the matrix $\boldsymbol{M}$ defined as

$$\boldsymbol{M}=\begin{pmatrix} -1 & k \\ 3 & -1 \end{pmatrix}$$

where $k \in \mathbb{R}$ is a constant.

The eigenvalues of $\boldsymbol{M}$ are 2 and -4.

(a)
Find the value of k.

[3 marks]

(b)
Find the eigenvectors of $\boldsymbol{M}$ that correspond to the two eigenvalues.

[3 marks]

(c)
Hence write $\boldsymbol{M}$ in the form $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{P}$ is a matrix of eigenvectors and $\boldsymbol{D}$ is a diagonal matrix of eigenvalues.

[2 marks]

Question 6

(a)
It is given that, for $n \times n$ matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$,

$$\boldsymbol{A}=\boldsymbol{BCB}^{-1}$$

Use the properties of matrices and matrix inverses to show that $\boldsymbol{A}^2=\boldsymbol{BC}^2\boldsymbol{B}^{-1}$.

[3 marks]

Consider the matrix $\boldsymbol{M}=\begin{pmatrix}3 & -2\\ p & 1\end{pmatrix}$, where $p\in\mathbb{R}$ is a constant and where it is given that $\begin{pmatrix}1\\2\end{pmatrix}$ is an eigenvector of $\boldsymbol{M}$.

(b)
Find the value of p.

[3 marks]

(c)
Hence, by first finding the eigenvalues and the other eigenvector of $\boldsymbol{M}$, write $\boldsymbol{M}$ in the form $\boldsymbol{M}=\boldsymbol{PDP}^{-1}$ for appropriate matrices $\boldsymbol{P}$ and $\boldsymbol{D}$.

[5 marks]

(d) (i) Use the result of part (c) to show that

$$\boldsymbol{M}^n=\frac{1}{3}\begin{pmatrix}2(5^n)+(-1)^n & -5^n+(-1)^n\\ -2(5^n)+2(-1)^n & 5^n+2(-1)^n\end{pmatrix}$$

(ii)
Show that the expression for $\boldsymbol{M}^n$ in part (d)(i) gives the expected result when $n=1$.

[4 marks]

Question 7

Exobiologists are studying two species of animals in a region of the distant planet Dirion. In the researchers' models the population of Heliors (a predator species) is indicated by h, while the population of Sklyveths (a competing predator species) is indicated by s.

If the respective populations at a particular point in time are h_n and s_n, then the researchers' data suggest that the populations one year later may be given by the following system of coupled equations:

$$h_{n+1}=1.06h_n-0.16s_n$$

$$s_{n+1}=-0.04h_n+0.94s_n$$

(a)
Represent the system of equations in the matrix form $\boldsymbol{x}_{n+1}=\boldsymbol{Mx}_n$.

[2 marks]

At the start of the study, there are 600 Heliors and 500 Sklyveths in the region.

(b)
Find the expected size of the respective populations after one year.

[2 marks]

(c)
By first finding the eigenvalues and corresponding eigenvectors of $\boldsymbol{M}$ write $\boldsymbol{M}$ in the form $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{P}$ is a matrix of eigenvectors and $\boldsymbol{D}$ is a diagonal matrix of eigenvalues.

[8 marks]

(d)
Hence show that the respective populations after n years are predicted by the model to be $h_n = 520(0.9^n) + 80(1.1^n)$ and $s_n = 520(0.9^n) - 20(1.1^n)$.

[3 marks]

(e)
Describe what the model predicts in the long term for the populations of the two species, and offer one criticism of the model based on this prediction.

[4 marks]