Question 1

Consider the 2×2 matrix $\boldsymbol{A}$ defined by

$$\boldsymbol{A} = \begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix}$$

(a) (i) Find the characteristic polynomial of $\boldsymbol{A}$.

(ii) By solving an appropriate equation with the characteristic polynomial, find the eigenvalues λ_1 and λ_2 of $\boldsymbol{A}$.

[3]

Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be the eigenvectors of $\boldsymbol{A}$ corresponding to λ_1 and λ_2 respectively.

(b) By solving the eigenvector equations $\boldsymbol{A}\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$ and $\boldsymbol{A}\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2$, find eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

[4]

(c) Show that the answers to part (b) could alternatively have been found by solving the equations $(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{x}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where $\boldsymbol{I}$ is the 2×2 identity matrix.

[3]

(a) (i) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = |A| = ad - bc$ ← formula booklet

$$\det\begin{pmatrix} 0.1-\lambda & 0.4 \\ 0.9 & 0.6-\lambda \end{pmatrix} = (0.1-\lambda)(0.6-\lambda) - (0.4)(0.9)$$

$$= 0.06 - 0.6\lambda - 0.1\lambda + \lambda^2 - 0.36$$

$$\boxed{\lambda^2 - 0.7\lambda - 0.3}$$

(ii) $\lambda^2 - 0.7\lambda - 0.3 = 0$

$(\lambda - 1)(\lambda + 0.3) = 0$

$$\boxed{\lambda_1 = 1, \quad \lambda_2 = -0.3}$$

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[3]

(b) For x_1:

$$\begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (1)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 0.1x + 0.4y \\ 0.9x + 0.6y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 0.9x - 0.4y \\ 0.9x - 0.4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, $9x = 4y$ $\quad \boxed{x_1 = \begin{pmatrix} 4 \\ 9 \end{pmatrix} \text{ (or any multiple of)}}$

For x_2:

$$\begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (-0.3)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 0.1x + 0.4y \\ 0.9x + 0.6y \end{pmatrix} = \begin{pmatrix} -0.3x \\ -0.3y \end{pmatrix}$$

$$\begin{pmatrix} 0.4x + 0.4y \\ 0.9x + 0.9y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, $x = -y$ $\quad \boxed{x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ (or any multiple of)}}$

Consider the 2×2 matrix $\boldsymbol{A}$ defined by

$$\boldsymbol{A} = \begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix}$$

(a) (i) Find the characteristic polynomial of $\boldsymbol{A}$.

(ii) By solving an appropriate equation with the characteristic polynomial, find the eigenvalues λ_1 and λ_2 of $\boldsymbol{A}$.

$\boxed{\lambda_1 = 1, \quad \lambda_2 = -0.3}$

[3]

Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be the eigenvectors of $\boldsymbol{A}$ corresponding to λ_1 and λ_2 respectively.

(b) By solving the eigenvector equations $\boldsymbol{A}\boldsymbol{x}_1 = \lambda_1\boldsymbol{x}_1$ and $\boldsymbol{A}\boldsymbol{x}_2 = \lambda_2\boldsymbol{x}_2$, find eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

[4]

(c) Show that the answers to part (b) could alternatively have been found by solving the equations $(\boldsymbol{A} - \lambda_1\boldsymbol{I})\boldsymbol{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $(\boldsymbol{A} - \lambda_2\boldsymbol{I})\boldsymbol{x}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where $\boldsymbol{I}$ is the 2×2 identity matrix.

[3]

(c) x_1: $\left(\begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix} - (1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\left(\begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -0.9 & 0.4 \\ 0.9 & -0.4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow 9x = 4y \quad \Rightarrow \quad \boxed{x_1 = \begin{pmatrix} 4 \\ 9 \end{pmatrix}}$

x_2: $\left(\begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix} - (-0.3)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\left(\begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix} - \begin{pmatrix} -0.3 & 0 \\ 0 & -0.3 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0.4 & 0.4 \\ 0.9 & 0.9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x = -y \quad \Rightarrow \quad \boxed{x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

Question 2

Find the eigenvalues and corresponding eigenvectors for the matrix $\boldsymbol{A}$ defined as

$$\boldsymbol{A} = \begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix}$$

[6]

Find the characteristic polynomial and solve it to find the eigenvalues

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det \boldsymbol{A} = |\boldsymbol{A}| = ad - bc$$ ← Formula booklet

$$\det\begin{pmatrix} (-1-\lambda) & 4 \\ 1 & (2-\lambda) \end{pmatrix} = (-1-\lambda)(2-\lambda) - (1)(4)$$

$$= -2 - 2\lambda + \lambda + \lambda^2 - 4$$

$$= \lambda^2 - \lambda - 6$$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$$\boxed{\lambda = 3 \qquad \lambda = -2}$$

Use $Ax = \lambda x$ or $(A - \lambda I)x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to find the corresponding eigenvectors

$\lambda_1 = 3$

$$\left(\begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix} - (3)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x = y$$

$$\Rightarrow \boxed{x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ (or any multiple of)}}$$

$\lambda_2 = -2$

$$\left(\begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix} - (-2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x = -4y$

$\Rightarrow \boxed{x_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \text{ (or any multiple of)}}$

Question 3

Consider the matrix $\boldsymbol{B}$ defined as

$$\boldsymbol{B} = \begin{pmatrix} 4 & -6 \\ 1 & -2 \end{pmatrix}$$

Find the eigenvalues and corresponding eigenvectors of $\boldsymbol{B}$.

[6]

Find the characteristic polynomial and solve it to find the eigenvalues

$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det \boldsymbol{A} = |\boldsymbol{A}| = ad - bc$ ← Formula booklet

$$\det\begin{pmatrix} (4-\lambda) & -6 \\ 1 & (-2-\lambda) \end{pmatrix} = (4-\lambda)(-2-\lambda) - (1)(-6)$$

$$= -8 + 2\lambda - 4\lambda + \lambda^2 + 6$$

$$= \lambda^2 - 2\lambda - 2$$

Completing the square helps you to get exact values

$$\lambda^2 - 2\lambda - 2 = 0$$

$$(\lambda - 1)^2 - 3 = 0$$

$$\lambda = 1 \pm \sqrt{3}$$

$$\boxed{\begin{array}{l} \lambda = 1 + \sqrt{3} \\ \lambda = 1 - \sqrt{3} \end{array}}$$

Use $A\mathbf{x} = \lambda \mathbf{x}$ or $(A - \lambda I)\mathbf{x} = \begin{pmatrix}0\\0\end{pmatrix}$ to find the corresponding eigenvectors

$\lambda_1 = 1 + \sqrt{3}$

$$\left(\begin{pmatrix}4 & -6\\1 & -2\end{pmatrix} - (1+\sqrt{3})\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\left(\begin{pmatrix}4 & -6\\1 & -2\end{pmatrix} - \begin{pmatrix}(1+\sqrt{3}) & 0\\0 & (1+\sqrt{3})\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}(3-\sqrt{3}) & -6\\1 & (-3-\sqrt{3})\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Be very careful with your negatives!

$$\begin{pmatrix}(3-\sqrt{3})x - 6y\\x + (-3-\sqrt{3})y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$(3-\sqrt{3})x - 6y = 0 \Rightarrow x = \frac{6}{3-\sqrt{3}}y$$

$$\Rightarrow x = (3+\sqrt{3})y$$

$$\Rightarrow \boxed{\mathbf{x}_1 = \begin{pmatrix}3+\sqrt{3}\\1\end{pmatrix} \text{ (or any multiple of)}}$$

$\lambda_1 = 1 - \sqrt{3}$

$$\left(\begin{pmatrix}4 & -6\\1 & -2\end{pmatrix} - (1-\sqrt{3})\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\left(\begin{pmatrix}4 & -6\\1 & -2\end{pmatrix} - \begin{pmatrix}(1-\sqrt{3}) & 0\\0 & (1-\sqrt{3})\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}(3+\sqrt{3}) & -6\\1 & (-3+\sqrt{3})\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Be very careful with your negatives!

$$\begin{pmatrix}(3+\sqrt{3})x - 6y\\x + (-3+\sqrt{3})y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$(3+\sqrt{3})x - 6y = 0 \Rightarrow x = \frac{6}{3+\sqrt{3}}y$$

$$\Rightarrow x = (3-\sqrt{3})y$$

$$\Rightarrow \boxed{\mathbf{x}_1 = \begin{pmatrix}3-\sqrt{3}\\1\end{pmatrix} \text{ (or any multiple of)}}$$

Question 4

Find the eigenvalues for each of the following matrices:

(a)

$$C = \begin{pmatrix} -2 & 13 \\ -1 & 2 \end{pmatrix}$$

[3]

(b)

$$D = \begin{pmatrix} 6 & -1 \\ 17 & -2 \end{pmatrix}$$

[3]

(a) Find the characteristic polynomial and solve it to find the eigenvalues

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = |A| = ad - bc$ ← Formula booklet

$$\det \begin{pmatrix} (-2-\lambda) & 13 \\ -1 & (2-\lambda) \end{pmatrix} = (-2-\lambda)(2-\lambda) - (-1)(13)$$

$$= -4 - 2\lambda + 2\lambda + \lambda^2 + 13$$

$$= \lambda^2 + 9$$

$$\lambda^2 + 9 = 0$$

$$\lambda = \pm\sqrt{-9}$$

$$\boxed{\lambda = 3i, -3i}$$

Find the eigenvalues for each of the following matrices:

(a)

$$C = \begin{pmatrix} -2 & 13 \\ -1 & 2 \end{pmatrix}$$

[3]

(b)

$$D = \begin{pmatrix} 6 & -1 \\ 17 & -2 \end{pmatrix}$$

[3]

(b) Find the characteristic polynomial and solve it to find the eigenvalues

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = |A| = ad - bc$ ← Formula booklet

$$\det \begin{pmatrix} (6-\lambda) & -1 \\ 17 & (-2-\lambda) \end{pmatrix} = (6-\lambda)(-2-\lambda) - (-1)(17)$$

$$= -12 + 2\lambda - 6\lambda + \lambda^2 + 17$$

$$= \lambda^2 - 4\lambda + 5$$

$$\lambda^2 - 4\lambda + 5 = 0$$

$$(\lambda - 2)^2 + 1 = 0$$

$$\lambda = 2 \pm \sqrt{-1}$$

$$\boxed{\lambda = 2 + i, \ 2 - i}$$

Question 5

Consider the matrix $\boldsymbol{M}$ defined as

$$\boldsymbol{M} = \begin{pmatrix} -1 & k \\ 3 & -1 \end{pmatrix}$$

where $k \in \mathbb{R}$ is a constant.

The eigenvalues of $\boldsymbol{M}$ are 2 and -4.

(a) Find the value of k. [3]

(b) Find the eigenvectors of $\boldsymbol{M}$ that correspond to the two eigenvalues. [3]

(c) Hence write $\boldsymbol{M}$ in the form $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{P}$ is a matrix of eigenvectors and $\boldsymbol{D}$ is a diagonal matrix of eigenvalues. [2]

(a) Find the characteristic polynomial and solve it using the given eigenvalues to find k

$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = |A| = ad - bc$ ← Formula booklet

$$\det\begin{pmatrix} (-1-\lambda) & k \\ 3 & (-1-\lambda) \end{pmatrix} = (-1-\lambda)(-1-\lambda) - 3k$$

$$= 1 + \lambda + \lambda + \lambda^2 - 3k$$

$$= \lambda^2 + 2\lambda - 3k + 1$$

Set equation to 0 and substitute in either value of λ

$$(2)^2 + 2(2) - 3k + 1 = 0$$

$$9 - 3k = 0$$

$$\boxed{k = 3}$$

Consider the matrix $\boldsymbol{M}$ defined as

$$\boldsymbol{M} = \begin{pmatrix} -1 & k \\ 3 & -1 \end{pmatrix}$$

where $k \in \mathbb{R}$ is a constant.

The eigenvalues of $\boldsymbol{M}$ are 2 and -4.

(a) Find the value of k. $\boxed{k = 3}$ [3]

(b) Find the eigenvectors of $\boldsymbol{M}$ that correspond to the two eigenvalues. [3]

(c) Hence write $\boldsymbol{M}$ in the form $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{P}$ is a matrix of eigenvectors and $\boldsymbol{D}$ is a diagonal matrix of eigenvalues. [2]

(b) Use $Ax = \lambda x$ or $(A - \lambda I)x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to find the corresponding eigenvectors

$\lambda_1 = 2$

$$\left(\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} - (2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3x + 3y \\ 3x - 3y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 3x = 3y$

$\Rightarrow x = y$

$\Rightarrow$ $\boxed{x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ (or any multiple of)}}$

$\lambda_1 = -4$

$$\left(\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} - (-4)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3x + 3y \\ 3x + 3y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 3x = -3y$

$\Rightarrow x = -y$

$$\Rightarrow \boxed{x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ (or any multiple of)}}$$

Consider the matrix $\boldsymbol{M}$ defined as

$$\boldsymbol{M} = \begin{pmatrix} -1 & k \\ 3 & -1 \end{pmatrix}$$

where $k \in \mathbb{R}$ is a constant.

The eigenvalues of $\boldsymbol{M}$ are 2 and -4.

(a) Find the value of k.

[3]

(b) Find the eigenvectors of $\boldsymbol{M}$ that correspond to the two eigenvalues.

[3]

(c) Hence write $\boldsymbol{M}$ in the form $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{P}$ is a matrix of eigenvectors and $\boldsymbol{D}$ is a diagonal matrix of eigenvalues.

[2]

(c) $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ or $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},\ ad \neq bc$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = |A| = ad - bc$$

Formula booklet

$$P^{-1} = \frac{1}{((1)(-1) - (1)(1))} \times \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$$

$D = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$ or $D = \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}$

$$\boxed{M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}}$$

or

$$\boxed{M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}}$$

Question 6

(a) It is given that, for $n \times n$ matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$,

$$\boldsymbol{A} = \boldsymbol{BCB}^{-1}$$

Use the properties of matrices and matrix inverses to show that $\boldsymbol{A}^2 = \boldsymbol{BC}^2\boldsymbol{B}^{-1}$.

[3]

Consider the matrix $\boldsymbol{M} = \begin{pmatrix} 3 & -2 \\ p & 1 \end{pmatrix}$, where $p \in \mathbb{R}$ is a constant and where it is given that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\boldsymbol{M}$.

(b) Find the value of p.

[3]

(c) Hence, by first finding the eigenvalues and the other eigenvector of $\boldsymbol{M}$, write $\boldsymbol{M}$ in the form $\boldsymbol{M} = \boldsymbol{PDP}^{-1}$ for appropriate matrices $\boldsymbol{P}$ and $\boldsymbol{D}$.

[5]

(d) (i) Use the result of part (c) to show that

$$\boldsymbol{M}^n = \frac{1}{3}\begin{pmatrix} 2(5^n) + (-1)^n & -5^n + (-1)^n \\ -2(5^n) + 2(-1)^n & 5^n + 2(-1)^n \end{pmatrix}$$

(ii) Show that the expression for $\boldsymbol{M}^n$ in part (d)(i) gives the expected result when $n = 1$.

[4]

(a) $A^2 = (BCB^{-1})(BCB^{-1})$

$= (BC)(B^{-1}B)(CB^{-1})$ ← Associative property $A(BC) = (AB)C$

$= (BC)(I)(CB^{-1})$ ← Property of inverses $AA^{-1} = I$

$= (BC)(IC)(B^{-1})$ ← Associative property $A(BC) = (AB)C$

$= BCCB^{-1}$ ← Property of Identity matrix

$\boxed{BC^2B^{-1}}$

(a) It is given that, for $n \times n$ matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$,

$$\boldsymbol{A} = \boldsymbol{BCB}^{-1}$$

Use the properties of matrices and matrix inverses to show that $\boldsymbol{A}^2 = \boldsymbol{BC}^2\boldsymbol{B}^{-1}$.

[3]

Consider the matrix $\boldsymbol{M} = \begin{pmatrix} 3 & -2 \\ p & 1 \end{pmatrix}$, where $p \in \mathbb{R}$ is a constant and where it is given that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\boldsymbol{M}$.

(b) Find the value of p.

[3]

(c) Hence, by first finding the eigenvalues and the other eigenvector of $\boldsymbol{M}$, write $\boldsymbol{M}$ in the form $\boldsymbol{M} = \boldsymbol{PDP}^{-1}$ for appropriate matrices $\boldsymbol{P}$ and $\boldsymbol{D}$.

[5]

(d) (i) Use the result of part (c) to show that

$$\boldsymbol{M}^n = \frac{1}{3}\begin{pmatrix} 2(5^n) + (-1)^n & -5^n + (-1)^n \\ -2(5^n) + 2(-1)^n & 5^n + 2(-1)^n \end{pmatrix}$$

(ii) Show that the expression for $\boldsymbol{M}^n$ in part (d)(i) gives the expected result when $n = 1$.

[4]

(b) Use $Ax = \lambda x$ or $(A - \lambda I)x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the given eigenvector

$$\left(\begin{pmatrix} 3 & -2 \\ p & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 3 & -2 \\ p & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} (3-\lambda) & -2 \\ p & (1-\lambda) \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow (3-\lambda) - 4 = 0$

$\lambda = -1$

Put $\lambda = -1$ into second equation to find p

$\Rightarrow p + 2(1-(-1)) = 0$

$\boxed{p = -4}$

(a) It is given that, for $n \times n$ matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$,

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^{-1}$$

Use the properties of matrices and matrix inverses to show that $\boldsymbol{A}^2 = \boldsymbol{B}\boldsymbol{C}^2\boldsymbol{B}^{-1}$.

[3]

Consider the matrix $\boldsymbol{M} = \begin{pmatrix} 3 & -2 \\ p & 1 \end{pmatrix}$, where $p \in \mathbb{R}$ is a constant and where it is given that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\boldsymbol{M}$.

$\boxed{\lambda = -1}$

(b) Find the value of p.

[3]

(c) Hence, by first finding the eigenvalues and the other eigenvector of $\boldsymbol{M}$, write $\boldsymbol{M}$ in the form $\boldsymbol{M} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ for appropriate matrices $\boldsymbol{P}$ and $\boldsymbol{D}$.

[5]

(d) (i) Use the result of part (c) to show that

$$\boldsymbol{M}^n = \frac{1}{3}\begin{pmatrix} 2(5^n) + (-1)^n & -5^n + (-1)^n \\ -2(5^n) + 2(-1)^n & 5^n + 2(-1)^n \end{pmatrix}$$

(ii) Show that the expression for $\boldsymbol{M}^n$ in part (d)(i) gives the expected result when $n = 1$.

[4]

(c) Substitute $p = -4$ into the characteristic equation to find the other eigenvalue

$$\det\begin{pmatrix} (3-\lambda) & -2 \\ -4 & (1-\lambda) \end{pmatrix} = 0$$

$$(3-\lambda)(1-\lambda) - (-2)(-4) = 0$$

$$3 - 4\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0 \quad \Rightarrow \lambda_2 = 5$$

Find x_2 $\left(\begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix} - (5)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -2 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x = -y \quad \Rightarrow x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

P is the matrix of eigenvectors

$$P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \quad \text{or} \quad P = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad \neq bc$$ ← Formula booklet

$$\Rightarrow P^{-1} = \frac{1}{(1)(-1) - (1)(2)}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\text{or} \quad P^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

D is the diagonal matrix of eigenvalues

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \quad \text{or} \quad D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

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$$M = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

or

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

(a) It is given that, for $n \times n$ matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$,

$$\boldsymbol{A} = \boldsymbol{BCB}^{-1}$$

Use the properties of matrices and matrix inverses to show that $\boldsymbol{A}^2 = \boldsymbol{BC}^2\boldsymbol{B}^{-1}$.

[3]

Consider the matrix $\boldsymbol{M} = \begin{pmatrix} 3 & -2 \\ p & 1 \end{pmatrix}$, where $p \in \mathbb{R}$ is a constant and where it is given that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\boldsymbol{M}$.

(b) Find the value of p.

[3]

(c) Hence, by first finding the eigenvalues and the other eigenvector of $\boldsymbol{M}$, write $\boldsymbol{M}$ in the form $\boldsymbol{M} = \boldsymbol{PDP}^{-1}$ for appropriate matrices $\boldsymbol{P}$ and $\boldsymbol{D}$.

$$M = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

[5]

(d) (i) Use the result of part (c) to show that

$$\boldsymbol{M}^n = \frac{1}{3}\begin{pmatrix} 2(5^n) + (-1)^n & -5^n + (-1)^n \\ -2(5^n) + 2(-1)^n & 5^n + 2(-1)^n \end{pmatrix}$$

(ii) Show that the expression for $\boldsymbol{M}^n$ in part (d)(i) gives the expected result when $n = 1$.

[4]

(d) (i) $M = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$

$$M = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Apply the matrix power formula

$M^n = PD^nP^{-1}$ ← Formula booklet

$$M^n = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}^n\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Make sure you put brackets around any negatives that are raised to the power n

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} (-1)^n & 5^n \\ 2(-1)^n & -5^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Careful when multiplying expressions involving powers

E.g. $a \times b^n = a(b^n)$

$a \times b^n \neq (ab)^n$

$$\frac{1}{3}\begin{pmatrix} 2(5^n) + (-1)^n & -5^n + (-1)^n \\ -2(5^n) + 2(-1)^n & 5^n + 2(-1)^n \end{pmatrix}$$

(ii) $M^1 = \frac{1}{3}\begin{pmatrix} 2(5^1) + (-1)^1 & -5^1 + (-1)^1 \\ -2(5^1) + 2(-1)^1 & 5^1 + 2(-1)^1 \end{pmatrix}$

$$= \frac{1}{3}\begin{pmatrix} 9 & -6 \\ -12 & 3 \end{pmatrix}$$

$$M = \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}$$

Question 7

Exobiologists are studying two species of animals in a region of the distant planet Dirion. In the researchers' models the population of Heliors (a predator species) is indicated by h, while the population of Sklyveths (a competing predator species) is indicated by s.

If the respective populations at a particular point in time are h_n and s_n, then the researchers' data suggest that the populations one year later may be given by the following system of coupled equations:

$$h_{n+1} = 1.06h_n - 0.16s_n$$

$$s_{n+1} = -0.04h_n + 0.94s_n$$

(a) Represent the system of equations in the matrix form $\boldsymbol{x}_{n+1} = \boldsymbol{M}\boldsymbol{x}_n$.

[2]

At the start of the study, there are 600 Heliors and 500 Sklyveths in the region.

(b) Find the expected size of the respective populations after one year.

[2]

(c) By first finding the eigenvalues and corresponding eigenvectors of $\boldsymbol{M}$ write $\boldsymbol{M}$ in the form $\boldsymbol{PDP}^{-1}$, where $\boldsymbol{P}$ is a matrix of eigenvectors and $\boldsymbol{D}$ is a diagonal matrix of eigenvalues.

[8]

(d) Hence show that the respective populations after n years are predicted by the model to be $h_n = 520(0.9^n) + 80(1.1^n)$ and $s_n = 520(0.9^n) - 20(1.1^n)$.

[3]

(e) Describe what the model predicts in the long term for the populations of the two species, and offer one criticism of the model based on this prediction.

[4]

(a) $$\boxed{\begin{pmatrix} h_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} 1.06 & -0.16 \\ -0.04 & 0.94 \end{pmatrix}\begin{pmatrix} h_n \\ s_n \end{pmatrix}}$$

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[4]

(b) $$\begin{pmatrix} h_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} 1.06 & -0.16 \\ -0.04 & 0.94 \end{pmatrix}\begin{pmatrix} 600 \\ 500 \end{pmatrix}$$

$$= \begin{pmatrix} (1.06)(600) + (-0.16)(500) \\ (-0.04)(600) + (0.94)(500) \end{pmatrix}$$

$$= \begin{pmatrix} 556 \\ 446 \end{pmatrix}$$

$\boxed{\text{556 Heliors, 446 Sklyvelths}}$

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[4]

(c) Find and solve the characteristic polynomial of M for λ

$$\det\begin{pmatrix} (1.06-\lambda) & -0.16 \\ -0.04 & (0.94-\lambda) \end{pmatrix} = 0$$

$$(1.06-\lambda)(0.94-\lambda) - (-0.04)(-0.16) = 0$$

$$0.9964 - 0.94\lambda - 1.06\lambda + \lambda^2 - 0.0064 = 0$$

$$\lambda^2 - 2\lambda + 0.99 = 0$$

$$(\lambda - 1.1)(\lambda - 0.9) = 0$$

$$\lambda = 0.9,\ 1.1$$

Find the corresponding eigenvectors using $Ax = \lambda x$ or $(A - \lambda I)x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\lambda = 0.9$
$$\left(\begin{pmatrix} 1.06 & -0.16 \\ -0.04 & 0.94 \end{pmatrix} - (0.9)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 1.06 & -0.16 \\ -0.04 & 0.94 \end{pmatrix} - \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0.16 & -0.16 \\ -0.04 & 0.04 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x = y$

$\Rightarrow x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or any multiple of)

$\lambda = 1.1$
$$\left(\begin{pmatrix} 1.06 & -0.16 \\ -0.04 & 0.94 \end{pmatrix} - (1.1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 1.06 & -0.16 \\ -0.04 & 0.94 \end{pmatrix} - \begin{pmatrix} 1.1 & 0 \\ 0 & 1.1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.04 & -0.16 \\ -0.04 & -0.16 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x = -4y$

$\Rightarrow x_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ (or any multiple of)

P is the matrix of eigenvectors

$P = \begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix}$ or $P = \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}$

$\Rightarrow P^{-1} = \dfrac{1}{(1)(-1)-(1)(4)}\begin{pmatrix} -1 & -4 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.8 \\ 0.2 & -0.2 \end{pmatrix}$

or $P^{-1} = \begin{pmatrix} 0.2 & -0.2 \\ 0.2 & 0.8 \end{pmatrix}$

D is the diagonal matrix of eigenvalues

$D = \begin{pmatrix} 0.9 & 0 \\ 0 & 1.1 \end{pmatrix}$ or $D = \begin{pmatrix} 1.1 & 0 \\ 0 & 0.9 \end{pmatrix}$

$$\boxed{M = \begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0.9 & 0 \\ 0 & 1.1 \end{pmatrix}\begin{pmatrix} 0.2 & 0.8 \\ 0.2 & -0.2 \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1.1 & 0 \\ 0 & 0.9 \end{pmatrix}\begin{pmatrix} 0.2 & -0.2 \\ 0.2 & 0.8 \end{pmatrix}}$$

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(d) $M^n = PD^nP^{-1}$ Formula booklet

$M^n = \begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0.9 & 0 \\ 0 & 1.1 \end{pmatrix}^n\begin{pmatrix} 0.2 & 0.8 \\ 0.2 & -0.2 \end{pmatrix}\begin{pmatrix} 600 \\ 500 \end{pmatrix}$

$= \begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0.9^n & 0 \\ 0 & 1.1^n \end{pmatrix}\begin{pmatrix} 0.2 & 0.8 \\ 0.2 & -0.2 \end{pmatrix}\begin{pmatrix} 600 \\ 500 \end{pmatrix}$

$= \begin{pmatrix} 0.9^n & 4(1.1^n) \\ 0.9^n & -(1.1^n) \end{pmatrix}\begin{pmatrix} 0.2 & 0.8 \\ 0.2 & -0.2 \end{pmatrix}\begin{pmatrix} 600 \\ 500 \end{pmatrix}$

$= \begin{pmatrix} 0.2(0.9^n) + 0.8(1.1^n) & 0.8(0.9^n) - 0.8(1.1^n) \\ 0.2(0.9^n) - 0.2(1.1^n) & 0.8(0.9^n) + 0.2(1.1^n) \end{pmatrix}\begin{pmatrix} 600 \\ 500 \end{pmatrix}$

$= \begin{pmatrix} 600(0.2(0.9^n) + 0.8(1.1^n)) + 500(0.8(0.9^n) - 0.8(1.1^n)) \\ 600(0.2(0.9^n) - 0.2(1.1^n)) + 500(0.8(0.9^n) + 0.2(1.1^n)) \end{pmatrix}$

$= \begin{pmatrix} 120(0.9^n) + 400(0.9^n) + 480(1.1^n) - 400(1.1^n) \\ 120(0.9^n) + 400(0.9^n) - 120(1.1^n) + 100(1.1^n) \end{pmatrix}$

Careful when multiplying expressions involving powers

E.g.

$a \times b^n = a(b^n)$

$a \times b^n \neq (ab)^n$

$$\boxed{\begin{aligned} h_n &= 520(0.9^n) + 80(1.1^n) \\ s_n &= 520(0.9^n) - 20(1.1^n) \end{aligned}}$$

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[4]

(e)

As n gets larger 1.1^n also gets larger but $0.9^n \to 0$, so the 1.1^n expressions will dominate as time goes on.

In the longterm, the Helior population is going to increase but the Sklyveth population will decrease.

At some point the Sklyveth population will reach zero but the Helior population will continue to increase with no bounds which is unlikely in a real ecosystem.