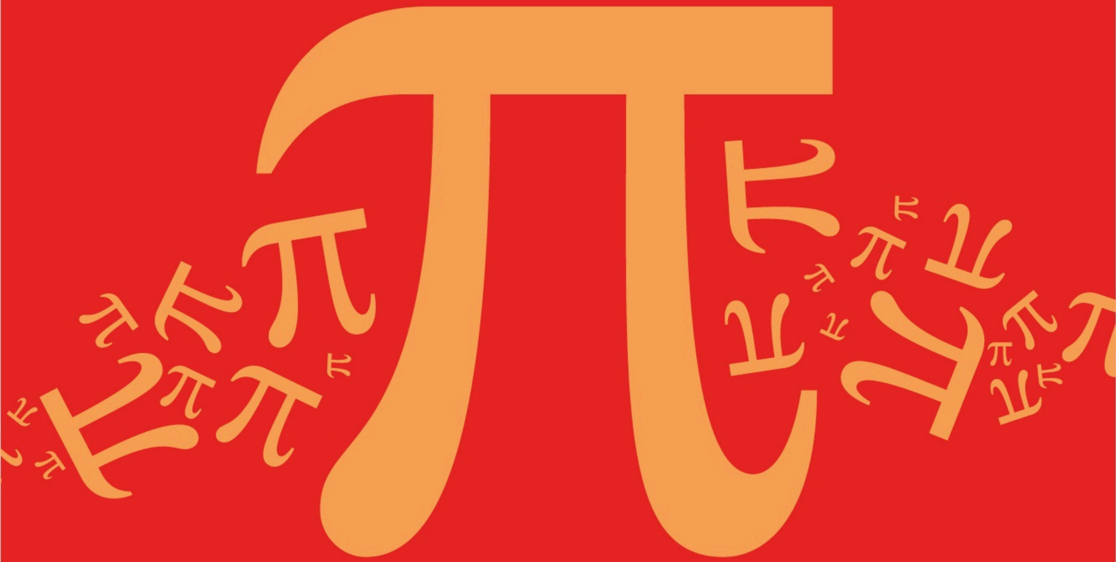
Pearson Edexcel Level 3 Advanced GCE Topic Answers



Further Mathematics (9FM0)

Core Pure, Further Statistics 1, Further Decisions 1

Suitable for Students Studying Further Mathematics (9FM0)

AS and A Edexcel Further Maths – Skills Revision

Core Pure, Further Statistics 1, Further Decision 1

This pack is intended for students to use once they have covered the AS content, either in preparation for their AS exam, or more likely alongside year 2 of the course to improve fluency, recall and pace on AS topics. It could be used as lesson starters, or supplied to students for independent use.

Sheet 1

Find a vector equation of the straight line that passes through the points A and B, with coordinates $(4,5,-1)$ and $(6,3,2)$ respectively.

The list of numbers below is to be sorted into **ascending** order. Perform a bubble sort to obtain the sorted list, giving the state of the list after each completed pass

45 56 37 79 46 18 90 81 51

Simplify $(7-4i)^2$

Solve the equation: $x^2+9=0$

The random variable X has the following probability distribution:

x	1	2	3	4
$\mathrm{P}(X=x)$	0.2	a	b	0.4

Given that E(X) = 2.8 find the values of a and b.

Find a vector equation of the straight line that passes through the points A and B, with coordinates $(4,5,-1)$ and $(6,3,2)$ respectively.

Sheet 1

$$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$$r = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

could be $\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$

could be $\begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}$

Sheet 1

Simplify $(7-4i)^2$

$(7-4i)(7-4i)$

$= 49 - 28i - 28i + 16i^2$

$= 49 - 16 - 56i$

$= \underline{\underline{33 - 56i}}$.

Solve the equation: $x^2 + 9 = 0$

$x^2 = -9$

$x = \pm\sqrt{-9}$

$x = \underline{\underline{\pm 3i}}$.

The list of numbers below is to be sorted into **ascending** order. Perform a bubble sort to obtain the sorted list, giving the state of the list after each completed pass

Sheet 1

45 56 37 79 46 18 90 81 51

(45 56)
37 — 56
(56 79)
46 — 79
18 — 79
(79 90)
81 — 90
51 — 90

End of 1st pass
45 37 56 46 18 79 81 51 90

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37 — 45
(45 56)
46 — 56
18 — 56
(56 79)
(79 81)
51 — 81
(81 90)

End of 2nd pass
37 45 46 18 56 79 51 81 90

(37 45)
(45 46)
18 — 46
(46 56)
(56 79)
51 — 79
(79 81)
(81 90)

37 45 18 46 56 51 79 81 90 ← end of 3rd pass

(37 45)
18 — 45
(45 46)
(46 56)

The random variable X has the following probability distribution:

x	1	2	3	4
$P(X = x)$	0.2	a	b	0.4

Given that E(X) = 2.8 find the values of a and b.

$$E(X) = \Sigma x . P(X=x)$$

$$2.8 = 0.2 + 2a + 3b + 1.6$$

$$2.8 = 1.8 + 2a + 3b$$

$$① \quad 1 = 2a + 3b$$

$$\Sigma P(X=x) = 1$$

$$0.6 + a + b = 1$$

$$a + b = 0.4 \quad ②$$

$$2a + 2b = 0.8 \quad ③ \quad \} \times 2$$

$$① - ③ \rightarrow \quad \underline{\underline{b = 0.2}} \quad \text{Sub } ② \quad \underline{\underline{a = 0.2}}$$

Sheet 2

Represent the following complex numbers on an Argand diagram:

$$z_1 = 2 + 5i$$

$$z_2 = 3 - 4i$$

$$z_3 = -4 + i$$

Find the magnitude of |OA|, |OB| and |OC|, where O is the origin of the Argand diagram, and A, B and C are z_1, z_2 and z_3 respectively

The following list gives the names of some students who have represented Britain in the International Mathematics Olympiad.

Roper (*R*), Palmer (*P*), Boase (*B*), Young (*Y*), Thomas (*T*), Kenney (*K*), Morris (*M*), Halliwell (*H*), Wicker (*W*), Garesalingam (*G*).

Use the quick sort algorithm to sort the names above into alphabetical order.

The straight line l has vector equation:

$$\boldsymbol{r} = (3\boldsymbol{i} + 2\boldsymbol{j} - 5\boldsymbol{k}) + t(\boldsymbol{i} - 6\boldsymbol{j} - 2\boldsymbol{k})$$

Given that the point $(a, b, 0)$ lies on l, find the value of a and the value of b.

A discrete random variable X has the following probability distribution:

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{20}$

Find the probability distribution of X^2.

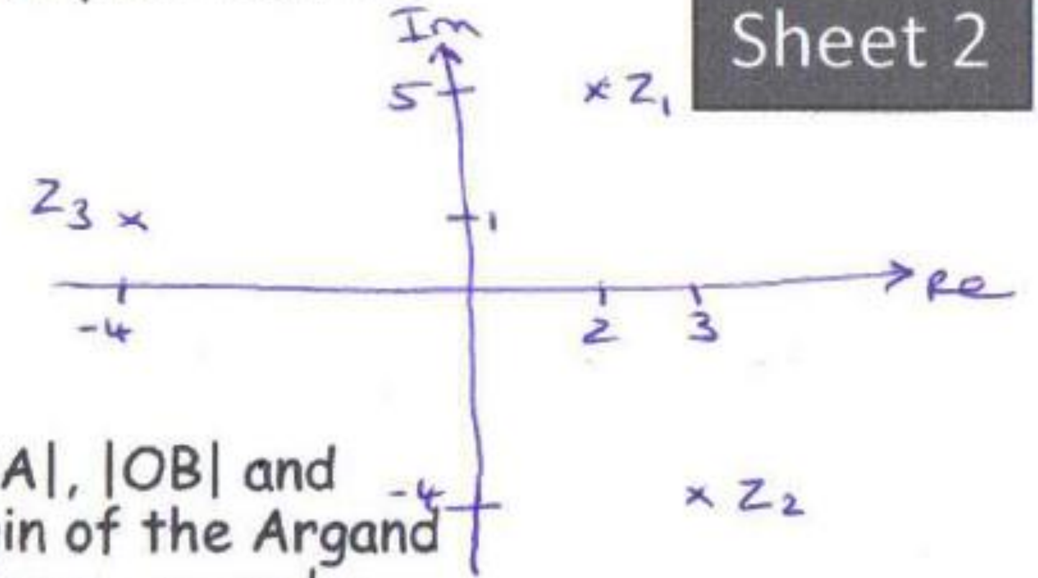
Sheet 2

Represent the following complex numbers on an Argand diagram:

$$z_1 = 2 + 5i$$

$$z_2 = 3 - 4i$$

$$z_3 = -4 + i$$



Find the magnitude of |OA|, |OB| and |OC|, where O is the origin of the Argand diagram, and A, B and C are z_1, z_2 and z_3 respectively

$|OA| = \sqrt{2^2+5^2} = \sqrt{29}$

$|OB| = \sqrt{3^2+4^2} = 5$

$|OC| = \sqrt{4^2+1^2} = \sqrt{17}$.

The straight line l has vector equation:

$$\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) + t(\mathbf{i} - 6\mathbf{j} - 2\mathbf{k})$$

Sheet 2

Given that the point $(a, b, 0)$ lies on l, find the value of a and the value of b.

$$r = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -6 \\ -2 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

$$-5 - 2t = 0$$
$$-5 = 2t$$
$$\underline{\underline{-5/2 = t}}.$$

$$a = 3 + t = 3 - 5/2 = \underline{\underline{1/2}}.$$

$$b = 2 - 6t = 2 - 6(-5/2)$$
$$= 2 + 15 = \underline{\underline{17}}.$$

Sheet 2

The following list gives the names of some students who have represented Britain in the International Mathematics Olympiad.

Roper (*R*), Palmer (*P*), Boase (*B*), Young (*Y*), Thomas (*T*), Kenney (*K*), Morris (*M*), Halliwell (*H*), Wicker (*W*), Garesalingam (*G*).

Use the quick sort algorithm to sort the names above into alphabetical order.

R	P	B	Y	T	Ⓚ	M	H	W	G
B	Ⓗ	G	$\underline{K}$	R	P	Ⓨ	T	M	W
B	Ⓖ	$\underline{H}$	$\underline{K}$	R	P	Ⓣ	M	W	$\underline{Y}$
B	$\underline{G}$	$\underline{H}$	$\underline{K}$	R	Ⓟ	M	$\underline{T}$	W	$\underline{Y}$
B	$\underline{G}$	$\underline{H}$	$\underline{K}$	M	$\underline{P}$	R	$\underline{T}$	W	$\underline{Y}$

A discrete random variable X has the following probability distribution:

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{20}$

Find the probability distribution of X^2.

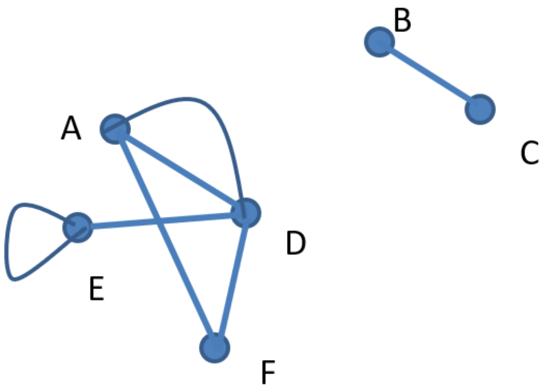
Let $y = x^2$

y	1	4	9	16	25	36
$P(Y=y)$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{20}$

Sheet 3

Find a Cartesian equation of the line with equation:

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$$



What sort of graph is this?
No. edges?
Order of each node?

Show that:

$$\sum_{r=1}^{n} (7r - 4) = \frac{n}{2}(7n - 1)$$

A discrete random variable X has the following probability distribution:

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{20}$

Find E(X) and Var (X)

Find a Cartesian equation of the line with equation:

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$$

$4 - \lambda = x$

$4 - x = \lambda$

$3 + 2\lambda = y$

$2\lambda = y - 3$

$\lambda = \frac{y-3}{2}$

$-2 + 5\lambda = z$

$5\lambda = z + 2$

$\lambda = \frac{z+2}{5}$

$$4 - x = \frac{y-3}{2} = \frac{z+2}{5}$$

Show that:

$$\sum_{r=1}^{n}(7r-4) = \frac{n}{2}(7n-1)$$

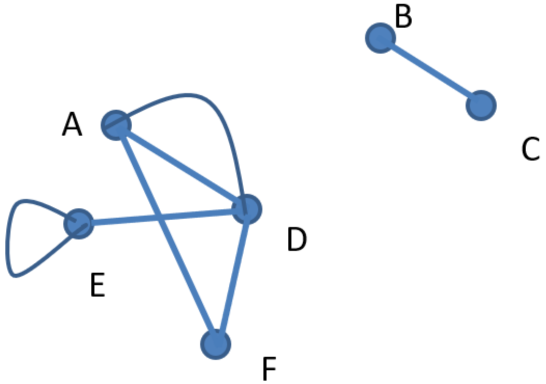
$$\sum(7r-4) = 7\sum_{1}^{n} r - \sum_{1}^{n} 4$$

$$= 7\left(\frac{1}{2}n(n+1)\right) - 4n$$

$$= \frac{7}{2}n^2 + \frac{7}{2}n - 4n$$

$$= \frac{7}{2}n^2 - \frac{1}{2}n$$

$$= \frac{n}{2}(7n-1)$$



total order = 2 x total edges

"handshake Lemma"

Every edge has 2 ends(ie edge x 2 = Σ nod ord

What sort of graph is this? Disconnected, non-Eulerian

No. edges? 7 edges (or "arcs")

Order of each node?

node	A	B	C	D	E	F
order	3	1	1	4	3	2 → Σ = 14

A discrete random variable X has the following probability distribution:

x^2	1	4	9	16	25	36
x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{20}$
$x . P(x=x)$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{10}$
$x^2 P(x=x)$	$\frac{1}{3}$	$\frac{4}{5}$	$\frac{9}{2}$	8	$\frac{25}{8}$	$\frac{9}{5}$

Find E(X) and Var (X)

$$E(x) = \sum x \, p(x=x) = \frac{1}{3} + \frac{2}{5} + 1 + \frac{5}{8} + \frac{3}{10} = \frac{319}{120} = \underline{\underline{2.66 \text{ (3sf)}}}$$

$$E(x^2) = \sum x^2 \, P(x=x) = \frac{1}{3} + \frac{4}{5} + \frac{9}{2} + 8 + \frac{25}{8} + \frac{9}{5} = \frac{1227}{120}$$

$$= 18.6 \text{ (3sf)}$$

$$Var(x) = E(x^2) - [E(x)]^2 = \frac{1227}{120} - \left(\frac{319}{120}\right)^2 = 11.49159\ldots$$

$$= \underline{\underline{11.5}} \text{ (3sf)}$$

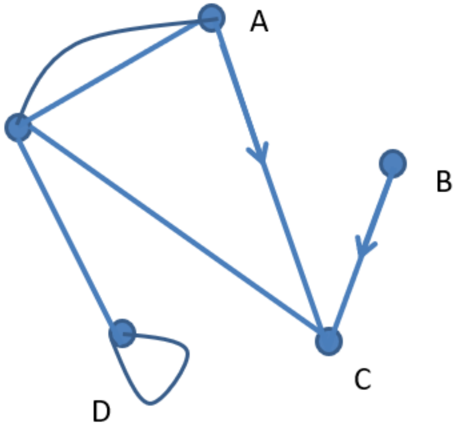
Sheet 4

Find, in the form $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$, an equation of the plane that passes through the points $A(2,2,-1)$, $B(3,2,-1)$ and $C(4,3,5)$

Draw the adjacency matrix (aka incidence matrix) for this graph.

To

From



Given that:

$$\sum_{r=1}^{n}(7r-4) = \frac{n}{2}(7n-1)$$

calculate the value of:

$$\sum_{r=20}^{50}(7r-4)$$

A fair 4-sided dice is rolled. Find E(X) and Var (X).

Find, in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$, an equation of the plane that passes through the points $A(2,2,-1)$, $B(3,2,-1)$ and $C(4,3,5)$

Find, in the form $\boldsymbol{r} = \overset{\text{point A}}{\boldsymbol{a}} + \lambda \overset{\overrightarrow{AB}}{\boldsymbol{b}} + \mu \overset{\overrightarrow{BC}}{\boldsymbol{c}}$, an equation of the plane that passes through the points $A(2,2,-1)$, $B(3,2,-1)$ and $C(4,3,5)$

Sheet 4

lots of possibilities!

$$\overrightarrow{AB} = \begin{pmatrix}3\\2\\-1\end{pmatrix} - \begin{pmatrix}2\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \overrightarrow{BC} = \begin{pmatrix}4\\3\\5\end{pmatrix} - \begin{pmatrix}3\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\6\end{pmatrix}$$

$$r = \begin{pmatrix}2\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\0\end{pmatrix} + \mu\begin{pmatrix}1\\1\\6\end{pmatrix}$$

Given that:

$$\sum_{r=1}^{n}(7r-4) = \frac{n}{2}(7n-1)$$

calculate the value of:

$$\sum_{r=2}^{50}(7r-4)$$

$$\sum_{2}^{50}(7r-4) = \sum_{1}^{50}(7r-4) - [7r-4]_1$$

Sum(2nd → 50th) = Sum($\overset{\text{first}}{50}$) − First value

$$= \frac{50}{2}(7(50)-1) - (7-4)$$

$$= 25(349) - (3)$$

$$= \underline{8722}$$

checked on classwiz.

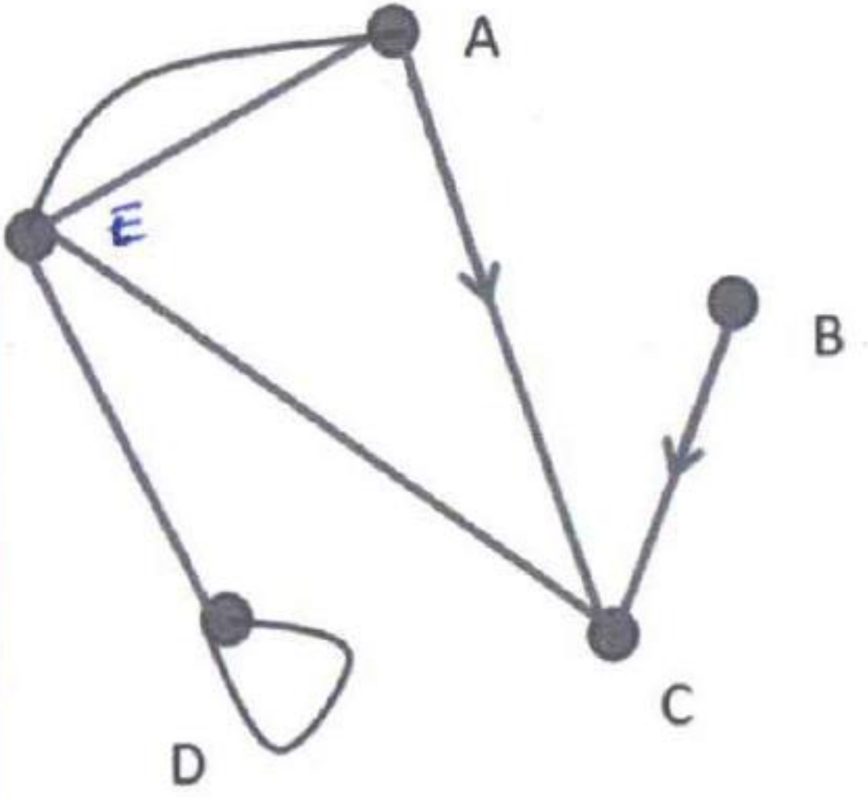
The 20 in this question got chopped to a 2. Can you use your calculator and the sigma button to check your answer?

Draw the adjacency matrix (aka incidence matrix) for this graph.

To

From

	A	B	C	D	E
A	0	0	1	0	2
B	0	0	1	0	0
C	0	0	0	0	1
D	0	0	0	2	1
E	2	0	1	1	0



It's not a simple graph. Simple graphs only have 0s or 1s in the adjacency matrix.

A fair 4-sided dice is rolled. Find E(X) and Var (X).

x	1	2	3	4
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$xP(X=x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
x^2	1	4	9	16
$x^2P(X=x)$	$\frac{1}{4}$	$\frac{4}{4}$	$\frac{9}{4}$	$\frac{16}{4}$

$\Sigma = \frac{10}{4} = 2.5$

$\Sigma = \frac{30}{4} = 7.5$

$E(X) = \underline{\underline{2.5}}$.

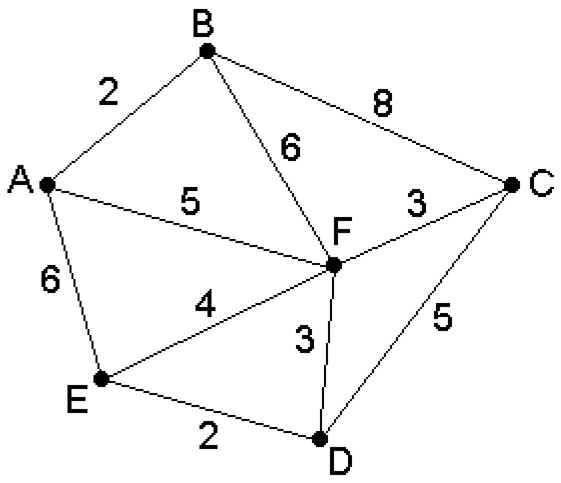
$Var(X) = E(X^2) - (E(X))^2 = 7.5 - (2.5)^2 = \frac{5}{4} = \underline{\underline{1.25}}$.

Sheet 5

Verify that the point P with position vector $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ lies in the plane with vector equation:

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Use Kruskal's Algorithm to find the MST showing clearly the order in which you include the edges. Draw the MST, and state its weight.



Find, to two decimal places, the modulus and argument of z = -2 + 4i

Find, to two decimal places, the modulus and argument of z = -3 - 3i

A discrete random variable X has the following probability distribution:

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{20}$

Y = 2X + 1. Find E(Y) and Var(Y).

Verify that the point P with position vector $\begin{pmatrix}2\\2\\-1\end{pmatrix}$ lies in the plane with vector equation:

$$\boldsymbol{r}=\begin{pmatrix}3\\4\\-2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\1\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

Assume it does:

$2 = 3 + 2\lambda + \mu$ ①
$2 = 4 + \lambda - \mu$ ②
$-1 = -2 + \lambda + 2\mu$ ③

① + ② $4 = 7 + 3\lambda$
$3\lambda = -3$
$\underline{\underline{\lambda = -1}}$

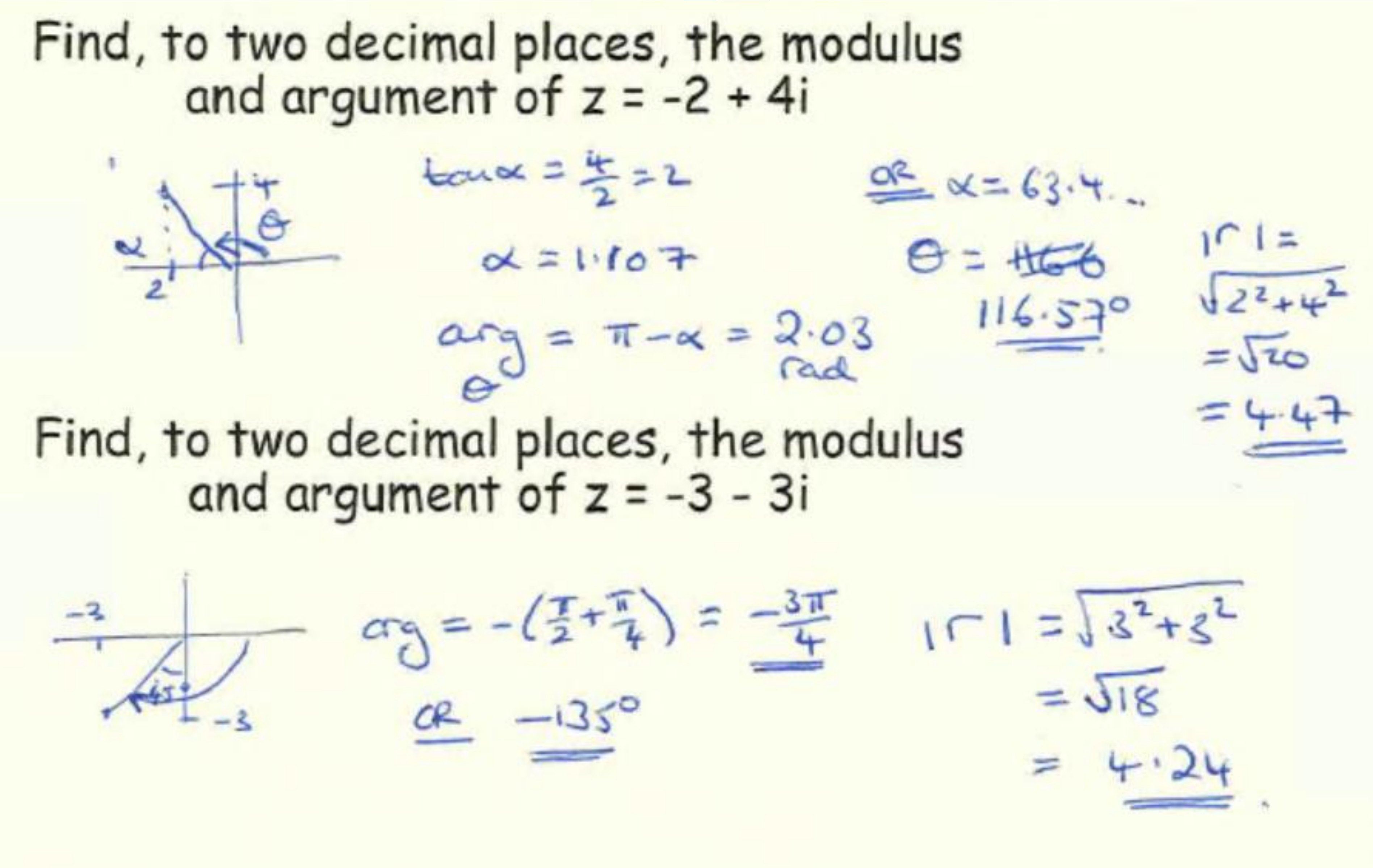
Sub ① ↘

$2 = 3 - 2 + \mu$
$2 = 1 + \mu$ $\underline{\underline{\mu = 1}}$

Sub ③

$-1 = -2 + (-1) + 2(1)$
$-1 = -2 - 1 + 2$
$-1 = -3 + 2$
$-1 = -1$
True

3 equations are consistent
∴ P lies in the plane.



Find, to two decimal places, the modulus and argument of $z = -2 + 4i$

$\tan\alpha = \frac{4}{2} = 2$

$\alpha = 1.107$

$\arg \theta = \pi - \alpha = 2.03$ rad

OR $\alpha = 63.4\ldots$

$\theta =$ ~~116~~ $116.57°$

$|r| = \sqrt{2^2+4^2}$

$= \sqrt{20}$

$= 4.47$

Find, to two decimal places, the modulus and argument of $z = -3 - 3i$

$\arg = -\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = -\frac{3\pi}{4}$

OR $-135°$

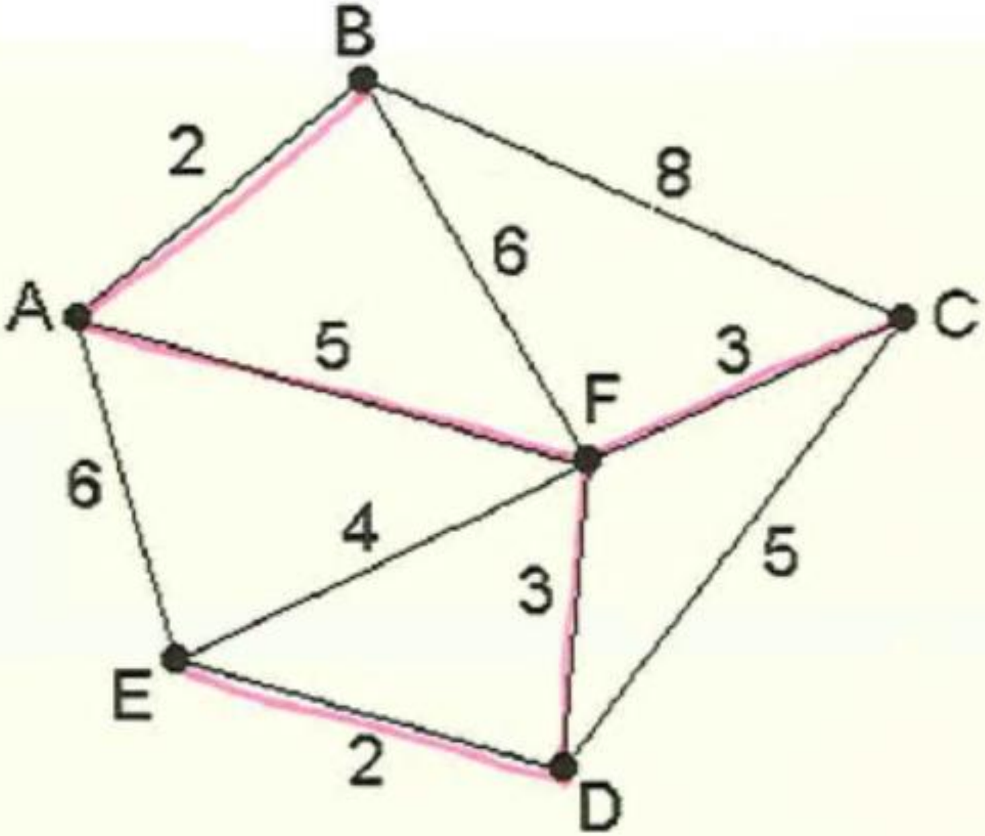
$|r| = \sqrt{3^2+3^2}$

$= \sqrt{18}$

$= 4.24$

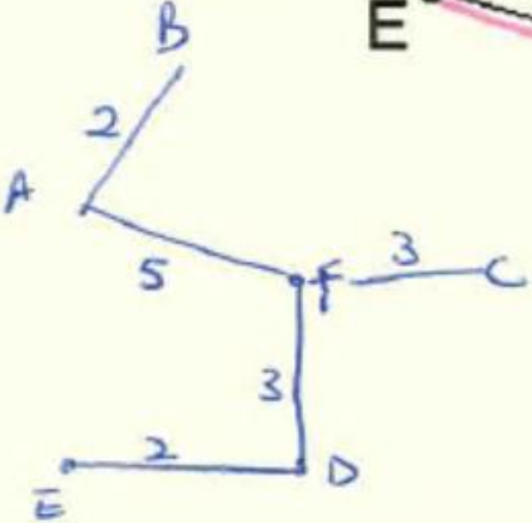
Disconnected .

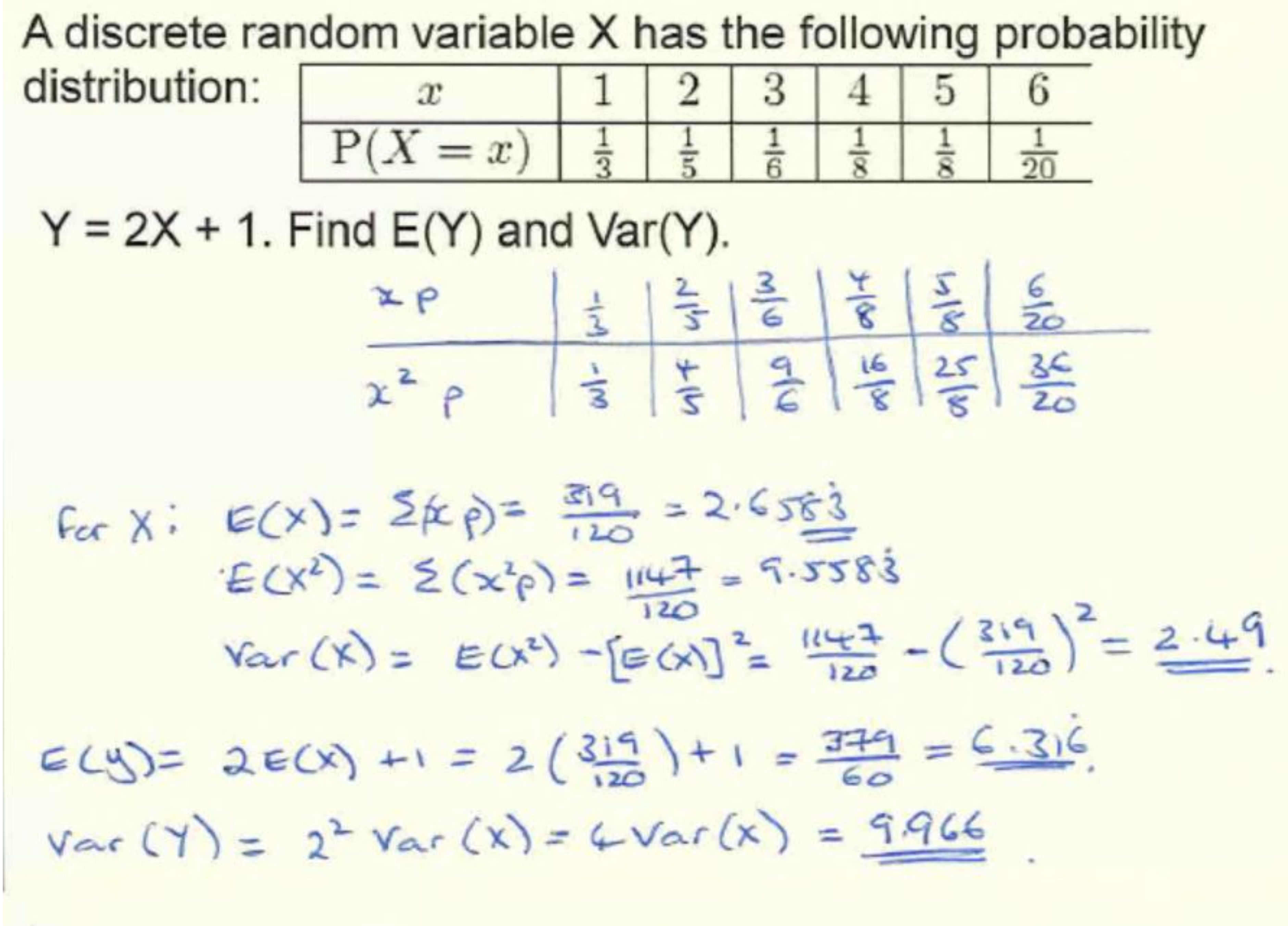
Use Kruskal's Algorithm to find the MST showing clearly the order in which you include the edges. Draw the MST, and state its weight.



ED = 2
AB = 2
FC = 3
FD = 3
AF = 5

15





A discrete random variable X has the following probability distribution:

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{20}$

Y = 2X + 1. Find E(Y) and Var(Y).

xp	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{6}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{20}$
x^2p	$\frac{1}{3}$	$\frac{4}{5}$	$\frac{9}{6}$	$\frac{16}{8}$	$\frac{25}{8}$	$\frac{36}{20}$

For X: $E(X) = \Sigma(xp) = \frac{319}{120} = 2.658\dot{3}$

$E(X^2) = \Sigma(x^2p) = \frac{1147}{120} = 9.558\dot{3}$

$Var(X) = E(X^2) - [E(X)]^2 = \frac{1147}{120} - \left(\frac{319}{120}\right)^2 = \underline{\underline{2.49}}$

$E(Y) = 2E(X) + 1 = 2\left(\frac{319}{120}\right) + 1 = \frac{379}{60} = \underline{\underline{6.31\dot{6}}}$

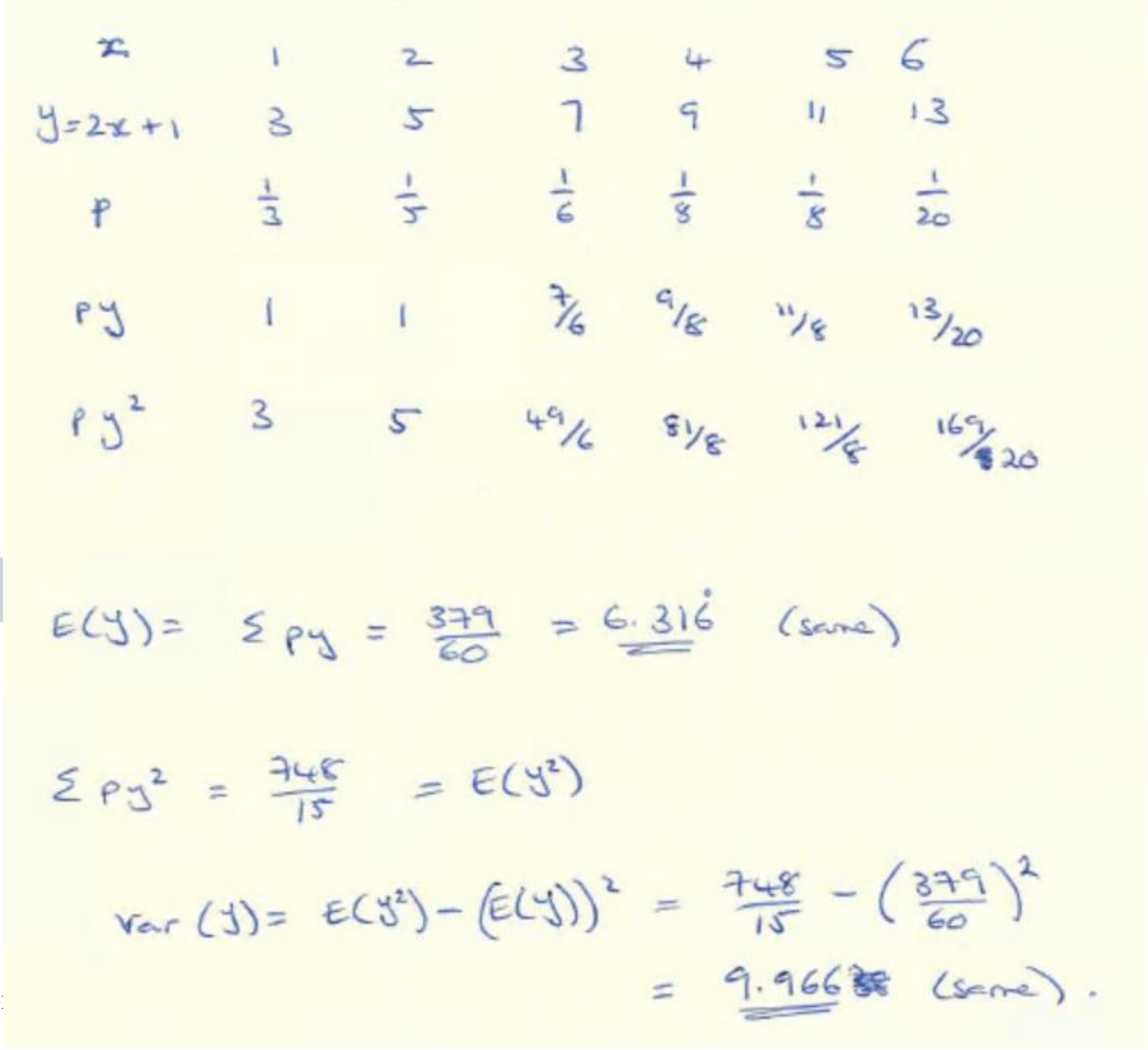
$Var(Y) = 2^2 Var(X) = 4 Var(X) = \underline{\underline{9.966}}$

A discrete random variable X has the following probability distribution:

x	1	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{20}$

Y = 2X + 1. Find E(Y) and Var(Y).

Alternative:



x	1	2	3	4	5	6
$y=2x+1$	3	5	7	9	11	13
p	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{20}$
py	1	1	$7/6$	$9/8$	$11/8$	$13/20$
py^2	3	5	$49/6$	$81/8$	$121/8$	$169/\cancel{8}\,20$

$E(y) = \Sigma py = \frac{379}{60} = \underline{\underline{6.31\dot{6}}}$ (same)

$\Sigma py^2 = \frac{748}{15} = E(y^2)$

$\text{var}(y) = E(y^2) - (E(y))^2 = \frac{748}{15} - \left(\frac{379}{60}\right)^2$

$= \underline{\underline{9.966}}$ (same).

The plane Π is perpendicular to the normal vector $\boldsymbol{n} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and passes through the point P with position vector $8\boldsymbol{i} + 4\boldsymbol{j} - 7\boldsymbol{k}$. Find a Cartesian equation of Π.

Sheet 6

Draw K_5.

By considering the number of edges in K_1 to K_5, write a formula for the number of edges in K_n.

Find the acute angle between the planes with equations

$$\boldsymbol{r}.\begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix} = 13 \text{ and } \boldsymbol{r}.\begin{pmatrix} 7 \\ -4 \\ 4 \end{pmatrix} = 6.$$

The random variable W has a mean of 5 and a variance of 12.

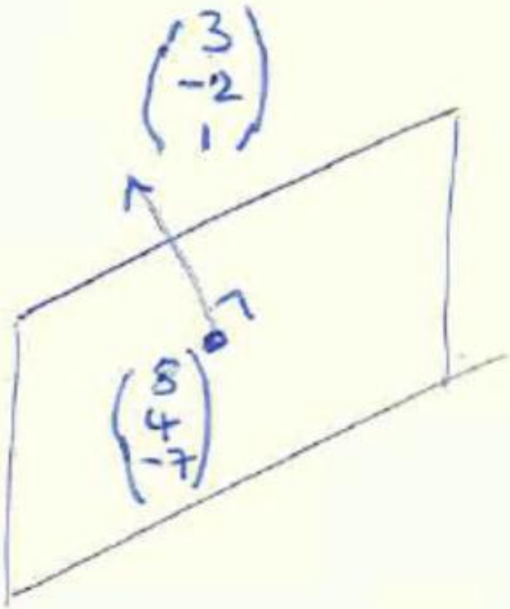
(a) Find E(3W - 1).

(b) Find E(3 - 4W).

(c) Find Var(3W + 1).

(d) Find E(W^2).

The plane Π is perpendicular to the normal vector $\boldsymbol{n} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and passes through the point P with position vector $8\boldsymbol{i} + 4\boldsymbol{j} - 7\boldsymbol{k}$. Find a Cartesian equation of Π.



$$\underline{r} \cdot n = k$$

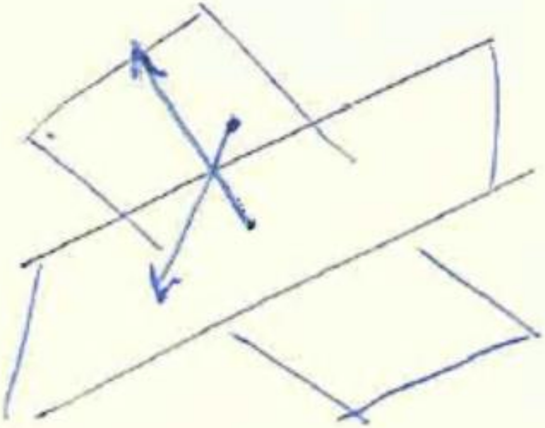
$$\begin{pmatrix} 8 \\ 4 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 24 - 8 - 7 = 9$$

$$\underline{r} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 9$$

$$3x - 2y + z = 9.$$

Find the acute angle between the planes with equations

$$\boldsymbol{r}.\begin{pmatrix}4\\4\\-7\end{pmatrix} = 13 \text{ and } \boldsymbol{r}.\begin{pmatrix}7\\-4\\4\end{pmatrix} = 6.$$



angle between normals

$\cos\theta$ ~~$a\cdot b$~~ $= \frac{a \cdot b}{|a||b|}$

$$a \cdot b = \begin{pmatrix}4\\4\\-7\end{pmatrix} \cdot \begin{pmatrix}7\\-4\\4\end{pmatrix} = 28 - 16 - 28$$

$$= -16$$

$$|a| = \sqrt{4^2+4^2+7^2} = \sqrt{81} = 9$$

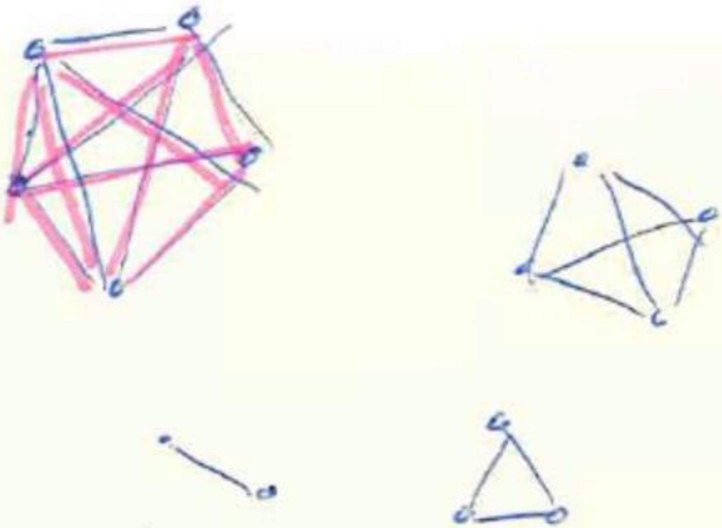
$$|b| = 9$$

$$\cos\theta = \frac{-16}{9 \times 9} = \frac{-16}{81} = 101.39^\circ$$

$$\text{acute angle} = 180 - 101.39^\circ = \underline{\underline{78.6^\circ}}.$$

Draw K_5.

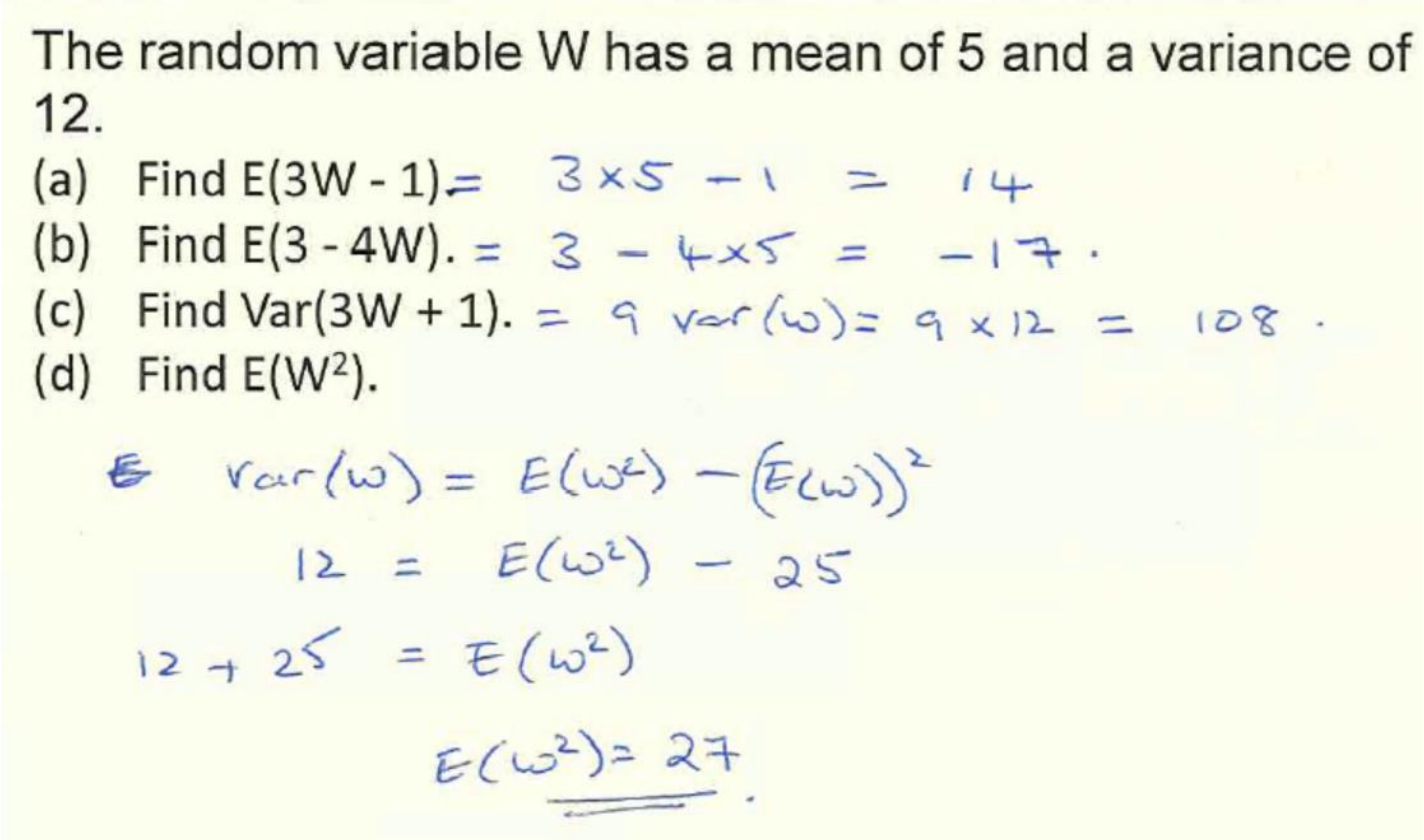
By considering the number of edges in K_1 to K_5, write a formula for the number of edges in K_n.



	K_1	K_2	K_3	K_4	K_5	K_n
edges	0	1	3	6	10	
				↑ $3+2+1$	↑ $4+3+2+1$	↖ $(n-1)+(n-2)+\cdots$

Triangular numbers

$$\text{edges in } K_n = \frac{n(n-1)}{2}$$



The random variable W has a mean of 5 and a variance of 12.

(a) Find E(3W - 1). $= 3 \times 5 - 1 = 14$

(b) Find E(3 - 4W). $= 3 - 4 \times 5 = -17$.

(c) Find Var(3W + 1). $= 9\ \mathrm{var}(w) = 9 \times 12 = 108$.

(d) Find $E(W^2)$.

~~E~~ $\mathrm{var}(w) = E(w^2) - (E(w))^2$

$12 = E(w^2) - 25$

$12 + 25 = E(w^2)$

$\underline{\underline{E(w^2) = 27}}$.

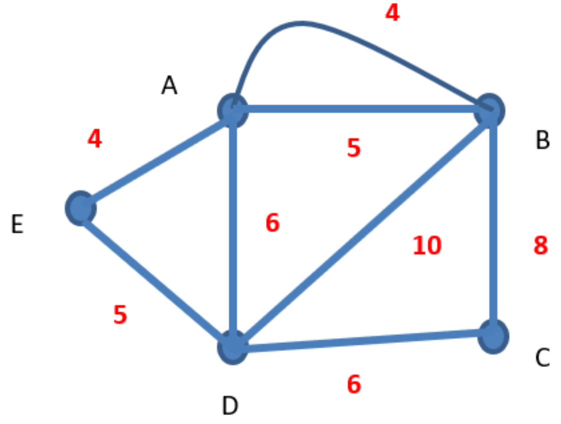
Sheet 7

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 4 & 1 \\ 0 & -2 \end{bmatrix}$$

(a) Calculate the value of **AB**

(b) Show that matrix multiplication is not commutative

A traffic police car, based at town *A*, has to travel along each of the roads at least once before returning to base at *A*. Find the minimum total distance the driver must travel, and a possible route.



Evaluate, using known results: $$\sum_{r=1}^{50} r$$ and: $$\sum_{r=1}^{30} r^2$$

X is a discrete random variable.

The random variable Y is defined by $Y = \frac{4-3X}{2}$

You are given that E(Y) = -1 and V ar(Y) = 9.

Find E(X) and Var(X).

$\boldsymbol{A} = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} 4 & 1 \\ 0 & -2 \end{bmatrix}$

Sheet 7

(a) Calculate the value of **AB**

(b) Show that matrix multiplication is not commutative

a) $AB = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -4 & -1 \\ 8 & -4 \end{pmatrix}$

b) $BA = \begin{pmatrix} 4 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -4 & -6 \end{pmatrix}$

$AB \neq BA$ so matrix multiplication is not commutative.

Evaluate, using known results: $\sum_{r=1}^{50} r$ and: $\sum_{r=1}^{30} r^2$

$$\sum_{r=1}^{50} r = \frac{1}{2}n(n+1) = \frac{1}{2}50(51) = 1275$$

$$\sum_{r=1}^{30} r^2 = \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(30)(31)(61) = 9455$$

A traffic police car, based at town *A*, has to travel along each of the roads at least once before returning to base at *A*. Find the minimum total distance the driver must travel, and a possible route.

Route inspection

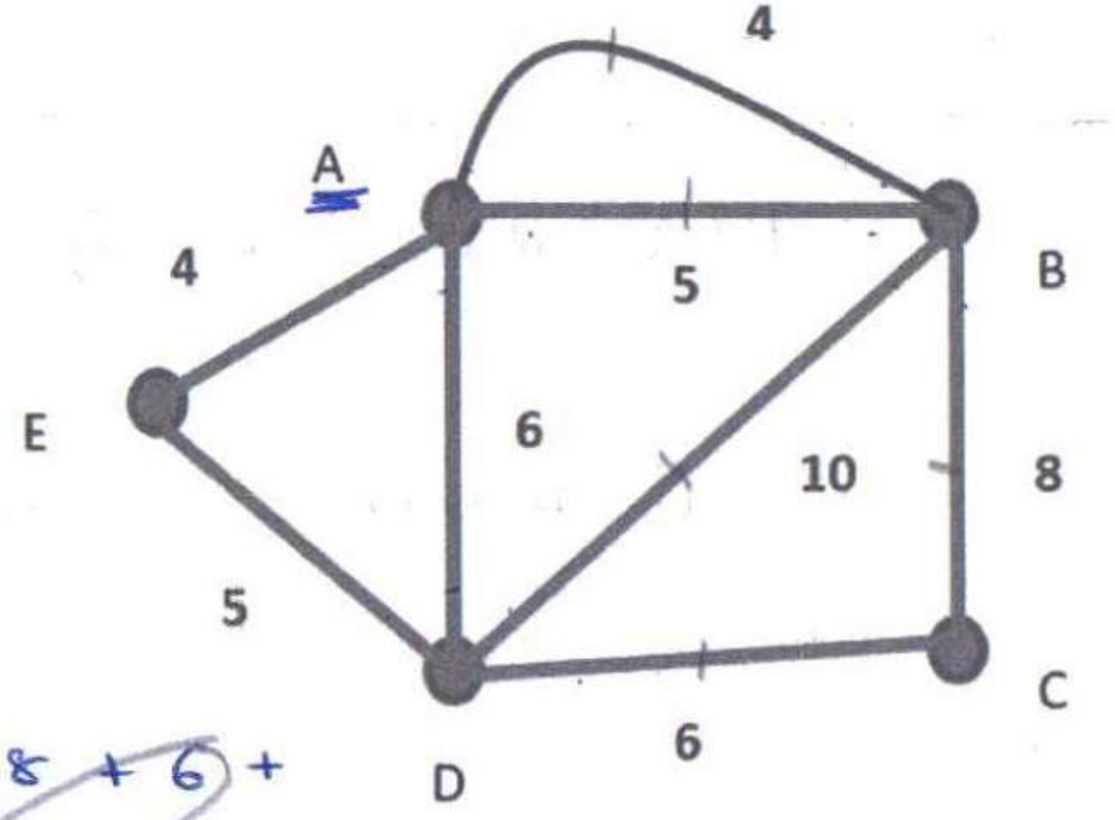
A	B	C	D	E
4	4	2	4	2

Fully Eulerian

A B C D B A D E A

Distance = 4 + 5 + 10 + 8 + 6 + 6 + 5 + 4

= 48



X is a discrete random variable.

The random variable Y is defined by $Y = \frac{4-3X}{2}$

You are given that E(Y) = -1 and V ar(Y) = 9.

Find E(X) and Var(X).

$2y = 4 - 3x$

$3x = 4 - 2y$

$x = \frac{4 - 2y}{3}$

$x = \frac{4}{3} - \frac{2}{3}y$

$E(x) = \frac{4}{3} - \frac{2}{3}(-1) = \frac{4}{3} + \frac{2}{3} = \frac{6}{3} = 2$

$Var(x) = \left(\frac{2}{3}\right)^2 \text{var } y$

$= \frac{4}{9} \times 9 = 4$

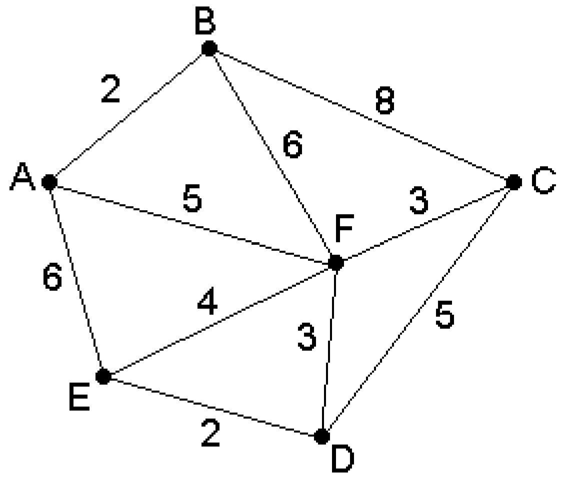
$E(x) = 2, \quad Var(x) = 4$

Sheet 8

$$\boldsymbol{A} = \begin{bmatrix} 4 & p+2 \\ -1 & 3-p \end{bmatrix}$$

Given that $\boldsymbol{A}$ is singular, find the value of p.

Use Prim's Algorithm starting at node A to find the MST of the network below, showing clearly the order in which you include the edges. Draw the MST, and state its weight



Find the acute angle between the line $\boldsymbol{l}$ with equation:

$$\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k} + \lambda(3\boldsymbol{i} + 4\boldsymbol{j} - 12\boldsymbol{k})$$

and the plane with equation:

$$\boldsymbol{r}.(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = 2$$

A discrete random variable X has the following probability distribution:

x	-2	-1	0	1	2
$\mathrm{P}(X=x)$	a	b	c	b	a

Y is the discrete random variable such that Y = $(X + 1)^2$.
Given that E(Y) = 2.4 and P(Y > 2) = 0.4, find a, b and c.

$$A = \begin{bmatrix} 4 & p+2 \\ -1 & 3-p \end{bmatrix}$$

Sheet 8

Given that A is singular, find the value of p.

$$|A| = 0 \Rightarrow 4(3-p) - (-1)(p+2) = 0$$

$$12 - 4p + p + 2 = 0$$

$$14 - 3p = 0$$

$$3p = 14$$

$$p = 14/3.$$

Find the acute angle between the line $\boldsymbol{l}$ with equation:

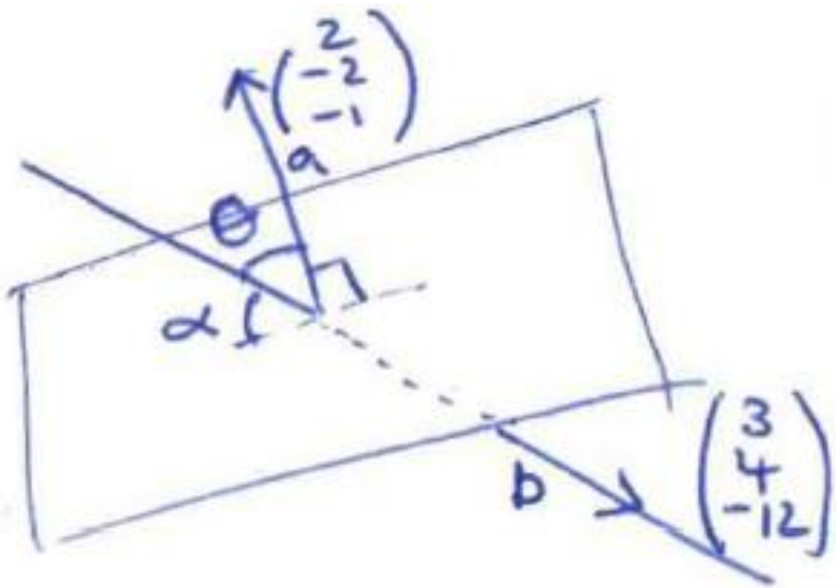
$$\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k} + \lambda(3\boldsymbol{i} + 4\boldsymbol{j} - 12\boldsymbol{k})$$

and the plane with equation:

$$\boldsymbol{r}.(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = 2$$

$$r = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ -12 \end{pmatrix}$$

$$r . \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = 2$$



$$\overset{a}{\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}} . \overset{b}{\begin{pmatrix} 3 \\ 4 \\ -12 \end{pmatrix}} = 6 - 8 + 12$$
$$= 10$$

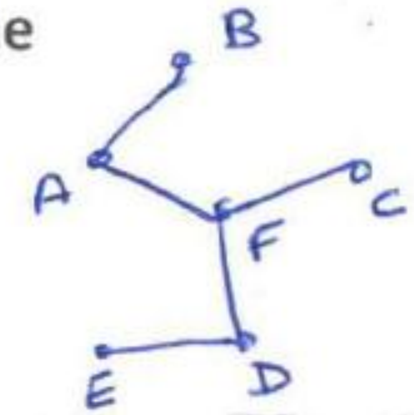
$$|a| = \sqrt{4+4+1} = \sqrt{9} = 3$$
$$|b| = \sqrt{9+16+144} = 13$$

$$\cos\theta = \frac{a.b}{|a|\,|b|}$$

$$\cos\theta = \frac{10}{39}$$

$$\theta = 75.1^\circ$$
$$\alpha = 90 - \theta$$
$$= \underline{\underline{14.9^\circ}}$$

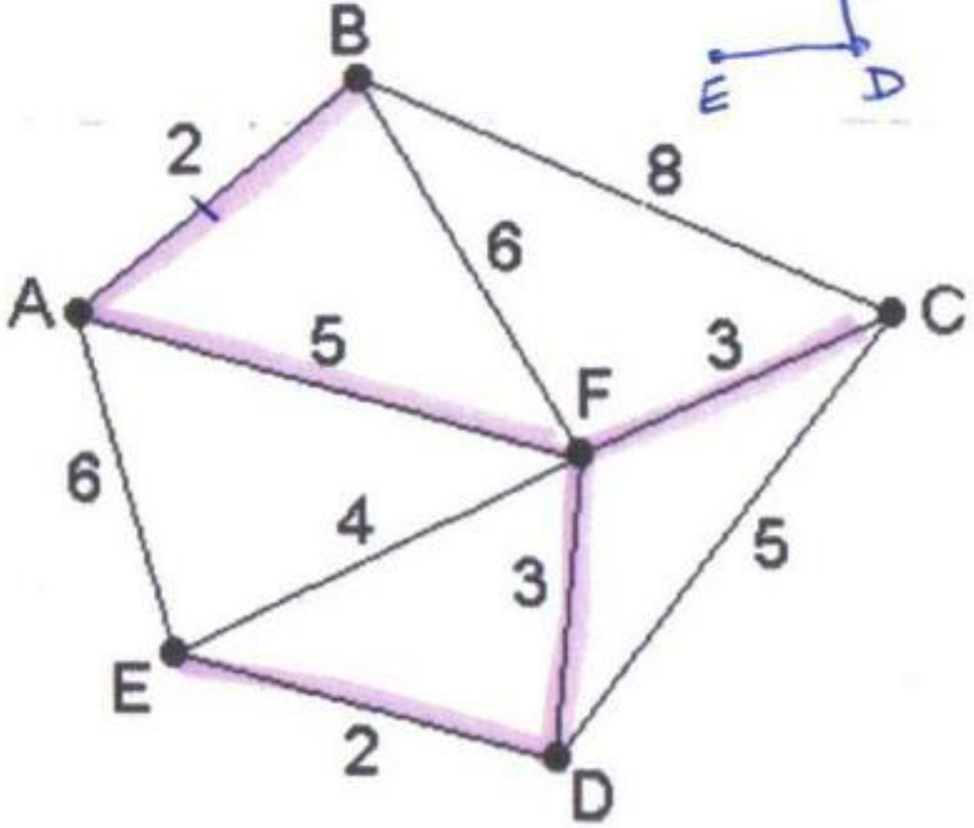
Use Prim’s Algorithm starting at node A to find the MST of the network below, showing clearly the order in which you include the edges. Draw the MST, and state its weight



Prim = Connected .

AB = 2
AF = 5
FC = 3
FD = 3
DE = 2 +

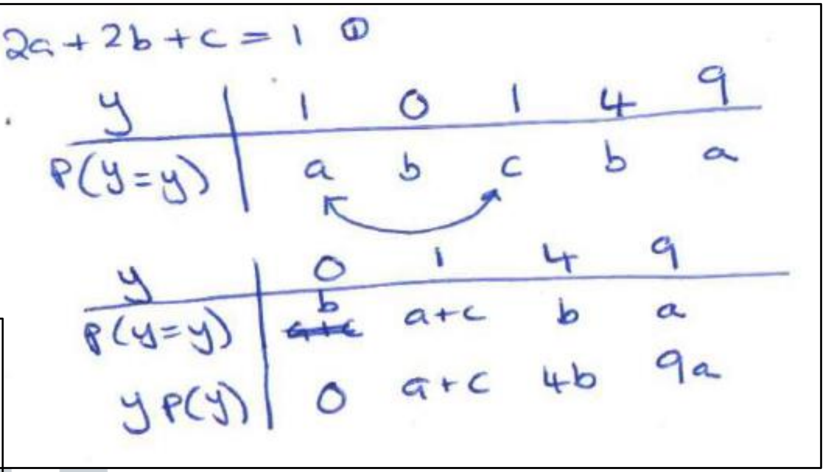
15 = weight



A discrete random variable X has the following probability distribution:

x	-2	-1	0	1	2
$P(X = x)$	a	b	c	b	a

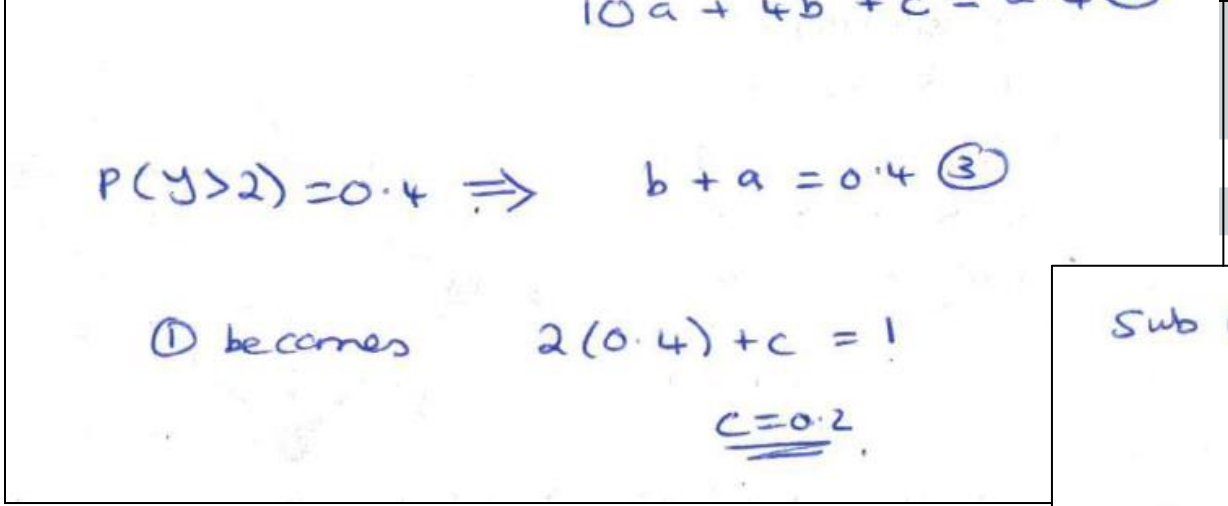
Y is the discrete random variable such that $Y = (X + 1)^2$.
Given that $E(Y) = 2.4$ and $P(Y > 2) = 0.4$, find a, b and c.



$\Sigma p = 1$ $\quad 2a + 2b + c = 1$ ①

y	1	0	1	4	9
$P(Y=y)$	a	b	c	b	a

y	0	1	4	9
$P(Y=y)$	b ~~a+c~~	$a+c$	b	a
$yP(y)$	0	$a+c$	$4b$	$9a$



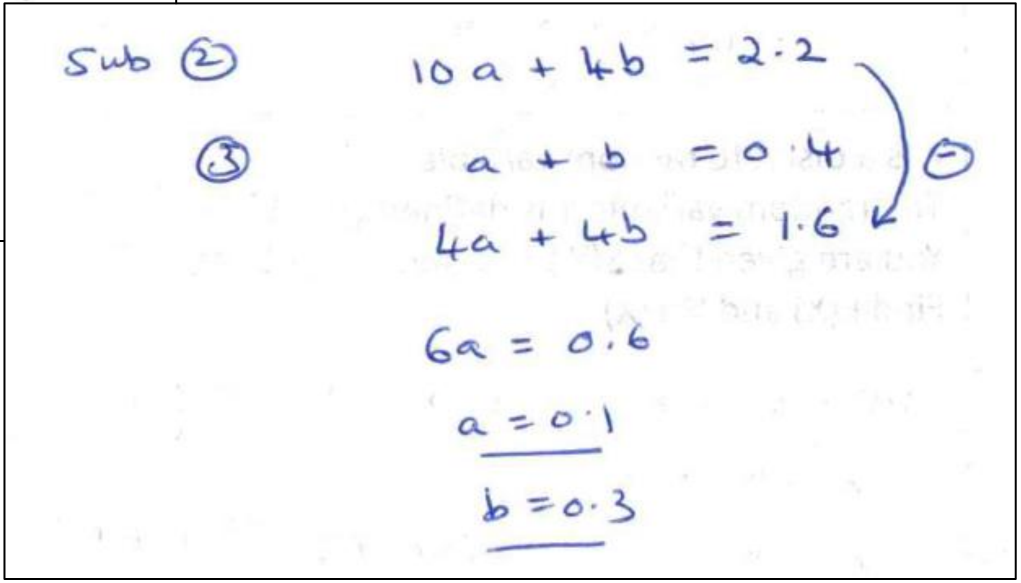
$E(Y) = \Sigma y . P(Y=y) = 0 + a + c + 4b + 9a$

$10a + 4b + c = 2.4$ ②

$P(Y > 2) = 0.4 \Rightarrow b + a = 0.4$ ③

① becomes $2(0.4) + c = 1$

$\underline{\underline{c = 0.2}}$



Sub ② $\quad 10a + 4b = 2.2$

③ $\quad a + b = 0.4$ ⊖

$4a + 4b = 1.6$

$6a = 0.6$

$\underline{a = 0.1}$

$\underline{b = 0.3}$

Find the value of $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix}$

Sheet 9

Use the quicksort algorithm to rearrange the following numbers into ascending order. Indicate clearly the pivots that you use.

18 23 12 7 26 19 16 24

The lines l_1 and l_2 have equations:

$$\frac{x-2}{4} = \frac{y+3}{2} = z - 1$$

and

$$\frac{x+1}{5} = \frac{y}{4} = \frac{z-4}{-2}$$

respectively.

Prove that l_1 and l_2 are skew.

A discrete random variable X has the following probability distribution:

x	-2	-1	0	1	2
$\mathrm{P}(X = x)$	a	b	c	b	a

Y is the discrete random variable such that $Y = (X + 1)^2$.

E(Y) = 2.4, a = 0.1, b = 0.3 and c = 0.2. Find P(2X + 3 ≤ Y).

Find the value of $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix}$

Sheet 9

$$1\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} + 4\begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix}$$

$$= 1\,(6-4) - 2\,(9+1) + 4\,(12+2)$$

$$= (2) - 20 + 56$$

$$= \underline{\underline{38}}.$$

(Check on classwiz / CG50 ✓ also get 38)

Sheet 9 l_1, l_2 skew. Suppose they do meet.

l_1: $x = 4\lambda + 2$
$y = 2\lambda + 3$
$z = \lambda + 1$

l_2: $x = 5\mu - 1$
$y = 4\mu$
$z = -2\mu + 4$

x: $4\lambda + 2 = 5\mu - 1$
$4\lambda = 5\mu - 3$

y: $2\lambda + 3 = 4\mu$
$2\lambda = 4\mu - 3$
$4\lambda = 8\mu - 6$

$\rightarrow$ $5\mu - 3 = 8\mu + 6$
$-9 = 3\mu$
$\underline{-3 = \mu}$

$2\lambda = 4(-3) - 3$
$= -12 - 3 = -15$
$\underline{\lambda = -15/2}$

sub for z, l_1: $z = -7.5 + 1$
$z = -6.5$ $\neq$

l_2: $-2(-3) + 4 = z$
$6 + 4 = z$
$z = 10$

$\therefore$ there is no point of intersection

$\therefore$ the lines are skew.

Use the quicksort algorithm to rearrange the following numbers into ascending order. Indicate clearly the pivots that you use.

$18 \quad 23 \quad 12 \quad 7 \quad \boxed{26} \quad 19 \quad 16 \quad 24$

$18 \quad 23 \quad 12 \quad \boxed{7} \quad 19 \quad 16 \quad 24 \quad \underline{26}$

$\underline{7} \quad 18 \quad 23 \quad 12 \quad \boxed{19} \quad 16 \quad 24 \quad \underline{26}$

$\underline{7} \quad 18 \quad \boxed{12} \quad 16 \quad \underline{19} \quad 23 \quad \boxed{24} \quad \underline{26}$

$\underline{7} \quad \underline{12} \quad 18 \quad \boxed{16} \quad \underline{19} \quad 23 \quad \underline{24} \quad \underline{26}$

$\underline{7} \quad \underline{12} \quad \underline{16} \quad 18 \quad \underline{19} \quad 23 \quad \underline{24} \quad \underline{26}$.

A discrete random variable X has the following probability distribution:

x	-2	-1	0	1	2
$P(X = x)$	a	b	c	b	a

Y is the discrete random variable such that $Y = (X + 1)^2$.

$E(Y) = 2.4$, $a = 0.1$, $b = 0.3$ and $c = 0.2$. Find $P(2X + 3 \le Y)$.

x	-2	-1	0	1	2
$2x+3$	-1	1	3	5	7
$(x+1)^2$	1	0	1	4	9
$2x+3 \le y$?	✓ a	×	×	×	✓ a

$$P(2x+3 \le y) = a + a = 2a = \underline{\underline{0.2}}$$

Sheet 10

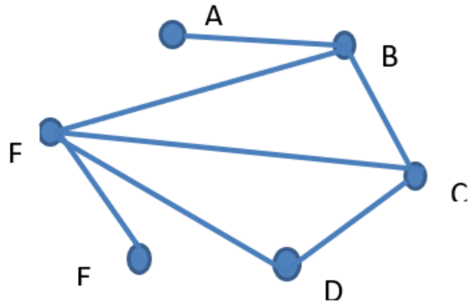
Express the numbers following numbers in the modulus argument form:

$$z_1 = 1 + i\sqrt{3}$$

$$z_2 = -3 - 3i$$

Draw a tree with 6 nodes

Identify three cycles in this graph



Given that $\boldsymbol{a} = \begin{pmatrix} 8 \\ -5 \\ -4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$.

a) Find $\boldsymbol{a}.\boldsymbol{b}$

b) Find the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, giving your answer in degrees to 1 decimal place

The number of demands for taxis to a taxi firm is Poisson distributed with, on average, four demands every thirty minutes.

(a) Find the probability of no demand in 30 minutes.

(b) Find the probability of 1 demand in 1 hour.

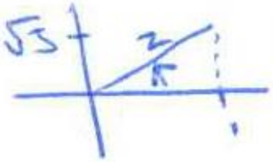
(c) Find the probability of fewer than 2 demands in 15 minutes.

Express the numbers following numbers in the modulus argument form:

Sheet 10

$$z_1 = 1 + i\sqrt{3}$$

$$z_2 = -3 - 3i$$

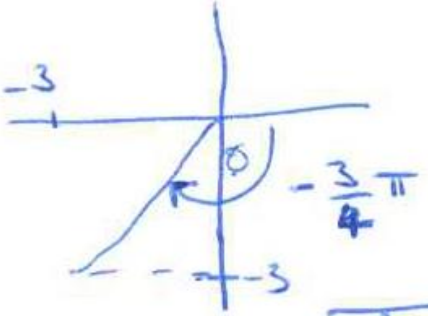


$|r| = \sqrt{1+3}$
$= 2$

$\theta = \tan^{-1}(\sqrt{3})$
$\theta = \pi/3$

$z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$z_2 = 3\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$



$r = \sqrt{3^2 + 3^2}$

$r = 3\sqrt{2}$

Given that $\boldsymbol{a} = \begin{pmatrix} 8 \\ -5 \\ -4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$.

a) Find $\boldsymbol{a}.\boldsymbol{b}$ $= 40 - 20 + 4 = \underline{\underline{24}}$.

b) Find the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, giving your answer in degrees to 1 decimal place

$$\cos\theta = \frac{a.b}{|a||b|} = \frac{24}{\sqrt{105}\sqrt{42}} = \frac{24}{21\sqrt{10}}$$

$$|a| = \sqrt{8^2+5^2+4^2} = \sqrt{105}$$

$$|b| = \sqrt{5^2+4^2+1^2} = \sqrt{42}$$

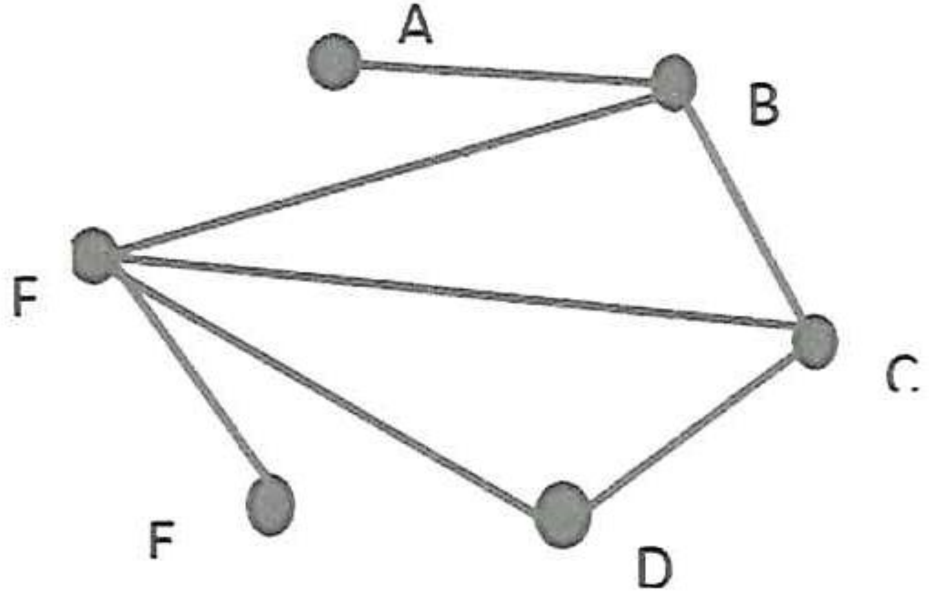
$\theta = 1.201$ radians

$= \underline{\underline{68.8^\circ}}$

Draw a tree with 6 nodes

eg)

Identify three cycles in this graph



BCF
BCDF
FCD.

The number of demands for taxis to a taxi firm is Poisson distributed with, on average, four demands every thirty minutes.

(a) Find the probability of no demand in 30 minutes.

(b) Find the probability of 1 demand in 1 hour.

(c) Find the probability of fewer than 2 demands in 15 minutes.

a) X = no. demands

$X \sim Po(4)$

$P(X=0) = 0{\cdot}0183$ (3sf).

b) $X \sim Po(8)$

$P(X=1) = 0{\cdot}00268$ (3sf)

c) $X \sim Po(2)$

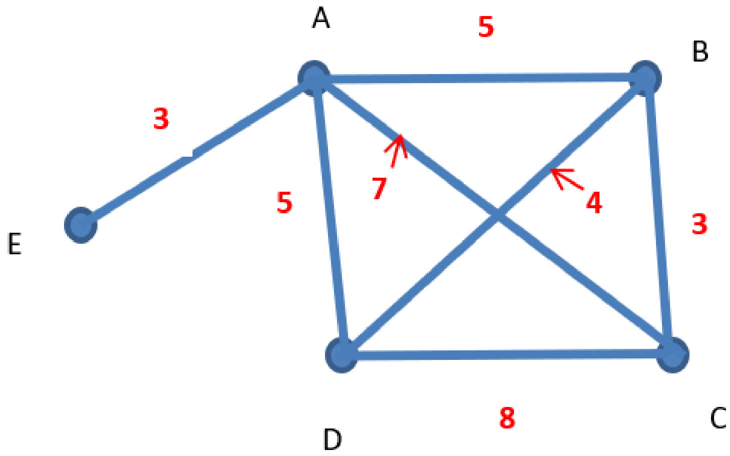
$P(X<2) = P(X \leq 1) =$ ~~0·06~~ $0{\cdot}406$ (3sf).

Sheet 11

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 1 & a \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} b & -1 \\ 2 & 4 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 3 & y \\ x & 3 \end{bmatrix}$$

Given that **A** + **B** = **C**, find the values of a, b, x and y

A snow plough leaves the depot at A and needs to travel down every road at least once before returning to the depot. Calculate the least distance it must cover and give a possible route it could use.



Given that $\boldsymbol{a} = -2\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}$, find a vector which is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$

The number of organic particles suspended in a volume V ml of water from a particular pond follows a Poisson distribution with mean 0.2V .

(a) Find the probability that a volume of 50 ml contains fewer than 8 particles.

(b) Find the probability that a volume of 30 ml contains more than 2 particles.

(c) Find the probability that a volume of 10 ml contains 3 particles.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 1 & a \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} b & -1 \\ 2 & 4 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 3 & y \\ x & 3 \end{bmatrix}$$

Given that **A** + **B** = **C**, find the values of a, b, x and y

$$2 + b = 3 \longrightarrow \underline{\underline{b = 1}}$$

$$3 - 1 = y \longrightarrow \underline{\underline{y = 2}}$$

$$1 + 2 = x \longrightarrow \underline{\underline{x = 3}}$$

$$a + 4 = 3 \longrightarrow \underline{\underline{a = -1}}$$

Given that $\boldsymbol{a} = -2\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} - 8\boldsymbol{j} + 5\boldsymbol{k}$, find a vector which is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$

Let $\underline{r}$ be $\perp$ to both $\underline{a}$ and $\underline{b}$.

$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\underline{a} \cdot \underline{r} = 0 \qquad -2x + 5y - 4z = 0$ ①

$\underline{b} \cdot \underline{r} = 0 \qquad 4x - 8y + 5z = 0$ ②

$2\times$① $+$ ② $\rightarrow \qquad 2y - 3z = 0$

$y = \frac{3z}{2}$

Sub ① $\qquad -2x + \frac{15z}{2} - 4z = 0$

$-2x + \frac{7}{2}z = 0 \qquad 2x = \frac{7}{2}z$

$x = \frac{7}{4}z$

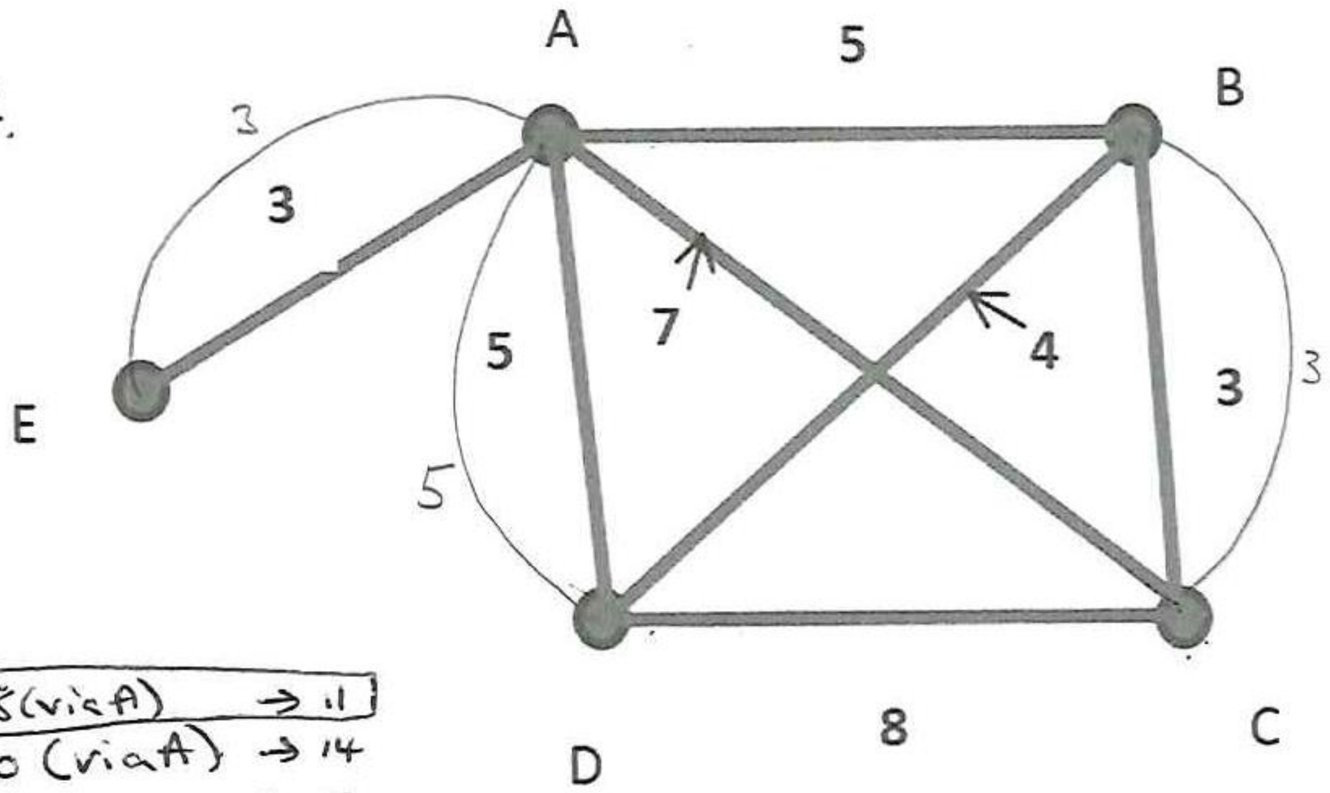
$\underline{r} = \begin{pmatrix} 7/4 \\ 3/2 \\ 1 \end{pmatrix}$ or any multiple

$\underline{r} = 7i + 6j + 4z$

A snow plough leaves the depot at A and needs to travel down every road at least once before returning to the depot. Calculate the least distance it must cover and give a possible route it could use.

Total weight = 46

Poss. route = EABCBDCADAE



A	B	C	D	E
4	3	3	3	1

BC = 3 DE = 8 (via A) → 11

BD = 4 CE = 10 (via A) → 14

BE = 8 (via A) CD = 8 → 16

The number of organic particles suspended in a volume V ml of water from a particular pond follows a Poisson distribution with mean 0.2V .

(a) Find the probability that a volume of 50 ml contains fewer than 8 particles.

(b) Find the probability that a volume of 30 ml contains more than 2 particles.

(c) Find the probability that a volume of 10 ml contains 3 particles.

(a) $X \sim Po(10)$ $\leftarrow 50 \times 0.2 = 10$

$P(X<8) = P(X \leq 7) = 0.2202$

(b) $X \sim Po(6)$ $P(X>2) = 1 - P(X \leq 2) = 0.9380$.

(c) $X \sim Po(2)$ $P(X=3) = 0.1804$

Sheet 12

Write z = 4 + 5i in modulus-argument form.

Express the following calculation in the form x + iy:

$$3\left(cos\frac{5\pi}{12}+isin\frac{5\pi}{12}\right)\times 4\left(cos\frac{\pi}{12}+isin\frac{\pi}{12}\right)$$

HINT: $z_1z_2 = r_1r_2\left(cos(\theta_1+\theta_2)+isin(\theta_1+\theta_2)\right)$

32 45 17 23 38 28 16 9 12 10

The numbers in the list above represent the lengths, in metres, of ten lengths of fabric. They are to be cut from rolls of fabric of length 60m.

(a) Calculate a lower bound for the number of rolls needed.

(b) Use the first-fit bin packing algorithm to determine how these ten lengths can be cut from rolls of length 60m.

(c) Use full bins to find an optimal solution that uses the minimum number of rolls.

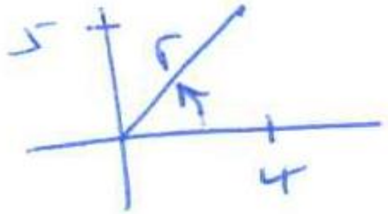
Given that the vectors $\boldsymbol{a} = 2\boldsymbol{i} - 6\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 5\boldsymbol{i} + 2\boldsymbol{j} + \lambda\boldsymbol{k}$ are perpendicular, find the value of λ.

The number of organic particles suspended in a volume V ml of water from a particular pond follows a Poisson distribution with mean 0.2V .

Find the smallest value of x such that the probability that there are more than x particles in a volume of 80ml is less than 0.15.

Write z = 4 + 5i in modulus-argument form.

Sheet 12



$\tan\theta = 5/4 \;;\; \theta = 0.896$

$r = \sqrt{4^2+5^2} = \sqrt{41}$

$r = \sqrt{41}\,(\cos 0.896 + i\sin 0.896)$.

Express the following calculation in the form x + iy:

$$3\left(cos\frac{5\pi}{12} + isin\frac{5\pi}{12}\right) \times 4\left(cos\frac{\pi}{12} + isin\frac{\pi}{12}\right) = 12\left(\overset{0}{\cos\frac{\pi}{2}} + i\overset{1}{\sin\frac{\pi}{2}}\right)$$

$$\uparrow$$

$$\frac{5\pi}{12} + \frac{\pi}{12} = \frac{\pi}{2}$$

$$= 0 + 12i$$

$$= \underline{\underline{12i}}.$$

HINT: $z_1z_2 = r_1r_2(cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2))$

Given that the vectors $\boldsymbol{a} = 2\boldsymbol{i} - 6\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 5\boldsymbol{i} + 2\boldsymbol{j} + \lambda\boldsymbol{k}$ are perpendicular, find the value of λ.

$$a \cdot b = 0 \quad \Rightarrow \quad 10 - 12 + \lambda = 0$$

$$-2 + \lambda = 0$$

$$\underline{\underline{\lambda = 2}}.$$

32 45 17 23 38 28 16 9 12 10

The numbers in the list above represent the lengths, in metres, of ten lengths of fabric. They are to be cut from rolls of fabric of length 60m.

(a) Calculate a lower bound for the number of rolls needed.

Sum = 230m $230/60 = 3.83\ldots \rightarrow$ 4 rolls (LB)

(b) Use the first-fit bin packing algorithm to determine how these ten lengths can be cut from rolls of length 60m.

(c) Use full bins to find an optimal solution that uses the minimum number of rolls.

(b) first fit

Bin 1	Bin 2	Bin 3	Bin 4	Bin 5
32	45	23	38	10
49 17	57 12	51 28	54 16	
58 9				

(c) full Bins

Bin 1	Bin 2	Bin 3	Bin 4
32	45	23	38
17	12	28	16.
10		9	

The number of organic particles suspended in a volume V ml of water from a particular pond follows a Poisson distribution with mean 0.2V .

Find the smallest value of x such that the probability that there are more than x particles in a volume of 80ml is less than 0.15.

$X \sim Po(0.2v)$

$X \sim Po(16)$

$P(X > x) < 0.15.$

$1 - P(X \leq x) < 0.15$

$0.85 < P(X \leq x)$

$P(X \leq 19) = 0.8122$

$P(X \leq 20) = 0.8681$

$\therefore \underline{\underline{x = 20}}.$

$[\text{Check } P(X > 20) = 1 - P(X \leq 20) = 0.1318 < 0.15]$

Sheet 13

Given that $|z-4|=5$

a) Sketch the locus of z on an Argand diagram

b) Find the values of z that satisfy:

i) $|z-4|=5$ and $Im(z)=0$

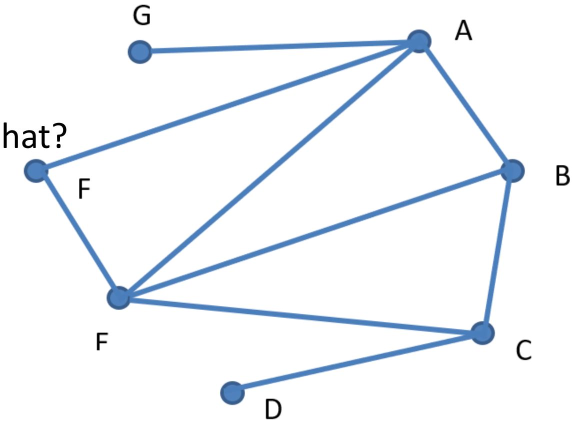
ii) $|z-4|=5$ and $Re(z)=0$

What sort of graph is this?

No. edges?

Order of each node?

Handshake Lemma means what?



The square S has coordinates (1,1), (3,1), (3,3) and (1,3).

Find the coordinates of the vertices of the image of S after the transformation given by the matrix:

$$\boldsymbol{M}=\begin{bmatrix}-1 & 2\\ 2 & 1\end{bmatrix}$$

Faulty components are detected at a rate of 2.5 per hour.

(a) Suggest a suitable model for the number of faulty components detected per hour.

(b) Describe, in the context of the question, two assumptions you have made in part a for this model to be suitable.

(c) Find the probability of 2 faulty components being detected in a 1-hour period.

(d) Find the probability of at least 6 faulty components being detected in a 3-hour period.

Sheet 13

Given that $|z-4|=5$

a) Sketch the locus of z on an Argand diagram

b) Find the values of z that satisfy:

i) $|z-4|=5$ and $Im(z)=0$ $z = -1, 9$

ii) $|z-4|=5$ and $Re(z)=0$ $z = -3i, 3i$.

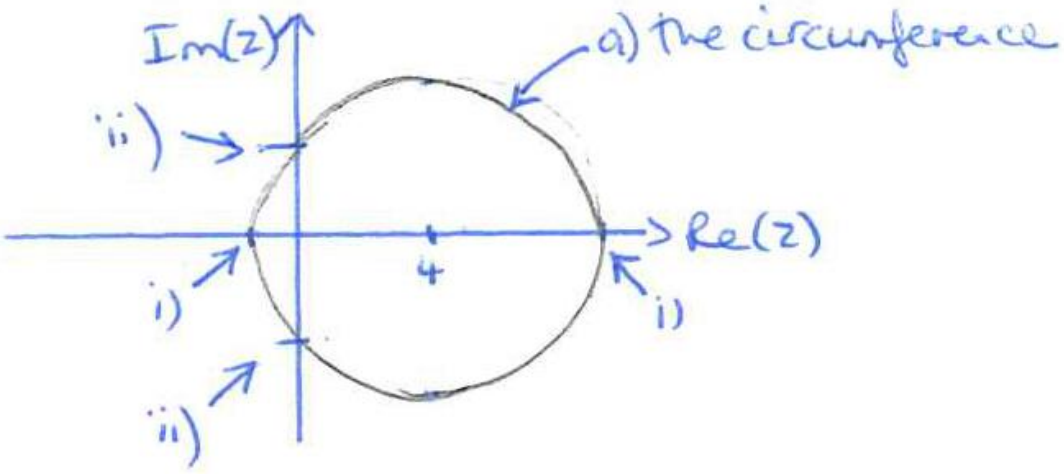
$(x-4)^2+y^2=25$

$(-4)^2+y^2=25$

$y^2=25-16$

$y^2=9$

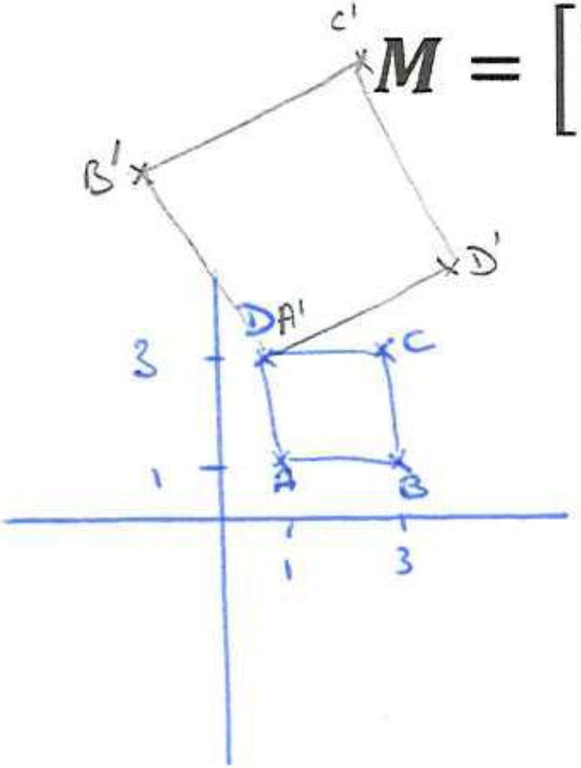
$y=\pm 3$



The square S has coordinates (1,1), (3,1), (3,3) and (1,3).

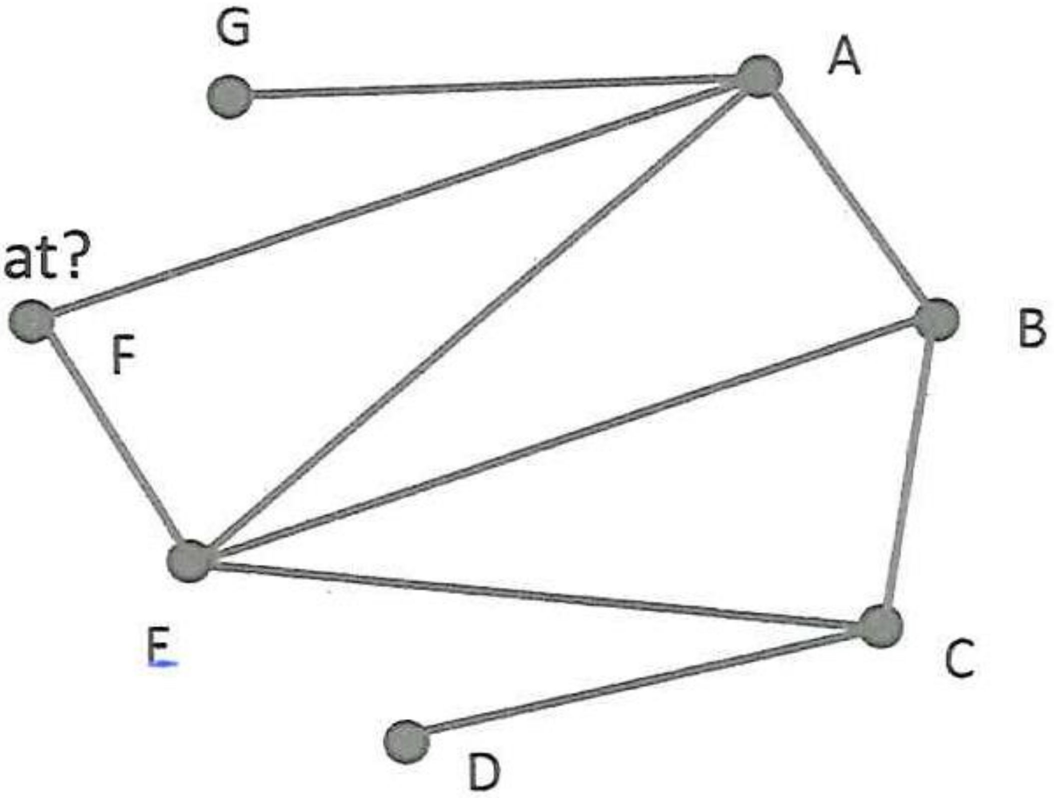
Find the coordinates of the vertices of the image of S after the transformation given by the matrix:

$$\boldsymbol{M} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} A & B & C & D \\ 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} A' & B' & C' & D' \\ 1 & -1 & 3 & 5 \\ 3 & 7 & 9 & 5 \end{pmatrix}$$



(1, 3), (-1, 7), (3, 9) & (5, 5)

What sort of graph is this?
No. edges?
Order of each node?
Handshake Lemma means what?



connected Graph.
Simple Graph

A	B	C	D	E	F	G
4	3	3	1	4	2	1

Total order = 18

No. arcs = 9

Handshake Lemma says total order of nodes = 2x no. arcs.

Faulty components are detected at a rate of 2.5 per hour.

(a) Suggest a suitable model for the number of faulty components detected per hour.

(b) Describe, in the context of the question, two assumptions you have made in part a for this model to be suitable.

(c) Find the probability of 2 faulty components being detected in a 1-hour period.

(d) Find the probability of at least 6 faulty components being detected in a 3-hour period.

(a) $X \sim Po(2.5)$ (b) constant average rate of faults, occur singly and independently.

(c) $P(X = 2) = 0.\underline{257}$ (3sf)

(d) $X \sim Po(7.5)$

$P(X \geq 6) = 1 - P(X \leq 5) = 0.\underline{759}$ (3sf)

Sheet 14

Given that the complex number
z = x + iy satisfies the equation:

$$|z - 12 - 5i| = 3$$

Find the minimum and maximum values of |z|

The list of numbers below is to be sorted into **descending** order. Perform a bubble sort to obtain the sorted list, giving the state of the list after each completed pass

52 48 50 45 64 47 53

Find the coordinates of the point of intersection of the line l and the plane Π where l has equation:

$$\boldsymbol{r} = -\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$$

And Π has equation:

$$\boldsymbol{r}.(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) = 4$$

The number of emissions per minutes from two radioactive sources are modelled by independent random variables X and Y which have Poisson distributions with means 5 and 8 respectively.

(a) Calculate the probability that the total number of emissions from the two sources is less than 6.

(b) Calculate the probability that in any second the total number of emissions from the two sources is greater than 1.

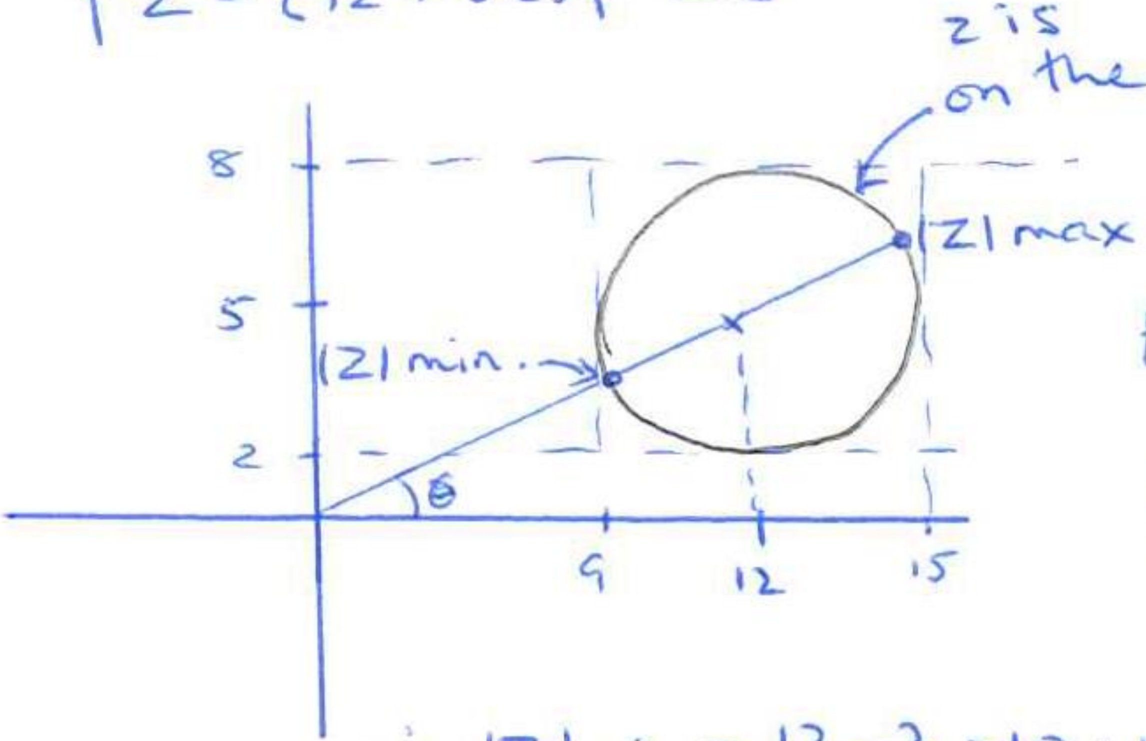
Given that the complex number
z = x + iy satisfies the equation:

Sheet 14

$$|z - 12 - 5i| = 3$$

Find the minimum and maximum values of $|z|$

$$|z - (12 + 5i)| = 3$$



~~tan θ = 5/12~~

~~θ =~~

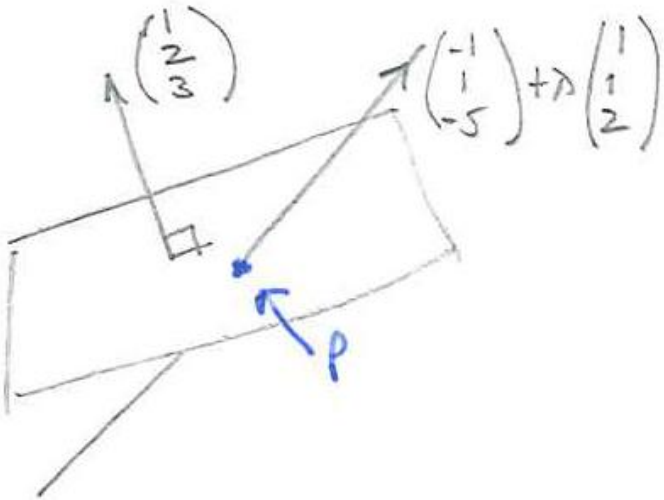
$\sqrt{5^2 + 12^2} = \underline{\underline{13}}$

↑
mod. of centre

$\therefore |z|_{min} = 13 - 3 = \underline{\underline{10}}\ ;\ |z|_{max} = 13 + 3 = \underline{\underline{16}}$.

Find the coordinates of the point of intersection of the line l and the plane Π where l has equation:

$$\boldsymbol{r} = -\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$$



And Π has equation:

$$\boldsymbol{r}.(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) = 4$$

at P

$$\begin{pmatrix}-1+\lambda\\1+\lambda\\-5+2\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}=4$$

$$-1+\lambda+2+2\lambda-15+6\lambda=4$$

$$-14+9\lambda=4$$

$$9\lambda=18$$

$$\underline{\underline{\lambda=2}}$$

$\therefore$ P $\underline{\underline{(1,3,-1)}}$

The list of numbers below is to be sorted into **descending** order. Perform a bubble sort to obtain the sorted list, giving the state of the list after each completed pass

52 48 50 45 64 47 53

52 — 48

(50 48)

48 — 45

(64 45)

(47 45)

(53 45)

52 50 48 64 47 53 45. ← end of pass 1

52 — 50

50 — 48

(64 48)

48 — 47

(53 47)

(47 45)

52 50 64 48 53 47 45 ← end of pass 2

Sheet 14
Bubble Sort ctd.

52 – 50
(64 50)
50 – 48
(53 48)
48 – 47
47 – 45
52 64 50 53 48 47 45 ← end of pass 3

64 – 52
(52 50)
53 — 50
50 – 48
48 — 47
47 — 45
64 52 53 50 48 47 45 ← end of pass 4

(64 52)
53 — 52
52 — 50
50 — 48
48 — 47
47 — 45
64 53 52 50 48 47 45 ← end of pass 5.

64 53 52 50 48 47 45 ← final pass.

The number of emissions per minutes from two radioactive sources are modelled by independent random variables X and Y which have Poisson distributions with means 5 and 8 respectively.

(a) Calculate the probability that the total number of emissions from the two sources is less than 6.

(b) Calculate the probability that in any second the total number of emissions from the two sources is greater than 1.

$X \sim Po(5)$ $\quad$ $Y \sim Po(8)$ $\quad$ $X + Y \sim Po(13)$ ← per minute

(a) $P(X+Y<6) = P(X+Y \le 5) = \underline{\underline{0.0107}}$.

(b) $X + Y \sim Po\left(\frac{13}{60}\right)$ ← per second

$P(X+Y>1) = 1 - P(X+Y \le 1)$

$= \underline{\underline{0.0203}}$.

Sheet 15

Given that **BA** = 0, calculate **AB** in terms of a.

$$\boldsymbol{A} = \begin{bmatrix} -1 \\ a \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} b & 2 \end{bmatrix}$$

By using Prim's Algorithm on the matrix below starting at node A, find the MST of the network. State clearly the order in which you included the edges, and draw the MST

	A	*B*	*C*	*D*	*E*	*F*
A	-	4	9	12	7	6
B	4	-	7	8	10	8
C	9	7	-	11	-	7
D	12	8	11	-	2	3
E	7	10	-	2	-	5
F	6	8	7	3	5	-

The lines l_1 and l_2 have vector equations:

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

and

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 1 \\ 4 \end{pmatrix}$$

Show that the lines intersect, and find their point of intersection.

A student is investigating the number of tulips, x, in each of 100 randomly selected squares within a field. The results can be summarised as: $\sum x = 143, \sum x^2 = 347.$

(a) Calculate the mean and variance of the number of tulips per square for the 100 squares.

(b) Explain why the results in part a suggest that a Poisson distribution may be a suitable model for the number of tulips per square for the 100 squares.

(c) Using a suitable value of , estimate the probability that exactly 3 tulips will be found in a randomly selected square.

Given that **BA** = 0, calculate **AB** in terms of a.

Sheet 15

$$\boldsymbol{A} = \begin{bmatrix} -1 \\ a \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} b & 2 \end{bmatrix}$$

$BA = 0 \qquad \begin{pmatrix} b & 2 \end{pmatrix}\begin{pmatrix} -1 \\ a \end{pmatrix} = -b + 2a = 0$

$\underline{\underline{b = 2a}}.$

$A = \begin{pmatrix} -1 \\ a \end{pmatrix} \quad B = \begin{pmatrix} 2a & 2 \end{pmatrix}$

$\ 2 \times \underline{1} \qquad\quad \underline{1} \times 2$

$AB = \begin{pmatrix} -2a & -2 \\ 2a^2 & 2a \end{pmatrix}.$

The lines l_1 and l_2 have vector equations:

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

and

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 1 \\ 4 \end{pmatrix}$$

Show that the lines intersect, and find their point of intersection.

Assume they intersect

$3 + \lambda = -5\mu$ ①

$1 - 2\lambda = -2 + \mu$ ②

$1 - \lambda = 3 + 4\mu$ ③

$5\mu + \lambda = -3$ ①

$-\mu - 2\lambda = -3$

$\mu + 2\lambda = 3$ ② $\times 4$

$4\mu + \lambda = -2$ ③

$4\mu + 8\lambda = 12$

⊖ $-7\lambda = -14$

$\underline{\underline{\lambda = 2}}$

Sub ② $\mu + 4 = 3$ $\underline{\underline{\mu = -1}}$

check ①

$5(-1) + 2 = 3$ ✓

consistent equations

∴ 1 pt of intersection

which is... $\underline{\underline{(5, -3, -1)}}$

By using Prim's Algorithm on the matrix below starting at node A, find the MST of the network. State clearly the order in which you included the edges, and draw the MST

	↓1 A	↓2 B	↓3 C	↓5 D	↓6 E	↓4 F
A	~~-~~	~~4~~	~~9~~	~~12~~	~~7~~	~~6~~
B	④	~~-~~	~~7~~	~~8~~	~~10~~	~~8~~
C	~~9~~	⑦	~~-~~	~~11~~	~~-~~	~~7~~
D	~~12~~	~~8~~	~~11~~	~~-~~	~~2~~	③
E	7	10	-	②	-	5
F	~~6~~	~~8~~	⑦	~~3~~	~~5~~	~~-~~

AB	4
BC	7
CF	7
FD	3
DE	2
	23 = MST

A student is investigating the number of tulips, x, in each of 100 randomly selected squares within a field. The results can be summarised as: $\sum x = 143, \sum x^2 = 347.$

(a) Calculate the mean and variance of the number of tulips per square for the 100 squares.

(b) Explain why the results in part a suggest that a Poisson distribution may be a suitable model for the number of tulips per square for the 100 squares.

(c) Using a suitable value of , estimate the probability that exactly 3 tulips will be found in a randomly selected square.

(a) $\bar{x} = \frac{143}{100} = 1.43$ var, $\cancel{\sigma}_s^2 = \frac{1}{\underset{100}{\cancel{n}}}\left(\underset{347}{\sum x^2} - \frac{(143)^2}{100}\right) = 1.4251$

(b) mean ≈ var $X \sim Po(1.43)$ single parameter "λ".

(c) $P(X=3) = \underline{\underline{0.1166}}$ (4dp).

Sheet 16

Describe fully the geometrical transformation represented by this matrix:

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Find a matrix to represent the transformation:
'Rotation of 45° anticlockwise about (0,0)'

29 52 73 87 74 47 38 61 41

The numbers in the list represent the lengths in minutes of nine radio programmes. They are to be recorded onto tapes which each store up to 100 minutes of programmes.

(*a*) Obtain a lower bound for the number of tapes needed to store the nine programmes.

(b) Use the first-fit bin packing algorithm to fit the programmes onto the tapes.

(*c*) Use the first-fit decreasing bin packing algorithm to fit the programmes onto the tapes.

The line l has equation:

$$\boldsymbol{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The point P has position vector:

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

Show that P does not line on l.

The probability that a patient has a particular disease is 0.008. One day 80 people go to their doctor.

(a) Let X = number of patients with the disease. State the distribution, with parameters, of X.

(b) Using a suitable approximation, what is the probability that exactly 2 of the patients have the disease?

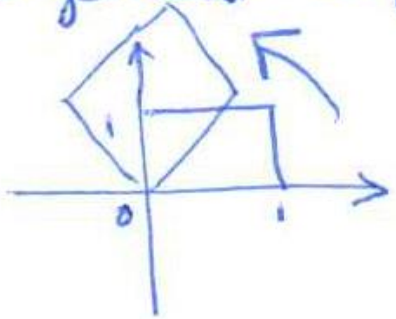
(c) What is the probability that 3 or more of them have the disease?

Describe fully the geometrical transformation represented by this matrix:

Sheet 16

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Enlargement s.f. 3 about (0,0).



use formula booklet.

Find a matrix to represent the transformation:
'Rotation of 45° anticlockwise about (0,0)'

$$\begin{pmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}.$$

The line l has equation:

$$r = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The point P has position vector:

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

Show that P does not line on l.

Assume P does lie on l:

$2 = -2 + \lambda$ ①

$1 = 1 - 2\lambda$ ②

$3 = 4 + \lambda$ ③

① → $\lambda = 4$

② → $2\lambda = 0$, $\lambda = 0$

③ $\lambda = -1$

inconsistent equations

∴ P does not lie on l.

29 52 73 87 74 47 38 61 41

The numbers in the list represent the lengths in minutes of nine radio programmes. They are to be recorded onto tapes which each store up to 100 minutes of programmes.

(*a*) Obtain a lower bound for the number of tapes needed to store the nine programmes. Total = 502 $\frac{502}{100} = 5.02$

$\therefore$ LB = 6 tapes.

(b) Use the first-fit bin packing algorithm to fit the programmes onto the tapes.

(*c*) Use the first-fit decreasing bin packing algorithm to fit the programmes onto the tapes.

(b)

tape1	tape2	tape3	tape4	tape5	tape6	tape7
29	73	87	74	47	61	41
52				38		
81				85		

(c) 87, 74, 73, 61, 52, 47, 41, 38, 29

Tape 1	Tape 2	Tape 3	tape 4	tape 5	tape 6
87	74	73	61	52	41
			38	47	29
			99	99	70

The probability that a patient has a particular disease is 0.008. One day 80 people go to their doctor.

(a) Let X = number of patients with the disease. State the distribution, with parameters, of X.

(b) Using a suitable approximation, what is the probability that exactly 2 of the patients have the disease?

(c) What is the probability that 3 or more of them have the disease?

(a) $X \sim Bin(80, 0.008)$

(b) Bin → Normal $\mu = np = 0.64$, $\sigma^2 = np(1-p)$

$\sigma^2 = 0.63488$

$P(x=2) \longrightarrow P(1.5 < X < 2.5)$ continuity correction

$= \underline{\underline{0.1304}}$ (4dp).

(c) $P(X \geq 3) = 1 - P(X \leq 2) = 1 - P(x < 2.5)$

$= \underline{\underline{0.00979}}$ (3sf)

Sheet 17

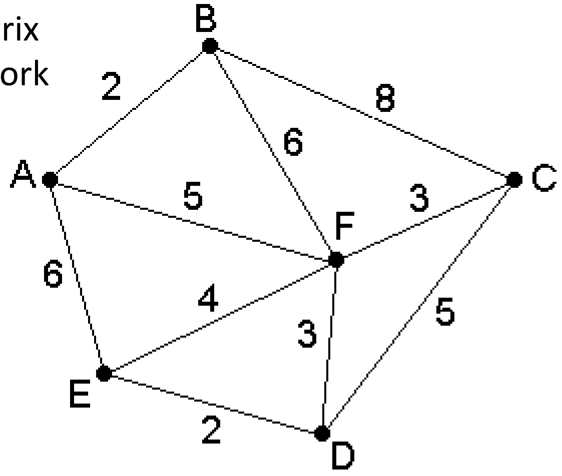
The matrix $A = \begin{bmatrix} 3 & k & 0 \\ -2 & 1 & 2 \\ 5 & 0 & k+3 \end{bmatrix}$,

where k is a constant.

a) Find $\det \boldsymbol{A}$ in terms of k

b) Given that A is singular, find the possible values of k

Construct the distance matrix corresponding to this network



Evaluate, using known results: $$\sum_{r=2}^{5}(r^2)$$ and: $$\sum_{r=20}^{40} r^3$$

Assume that for the city of Naples in Italy, the chance of an earthquake on a random day is 0.00002. I want to find the probability that there are at least two earthquakes between the start of 2001 and the end of 2010.

(a) Find the probability using a valid distribution.

(b) Find the probability using a Poisson approximation.

(c) What is the % difference in your answers?

The matrix $A = \begin{bmatrix} 3 & k & 0 \\ -2 & 1 & 2 \\ 5 & 0 & k+3 \end{bmatrix}$,

where k is a constant.

Sheet 17

a) Find $\det A$ in terms of k

$= 3\begin{vmatrix} 1 & 2 \\ 0 & k+3 \end{vmatrix} - k \begin{vmatrix} -2 & 2 \\ 5 & k+3 \end{vmatrix}$

b) Given that A is singular, find the possible values of k

$= 3(k+3) - k(-2k-6-10)$

$= 3k+9 - k(-2k-16)$

$= 2k^2+19k+9$

b) $2k^2+19k+9=0$

$(2k+1)(k+9)=0$

$k=-\frac{1}{2} \qquad k=-9$

Evaluate, using known results: $\sum_{r=2}^{5}(r^2)$ and: $\sum_{r=20}^{40} r^3$

$$\sum_{2}^{5}(r^2) = \sum_{1}^{5} r^2 - \sum_{1}^{1} r^2 = \frac{1}{6}5(6)(11) - (1^2) = 54$$

[OR $2^2+3^2+4^2+5^2 = 54$ checked ok]

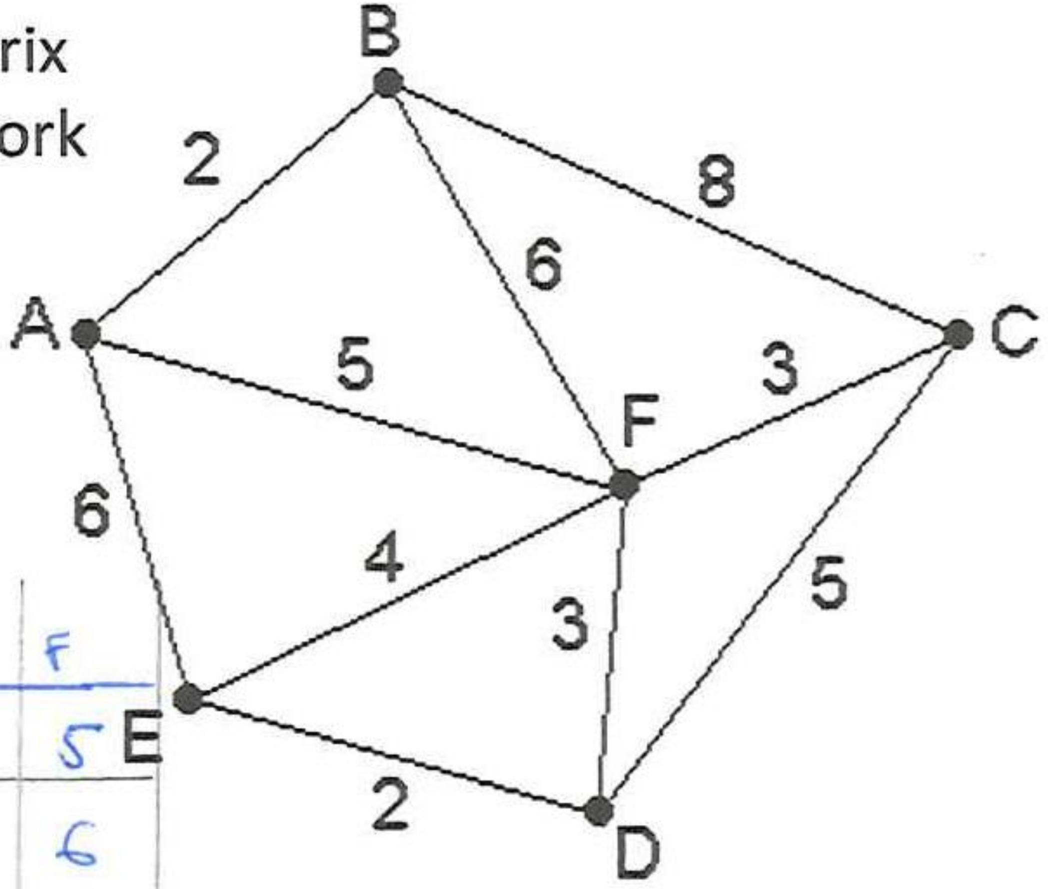
$$\sum_{20}^{40} r^3 = \sum_{1}^{40} r^3 - \sum_{1}^{19} r^3 = \left[\frac{1}{4}40^2(41)^2\right] - \left[\frac{1}{4}(19)^2(20)^2\right]$$

$$= 672400 - 36100$$

$$= \underline{\underline{636300}}$$

[$\sum_{20}^{40} r^3$ on classwizz = 636300 checked ok]

Construct the distance matrix corresponding to this network



from ↓

To →

	A	B	C	D	E	F
A	–	2	∞	∞	6	5
B	2	–	8	∞	∞	6
C	∞	8	–	5	∞	3
D	∞	∞	5	–	2	3
E	6	∞	∞	2	–	4
F	5	6	3	3	4	–

Assume that for the city of Naples in Italy, the chance of an earthquake on a random day is 0.00002. I want to find the probability that there are at least two earthquakes between the start of 2001 and the end of 2010.

(a) Find the probability using a valid distribution.

(b) Find the probability using a Poisson approximation.

(c) What is the % difference in your answers?

(a) $p = 0.00002$ in 1 day X = No earthquakes

n = no. days = $(366 \times 2) + (365 \times 8) = 3652$

$X \sim Bin(3652, 0.00002)$; $P(X \geq 2) = 1 - P(X \leq 1)$

$= \underline{0.00254}$ (3sf)

$= 0.002540395127$ Ⓐ

(b) $X \sim Po(0.07304) \leftarrow 3652 \times 0.00002$

$P(X \geq 2) = 1 - P(X \leq 1) = 0.002541024504$ Ⓑ

(c) $\dfrac{Ⓐ - Ⓑ}{Ⓐ} \times 100 = \underline{0.025\%}$ (2sf).

Sheet 18

Show that the shortest distance between the parallel lines with equations:

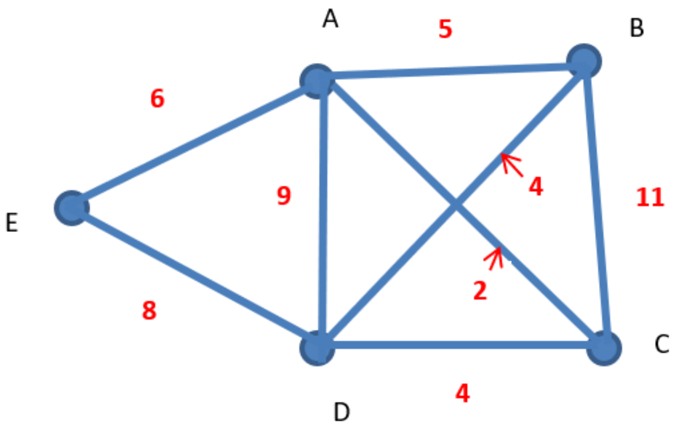
$$\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} + \lambda(5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$$

and

$$\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{k} + \mu(5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$$

is $\frac{21\sqrt{2}}{10}$

A council employee needs to check the condition of the roads. To do this she needs to start at her office at A, travel down each road at least once, and return to her office. She wishes to travel the least possible distance. Find the distance she must travel, and one possible route she could take.



The following matrices represent three different transformations:

$$\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \boldsymbol{Q} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{R} = \begin{bmatrix} 3 & 7 \\ -1 & -2 \end{bmatrix}$$

Find the matrix representing the transformation represented by **R**, followed by **Q**, followed by **P** and give a geometrical interpretation of this transformation.

The random variable X has a Poisson distribution believed to have a mean of 5. A single observation of X has the value 10. Test, at the 10% significance level, whether the mean is equal to 5.

Show that the shortest distance between the parallel lines with equations:

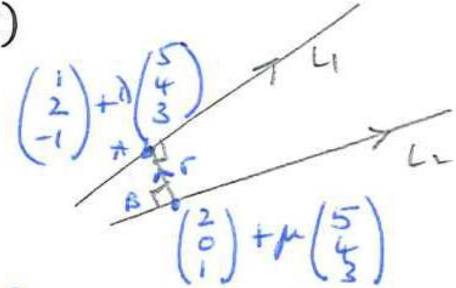
$$\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} + \lambda(5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$$

and

$$\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{k} + \mu(5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$$

is $\frac{21\sqrt{2}}{10}$

Sheet 18



They're parallel! $r.\begin{pmatrix}5\\4\\3\end{pmatrix} = 0$

dir^n vector of r. $\rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}\begin{pmatrix}5\\4\\3\end{pmatrix} = 0$ $\quad 5x + 4y + 3z = 0$

Let $A = \begin{pmatrix}1\\2\\-1\end{pmatrix}$ $\quad B = \begin{pmatrix}2+5\mu\\4\mu\\1+3\mu\end{pmatrix}$ $\quad \overrightarrow{AB} = \begin{pmatrix}1+5\mu\\4\mu-2\\2+3\mu\end{pmatrix}$ $\quad AB.\begin{pmatrix}5\\4\\3\end{pmatrix} = 0$

$5(1+5\mu) + 4(4\mu-2) + 3(2+3\mu) = 0$

$25\mu + 16\mu + 9\mu + 3 = 0$

$50\mu = -3$

$\mu = -3/50$

$\downarrow$

$\overrightarrow{AB} = \begin{pmatrix}0.7\\-2.24\\1.82\end{pmatrix}$

$|AB| = \sqrt{0.7^2 + 2.24^2 + 1.82^2}$

$|AB| = \sqrt{\frac{441}{50}}$

$|AB| = \frac{21\sqrt{2}}{10}.$

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The following matrices represent three different transformations:

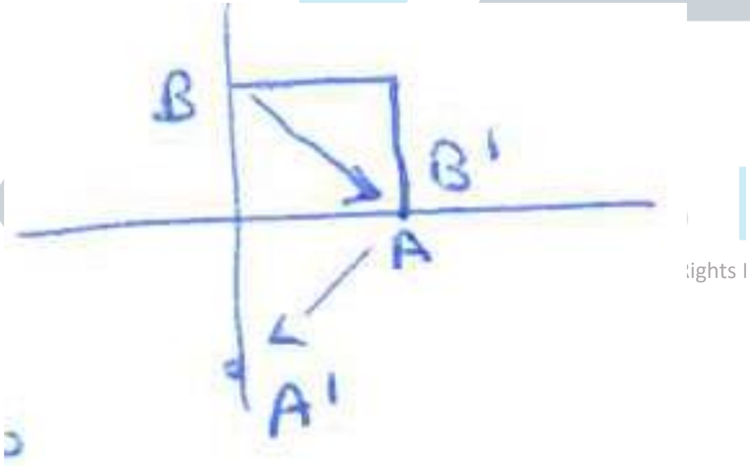
$$\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \boldsymbol{Q} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{R} = \begin{bmatrix} 3 & 7 \\ -1 & -2 \end{bmatrix}$$

Find the matrix representing the transformation represented by **R**, followed by **Q**, followed by **P** and give a geometrical interpretation of this transformation.

$$PQR = P\left[\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 7 \\ -1 & -2 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



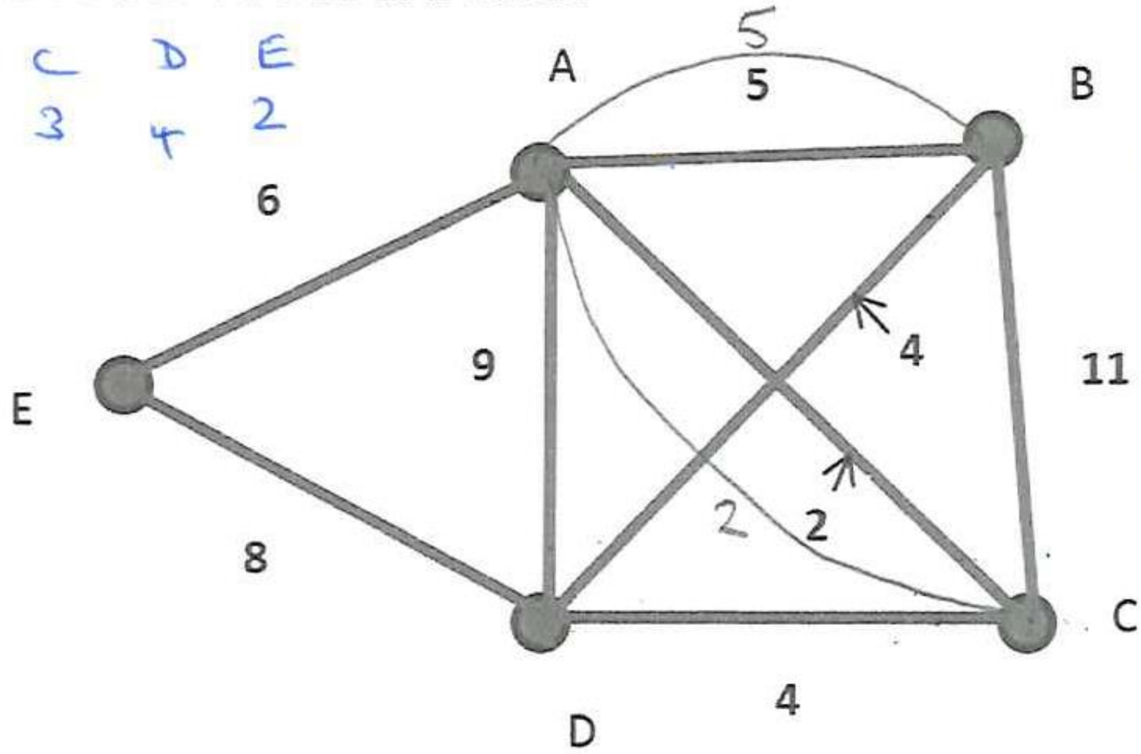
clockwise rotation 90° about (0,0)

A council employee needs to check the condition of the roads. To do this she needs to start at her office at A, travel down each road at least once, and return to her office. She wishes to travel the least possible distance. Find the distance she must travel, and one possible route she could take.

A	B	C	D	E
4	3	3	4	2

Total weight = <u>56</u>

Possible Route =
ABCACDBA..
..DEA.



The random variable X has a Poisson distribution believed to have a mean of 5. A single observation of X has the value 10. Test, at the 10% significance level, whether the mean is equal to 5.

$H_0: \lambda = 5$

$H_1: \lambda \neq 5$

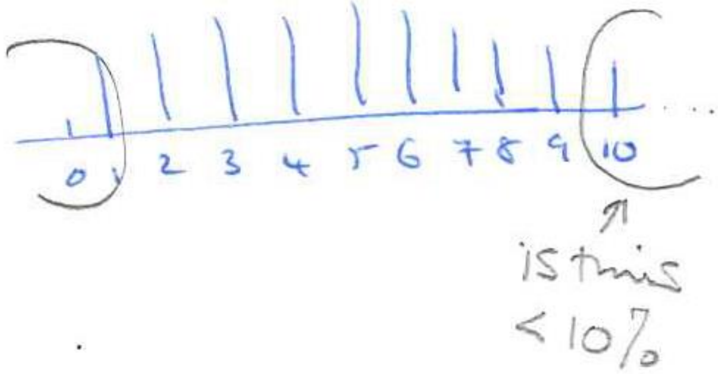
2 tail

$X \sim Po(5)$

$P(X \geq 10) = 1 - P(X \leq 9)$

$= 1 - 0.96817..$

$= 0.0318$



$0.0318 <$ ~~0.1~~ 0.05

∴ evidence to reject H_0, indicating that the pop^n mean may well not equal 5.

Sheet 19

The triangle T is rotated 90° anticlockwise around (0,0) and then the image T′ is reflected in the line y = x to obtain the triangle T″.

a) i) Find the matrix **P** such that **P**(T) = T′

ii) Find the matrix **Q** such that **Q**(T′) = T″

b) By finding a matrix product, find the single matrix that will perform a 90° anticlockwise rotation followed by a reflection in y = x

There are five mathematicians who are members of a committee

Newton (N), Euler (E), Descartes (D), Pythagoras (P) and Archimedes (A).

Use a bubble sort algorithm to rearrange the names into alphabetical order, showing the new arrangement after each comparison.

The points A, B and C have coordinates $(2,-1,1)$, $(5,1,7)$ and $(6,-3,1)$ respectively.

a) Find $\overrightarrow{AB}.\overrightarrow{AC}$

b) Hence, or otherwise, find the area of triangle ABC

In the past, an office printer has failed, on average, once every four weeks. A new, more expensive, printer is on trial. The manufacturer claims that it is more reliable. In the first 44 weeks of use, the new photocopier fails 5 times. Assuming that the failures of the printer occur independently and at random, test, at the 5% significance level, whether there is evidence that the new printer is more reliable than the old one.

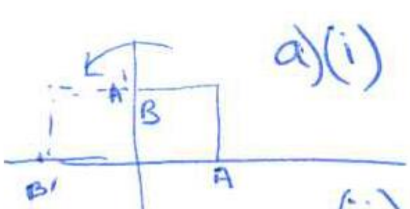
Sheet 19

The triangle T is rotated 90° anticlockwise around (0,0) and then the image T' is reflected in the line y = x to obtain the triangle T''.

a) i) Find the matrix P such that P(T) = T'

ii) Find the matrix Q such that Q(T') = T''

b) By finding a matrix product, find the single matrix that will perform a 90° anticlockwise rotation followed by a reflection in y = x

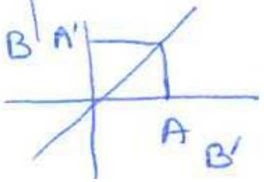


a)(i) $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = P$

b) $QP = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

$= \underline{\underline{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}}$

(ii)



$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = Q.$

The points A, B and C have coordinates $(2,-1,1)$, $(5,1,7)$ and $(6,-3,1)$ respectively.

a) Find $\overrightarrow{AB}.\overrightarrow{AC}$

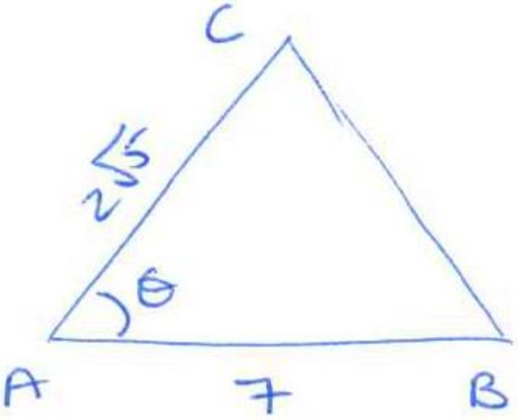
$$\overrightarrow{AB} = \begin{pmatrix} 5-2 \\ 1+1 \\ 7-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} 6-2 \\ -3+1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AB}.\overrightarrow{AC} = 12 - 4 = 8$$

b) Hence, or otherwise, find the area of triangle ABC

$$|AB| = \sqrt{3^2+2^2+6^2} = \sqrt{49} = 7$$

$$|AC| = \sqrt{4^2+2^2} = \sqrt{20} = 2\sqrt{5}.$$



$$\cos\theta = \frac{8}{14\sqrt{5}} \qquad \theta = 75.193\ldots$$

$$\text{Area} = \frac{1}{\cancel{2}} \times \cancel{2}\sqrt{5} \times 7 \times \sin(\theta)$$

$$= \underline{\underline{15.1 \text{ units}^2}} \ (3sf).$$

There are five mathematicians who are members of a committee

Newton (N), Euler (E), Descartes (D), Pythagoras (P) and Archimedes (A).

Use a bubble sort algorithm to rearrange the names into alphabetical order, showing the new arrangement after each comparison.

N E D P A
E – N
D – N
(N P)
A – P
e.o. pass1 → E D N A P

D – E
(E N)
A – N
(N P)
e.o. pass2 → D E A N P

(D E)
A – E
(E N)
(N P)
D A E N P ← e.o. pass3

A – D
(D E)
(E N)
(N P)
A D E N P e.o. pass4

A D E N P e.o. pass5

In the past, an office printer has failed, on average, once every four weeks. A new, more expensive, printer is on trial. The manufacturer claims that it is more reliable. In the first 44 weeks of use, the new photocopier fails 5 times. Assuming that the failures of the printer occur independently and at random, test, at the 5% significance level, whether there is evidence that the new printer is more reliable than the old one.

$H_0: \lambda = 11$

$H_1: \lambda < 11$

one tail.

1 in 4 weeks = 11 in 44 weeks.

$X \sim Po(11)$

$P(X \leq 5) = 0.0375$ (3sf)

$0.0375 < 0.05$

$\therefore$ reject H_0.

Evidence suggests the new printer is more reliable.

Sheet 20

Express the following calculation in the form x + iy:

$$2\left(cos\frac{\pi}{15}+isin\frac{\pi}{15}\right)\times 3\left(cos\frac{2\pi}{5}-isin\frac{2\pi}{5}\right)$$

HINT: $z_1z_2 = r_1r_2(cos(\theta_1+\theta_2)+isin(\theta_1+\theta_2))$

HINT : cos(-θ) = cosθ and sin(-θ) = -sinθ

Draw the network corresponding to this distance matrix

	A	B	C	D
A	-	14	11	-
B	14	-	7	-
C	11	7	-	20
D	-	-	20	-

Find a formula for the sum of the series:

$$\sum_{r=1}^{n} r(r+3)(2r-1)$$

A company manufactures 60-watt light bulbs and, under normal conditions, 5% of the light bulbs are faulty. They are packed in boxes of 280. A box that is randomly chosen on a random day has 20 faulty light bulbs in. Using a Poisson approximation to the binomial distribution and a 5% level of significance, test whether the percentage of faulty light bulbs on that day is different from 5%.

Express the following calculation in the form x + iy:

Sheet 2

$$2\left(cos\frac{\pi}{15}+isin\frac{\pi}{15}\right)\times 3\left(cos\frac{2\pi}{5}-isin\frac{2\pi}{5}\right)$$

HINT: $z_1z_2=r_1r_2\left(cos(\theta_1+\theta_2)+isin(\theta_1+\theta_2)\right)$

HINT : cos(-θ) = cosθ and sin(-θ) = -sinθ

$$2\left(\cos\frac{\pi}{15}+i\sin\frac{\pi}{15}\right)\times 3\left(\cos\left(-\frac{2\pi}{5}\right)+i\sin\left(-\frac{2\pi}{5}\right)\right)$$

$$=2\times 3\left(\cos\left(\frac{\pi}{15}-\frac{2\pi}{5}\right)+i\sin\left(\frac{\pi}{15}-\frac{2\pi}{5}\right)\right)$$

$$=6\left(\cos\left(-\pi/3\right)+i\sin\left(-\pi/3\right)\right)$$

$$=6\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=3-\sqrt{3}i.$$

Find a formula for the sum of the series:

$$\sum_{r=1}^{n} r(r+3)(2r-1) = \sum r\left(2r^2+5r-3\right)$$

$$= \sum 2r^3+5r^2-3r = 2\sum r^3 + 5\sum r^2 - 3\sum r$$

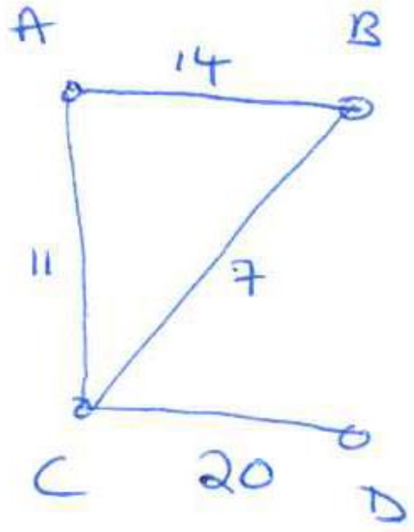
$$= \frac{1}{2}\cancel{\frac{2}{4}}\, n^2(n+1)^2 + \frac{5}{6}\, n(n+1)(2n+1) - \frac{3}{2}\, n(n+1)$$

$$= \frac{1}{6}\, n(n+1)\left[\, 3n(n+1) + 5(2n+1) - 9 \,\right]$$

$$= \frac{1}{6}\, n(n+1)\left[\, 3n^2+3n + 10n + 5 - 9 \,\right]$$

$$= \frac{1}{6}\, n(n+1)\left(3n^2+10n-4\right)$$

Draw the network corresponding to this distance matrix



	A	B	C	D
A	-	14	11	-
B	14	-	7	-
C	11	7	-	20
D	-	-	20	-

A company manufactures 60-watt light bulbs and, under normal conditions, 5% of the light bulbs are faulty. They are packed in boxes of 280. A box that is randomly chosen on a random day has 20 faulty light bulbs in. Using a Poisson approximation to the binomial distribution and a 5% level of significance, test whether the percentage of faulty light bulbs on that day is different from 5%.

2-tail (2.5%)

5% of 280 = 14 $X \sim Po(14)$

$H_0: p = 0.05$

$H_1: p \neq 0.05$

↑

% faulty.

$P(X \geq 20) = 1 - P(X \leq 19)$

$= 0.0765$

$0.0765 > 0.025$

∴ accept H_0. Insufficient evidence to suggest the % of faulty lightbulbs is different from 5%.

Sheet 21

The plane Π has equation:

$$\boldsymbol{r}.(\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k})=5$$

The point P has coordinates:

$$(1,3,-2)$$

Find the shortest distance between P and Π

Use the quick sort algorithm to sort these letters into alphabetical order, showing your pivots clearly at each stage

G, A, Z, C, M, T, B

Given

$$\sum_{r=1}^{n} r(r+3)(2r-1) = \frac{n(n+1)(3n^2+13n-4)}{6}$$

Calculate the following:

$$\sum_{r=11}^{40} r(r+3)(2r-1)$$

A company claims that it receives emails at a mean rate of four every 10 minutes.

(a) Using a 5% level of significance, find the critical region for a two-tailed test of the hypothesis that the mean number of emails received in a 10 minute period is not 4. The probability of rejection in each tail should be as close as possible to 0.025, but not larger.

(b) Find the actual level of significance of this test.

Sheet 21

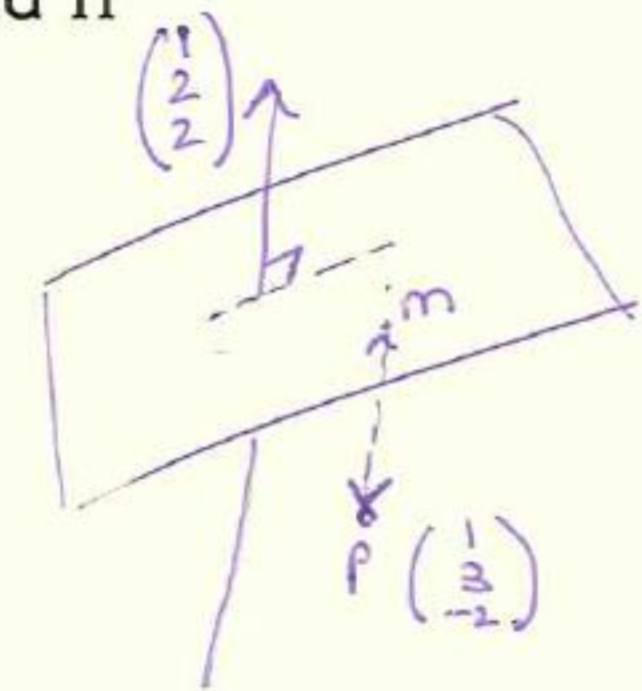
The plane Π has equation:

$$\boldsymbol{r}.(\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k})=5$$

The point P has coordinates:

$$(1,3,-2)$$

Find the shortest distance between P and Π



$$\underline{m}=\begin{pmatrix}1+\lambda\\3+2\lambda\\-2+2\lambda\end{pmatrix}\overset{*}{=}\begin{pmatrix}11/9\\31/9\\-14/9\end{pmatrix}$$

$$\underline{m}.\underline{r}=5\Rightarrow\begin{pmatrix}1+\lambda\\3+2\lambda\\-2+2\lambda\end{pmatrix}.\begin{pmatrix}1\\2\\2\end{pmatrix}=5$$

$$1+\lambda+6+4\lambda-4+4\lambda=5$$

$$9\lambda+3=5$$

$$9\lambda=2$$

$$\lambda=2/9^{*}$$

$$* \ \overrightarrow{Pm}=\begin{pmatrix}11/9-1\\31/9-3\\-14/9+2\end{pmatrix}=\begin{pmatrix}2/9\\4/9\\4/9\end{pmatrix}$$

$$|Pm|=\sqrt{\left(\tfrac{1}{9}\right)^2\left(2^2+4^2+4^2\right)}=\sqrt{\tfrac{4}{9}}\rightarrow\boxed{\tfrac{2}{3}}$$

Given

$$\sum_{r=1}^{n} r(r+3)(2r-1) = \frac{n(n+1)(3n^2+13n-4)}{6}$$

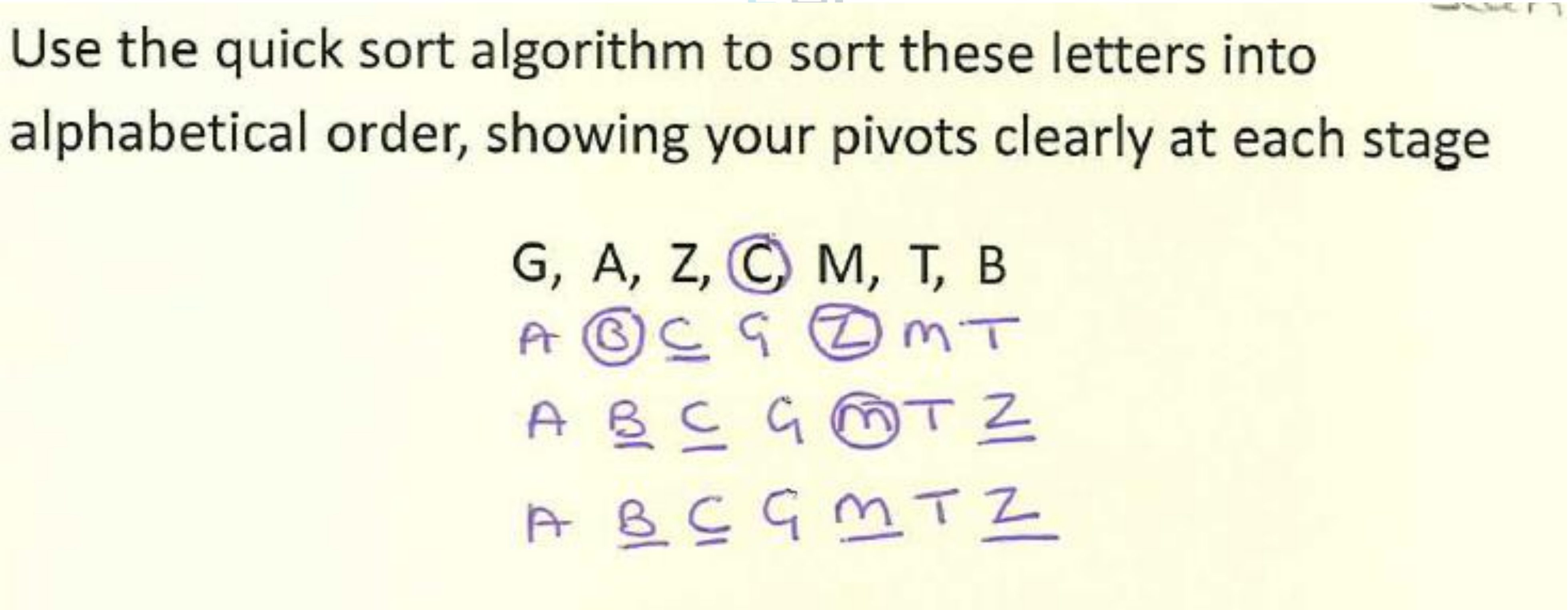
Calculate the following:

$$\text{inc. 11th} \rightarrow \sum_{r=11}^{40} r(r+3)(2r-1) = \sum_{1}^{40} - \sum_{1}^{10}$$

$$\sum_{1}^{40} = \frac{40(41)\left(3\times 40^2 + 13(40) - 4\right)}{6} = \frac{40(41)(5316)}{6} = 1453040$$

$$\sum_{1}^{10} = \frac{10(11)(300 + 130 - 4)}{6} = \frac{10(11)(426)}{6} = 7810$$

$$\sum_{11}^{40} = 1453040 - 7810 = \underline{\underline{1{,}445{,}230}}$$



Use the quick sort algorithm to sort these letters into alphabetical order, showing your pivots clearly at each stage

G, A, Z, C, M, T, B

A B C G Z M T

A B C G M T Z

A B C G M T Z

A company claims that it receives emails at a mean rate of four every 10 minutes.

(a) Using a 5% level of significance, find the critical region for a two-tailed test of the hypothesis that the mean number of emails received in a 10 minute period is not 4. The probability of rejection in each tail should be as close as possible to 0.025, but not larger. critical region $X=0 \cup X \geq 9$

(b) Find the actual level of significance of this test. $= 0.0183 + 0.0214$

$= 3.97\%$

$X \sim Po(4)$

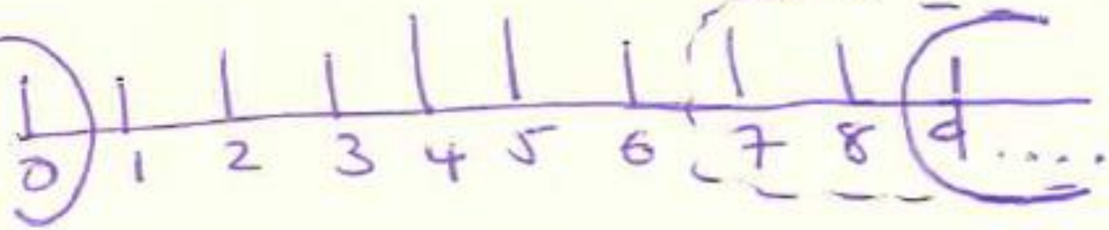
$H_0 : \lambda = 4$

$H_1 : \lambda \neq 4$

$P(X \geq 7) = 1 - P(X \leq 6)$

$= 1 - 0.889\ldots$

$= 0.11067 > 0.025$



$P(X \geq 9) = 1 - P(X \leq 8)$

$= 0.0214$ ✓

$P(X=0) = 0.0183 < 0.025$ ✓

$P(X \geq 8) = 0.0511$

$P(X \leq 1) = 0.092 > 0.025.$

Determine whether each of the following can be evaluated and if so, find the product:

Sheet 22

$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$

$\boldsymbol{B} = \begin{bmatrix} 3 & -2 \end{bmatrix}$

$\boldsymbol{C} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

a) **AB**

b) **BC**

c) **CA**

d) **BCA**

Draw a semi-Eulerian graph with 6 nodes

Draw a graph with 6 nodes and no odd nodes that is ***not*** Eulerian

Find, in terms of n:

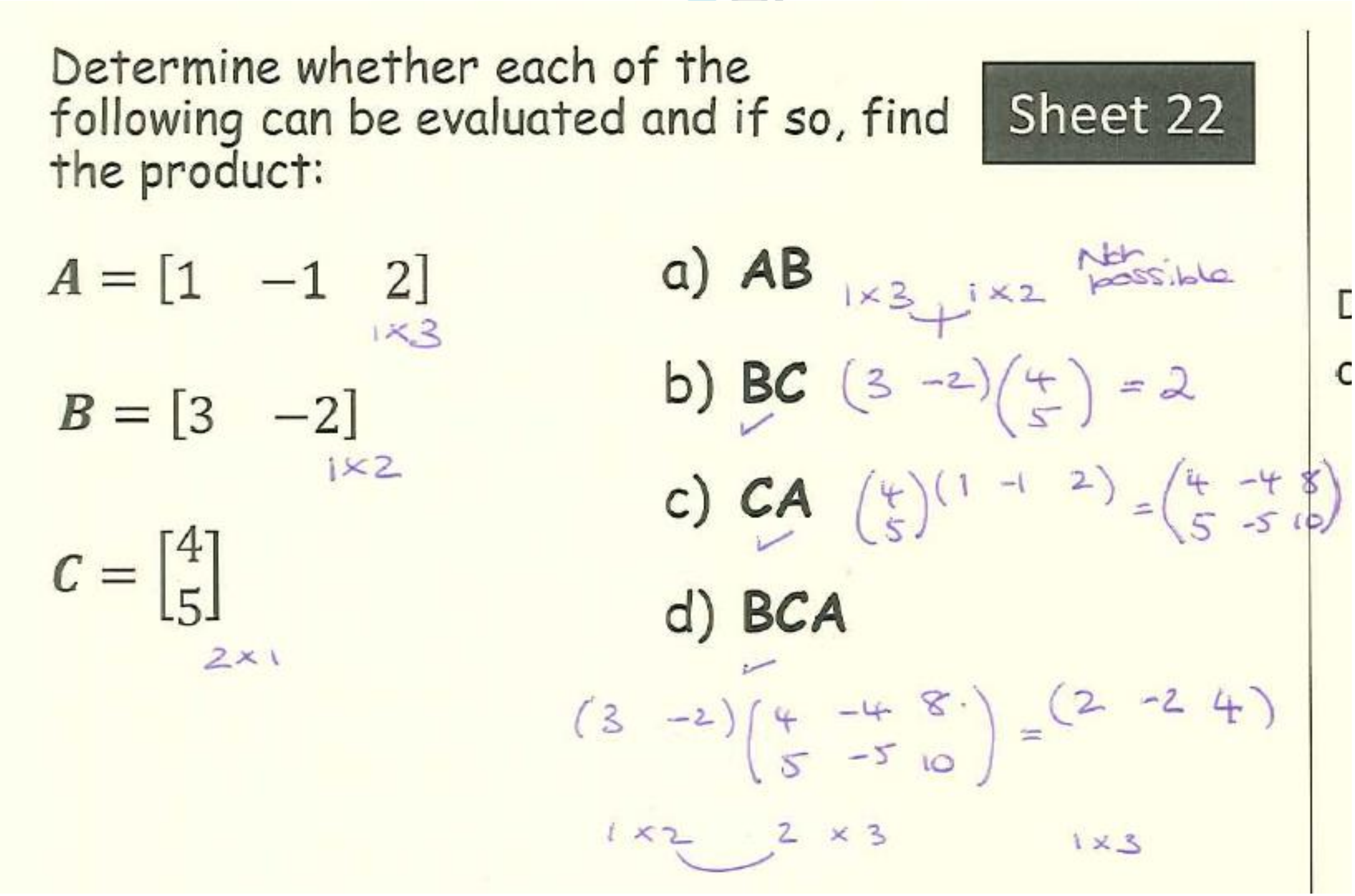
$$\sum_{r=n+1}^{2n} r^2$$

It is believed that the number of errors in a page of a manuscript word-processed by the school's secretary has a Poisson distribution with mean 1.4.

(a) Using a 5% level of significance, find the critical region for a one-tailed test of the hypothesis that the mean number of errors in a page of a manuscript word-processed by the school’s secretary is more than 1.4.

(b) Find the actual level of significance of this test.

(c) On a particular day, the headmaster counted 4 errors on a manuscript word-processed by the school's secretary. Comment on this observation in light of your critical region.



Determine whether each of the following can be evaluated and if so, find the product:

Sheet 22

$A = [1 \quad -1 \quad 2]$

1×3

$B = [3 \quad -2]$

1×2

$C = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

2×1

a) **AB** 1×3 1×2 Not possible

b) **BC** ✓ $\begin{pmatrix} 3 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = 2$

c) **CA** ✓ $\begin{pmatrix} 4 \\ 5 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -4 & 8 \\ 5 & -5 & 10 \end{pmatrix}$

d) **BCA** ✓

$$\underset{1\times 2}{\begin{pmatrix} 3 & -2 \end{pmatrix}}\underset{2\times 3}{\begin{pmatrix} 4 & -4 & 8 \\ 5 & -5 & 10 \end{pmatrix}} = \underset{1\times 3}{\begin{pmatrix} 2 & -2 & 4 \end{pmatrix}}$$

Find, in terms of n:

$$\sum_{r=n+1}^{2n} r^2$$

$$\sum_{n+1}^{2n} r^2 = \sum_{1}^{2n} r^2 - \sum_{1}^{n} r^2$$

$$= \cancel{\tfrac{1}{2}(2n)(2n+1) - \tfrac{1}{2}n(n+1)}$$

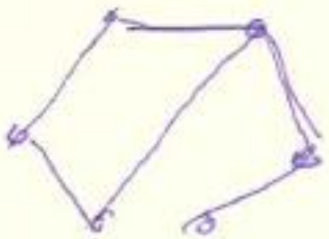
$$= \underline{\tfrac{1}{6}}(2\underline{n})\underline{(2n+1)}(2(2n)+1) - \underline{\tfrac{1}{6}}\,\underline{n}(n+1)\underline{(2n+1)}$$

$$= \tfrac{1}{6}n(2n+1)\left[2(4n+1) - (n+1)\right]$$

$$= \tfrac{1}{6}n(2n+1)(8n+2-n-1)$$

$$= \tfrac{1}{6}n(2n+1)(7n+1).$$

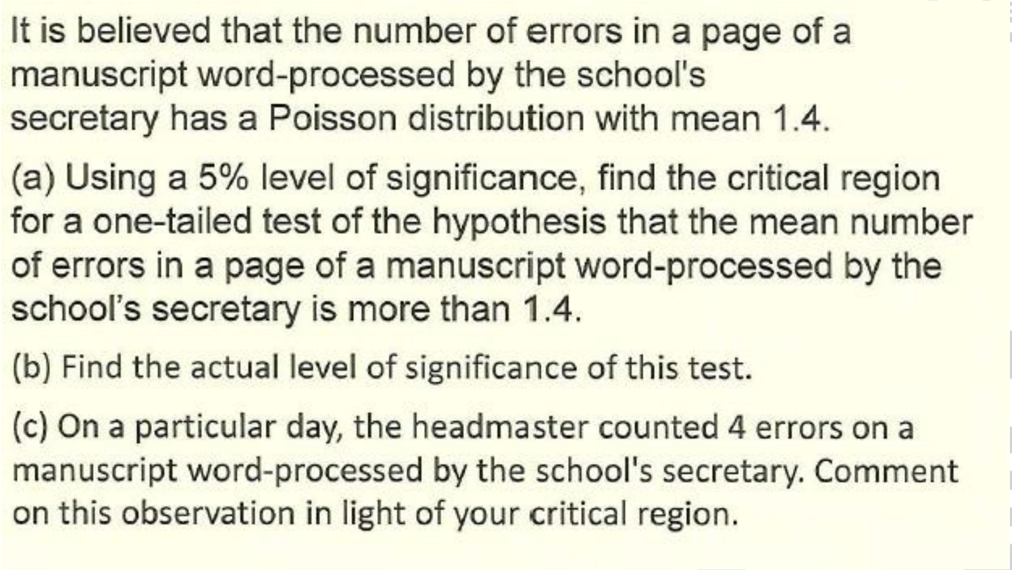
Draw a semi-Eulerian graph with 6 nodes



Many possible correct answers to this. Should have 6 nodes: 2 of odd order , 4 of even order.

Draw a graph with 6 nodes and no odd nodes that is *not* Eulerian

This would have to be a disconnected graph in, say, two parts.



It is believed that the number of errors in a page of a manuscript word-processed by the school's secretary has a Poisson distribution with mean 1.4.

(a) Using a 5% level of significance, find the critical region for a one-tailed test of the hypothesis that the mean number of errors in a page of a manuscript word-processed by the school's secretary is more than 1.4.

(b) Find the actual level of significance of this test.

(c) On a particular day, the headmaster counted 4 errors on a manuscript word-processed by the school's secretary. Comment on this observation in light of your critical region.

a) critical region: $x \geq 5$.

b) act sig level = 1.4%

c) 4 is not in the critical region

$\therefore$ no evidence to reject H_0.

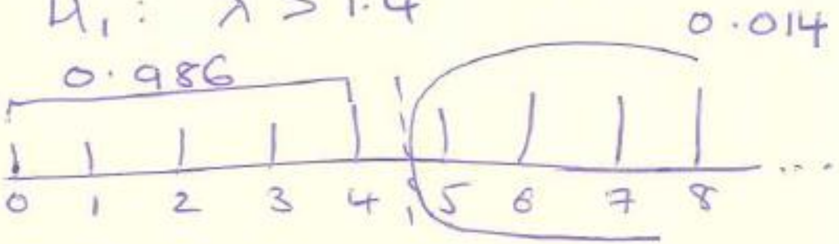
Accept that secretary's error rate remains at 1.4 per page.

X = no. errors 1 page

$X \sim Po(1.4)$

$H_0: \lambda = 1.4$

$H_1: \lambda > 1.4$



$P(X \geq x) < 0.05$

$1 - P(X \leq x) < 0.05$

$0.95 < P(X \leq x)$

$P(X \leq 4) = 0.986$ ok

$P(X \leq 3) = 0.946$ not ok.

$P(X \geq 5) = 1 - 0.986 = 0.014$

If: $\quad arg z = \frac{\pi}{4}$

Sketch the locus of P(x,y) which is represented by z on an Argand diagram. Then find the Cartesian equation of this locus algebraically.

Sheet 23

Nine pieces of wood are required to build a small cabinet. The lengths, in cm, of the pieces of wood are listed below.

20, 20, 20, 35, 40, 50, 60, 70, 75

Planks, one metre in length, can be purchased at a cost of £3 each.

Use the first fit algorithm to determine how many of these planks are to be purchased to make this cabinet. Find the total cost and the amount of wood wasted.

The line l has equation:
$$\frac{x-1}{2} = \frac{y-1}{-2} = \frac{z+3}{-1}$$

The point A has coordinates $(1,2,-1)$

a) Find the shortest distance between A and l.

b) Find a Cartesian equation of the line that is perpendicular to l, and passes through A.

During winter months, the number of emergency calls received by a power company occur randomly at a uniform rate of 6 per day. They believe that the rate of calls has changed recently. To test this, the number of incoming calls during a 3-day period is recorded.

(a) Using a 5% level of significance, find the critical region for a two-tailed test of this hypothesis.

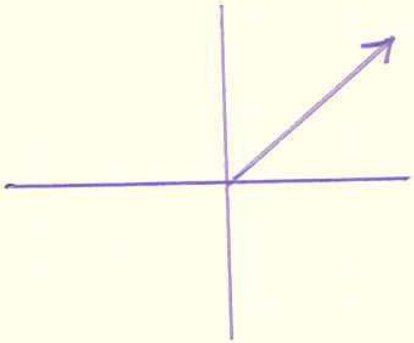
(b) Find the actual level of significance of this test.

(c) The actual number of calls recorded over the 3-day period was 9. Comment on this observation in light of your critical region.

If: $arg z = \frac{\pi}{4}$

Sheet 23

Sketch the locus of P(x,y) which is represented by z on an Argand diagram. Then find the Cartesian equation of this locus algebraically.



$y = x$

$x > 0$

(it's a half line)

The line l has equation: $$\frac{x-1}{2}=\frac{y-1}{-2}=\frac{z+3}{-1}$$

The point A has coordinates $(1,2,-1)$

a) Find the shortest distance between A and l.

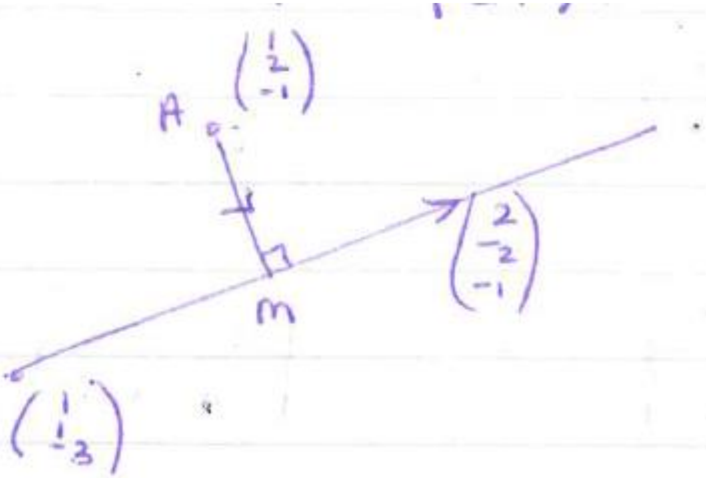
b) Find a Cartesian equation of the line that is perpendicular to l, and passes through A.

$x-1=2\lambda \qquad y-1=-2\lambda \qquad z+3=-\lambda$

$x=1+2\lambda \qquad y=1-2\lambda \qquad z=-3-\lambda$

$$r=\begin{pmatrix}1\\1\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\-1\end{pmatrix}$$

a)



$$m=\begin{pmatrix}1+2\lambda\\1-2\lambda\\-3-\lambda\end{pmatrix}$$

$$\overrightarrow{Am}=\begin{pmatrix}1+2\lambda\\1-2\lambda\\-3-\lambda\end{pmatrix}-\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}2\lambda\\-1-2\lambda\\-2-\lambda\end{pmatrix}$$

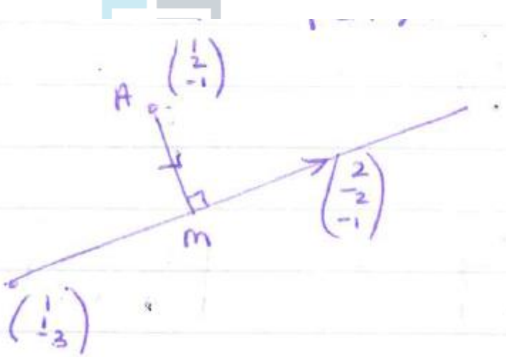
The line l has equation: $\frac{x-1}{2}=\frac{y-1}{-2}=\frac{z+3}{-1}$

The point A has coordinates $(1,2,-1)$

a) Find the shortest distance between A and l.

b) Find a Cartesian equation of the line that is perpendicular to l, and passes through A.

a)



$$m=\begin{pmatrix}1+2\lambda\\1-2\lambda\\-3-\lambda\end{pmatrix}$$

$$\overrightarrow{Am}=\begin{pmatrix}1+2\lambda\\1-2\lambda\\-3-\lambda\end{pmatrix}-\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}2\lambda\\-1-2\lambda\\-2-\lambda\end{pmatrix}$$

$$Am\cdot\begin{pmatrix}2\\-2\\-1\end{pmatrix}=0 \Rightarrow \begin{pmatrix}2\lambda\\-1-2\lambda\\-2-\lambda\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\-1\end{pmatrix}=0$$

$$4\lambda+2+4\lambda+2+\lambda=0$$

$$9\lambda+4=0$$

$$9\lambda=-4$$

$$\lambda=-4/9 \rightarrow \overrightarrow{Am}=\begin{pmatrix}-8/9\\-1/9\\-10/9\end{pmatrix}=\frac{1}{9}\begin{pmatrix}-8\\-1\\-10\end{pmatrix}$$

$$|Am|=\sqrt{\left(\tfrac{1}{9}\right)^2(8^2+1^2+10^2)}=\sqrt{\frac{55}{27}}=\frac{\sqrt{165}}{9}\quad(=1.427\ldots)$$

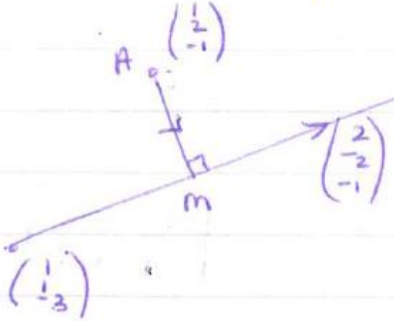
The line l has equation: $$\frac{x-1}{2}=\frac{y-1}{-2}=\frac{z+3}{-1}$$

The point A has coordinates $(1,2,-1)$

a) Find the shortest distance between A and l.

b) Find a Cartesian equation of the line that is perpendicular to l, and passes through A.

a)



$$m=\begin{pmatrix}1+2\lambda\\1-2\lambda\\-3-\lambda\end{pmatrix}$$

$$\overrightarrow{Am}=\begin{pmatrix}1+2\lambda\\1-2\lambda\\-3-\lambda\end{pmatrix}-\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}2\lambda\\-1-2\lambda\\-2-\lambda\end{pmatrix}$$

$$\overrightarrow{Am}=\begin{pmatrix}-8/9\\-1/9\\-10/9\end{pmatrix}$$

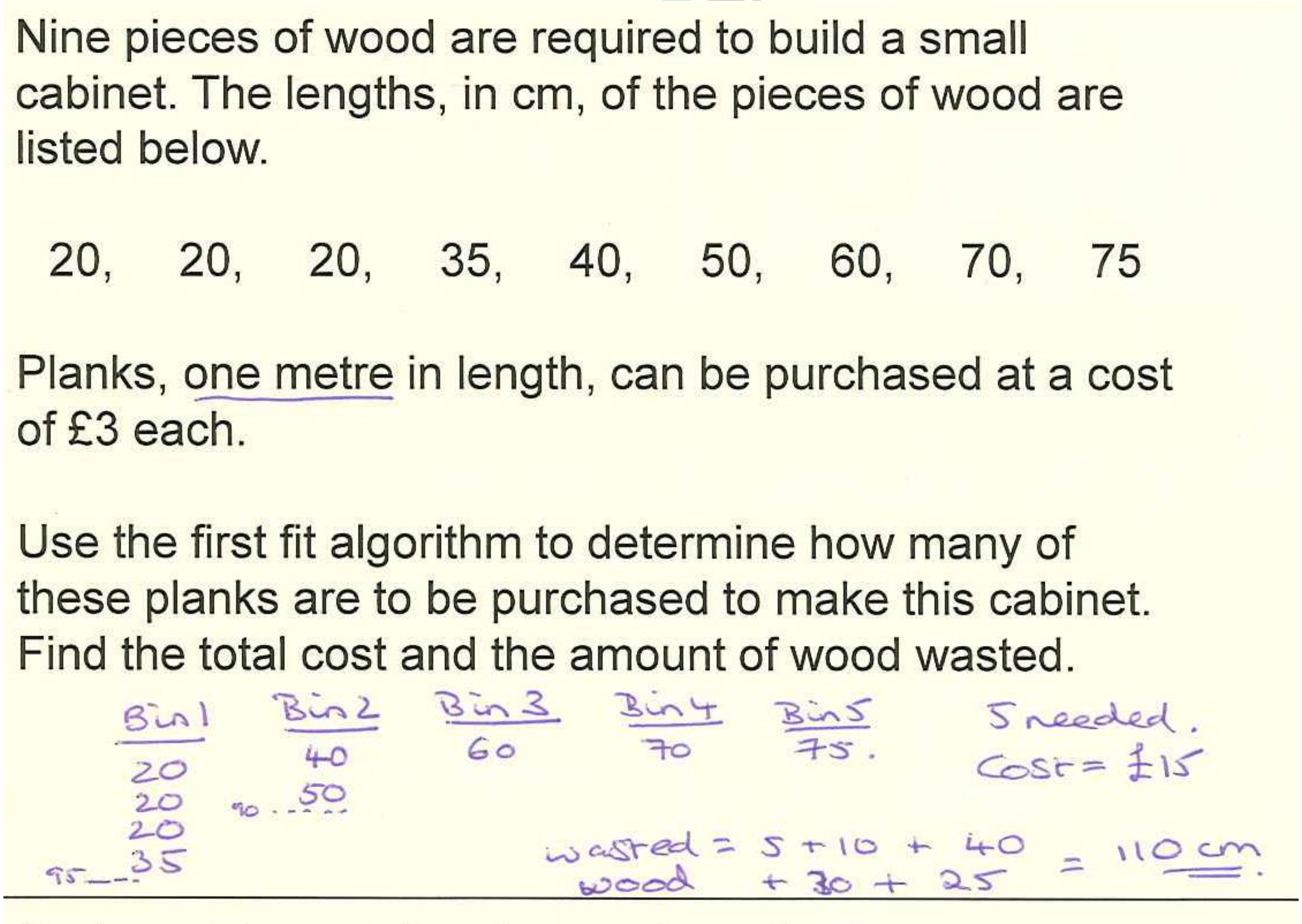
b) l_A: $\begin{pmatrix}1\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}8\\1\\10\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$

$x=1+8\lambda$ $\quad 2+\lambda=y$ $\quad -1+10\lambda=z$

$x-1=8\lambda$ $\quad \frac{y-2}{1}=\lambda$ $\quad 10\lambda=z+1$

$\frac{x-1}{8}=\lambda$ $\quad \lambda=\frac{z+1}{10}$

l_A: $\frac{x-1}{8}=\frac{y-2}{1}=\frac{z+1}{10}$



Nine pieces of wood are required to build a small cabinet. The lengths, in cm, of the pieces of wood are listed below.

20, 20, 20, 35, 40, 50, 60, 70, 75

Planks, one metre in length, can be purchased at a cost of £3 each.

Use the first fit algorithm to determine how many of these planks are to be purchased to make this cabinet. Find the total cost and the amount of wood wasted.

Bin 1	Bin 2	Bin 3	Bin 4	Bin 5
20	40	60	70	75.
20	90 .. 50			
20				
95 __ 35				

5 needed.
Cost = £15

wasted wood = 5 + 10 + 40 + 30 + 25 = 110 cm.

During winter months, the number of emergency calls received by a power company occur randomly at a uniform rate of 6 per day. They believe that the rate of calls has changed recently. To test this, the number of incoming calls during a 3-day period is recorded.

(a) Using a 5% level of significance, find the critical region for a two-tailed test of this hypothesis. $X \leq 9 \quad \cup \quad X \geq 27$

(b) Find the actual level of significance of this test. $1.5\% + 2.8\% = 4.3\%$

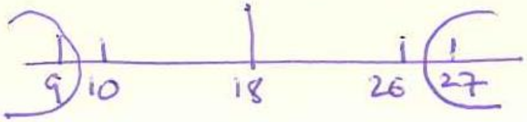
(c) The actual number of calls recorded over the 3-day period was 9. Comment on this observation in light of your critical region.

$X \sim Po(18)$

$H_0 : \lambda = 18$

$H_1 : \lambda \neq 18$

2.5% in each tail.



$P(X \leq 10) = 0.0303 \quad > 0.025$

$P(X \leq 9) = 0.015 \quad < 0.025$

$P(X \geq 26) = 0.044$

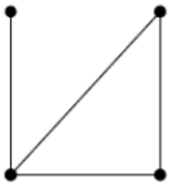
$P(X \geq 27) = 0.028$

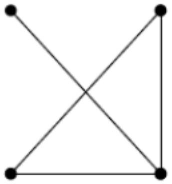
c) 9 is in the critical region ∴ evidence to reject Ho and conclude the number of calls per day has changed from 6.

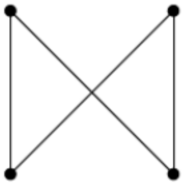
Sheet 24

Find the perpendicular distance from the point with coordinates $(3,2,-1)$ to the plane with equation $2x - 3y + z = 5$

Two of the three graphs below are isomorphic to each other. Which two?


A


B


C

For each of the matrices below, determine if they are singular and if they are not, find their inverse:

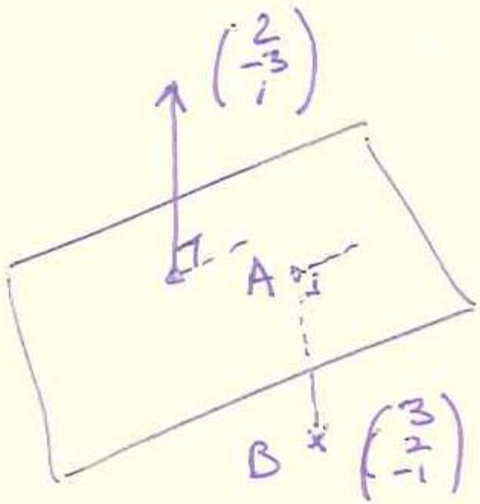
$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

A die is thrown 120 times. Carry out a hypothesis test at the significance level of 5% to see whether the data indicates that the die is fair.

Score	1	2	3	4	5	6
Observed Frequency	15	29	14	18	20	24

Sheet 24

Find the perpendicular distance from the point with coordinates $(3,2,-1)$ to the plane with equation $2x-3y+z=5$



~~$\overrightarrow{BA} = \begin{pmatrix}3\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\1\end{pmatrix} = \begin{pmatrix}3+2\lambda\\2-3\lambda\\-1+\lambda\end{pmatrix}$~~ not needed.

$\boxed{\text{use formula Book}}$

$$\text{distance} = \frac{|2(3)-3(2)+1(-1)-5|}{\sqrt{2^2+3^2+1^2}}$$

$$= \frac{6}{\sqrt{14}} = \frac{3\sqrt{14}}{7}.$$

$$\overset{n_1}{2x} - \overset{n_2}{3y} + \overset{n_3}{z} - \overset{d}{5} = 0$$

For each of the matrices below, determine if they are singular and if they are not, find their inverse:

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

$|A| = 3 - -2 = 5$

$|B| = 2 - 2 = 0$

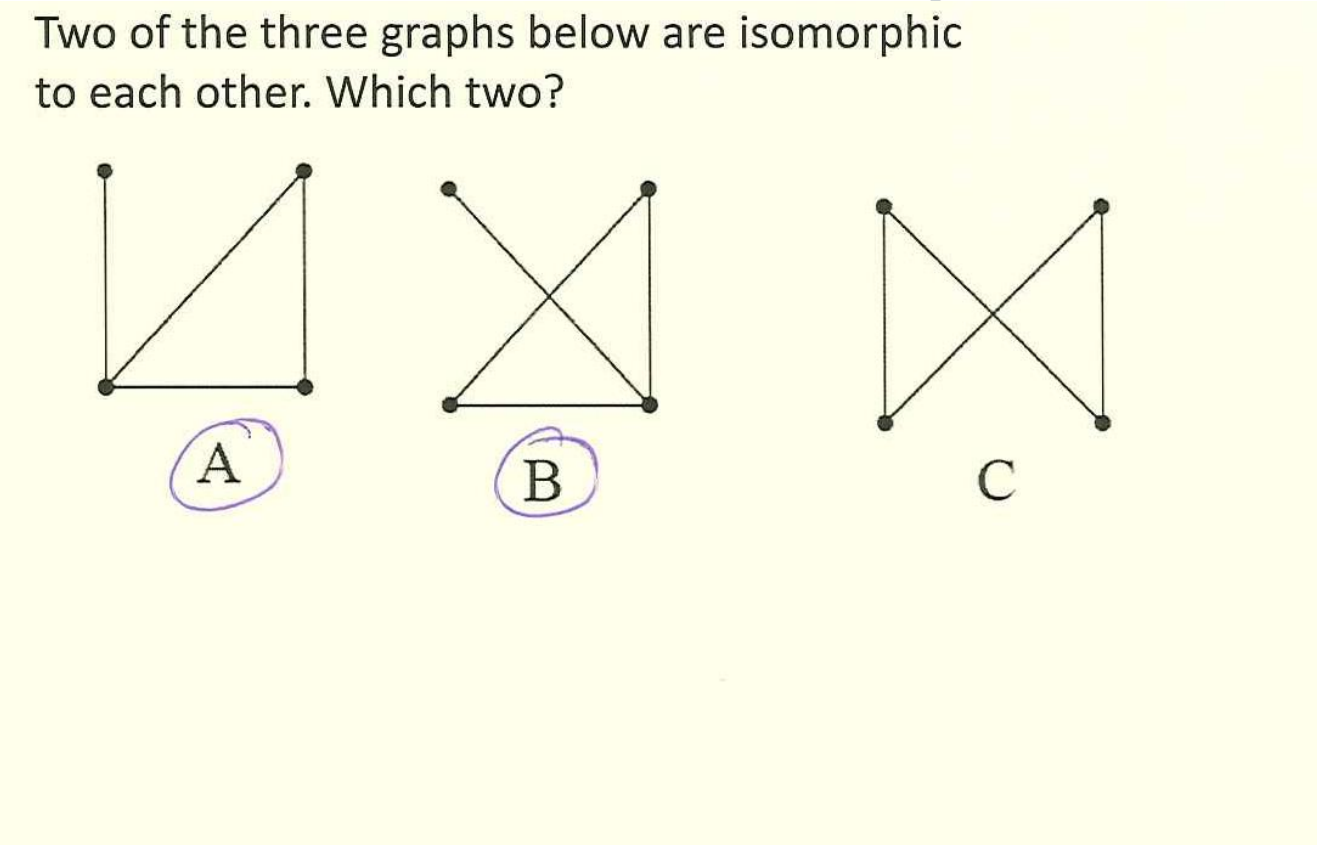
singular

no inverse

$|C| = -6$

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

$$C^{-1} = -\frac{1}{6}\begin{pmatrix} 0 & -3 \\ -2 & 1 \end{pmatrix}.$$



Two of the three graphs below are isomorphic to each other. Which two?

[✓ A] [✓ B] C

A die is thrown 120 times. Carry out a hypothesis test at the significance level of 5% to see whether the data indicates that the die is fair.

H_0 : uniform is good fit
H_1 : uniform is not a good fit

Score	1	2	3	4	5	6
Observed Frequency	15	29	14	18	20	24
Expected (E)	20	20	20	20	20	20
$\frac{(O-E)^2}{E}$	$\frac{25}{20}$	$\frac{81}{20}$	$\frac{36}{20}$	$\frac{4}{20}$	0	$\frac{16}{20}$

$\sum \frac{(O-E)^2}{E} = \frac{162}{20} = 8.1$ test statistic

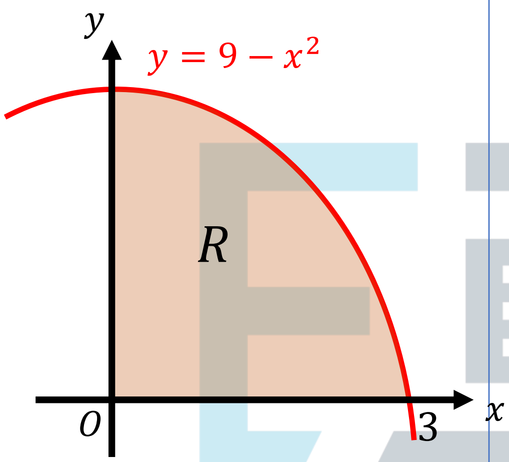
$\nu = 5$

$\chi^2_5(5\%) = 11.07$ c.v.

$8.1 < 11.07$
So the uniform distribution is a good fit.
Accept H_1. Conclude dice is fair.

Sheet 25

The diagram shows the region R which is bounded by the x-axis, the y-axis and the curve with equation $y = 9 - x^2$. The region is rotated through 360° about the x-axis. Find the exact volume of the solid generated.



The list of numbers below is to be sorted into **ascending** order. Perform a Quick Sort to obtain the sorted list, giving the state of the list after each pass, indicating the pivot elements.

45 32 51 75 56 47 61 70 28

Evaluate, using known results:

$$\sum_{r=1}^{3}(10r - 1)$$

and:

$$\sum_{r=1}^{25}(3r + 1)$$

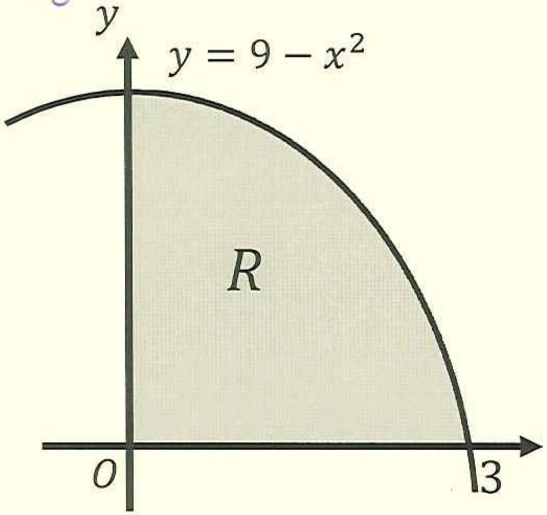
In genetic work it is predicted that the children with both parents of blood group AB will fall into blood groups AB, A, and B in the ratio of 2:1:1. Of a random sample of 100 such children 55 were blood group AB, 27 blood group A and 18 blood group B. Test at the 10% significance level whether the observed results agree with the theoretical prediction.

Sheet 25

The diagram shows the region R which is bounded by the x-axis, the y-axis and the curve with equation $y = 9 - x^2$. The region is rotated through 360° about the x-axis. Find the exact volume of the solid generated.

$$V = \int_0^3 \pi(9-x^2)^2\,dx$$

$$= \pi\int_0^3 81 - 18x^2 + x^4\,dx$$



$$V = \pi\int_a^b y^2\,dx$$

$$= \pi\int_0^3 (9-x^2)^2\,dx$$

$$= \pi\int_0^3 81 - 18x^2 + x^4\,dx$$

$$= \pi\left[81x - \frac{\overset{6}{\cancel{18}}x^3}{\cancel{3}} + \frac{x^5}{5}\right]_0^3$$

$$= \pi\left\{\left[243 - 162 + \frac{243}{5}\right] - [0]\right\}$$

$$= \frac{\pi . 648}{5}$$

$$= 129.6\,\pi$$

$$= 407 \text{ units}^3 \ (3sf)$$

__But__ wanted exact volume

$$\text{So } V = \frac{648\pi}{5}.$$

Evaluate, using known results: and:

1) $$\sum_{r=1}^{3}(10r-1)$$

2) $$\sum_{r=1}^{25}(3r+1)$$

1) $= 10\sum r - \sum 1$

$= 10\left(\frac{1}{2}n(n+1)\right) - n(1)$

$= \frac{10}{2}(3)(4) - 3$

$= 5(3)(4) - 3$

$= 60 - 3$

$= 57$

Check $9 + 19 + 29$ ✓

2) $3\sum r + \sum 1$

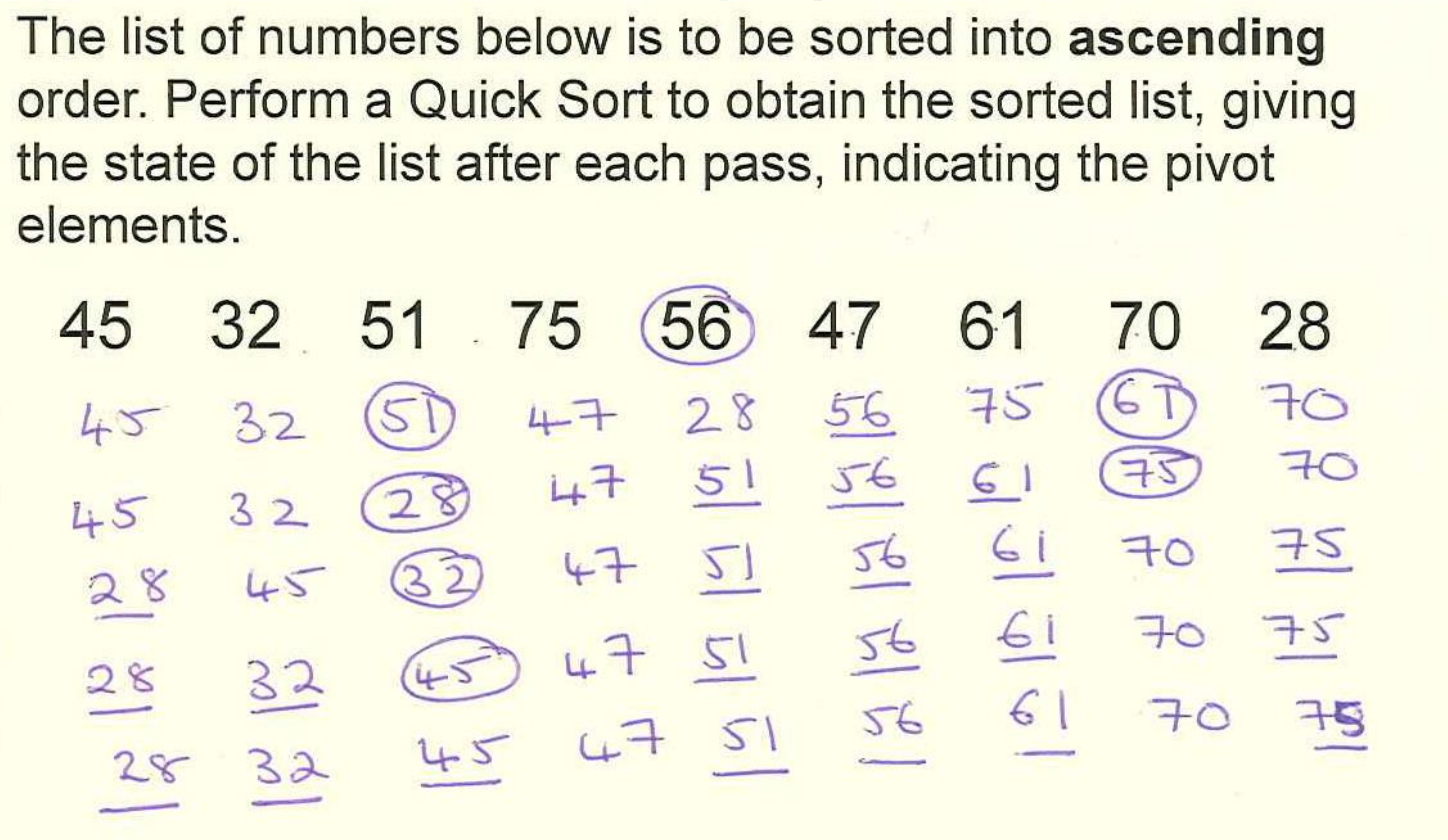
$= 3 \cdot \frac{1}{2}n(n+1) + n(1)$

$= \frac{3}{2}\, 25(26) + 25(1)$

$= \underline{1000}.$

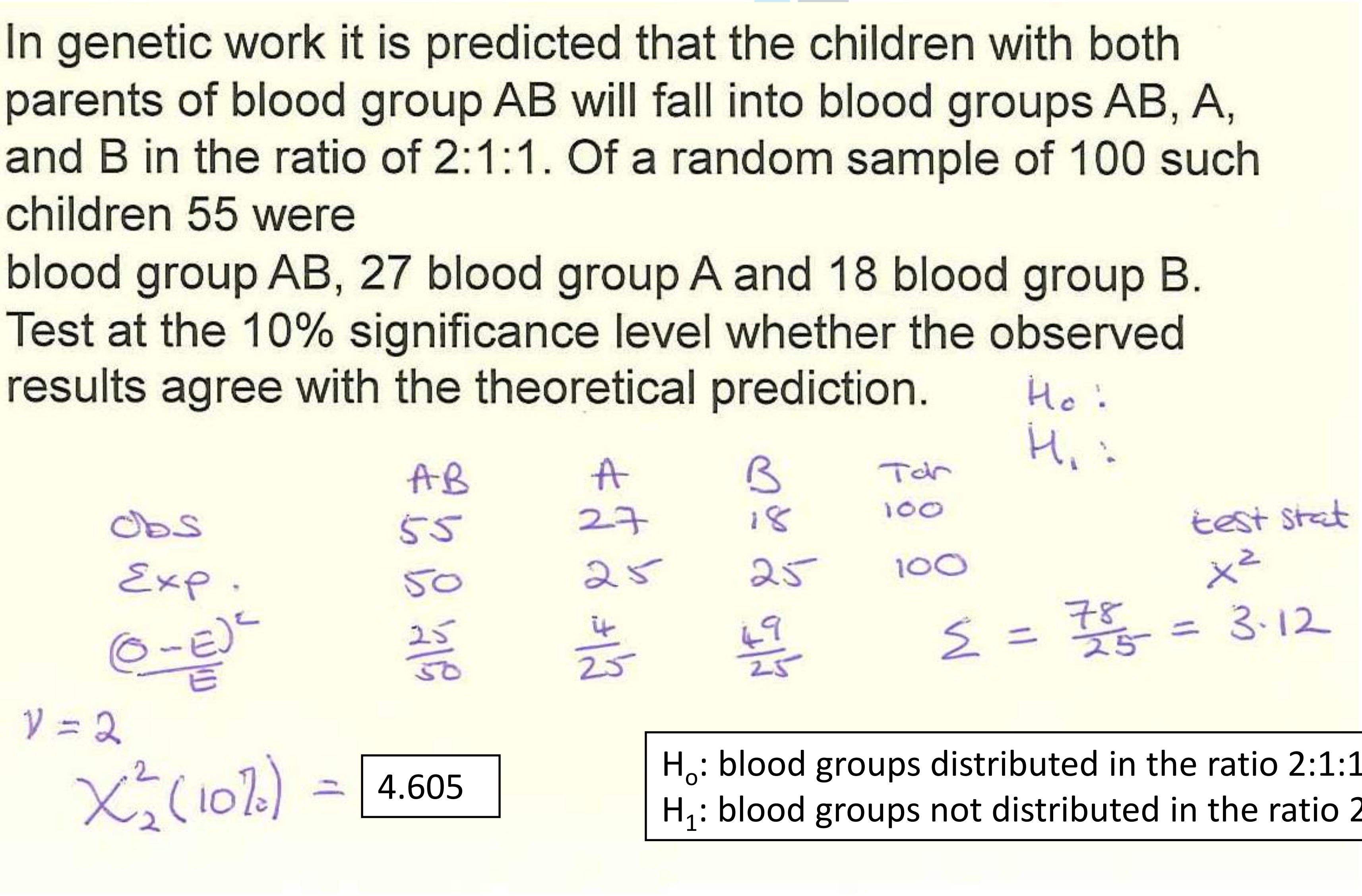
Check using $\sum$ button:

✓



The list of numbers below is to be sorted into **ascending** order. Perform a Quick Sort to obtain the sorted list, giving the state of the list after each pass, indicating the pivot elements.

45	32	51	75	$\boxed{56}$	47	61	70	28
45	32	$\boxed{51}$	47	28	$\underline{56}$	75	$\boxed{61}$	70
45	32	$\boxed{28}$	47	$\underline{51}$	$\underline{56}$	$\underline{61}$	$\boxed{75}$	70
$\underline{28}$	45	$\boxed{32}$	47	$\underline{51}$	$\underline{56}$	$\underline{61}$	70	$\underline{75}$
$\underline{28}$	$\underline{32}$	$\boxed{45}$	47	$\underline{51}$	$\underline{56}$	$\underline{61}$	70	$\underline{75}$
$\underline{28}$	$\underline{32}$	$\underline{45}$	47	$\underline{51}$	$\underline{56}$	$\underline{61}$	70	$\underline{75}$



In genetic work it is predicted that the children with both parents of blood group AB will fall into blood groups AB, A, and B in the ratio of 2:1:1. Of a random sample of 100 such children 55 were

blood group AB, 27 blood group A and 18 blood group B. Test at the 10% significance level whether the observed results agree with the theoretical prediction.

H_0 :

H_1 :

	AB	A	B	Tot
Obs	55	27	18	100
Exp.	50	25	25	100
$\frac{(O-E)^2}{E}$	$\frac{25}{50}$	$\frac{4}{25}$	$\frac{49}{25}$	

test stat

χ^2

$\Sigma = \frac{78}{25} = 3.12$

$\nu = 2$

$\chi^2_2(10\%) = \boxed{4.605}$

H_o: blood groups distributed in the ratio 2:1:1

H_1: blood groups not distributed in the ratio 2:1:1

3.12 (test statistic) < 4.605 (critical value)

We do not have enough evidence to reject H_0 so it seems reasonable to accept that the blood groups are in the ratio 2:1:1

Shade on an Argand diagram the region indicated by:

$$0 \leq arg(z-2-2i) \leq \frac{\pi}{4}$$

Sheet 26

Draw the graph which has this adjacency matrix (aka incidence matrix)

	A	**B**	**C**	**D**
A	0	1	2	0
B	1	2	1	0
C	2	1	0	1
D	0	0	1	0

A and **B** are 2 x 2 non-singular matrices such that **BAB** = **I**.

a) Prove that ***A*** = $\mathbf{B}^{-1}\mathbf{B}^{-1}$

b) Given that $\boldsymbol{B} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

Find the matrix ***A*** such that **BAB** = **I**

Is a binomial distribution B(4, ½) a good fit for the following data? Test at the 5% significance level.

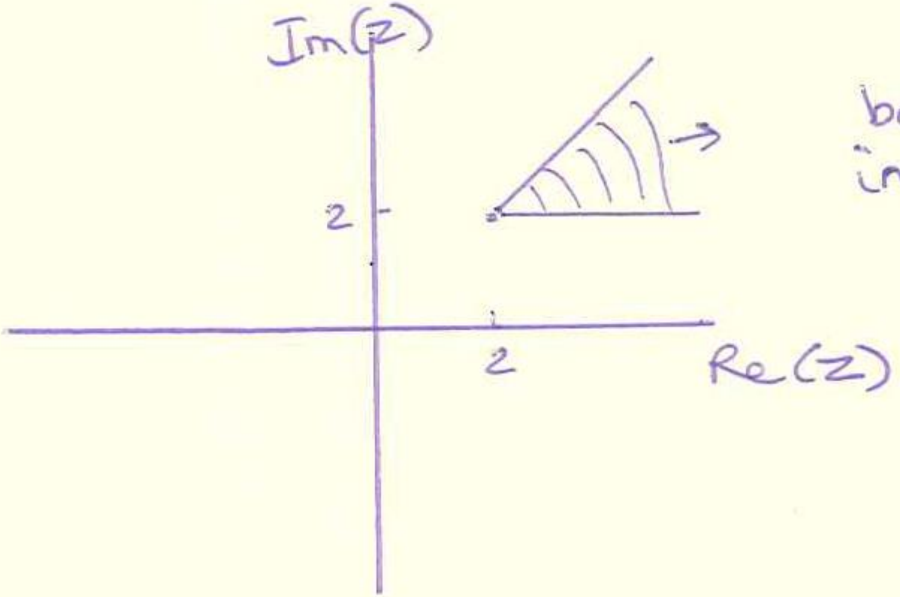
Number of heads	0	1	2	3	4
Frequency	15	46	54	35	10

Shade on an Argand diagram the region indicated by:

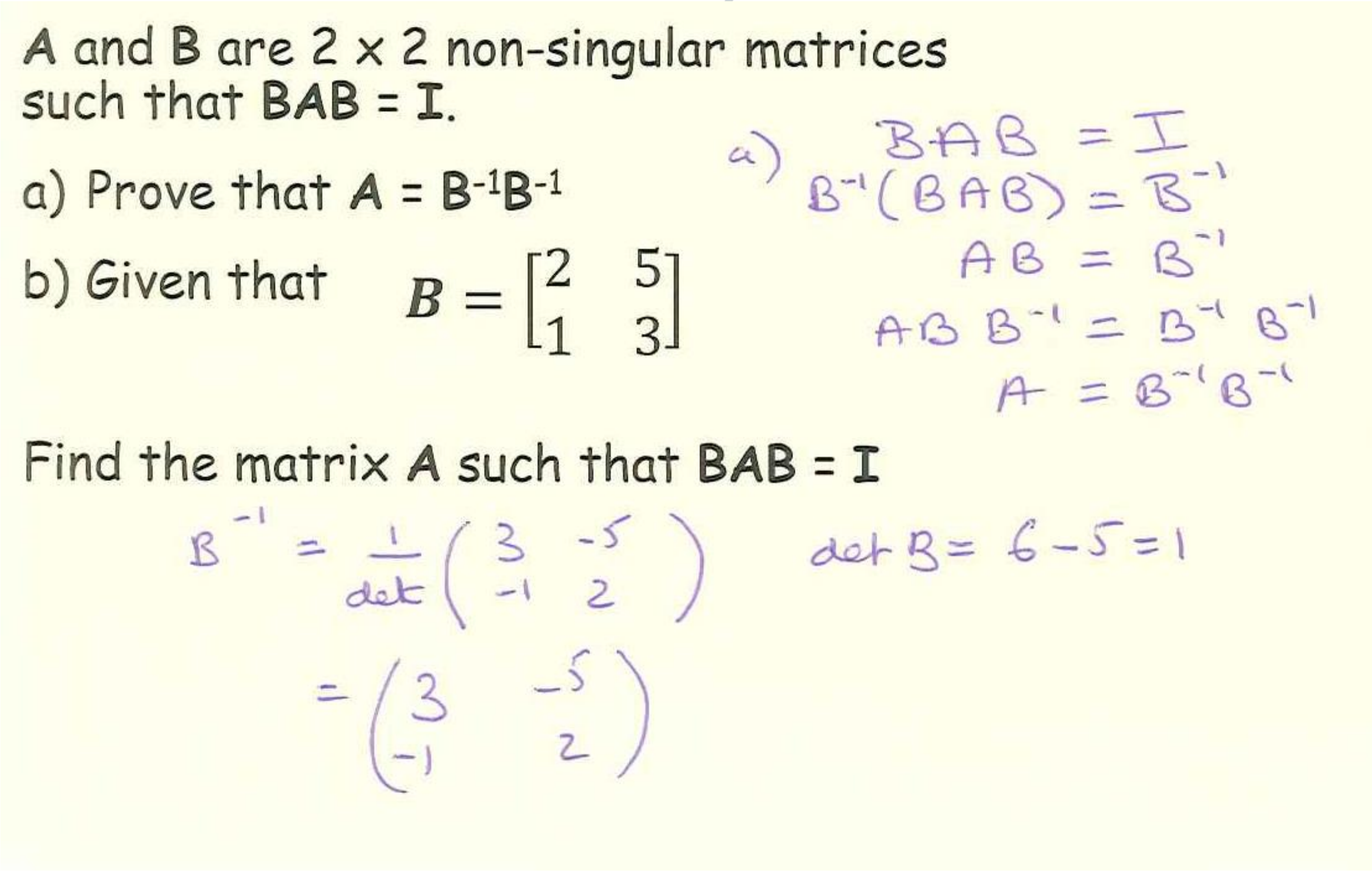
Sheet 26

$$0 \leq arg(z - 2 - 2i) \leq \frac{\pi}{4}$$

$z - (2+2i)$



boundary lines included



A and B are 2 x 2 non-singular matrices such that **BAB** = **I**.

a) Prove that **A** = **B**$^{-1}$**B**$^{-1}$

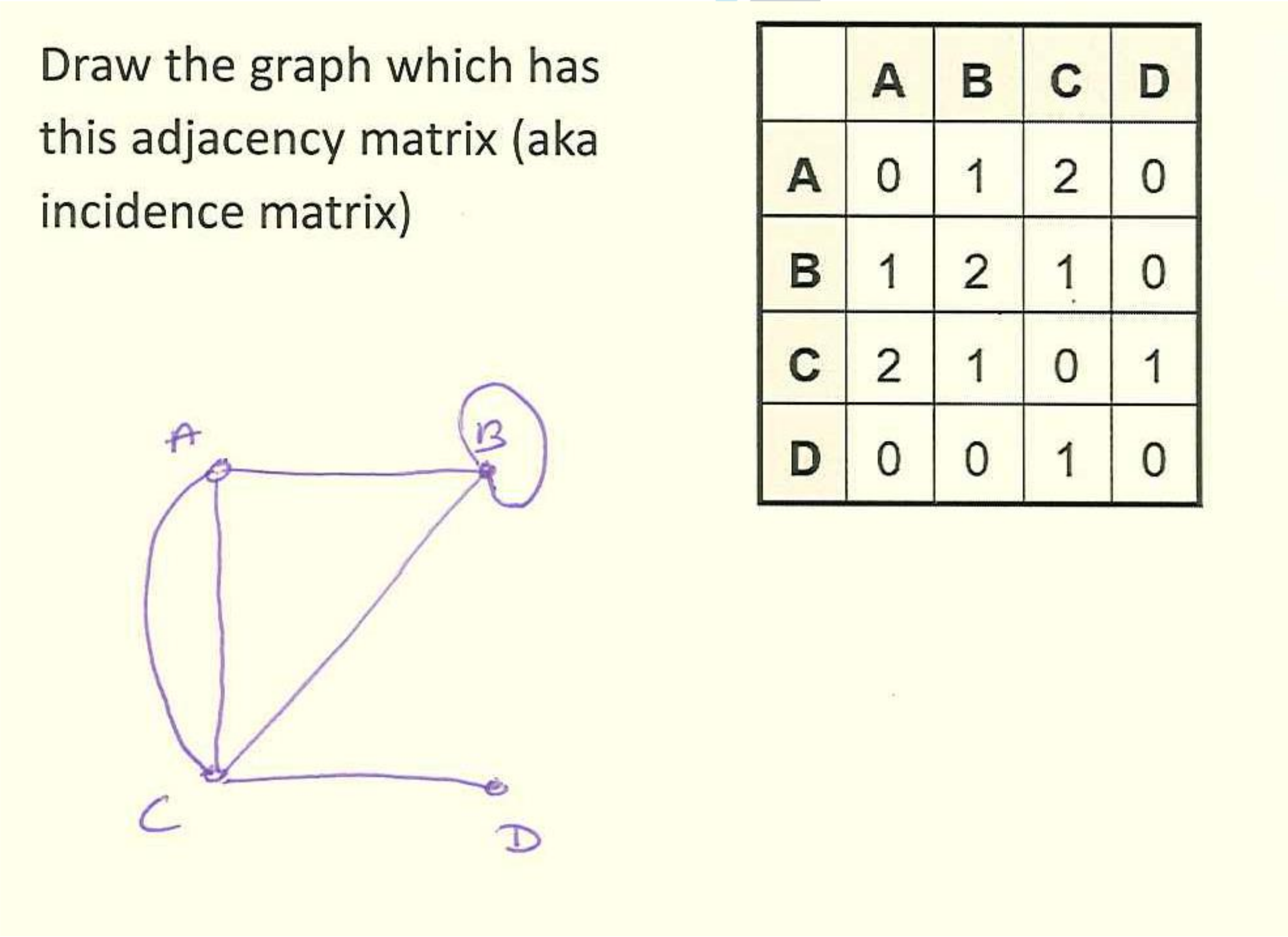
b) Given that $B = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

Find the matrix **A** such that **BAB** = **I**

a)
$$\cancel{B}AB = I$$
$$B^{-1}(BAB) = B^{-1}$$
$$AB = B^{-1}$$
$$ABB^{-1} = B^{-1}B^{-1}$$
$$A = B^{-1}B^{-1}$$

$$B^{-1} = \frac{1}{\det}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \qquad \det B = 6 - 5 = 1$$

$$= \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$



Draw the graph which has this adjacency matrix (aka incidence matrix)

	A	B	C	D
A	0	1	2	0
B	1	2	1	0
C	2	1	0	1
D	0	0	1	0

Is a binomial distribution B(4, ½) a good fit for the following data? Test at the 5% significance level

Number of heads	0	1	2	3	4	Total
Obs. Frequency	15	46	54	35	10	160
Exp. frequency.	10	40	60	40	10	160
$\frac{(O-E)^2}{E}$	$\frac{25}{10}$	$\frac{36}{40}$	$\frac{36}{60}$	$\frac{25}{40}$	0	$\frac{37}{8}$

$p(X=0) \times 160 = 10$

$X^2 = \frac{37}{8} = 4.625$

test statistic

$\nu = 5-1=4$

$\chi^2_4(5\%) = 9.488$

$4.625 < 9.488$

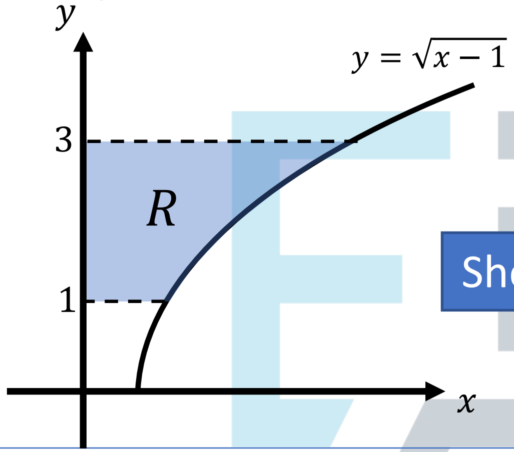
$\therefore$ accept H_0.

$B(4,\frac{1}{2})$ seems a good fit.

H_0 : $B(4,\frac{1}{2})$ is a good fit

H_1 : $B(4,\frac{1}{2})$ is not a good fit

The diagram shows the curve with equation $y = \sqrt{x-1}$. The region R is bounded by the curve, the y axis and the lines $y = 1$ and $y = 3$. The region is rotated 360° about the y axis. Find the volume of the solid generated.



The list of numbers below is to be sorted into **ascending** order.

8 4 13 2 17 9 15

Perform:

(i)a **bubble sort** to obtain the sorted list, giving the state of the list after each completed pass.

(ii)a **quick sort** to obtain the sorted list, giving the state of the list after each completed pass.

Sheet 27

The Matrix $\boldsymbol{M} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

a) Describe fully the transformation represented by matrix $\boldsymbol{M}$

b) A triangle T has vertices at (1,0), (4,0) and (4,2). Find the area of the triangle

c) Triangle T is transformed by using matrix $\boldsymbol{M}$. Find the area of the image of T.

In routine tests of germination rates, carrot seeds are planted in rows of 5 and the number of seeds which have germinated in each row after a fixed time interval is counted. The table below shows the results for 100 such rows.

Number of seeds germinated (r)	0	1	2	3	4	5
Number of rows (f_r)	0	0	8	23	43	26

(a) Use the data to estimate a value for p, the probability that a seed germinates.

(b) Calculate the expected frequencies for the model B(5, p). Hence, use a 2 goodness of fit test at the 5% significance level to test the suitability of the model B(5, p).

The diagram shows the curve with equation $y = \sqrt{x - 1}$. The region R is bounded by the curve, the y axis and the lines $y = 1$ and $y = 3$. The region is rotated 360° about the y axis. Find the volume of the solid generated.

$$V = \int \pi x^2 \, dy$$

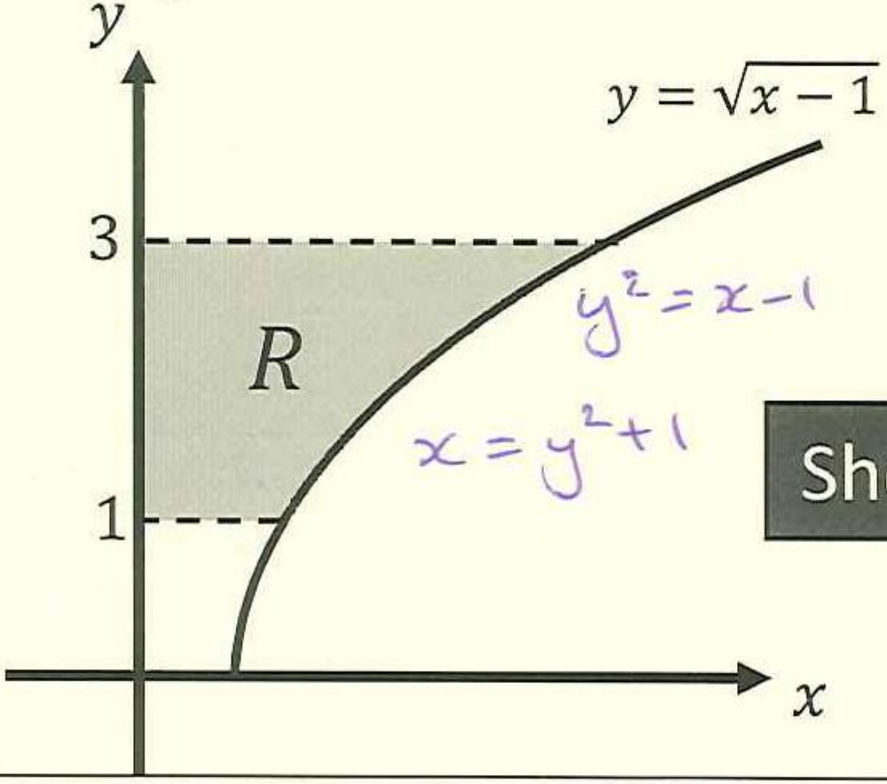
$$= \pi \int_1^3 y^2 + 1 \, dy$$

$$= \pi \left[\frac{y^3}{3} + y \right]_1^3$$

$$= \pi \left\{ [9 + 3] - \left[\frac{1}{3} + 1 \right] \right\}$$

$$= \pi \left\{ 12 - \frac{4}{3} \right\}$$

$$= \frac{32}{3} \pi$$



$y^2 = x - 1$

$x = y^2 + 1$

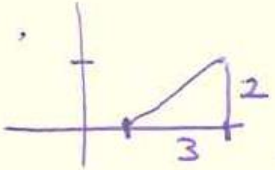
She

The Matrix $M = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

a) Describe fully the transformation represented by matrix M

Stretch s.f.2 parallel to x-axis and stretch s.f.4 parallel to y-axis

b) A triangle T has vertices at (1,0), (4,0) and (4,2). Find the area of the triangle



area = $\frac{1}{2} \times 6$

= 3 squares

c) Triangle T is transformed by using matrix M. Find the area of the image of T.

det m = 8

↑

area s.f.

∴ area T' = 3 × 8

= 24 squares

The list of numbers below is to be sorted into **ascending** order.

8 4 13 2 17 9 15

Perform:

(i)a **bubble sort** to obtain the sorted list, givin
of the list after each completed pass.

8 4 13 2 17 9 15

4 — 8

(8 13)

2 — 13

(13 17)

9 — 17

15 — 17 e.o. pass 1.

4 8 2 13 9 15 17

(4 8)

2 — 8

(8 13)

9 — 13

(13 15)

4 2 8 9 13 15 17 e.o. pass 2

2 — 4

(4 8)

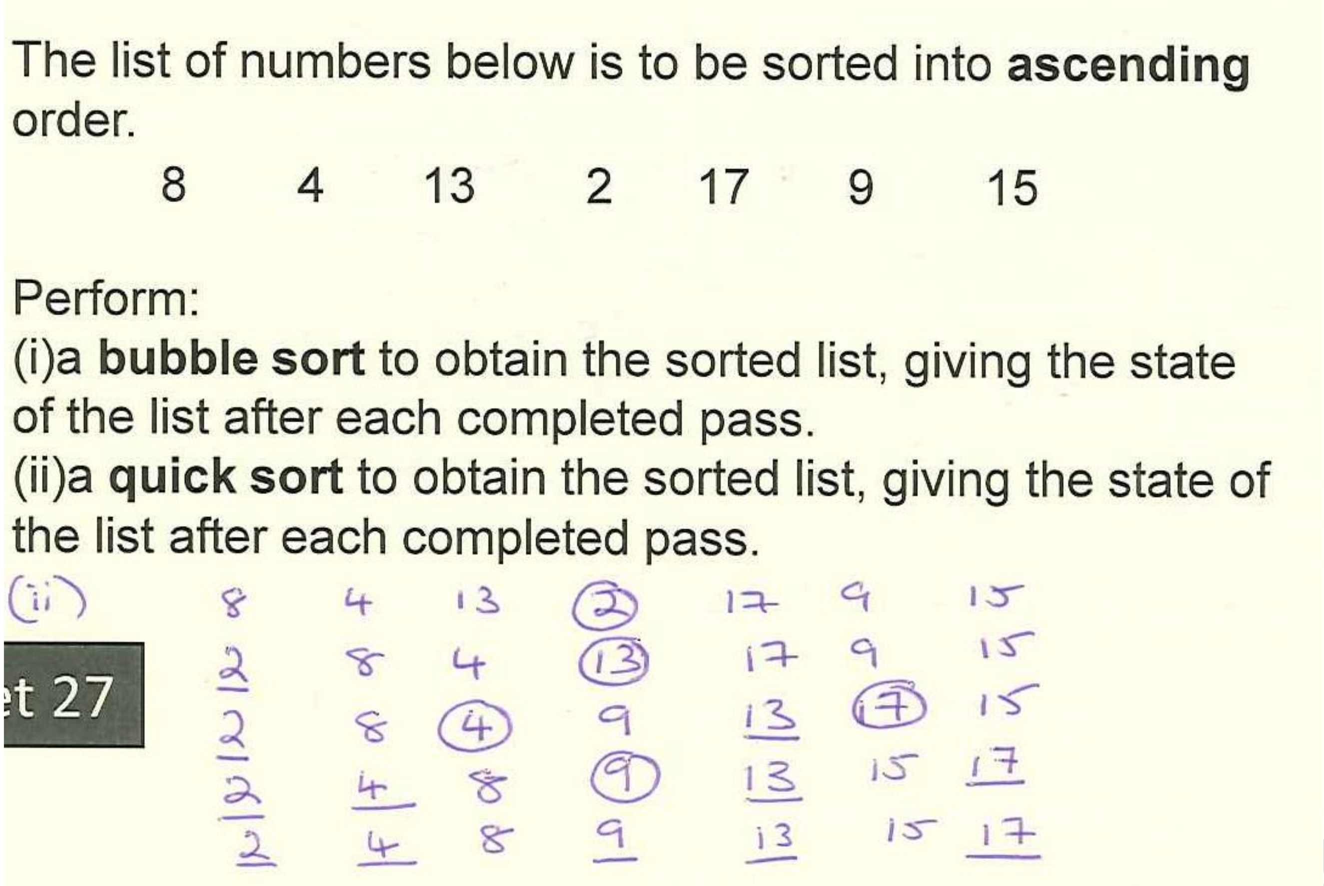
(8 9)

(9 13)

(13 15)

2 4 8 9 13 15 17 e.o. pass 3

2 4 8 9 13 15 17 e.o. pass 4



The list of numbers below is to be sorted into **ascending** order.

8 4 13 2 17 9 15

Perform:

(i)a **bubble sort** to obtain the sorted list, giving the state of the list after each completed pass.

(ii)a **quick sort** to obtain the sorted list, giving the state of the list after each completed pass.

(ii)	8	4	13	2	17	9	15
	2	8	4	13	17	9	15
	2	8	4	9	13	17	15
	2	4	8	9	13	15	17
	2	4	8	9	13	15	17

et 27

In routine tests of germination rates, carrot seeds are planted in rows of 5 and the number of seeds which have germinated in each row after a fixed time interval is counted. The table below shows the results for 100 such rows.

Number of seeds germinated (r)	0	1	2	3	4	5
Number of rows (f_r)	0	0	8	23	43	26

(a) Use the data to estimate a value for p, the probability that a seed germinates.

(b) Calculate the expected frequencies for the model B(5, p). Hence, use a 2 goodness of t test at the 5% significance level to test the suitability of the model B(5, p).

Total no. seeds = 100 × 5 = 500

Total which germinated = (8×2) + (23×3) + (43×4) + (26×5)

= 387

$p = \frac{387}{500} = 0.774$

No. seeds germinating	0	1	2	3	4	5
Obs.	0	0	8	23	43	26
Exp.	0.05896	1.008	6.915	~~[illegible]~~ 23.68	~~23.68~~ 40.55	27.78
$\frac{(O-E)^2}{E}$	$\underbrace{\qquad\qquad}$ $\frac{(7.98-8)^2}{7.98} = 5\times10^{-5}$			0.0195	0.148	0.114

$X \sim B(5, 0.774)$

$P(X=0) \times 100 = ~~58.96~~$

0.05896

$$X^2 = \sum \frac{(O-E)^2}{E}$$

= 0.2815

$\nu = 4 - 1 - 1 = 2$ $\quad \chi^2_2(5\%) = 5.991$

$0.2815 < 5.991$ ∴ accept H_0

H_0: $B(5,p)$ is a good fit

H_1: $B(5,p)$ is not a good fit

No reason to suspect that $B(5, 0.774)$ is not a good fit.

Sheet 28

The lines l_1 and l_2 have equations:

$$\boldsymbol{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix} \quad \boldsymbol{r} = \begin{pmatrix}-1\\3\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-1\end{pmatrix}$$

Find the shortest distance between these two lines.

8 7 14 9 6 9 5 15
6 7 8

The numbers represent the lengths, in cm, of pieces to be cut from 20cm rods
Use a first fit algorithm to identify the number of rods required and the wastage.

The roots of the quadratic equation $2x^2 - 5x - 4 = 0$ are α and β.
Without solving the equation, find the values of:

a) $\alpha + \beta$

b) $\alpha\beta$

c) $\frac{1}{\alpha} + \frac{1}{\beta}$

d) $\alpha^2 + \beta^2$

The numbers of defects in 60 printed circuit boards were recorded and the results are shown in the table below. Can these results be modelled by a Poisson distribution? Test at the 5% significance level.

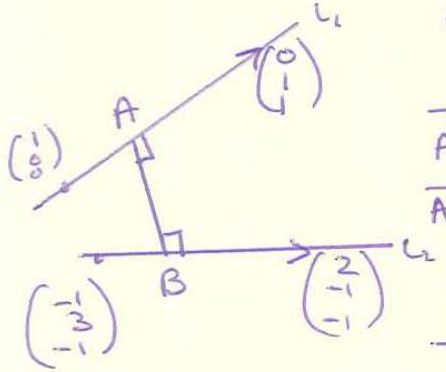
Number of observed defects (r)	0	1	2	3
Frequency (f_r)	32	15	9	4

Sheet 28

The lines l_1 and l_2 have equations:

$$\boldsymbol{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix} \quad \boldsymbol{r} = \begin{pmatrix}-1\\3\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-1\end{pmatrix}$$

Find the shortest distance between these two lines.



$A = \begin{pmatrix}1\\\lambda\\\lambda\end{pmatrix}$ $B = \begin{pmatrix}-1+2\mu\\3-\mu\\-1-\mu\end{pmatrix}$ $\overrightarrow{AB} = \begin{pmatrix}-2-2\mu\\3-\mu-\lambda\\-1-\mu-\lambda\end{pmatrix}$

$$\overrightarrow{AB}.l_2 = 0 \rightarrow \left.\begin{matrix}-4-4\mu\\-3+\mu+\lambda\\+1+\mu+\lambda\end{matrix}\right\} = 0$$

$$\overrightarrow{AB}.l_1 = 0$$

$$\downarrow$$

$$\begin{matrix}3-\mu-\lambda\\-1-\mu-\lambda\end{matrix} = 0$$

$$2 = 2\lambda + 2\mu$$

$$2\lambda - 2\mu = 6 \quad ①$$

$$\underline{2\lambda + 2\mu = 2 \quad ②} \; -$$

$$-4\mu = 4$$

$$\underline{\underline{\mu = -1}}$$

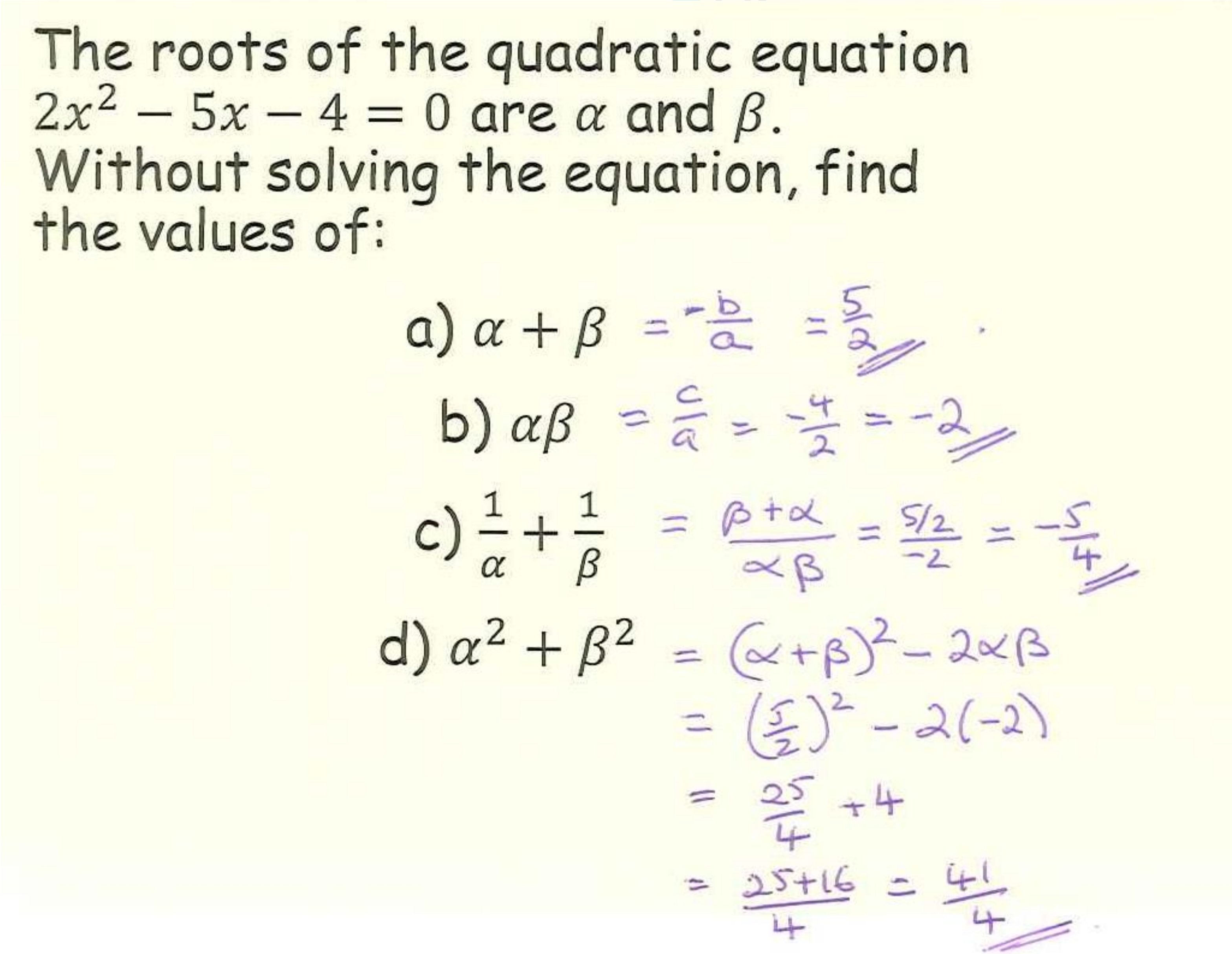
Sub ① $2\lambda + 2 = 6$

$$2\lambda = 4$$

$$\underline{\underline{\lambda = 2}}$$

$$\overrightarrow{AB} = \begin{pmatrix}-2-2(-1)\\3-(-1)-2\\-1-(-1)-2\end{pmatrix} = \begin{pmatrix}0\\2\\-2\end{pmatrix}$$

$$|AB| = \sqrt{2^2+2^2} = \sqrt{8} = \underline{\underline{2\sqrt{2}}}$$



The roots of the quadratic equation $2x^2 - 5x - 4 = 0$ are α and β. Without solving the equation, find the values of:

a) $\alpha + \beta = \frac{-b}{a} = \frac{5}{2}$

b) $\alpha\beta = \frac{c}{a} = \frac{-4}{2} = -2$

c) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{5/2}{-2} = -\frac{5}{4}$

d) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{5}{2}\right)^2 - 2(-2)$

$= \frac{25}{4} + 4$

$= \frac{25+16}{4} = \frac{41}{4}$

8 7 14 9 6 9 5 15

6 7 8

The numbers represent the lengths, in cm, of pieces to be cut from 20cm rods

Use a first fit algorithm to identify the number of rods required and the wastage.

Bin1	Bin2	Bin3	Bin4	Bin5	Bin6
8	14	9	15	6	8
7	6	9		7	
5					

(2) + (5) + (7) + (12) = 26

6 rods required. 26 cm is wasted.

The numbers of defects in 60 printed circuit boards were recorded and the results are shown in the table below. Can these results be modelled by a Poisson distribution? Test at the 5% significance level.

H_0 : Poisson is good fit
H_1 : Poisson not a good fit

Number of observed defects (r)	0	1	2	3
Obs Frequency (f_r)	32	15	9	4
Exp.	28.34	21.26	7.97	1.99

$\underbrace{7.97 \quad 1.99}_{9.96}$.

$\frac{(O-E)^2}{E}$ $\quad \nu = 3-1-1=1$ $\quad$ 0.473 $\quad$ 1.843 $\quad$ 0.928,

$\chi^2_1(5\%) = 3.841$ $\quad \Sigma = 3.24$

Total defects = 15 + 18 + 12 = 45. $\quad \frac{45}{60} = 0.75$

$\lambda \approx 0.75$ $\quad$ eg) $P(X=0) \times 60 = 28.34$

3.24 < 3.841
∴ accept H_0.
Poisson is a good fit.

Sheet 29

$$\boldsymbol{A} = \begin{bmatrix} a & 0 \\ 1 & 2 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 & b \\ 0 & 3 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 6 & 6 \\ 1 & c \end{bmatrix}$$

Given that **A** + 2**B** = **C**, find the values of a, b and c

8 7 14 9 6 9 5 15
6 7 8

The numbers represent the lengths, in cm, of pieces to be cut from 20cm rods
Use a first fit decreasing algorithm to identify the number of rods required and the wastage.

The straight line l has vector equation:

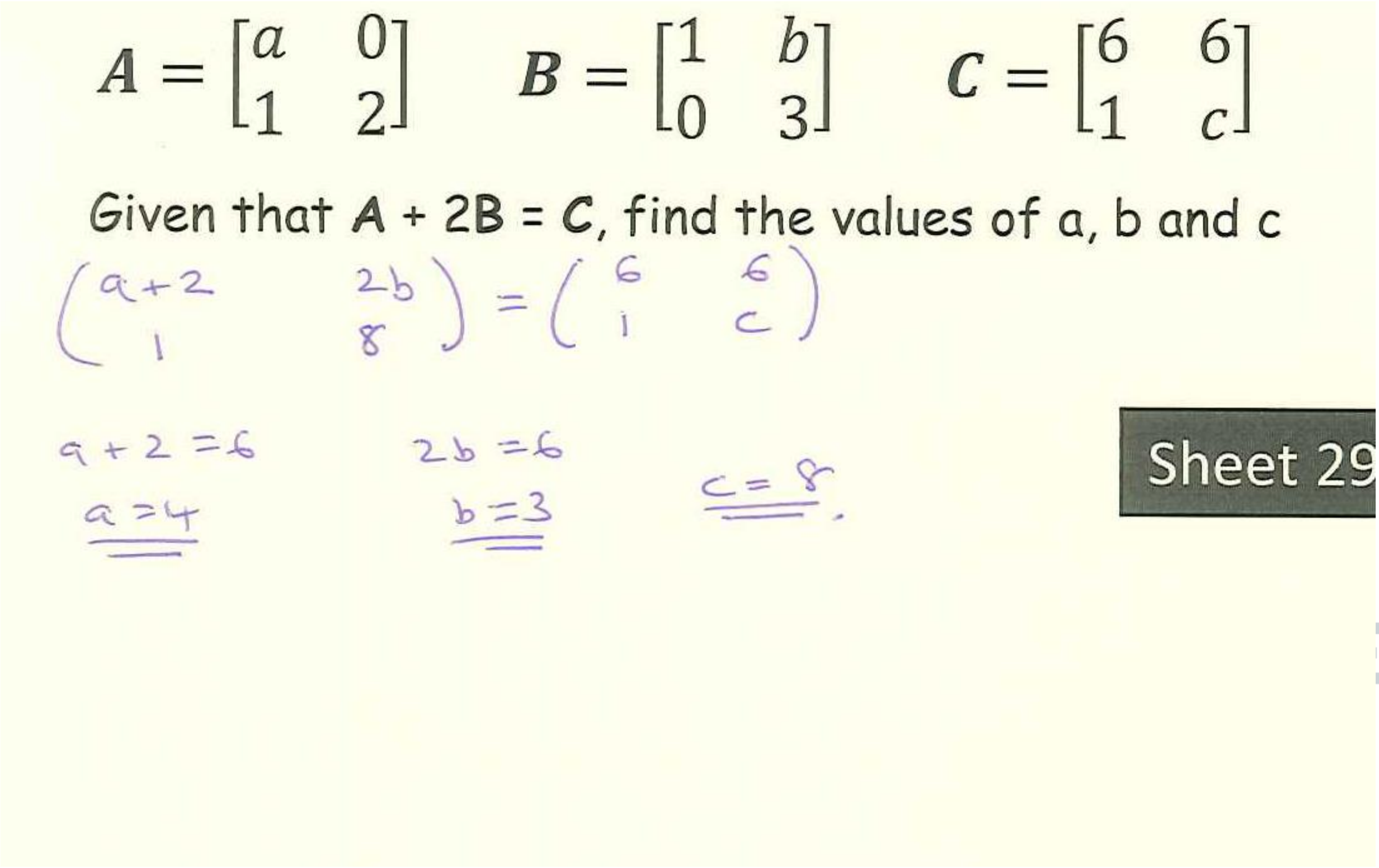
$$\boldsymbol{r} = (2\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}) + \lambda(6\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k})$$

Show that another vector equation of l is:

$$\boldsymbol{r} = (8\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) + \mu(3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$$

A public opinion poll surveyed a simple random sample of 1000 voters. Respondents were classified by their sex (male or female) and by their voting preference (Conservative, Labour, or others). Results are shown in the contingency table below. Conduct a goodness-of-t test, at the 5% significance level, to see whether voting preference and sex are independent of each other.

	Voting preference			Total
	Conservative	Labour	Others	
Male	200	150	50	400
Female	250	300	50	600
Total	450	450	100	1000



$$A = \begin{bmatrix} a & 0 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & b \\ 0 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 6 & 6 \\ 1 & c \end{bmatrix}$$

Given that **A** + 2**B** = **C**, find the values of a, b and c

$$\begin{pmatrix} a+2 & 2b \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 1 & c \end{pmatrix}$$

$a + 2 = 6$ $\qquad$ $2b = 6$

$\underline{\underline{a = 4}}$ $\qquad$ $\underline{\underline{b = 3}}$ $\qquad$ $\underline{\underline{c = 8}}$.

Sheet 29

The straight line l has vector equation:

$$\boldsymbol{r} = (2\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}) + \lambda(6\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}) \rightarrow \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}$$

Show that another vector equation of l is:

$$\boldsymbol{r} = (8\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) + \mu(3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}) \rightarrow \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

Direction ~~vectors are~~ is the same as $\begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} = 2 \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

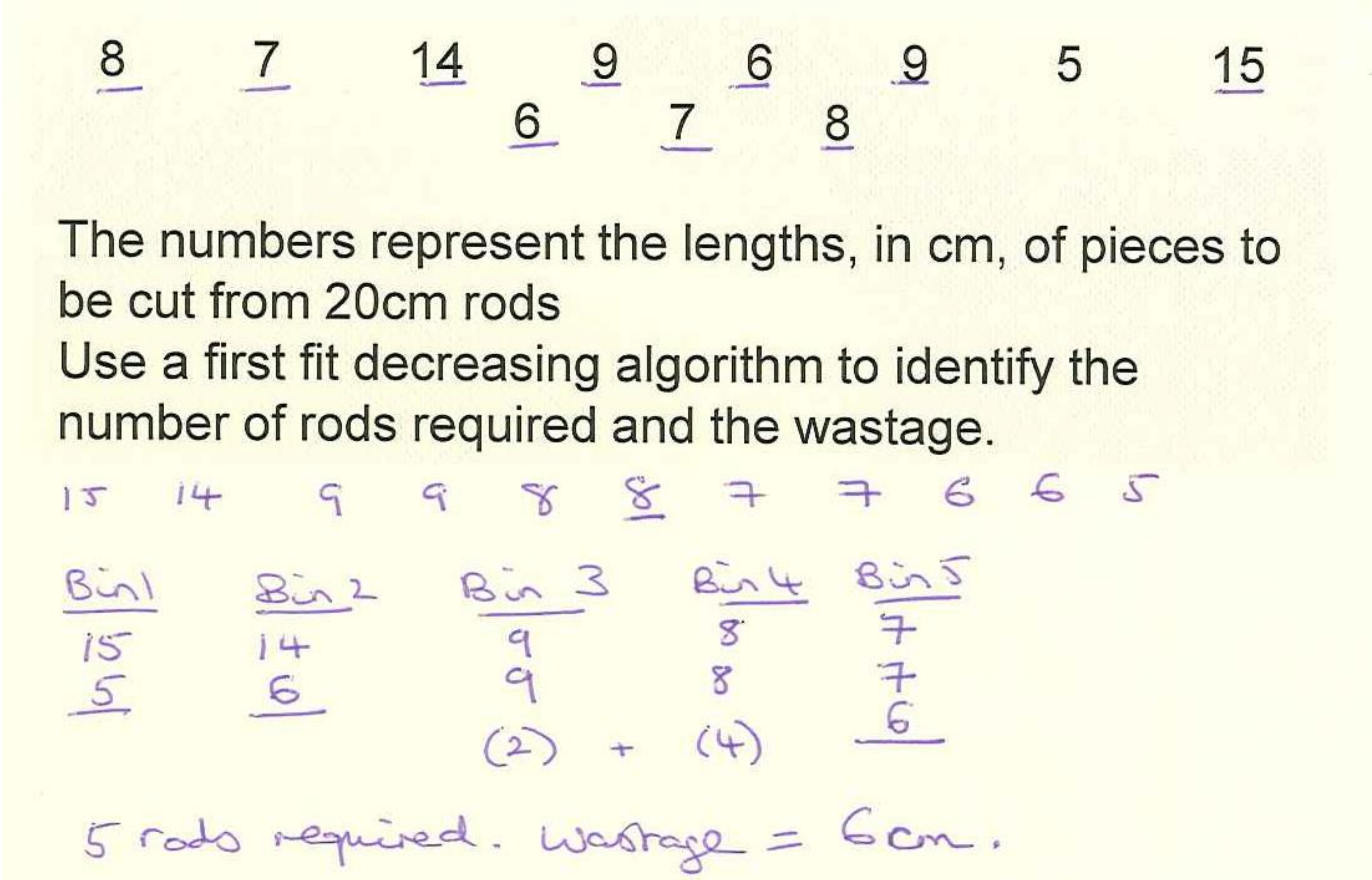
Need to show that $\begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix}$ is a point on the 1st line.

$2 + 6\lambda = 8 \rightarrow \lambda = 1$

$5 - 2\lambda = 3 \rightarrow \lambda = 1$

$-3 + 4\lambda = 1 \rightarrow \lambda = 1$

Consistent equations $\therefore (8, 3, 1)$ is on the 1st line and they have the same direction $\Rightarrow$ same lines



8 7 14 9 6 9 5 15

6 7 8

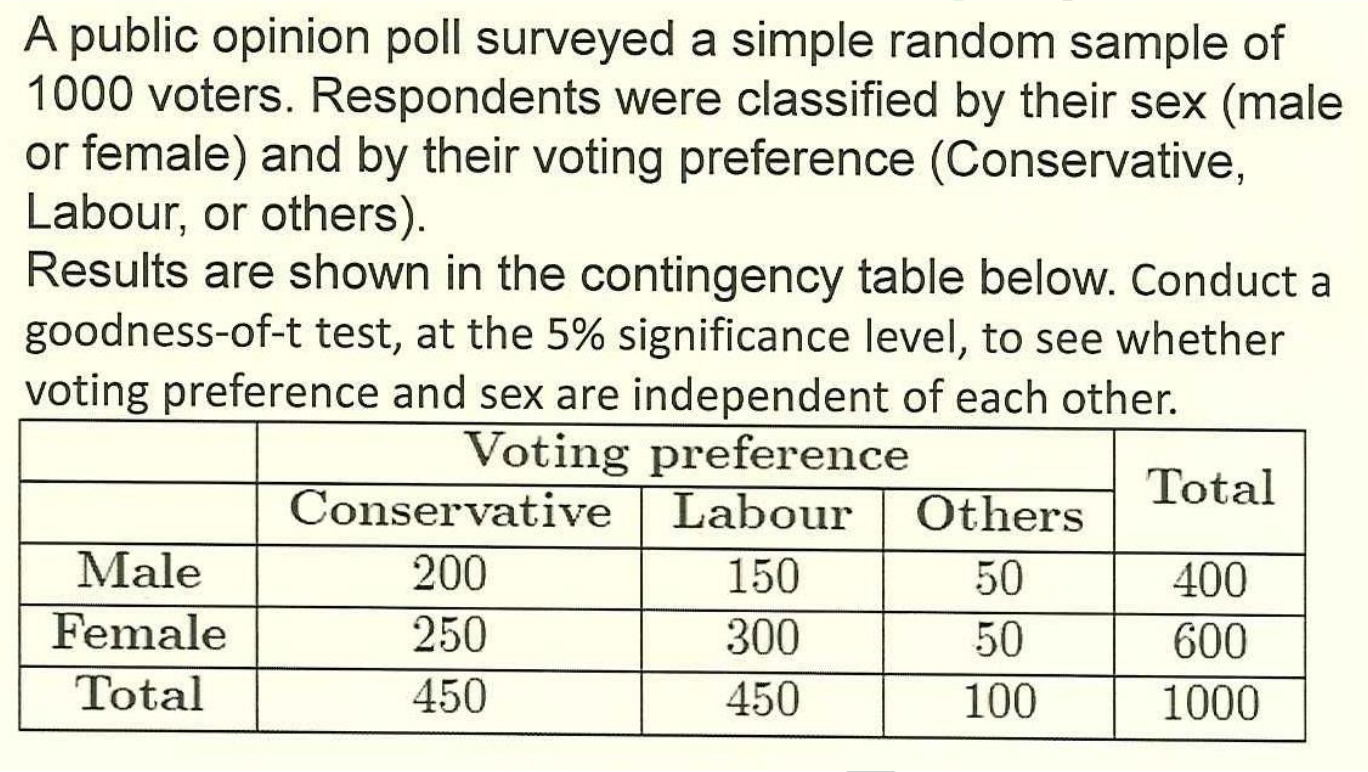
The numbers represent the lengths, in cm, of pieces to be cut from 20cm rods

Use a first fit decreasing algorithm to identify the number of rods required and the wastage.

15 14 9 9 8 8 7 7 6 6 5

Bin 1	Bin 2	Bin 3	Bin 4	Bin 5
15	14	9	8	7
5	6	9	8	7
		(2) +	(4)	6

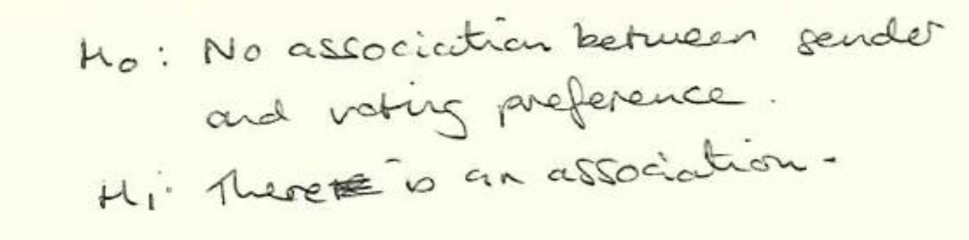
5 rods required. Wastage = 6cm.



A public opinion poll surveyed a simple random sample of 1000 voters. Respondents were classified by their sex (male or female) and by their voting preference (Conservative, Labour, or others).

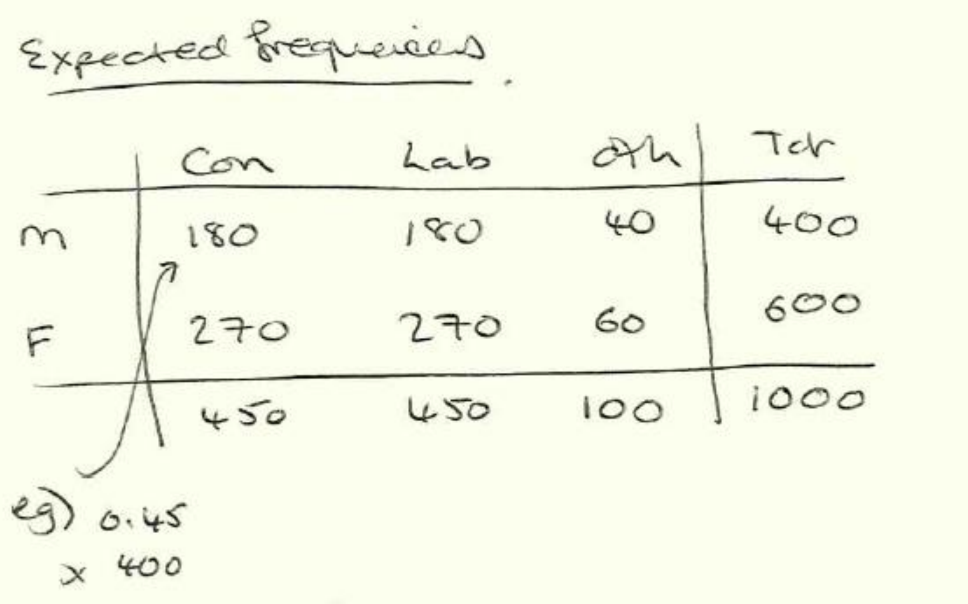
Results are shown in the contingency table below. Conduct a goodness-of-t test, at the 5% significance level, to see whether voting preference and sex are independent of each other.

	Voting preference			Total
	Conservative	Labour	Others	
Male	200	150	50	400
Female	250	300	50	600
Total	450	450	100	1000



H_0: No association between gender and voting preference.

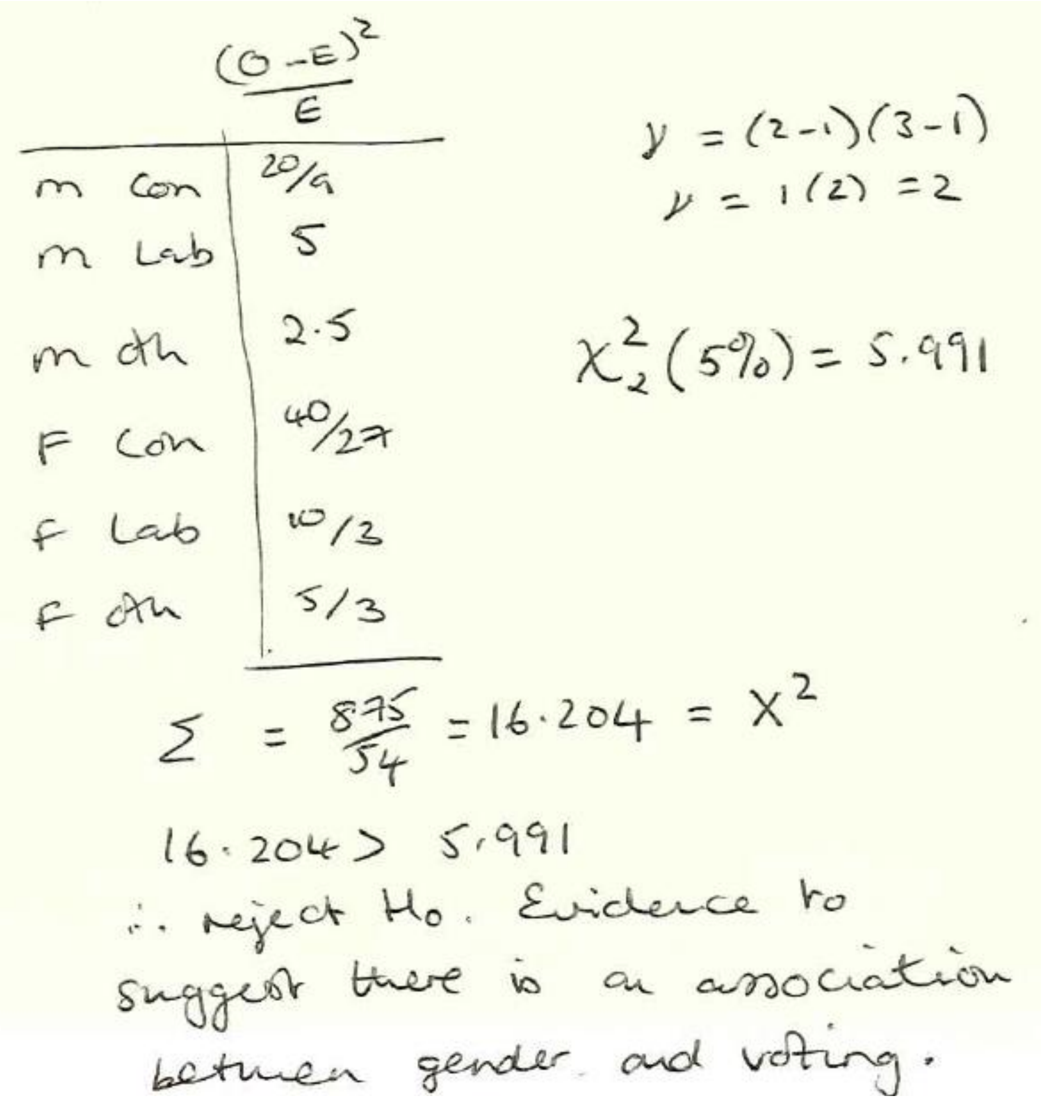
H_1: There~~e~~ is an association.



Expected frequencies

	Con	Lab	Oth	Tot
m	180	180	40	400
F	270	270	60	600
	450	450	100	1000

eg) 0.45 × 400



	$\frac{(O-E)^2}{E}$
m Con	$20/9$
m Lab	5
m Oth	2.5
F Con	$40/27$
F Lab	$10/3$
F Oth	$5/3$

$\nu = (2-1)(3-1)$

$\nu = 1(2) = 2$

$\chi^2_2(5\%) = 5.991$

$\Sigma = \frac{875}{54} = 16.204 = X^2$

$16.204 > 5.991$

∴ reject H_0. Evidence to suggest there is an association between gender and voting.

Prove by mathematical induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n}(2r-1) = n^2$$

Sheet 30

8	7	14	9	6	9	5	15
			6	7	8		

The numbers represent the lengths, in cm, of pieces to be cut from 20cm rods
Use a full-bin algorithm to identify the number of rods required and the wastage.

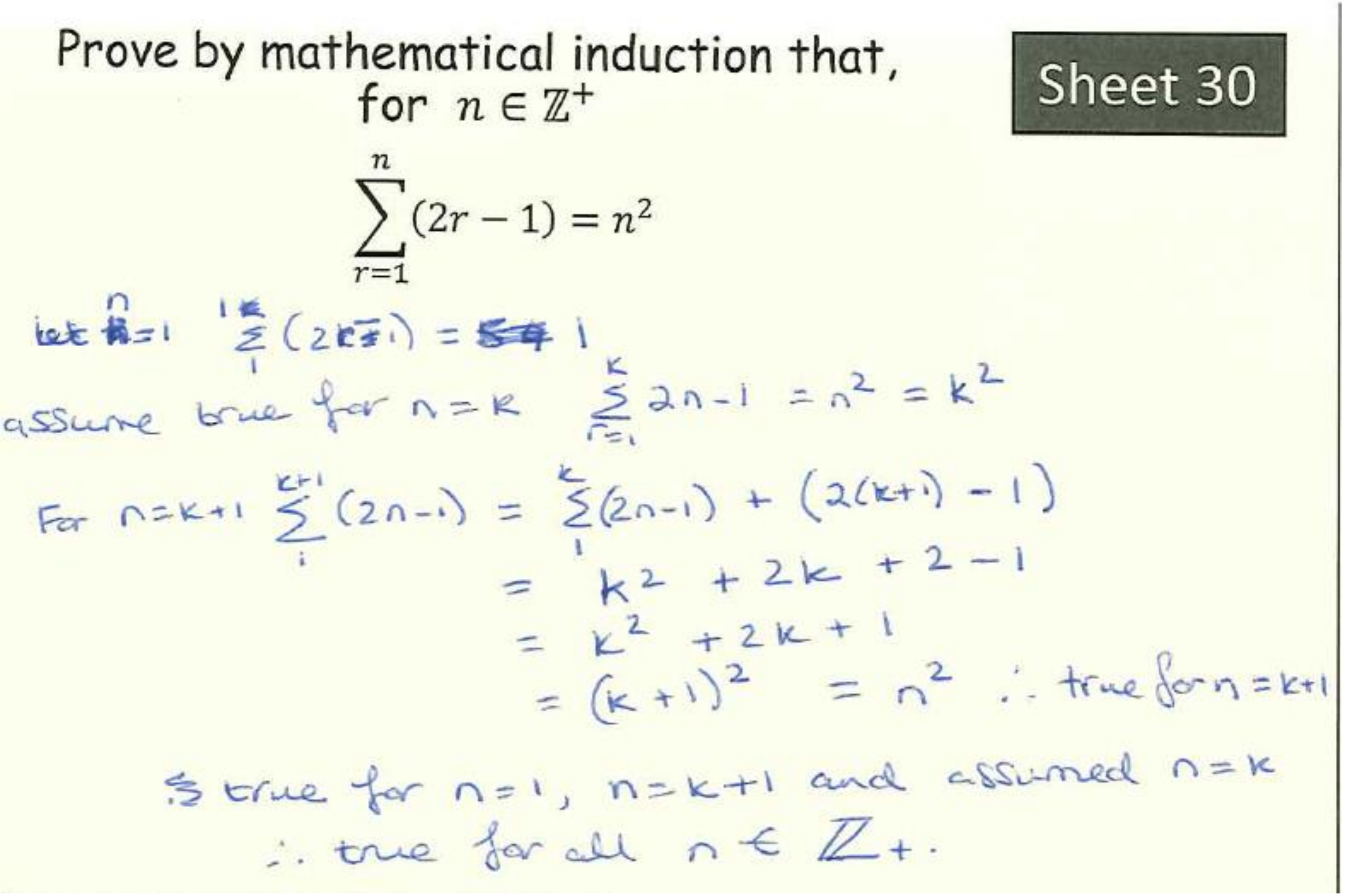
The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\alpha = -\frac{3}{2}$ and $\beta = \frac{5}{4}$. Find integer values for a, b and c.

$$\boldsymbol{M} = \begin{bmatrix} -2\sqrt{2} & -2\sqrt{2} \\ 2\sqrt{2} & -2\sqrt{2} \end{bmatrix}$$

The matrix $\boldsymbol{M}$ represents an enlargement with scale factor k followed by an anticlockwise rotation through angle θ about the origin.

a) Find the value of k

b) Find the value of θ



Prove by mathematical induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n}(2r-1) = n^2$$

Sheet 30

let ~~k~~ n = 1 $\sum_{1}^{1}(2\cancel{k}\cancel{-}1) = \cancel{5}\cancel{4}\ 1$

assume true for n = k $\sum_{r=1}^{k} 2n-1 = n^2 = k^2$

For n = k+1 $\sum_{1}^{k+1}(2n-1) = \sum_{1}^{k}(2n-1) + (2(k+1)-1)$

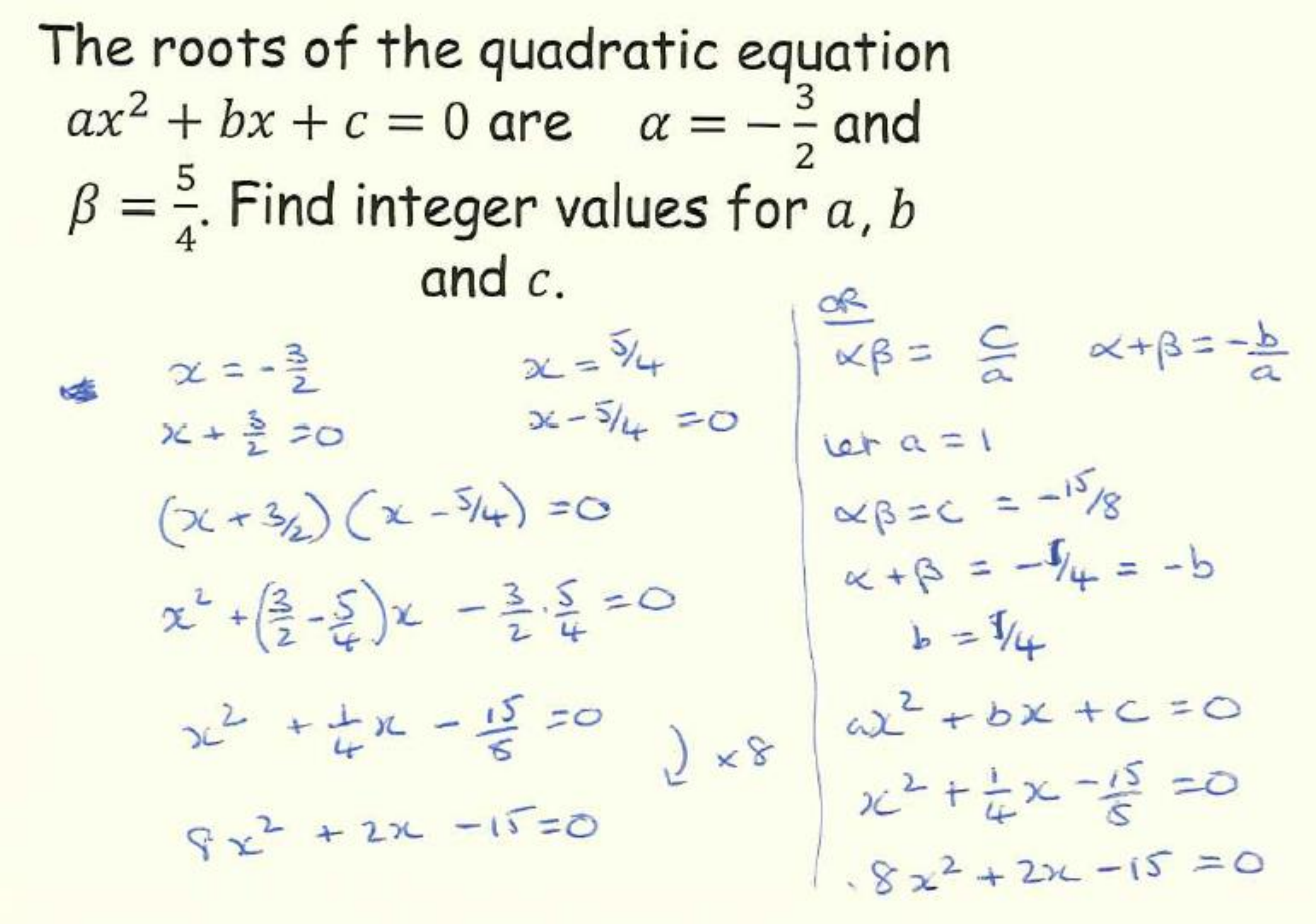
$= k^2 + 2k + 2 - 1$

$= k^2 + 2k + 1$

$= (k+1)^2 = n^2$ ∴ true for n = k+1

∴ true for n = 1, n = k+1 and assumed n = k

∴ true for all $n \in \mathbb{Z}+$.



The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\alpha = -\frac{3}{2}$ and $\beta = \frac{5}{4}$. Find integer values for a, b and c.

$x = -\frac{3}{2}$ $\quad$ $x = 5/4$

$x + \frac{3}{2} = 0$ $\quad$ $x - 5/4 = 0$

$(x + 3/2)(x - 5/4) = 0$

$x^2 + \left(\frac{3}{2} - \frac{5}{4}\right)x - \frac{3}{2} \cdot \frac{5}{4} = 0$

$x^2 + \frac{1}{4}x - \frac{15}{8} = 0$ $\quad$ $) \times 8$

$8x^2 + 2x - 15 = 0$

OR

$\alpha\beta = \frac{c}{a}$ $\quad$ $\alpha + \beta = -\frac{b}{a}$

let $a = 1$

$\alpha\beta = c = -15/8$

$\alpha + \beta = -1/4 = -b$

$b = 1/4$

$ax^2 + bx + c = 0$

$x^2 + \frac{1}{4}x - \frac{15}{8} = 0$

$8x^2 + 2x - 15 = 0$

8 7 14 9 6 9 5 15

6 7 8

The numbers represent the lengths, in cm, of pieces to be cut from 20cm rods

Use a full-bin algorithm to identify the number of rods required and the wastage.

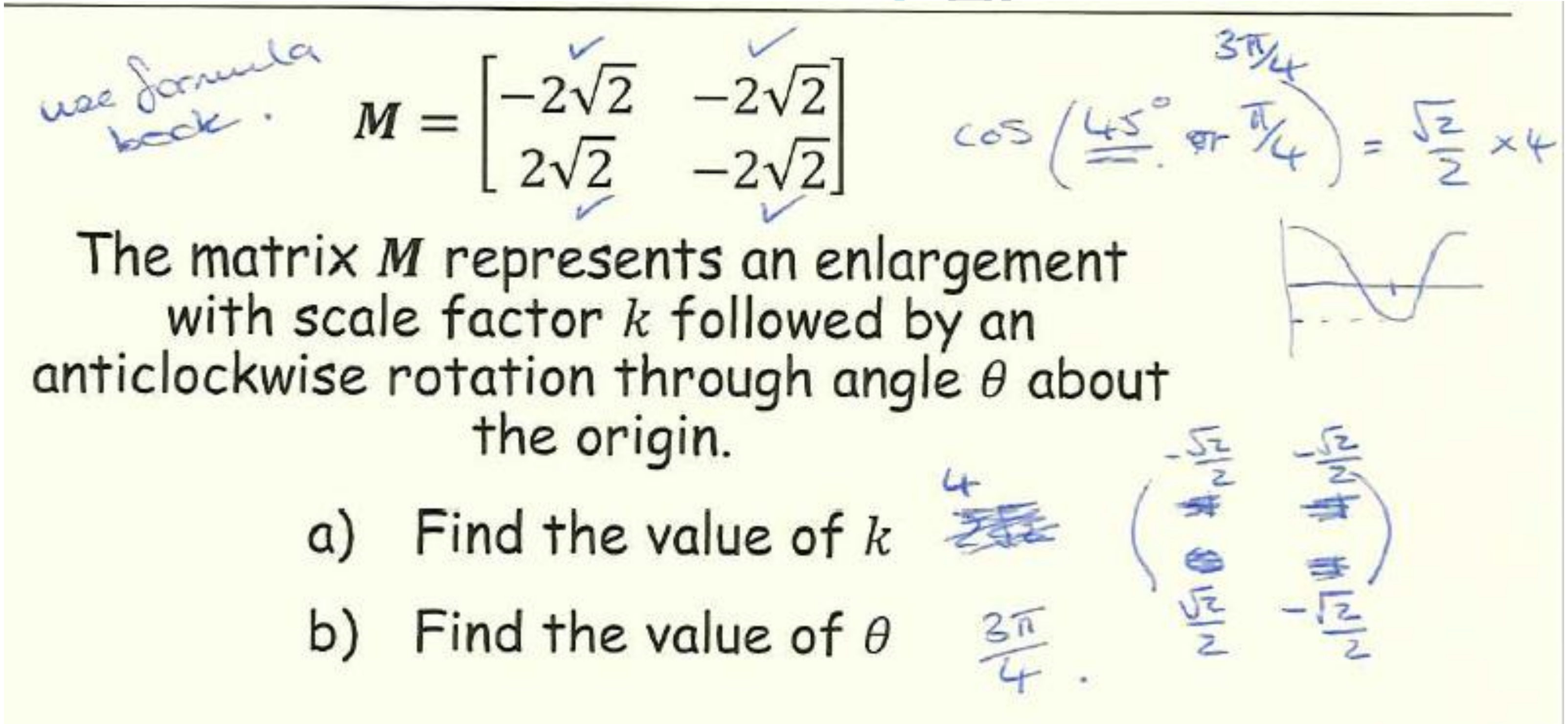
Sum = 94

94 ÷ 2 = 4·7

LB = 5 rods

Rod :	1	2	3	4	5
	15	14	7	9	8
	5	6	7	9	8
			6	18	16
				(2)	(4)

5 rods required, wastage = 6 cm.



use formula book.

$$M = \begin{bmatrix} -2\sqrt{2} & -2\sqrt{2} \\ 2\sqrt{2} & -2\sqrt{2} \end{bmatrix}$$

$3\pi/4$

$\cos\left(45^\circ \text{ or } \pi/4\right) = \frac{\sqrt{2}}{2} \times 4$

The matrix M represents an enlargement with scale factor k followed by an anticlockwise rotation through angle θ about the origin.

a) Find the value of k — 4

b) Find the value of θ — $\frac{3\pi}{4}$.

$$\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

Sheet 31

Describe fully the geometrical transformation represented by this matrix:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Describe fully the geometrical transformation represented by this matrix:

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

The lines l_1 and l_2 have vector equations:

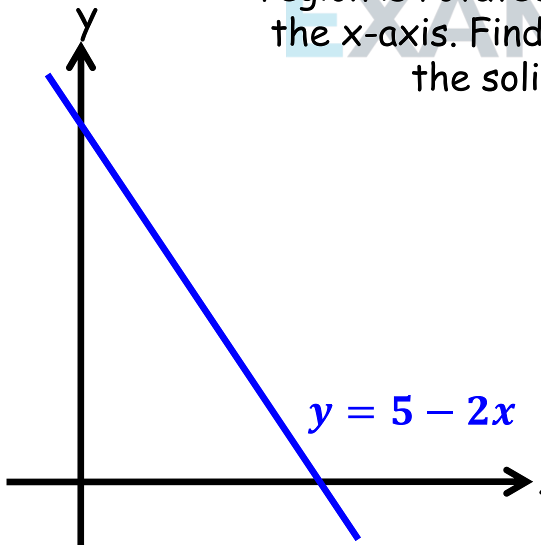
$$\boldsymbol{r} = (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + t(3\boldsymbol{i} - 8\boldsymbol{j} - \boldsymbol{k})$$

And

$$\boldsymbol{r} = (7\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}) + s(2\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$$

Given that l_1 and l_2 intersect, find the size of the acute angle between the lines, to 1 decimal place.

The region R is bounded by the line $y = 5 - 2x$, and the x and y axes. The region is rotated through 360° about the x-axis. Find the exact volume of the solid generated.



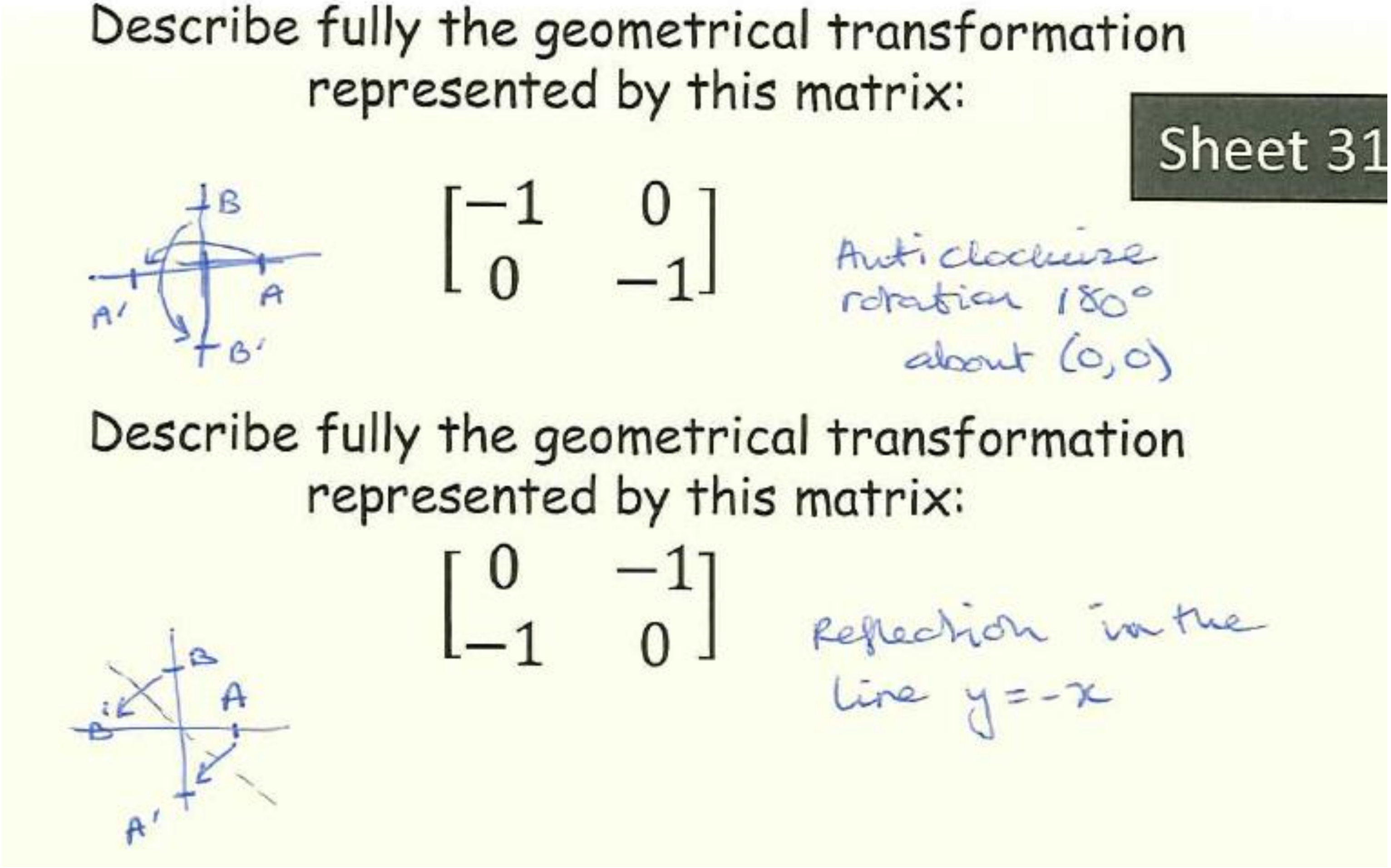
If α, β and γ are the roots of the equation $2x^3 + 3x^2 - 4x + 2 = 0$, find the values of:

a) $\alpha + \beta + \gamma$

b) $\alpha\beta + \beta\gamma + \gamma\alpha$

c) $\alpha\beta\gamma$

d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$



Sheet 31

Describe fully the geometrical transformation represented by this matrix:

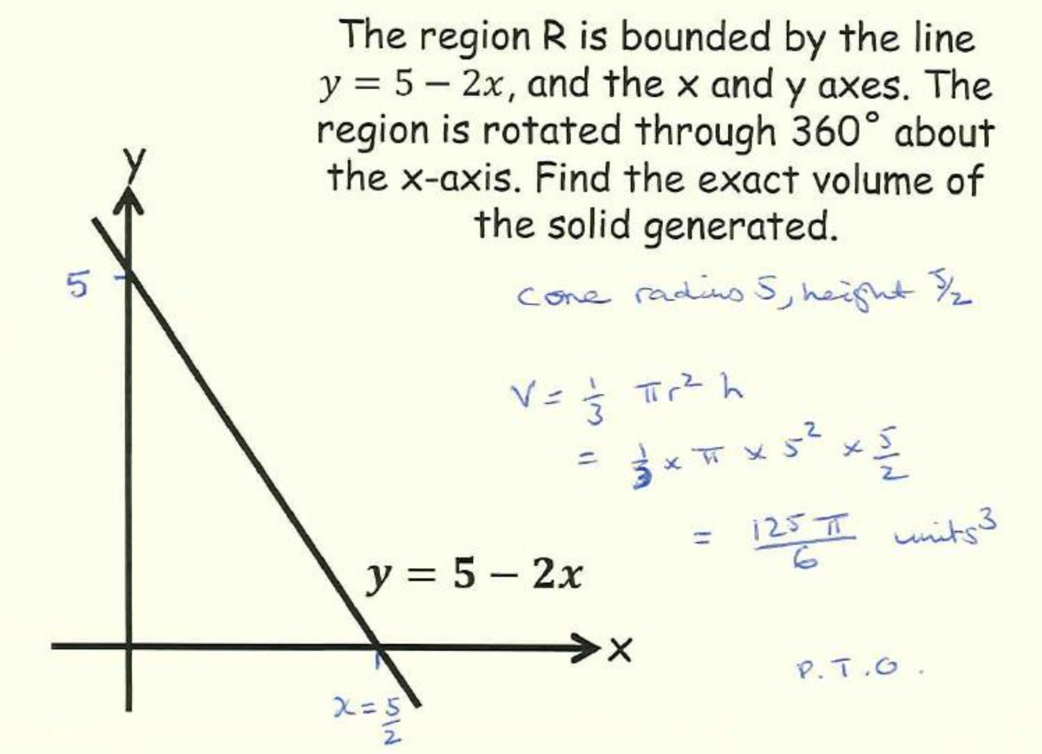
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Anticlockwise rotation 180° about (0,0)

Describe fully the geometrical transformation represented by this matrix:

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Reflection in the line $y=-x$

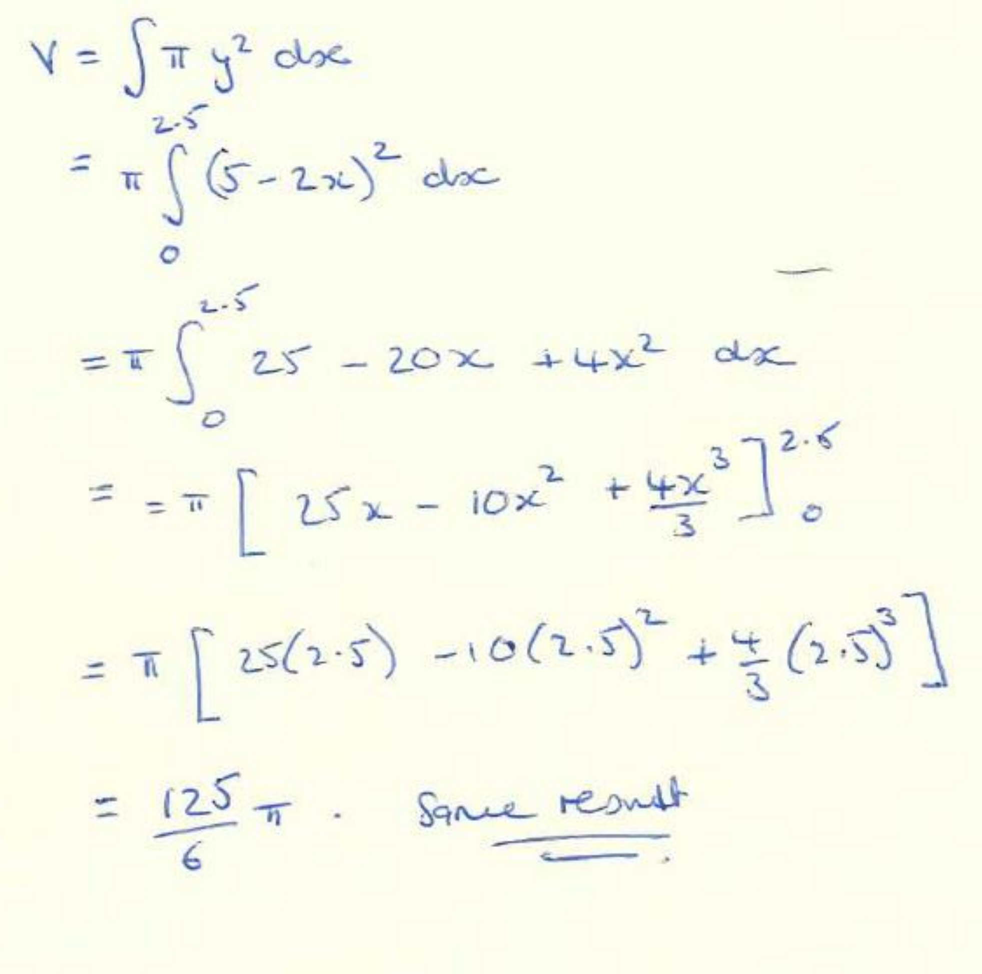


The region R is bounded by the line $y = 5 - 2x$, and the x and y axes. The region is rotated through 360° about the x-axis. Find the exact volume of the solid generated.

cone radius 5, height $5/2$

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \pi \times 5^2 \times \frac{5}{2}$$
$$= \frac{125\pi}{6} \text{ units}^3$$

P.T.O.



$$V = \int \pi y^2 \, dx$$
$$= \pi \int_0^{2.5} (5-2x)^2 \, dx$$
$$= \pi \int_0^{2.5} 25 - 20x + 4x^2 \, dx$$
$$= = \pi \left[25x - 10x^2 + \frac{4x^3}{3} \right]_0^{2.5}$$
$$= \pi \left[25(2.5) - 10(2.5)^2 + \frac{4}{3}(2.5)^3 \right]$$
$$= \frac{125}{6}\pi .$$

Same result

The lines l_1 and l_2 have vector equations:

$$\mathbf{r} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(3\mathbf{i} - 8\mathbf{j} - \mathbf{k})$$

direction = $\begin{pmatrix} 3 \\ -8 \\ -1 \end{pmatrix}$

And

$$\mathbf{r} = (7\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + s(2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

direction = $\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

Given that l_1 and l_2 intersect, find the size of the acute angle between the lines, to 1 decimal place.

$$\begin{pmatrix} 3 \\ -8 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = 6 - 16 - 3 = -13$$

$$|a| = \sqrt{3^2 + 8^2 + 1^2} = \sqrt{74}$$

$$|b| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{17}$$

$$\cos\theta = \frac{a.b}{|a||b|}$$

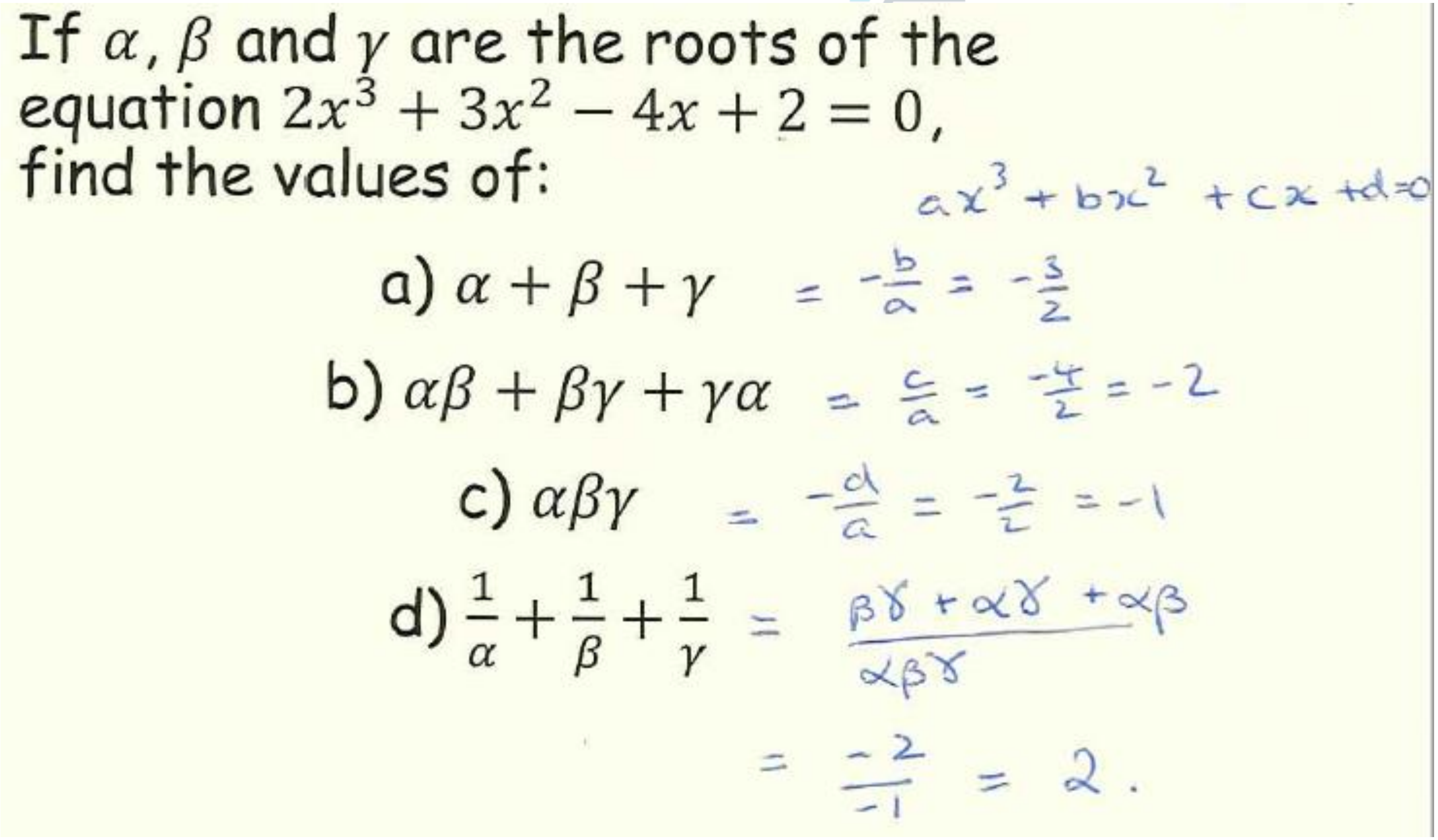
$$\cos\theta = -\frac{13}{\sqrt{74}\sqrt{17}}$$

$\downarrow \cos^{-1}$

$$\theta = 113.8°$$

$$\alpha = 180 - \theta$$
$$= 66.2°$$

(1dp)



If α, β and γ are the roots of the equation $2x^3 + 3x^2 - 4x + 2 = 0$, find the values of:

$ax^3 + bx^2 + cx + d = 0$

a) $\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{3}{2}$

b) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{-4}{2} = -2$

c) $\alpha\beta\gamma = -\frac{d}{a} = -\frac{2}{2} = -1$

d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{-2}{-1} = 2.$

The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = 1 - 2i$, $\beta = 1 + 2i$ and $\gamma = 2$.

Find integer values for a, b, c and d.

Sheet 32

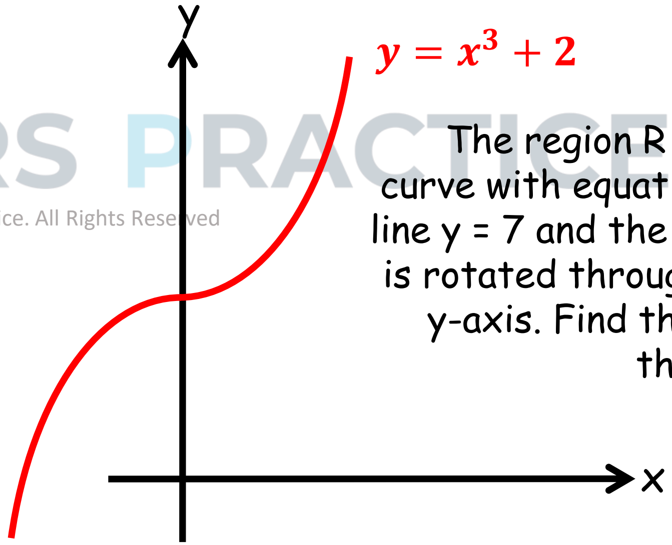
Prove, by induction, that the expression '$n^3 - 7n + 9$' is divisible by 3 for all positive integers $n \in \mathbb{Z}^+$

The plane Π passes through the point A and is perpendicular to the vector $\boldsymbol{n}$.

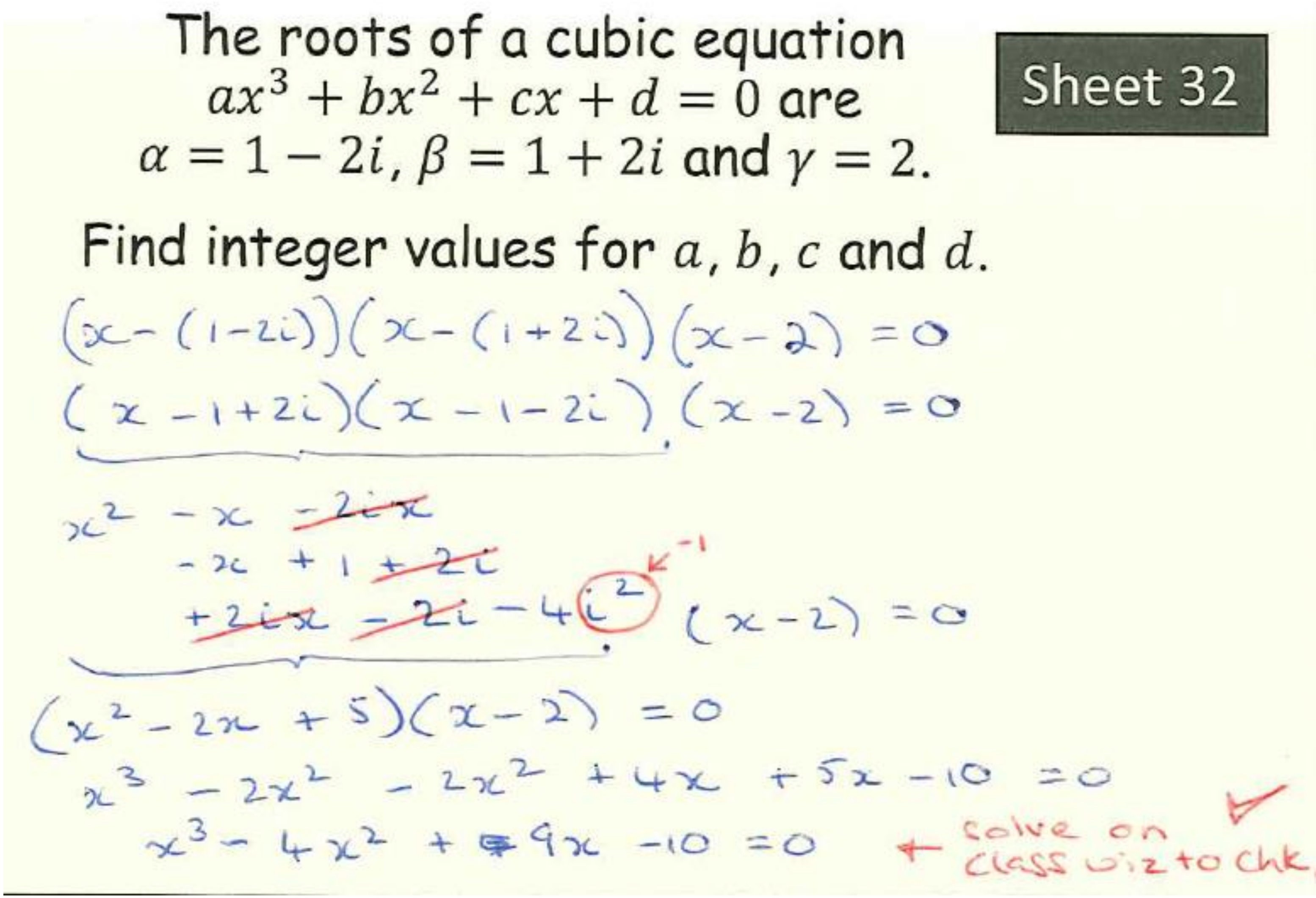
Given that $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ and $\boldsymbol{n} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$, with O being the origin, find an equation of the plane:

a) In scalar product form

b) In Cartesian form



The region R is bounded by the curve with equation $y = x^3 + 2$, the line y = 7 and the y axis. The region is rotated through 360° about the y-axis. Find the exact volume of the solid generated.



Sheet 32

The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = 1 - 2i$, $\beta = 1 + 2i$ and $\gamma = 2$.

Find integer values for a, b, c and d.

$$(x-(1-2i))(x-(1+2i))(x-2)=0$$

$$\underbrace{(x-1+2i)(x-1-2i)}(x-2)=0$$

$$\underbrace{x^2 - x \cancel{-2ix} - x + 1 \cancel{+2i} \cancel{+2ix} \cancel{-2i} - 4\boxed{i^2}}(x-2)=0$$

-1

$$(x^2-2x+5)(x-2)=0$$

$$x^3-2x^2-2x^2+4x+5x-10=0$$

$$x^3-4x^2+9x-10=0$$

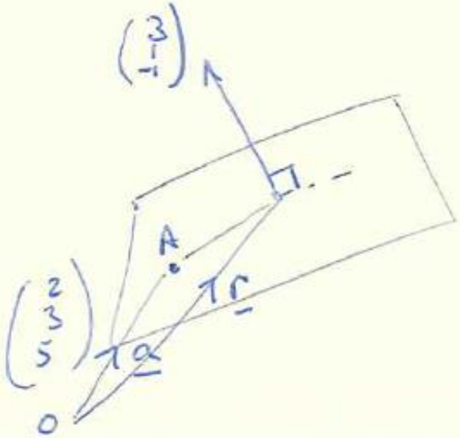
← solve on Class wiz to chk ✓

The plane Π passes through the point A and is perpendicular to the vector $\boldsymbol{n}$.

Given that $\overrightarrow{OA} = \begin{pmatrix}2\\3\\-5\end{pmatrix}$ and $\boldsymbol{n} = \begin{pmatrix}3\\1\\-1\end{pmatrix}$, with O being the origin, find an equation of the plane:

a) In scalar product form

b) In Cartesian form



$(\underline{r}-\underline{a})\cdot\underline{n}=0$

$\underline{r}\cdot\underline{n}-\underline{a}\cdot\underline{n}=0$

$\underline{r}\cdot\underline{n}=\underline{a}\cdot\underline{n}$

$\underline{r}\cdot\underline{n} = \begin{pmatrix}2\\3\\-5\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-1\end{pmatrix} = 6+3+5$
$= 14$

a)

$\Pi : r\cdot\begin{pmatrix}3\\1\\-1\end{pmatrix} = 14.$

b)

$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-1\end{pmatrix} = 14$

$3x + y - z = 14$

Prove, by induction, that the expression 'n³ - 7n + 9' is divisible by 3 for all positive integers $n \in \mathbb{Z}^+$

$n=1 \quad n^3 - 7n + 9 = 1 - 7 + 9 = 3$ is divisible by 3.

assume true for $n=k$ ie) $k^3 - 7k + 9 = 3m, \; m \in \mathbb{Z}^+$

if $n = k+1 \quad (k+1)^3 - 7(k+1) + 9$

$$= k^3 + 3k^2 + 3k + 1 - 7k - 7 + 9$$

$$= k^3 + 3k^2 \mp 4k + 3$$

$$= k(k^2 + 3k \mp 4) + 3$$

$$= \underbrace{k(k-1)(k+4)}_{\text{one of these will be a multiple of 3.}} + 3 \leftarrow \text{a multiple of 3}$$

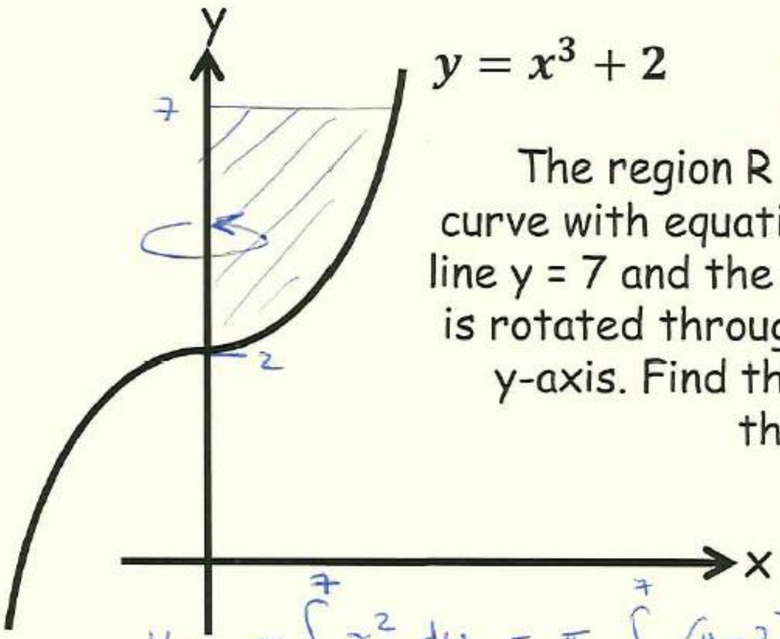
2	3	1	2	3	1
3	1	3	3	1	2
1	2	3	1	2	3
k-1	k	_	_	_	k+4

$\longleftrightarrow$

6 apart.

∴ true for k+1

∴ true for all $n \in \mathbb{Z}^+$.



$y = x^3 + 2$

$x^3 = y - 2$

$x = (y-2)^{1/3}$

The region R is bounded by the curve with equation $y = x^3 + 2$, the line y = 7 and the y axis. The region is rotated through 360° about the y-axis. Find the exact volume of the solid generated.

$$V = \pi\int_2^7 x^2\,dy = \pi\int_2^7 (y-2)^{2/3}\,dy = \frac{3\pi}{5}\left[(y-2)^{5/3}\right]_2^7$$

<u>guess</u>

$$(y-2)^{5/3} \xrightarrow{\text{diff}} \frac{5}{3}(y-2)^{2/3} \times 1$$

so $\frac{3}{5}(y-2)^{5/3} \xrightarrow[\text{diff}]{} (y-2)^{2/3}$.

$$V = \frac{3\pi}{5}\left\{5^{5/3} - 0^{5/3}\right\}$$

$$= \frac{3\pi}{5} \times 5^{5/3}$$

$$= 3\pi \times 5^{(5/3-1)}$$

$$= 3\pi \times 5^{2/3}$$

$$= \underline{\underline{3\sqrt[3]{5^2}\,\pi}}$$

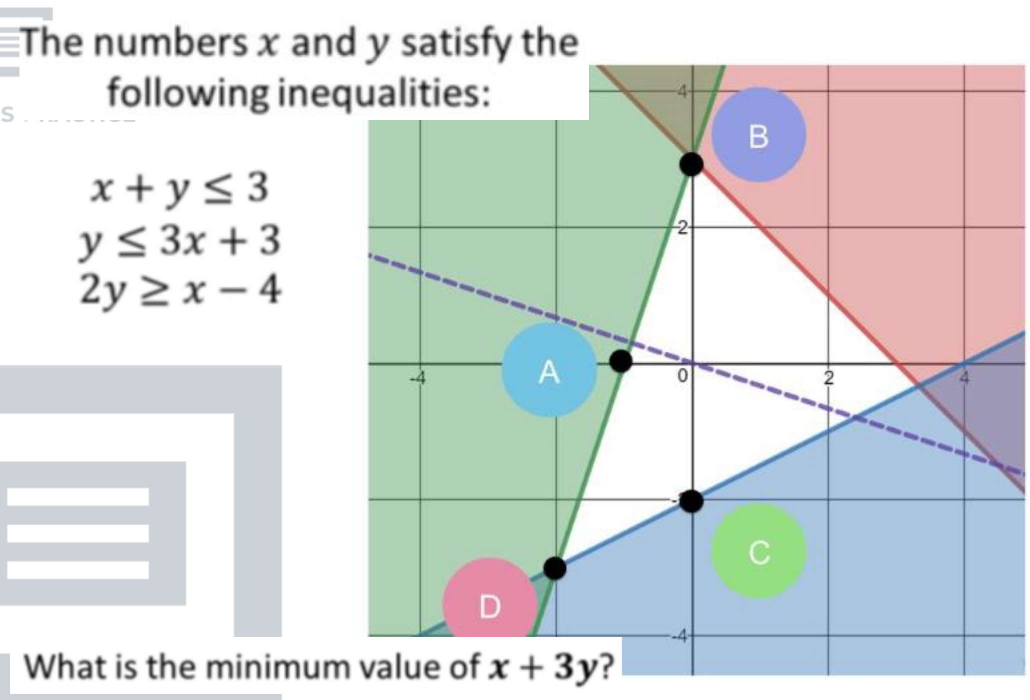
Sheet 34

Write √-36 in terms of i

Expand & simplify $(-7-22i)(1+3i)$

Find z + z*, and zz*, given that:

z = 2 - 7i



The numbers x and y satisfy the following inequalities:

$$x + y \leq 3$$
$$y \leq 3x + 3$$
$$2y \geq x - 4$$

What is the minimum value of $x + 3y$?

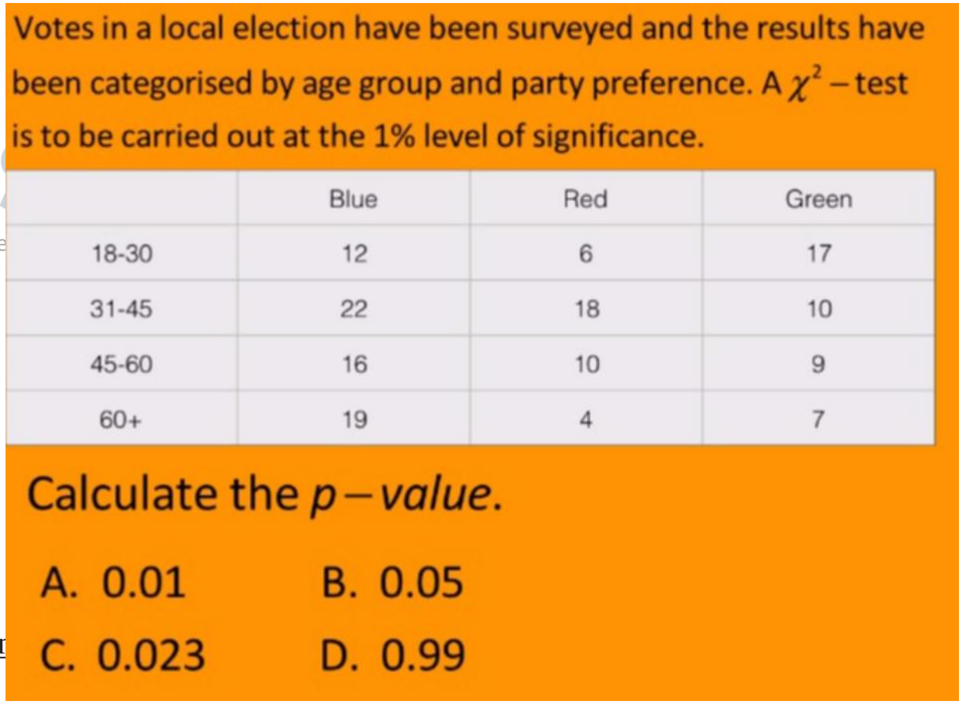
The line l has equation:

$$\boldsymbol{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The point P has position vector:

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

Show that P does not line on l.



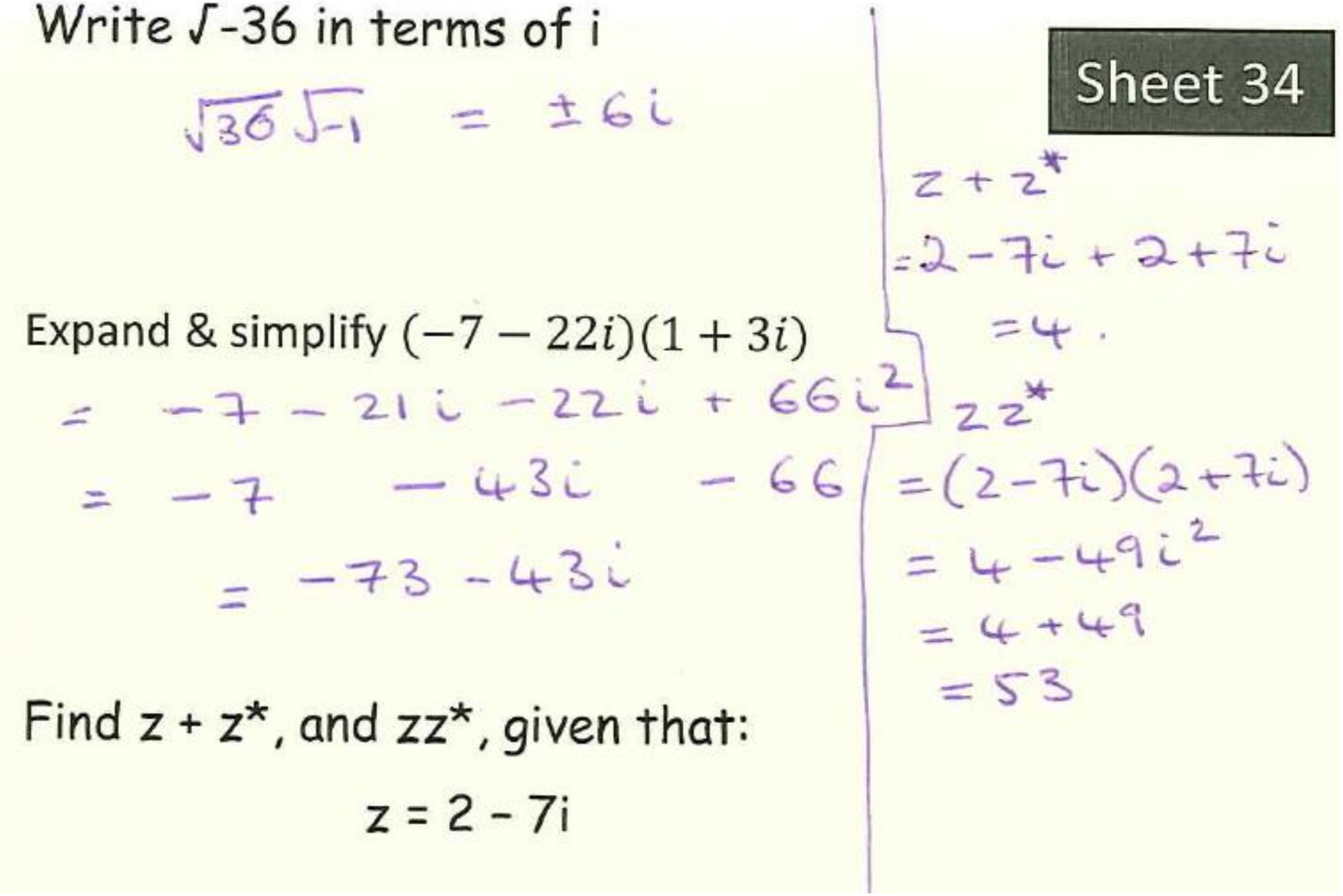
Votes in a local election have been surveyed and the results have been categorised by age group and party preference. A χ^2 – test is to be carried out at the 1% level of significance.

	Blue	Red	Green
18-30	12	6	17
31-45	22	18	10
45-60	16	10	9
60+	19	4	7

Calculate the $p-value$.

A. 0.01 B. 0.05

C. 0.023 D. 0.99



Sheet 34

Write $\sqrt{-36}$ in terms of i

$\sqrt{36}\sqrt{-1} = \pm 6i$

Expand & simplify $(-7-22i)(1+3i)$

$= -7 - 21i - 22i + 66i^2$

$= -7 - 43i - 66$

$= -73 - 43i$

Find z + z*, and zz*, given that:

$z = 2 - 7i$

$z + z^*$

$= 2 - 7i + 2 + 7i$

$= 4$.

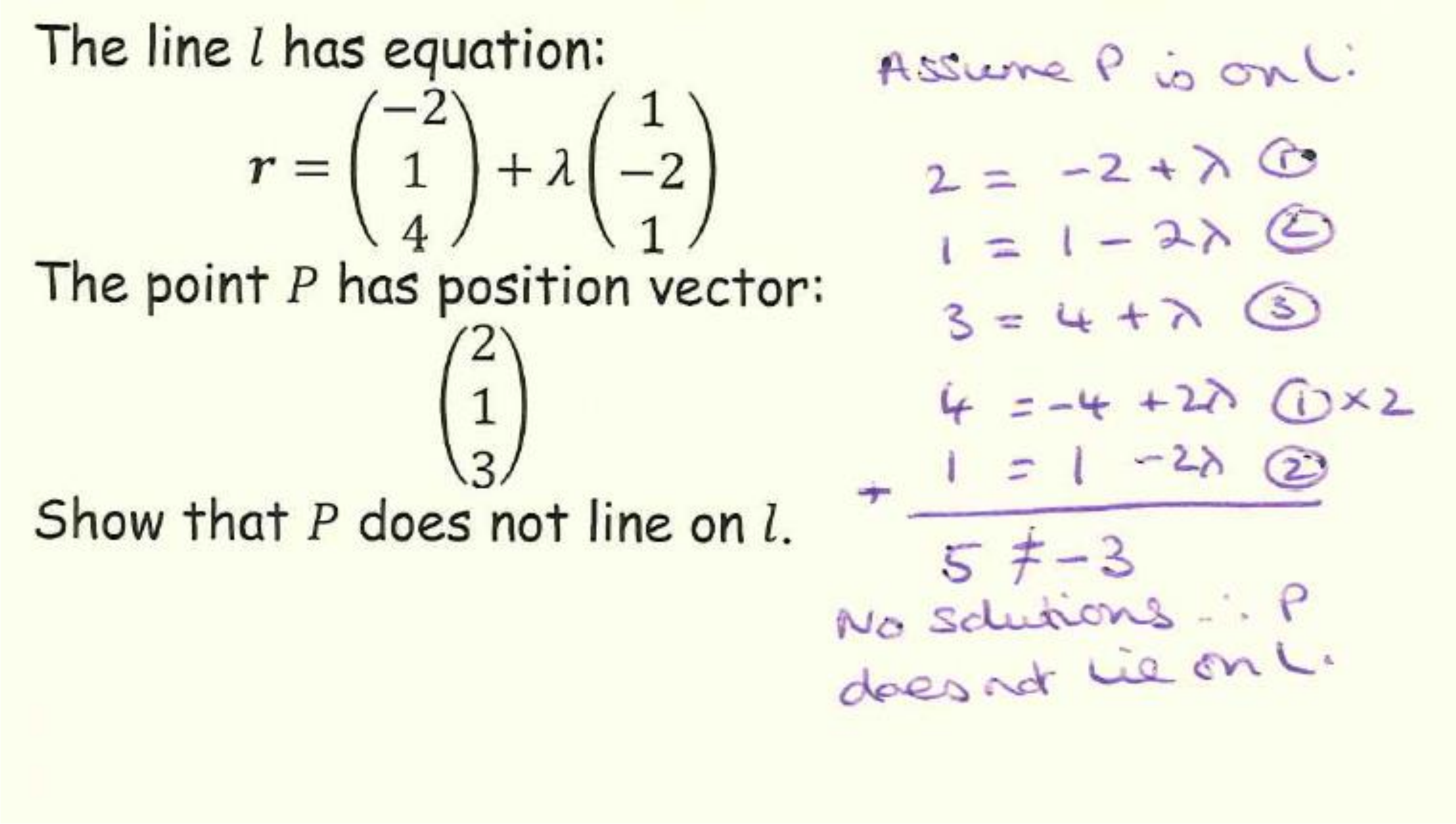
zz^*

$= (2-7i)(2+7i)$

$= 4 - 49i^2$

$= 4 + 49$

$= 53$



The line l has equation:

$$\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The point P has position vector:

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

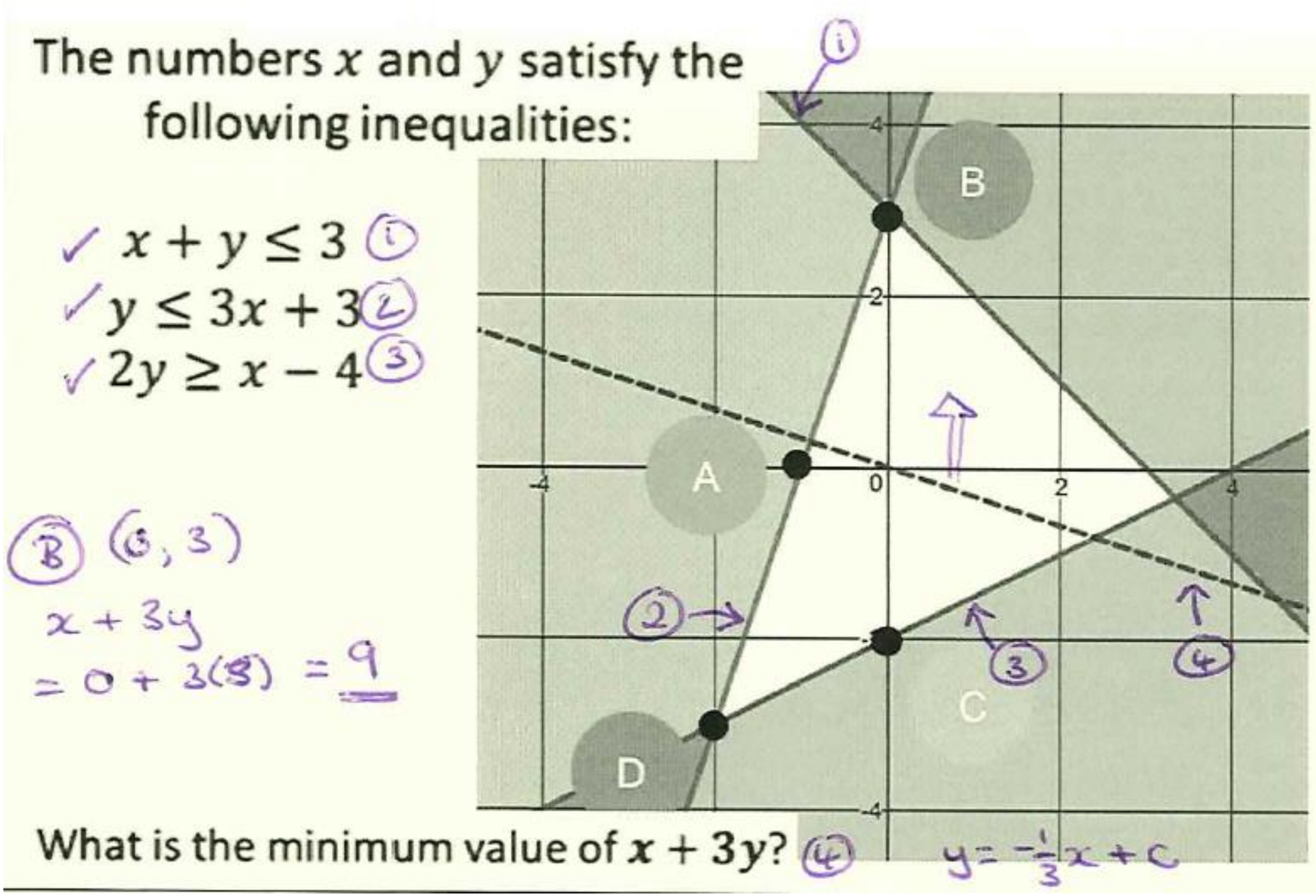
Show that P does not line on l.

Assume P is on l:

$$2 = -2 + \lambda \quad ①$$
$$1 = 1 - 2\lambda \quad ②$$
$$3 = 4 + \lambda \quad ③$$

$$4 = -4 + 2\lambda \quad ①\times 2$$
$$+ \quad 1 = 1 - 2\lambda \quad ②$$
$$5 \neq -3$$

No solutions ∴ P does not lie on l.



The numbers x and y satisfy the following inequalities:

✓ $x + y \le 3$ ⓘ

✓ $y \le 3x + 3$ ②

✓ $2y \ge x - 4$ ③

Ⓑ $(0, 3)$

$x + 3y$

$= 0 + 3(3) = \underline{9}$

What is the minimum value of $x + 3y$? ④

$y = -\frac{1}{3}x + c$

Votes in a local election have been surveyed and the results have been categorised by age group and party preference. A χ^2 – test is to be carried out at the 1% level of significance.

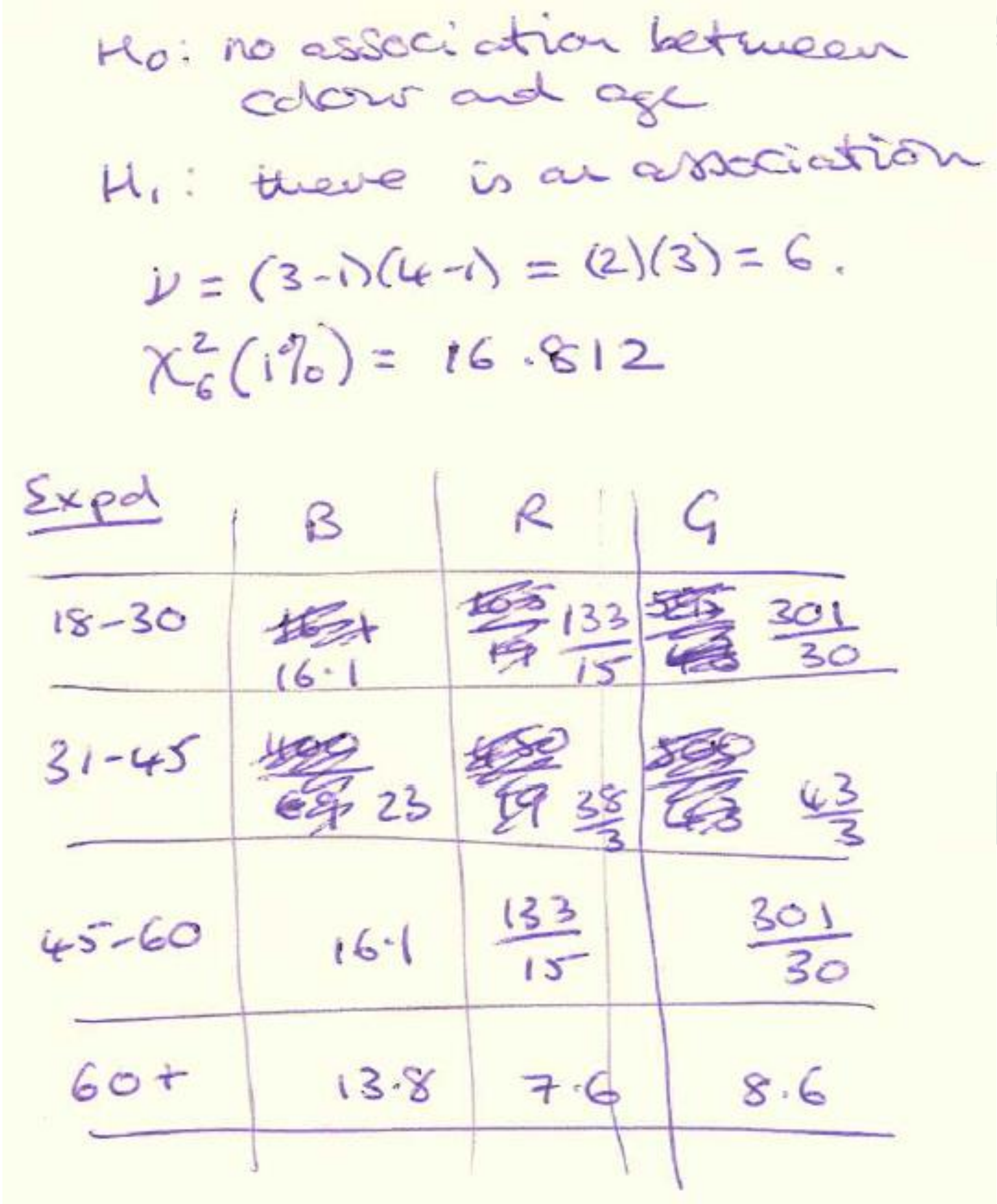
	Blue	Red	Green	
18-30	12	6	17	35
31-45	22	18	10	50
45-60	16	10	9	35
60+	19	4	7	30
	69	38	43	150

Calculate the *p – value.*

A. [✓ 0.01]　　B. 0.05

C. 0.023　　D. 0.99

Not asked for but here's what the full test should look like….



H_0: no association between colour and age

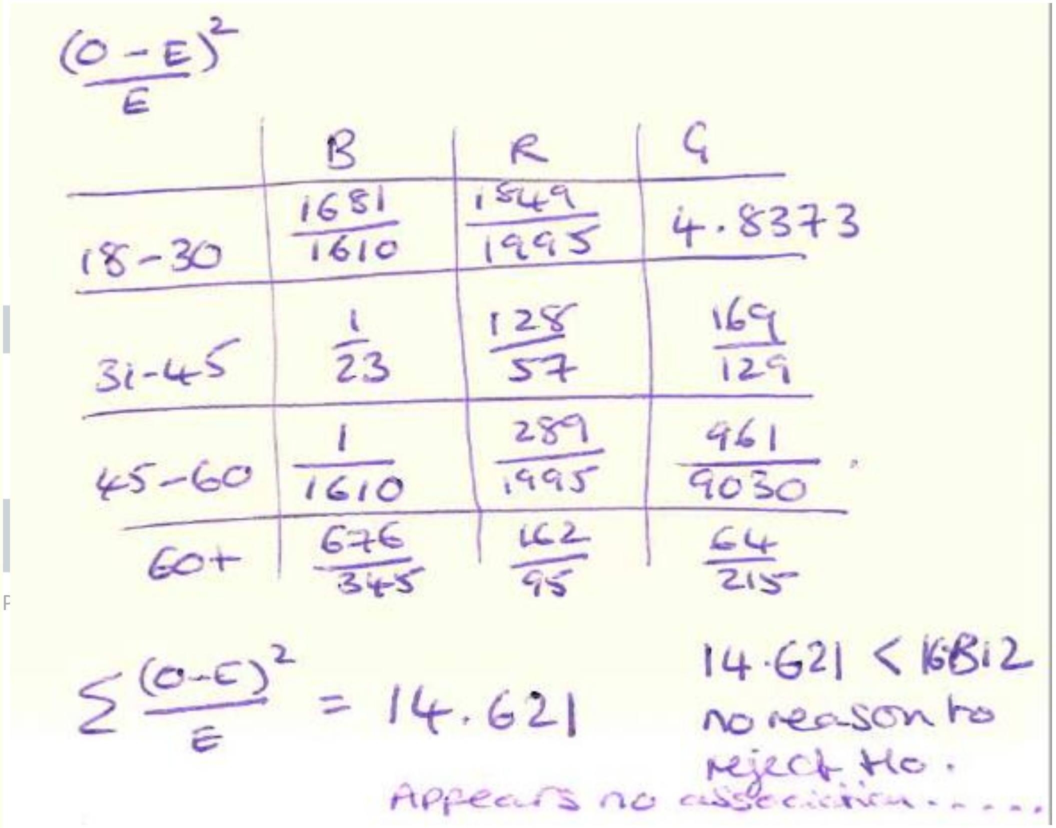
H_1: there is an association

$\nu = (3-1)(4-1) = (2)(3) = 6.$

$\chi^2_6(1\%) = 16.812$

Expd

	B	R	G
18-30	16·1	$\frac{133}{15}$	$\frac{301}{30}$
31-45	23	$\frac{38}{3}$	$\frac{43}{3}$
45-60	16·1	$\frac{133}{15}$	$\frac{301}{30}$
60+	13·8	7·6	8·6



$\frac{(O-E)^2}{E}$

	B	R	G
18-30	$\frac{1681}{1610}$	$\frac{1849}{1995}$	4.8373
31-45	$\frac{1}{23}$	$\frac{128}{57}$	$\frac{169}{129}$
45-60	$\frac{1}{1610}$	$\frac{289}{1995}$	$\frac{961}{9030}$
60+	$\frac{676}{345}$	$\frac{162}{95}$	$\frac{64}{215}$

$\sum \frac{(O-E)^2}{E} = 14.621$

$14.621 < 16.812$

no reason to reject Ho.

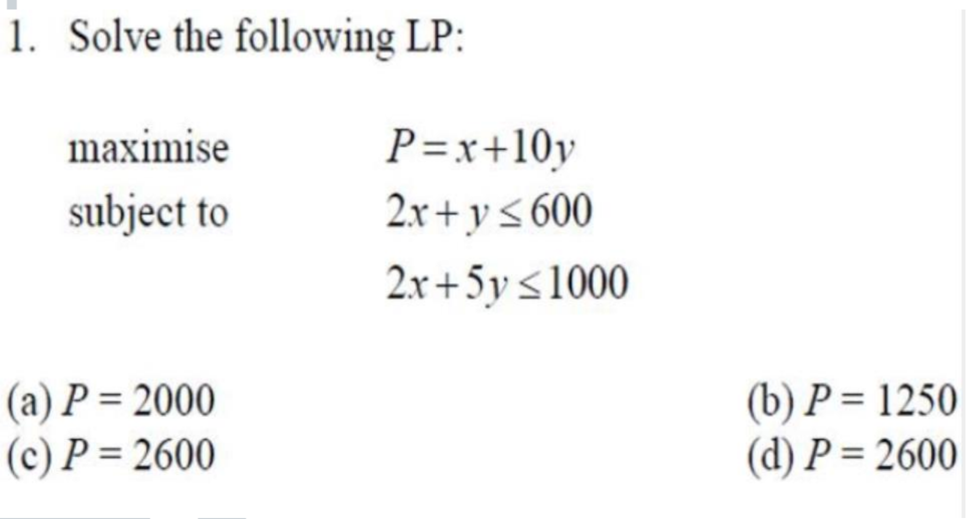
Appears no association.....

Sheet 35

Write √-28 in terms of i

Expand & simplify $(2-3i)(4-5i)(1+3i)$

Solve the equation: $x^2+6x+25=0$



1. Solve the following LP:

maximise $P=x+10y$

subject to $2x+y\leq 600$

$2x+5y\leq 1000$

(a) $P=2000$ (b) $P=1250$

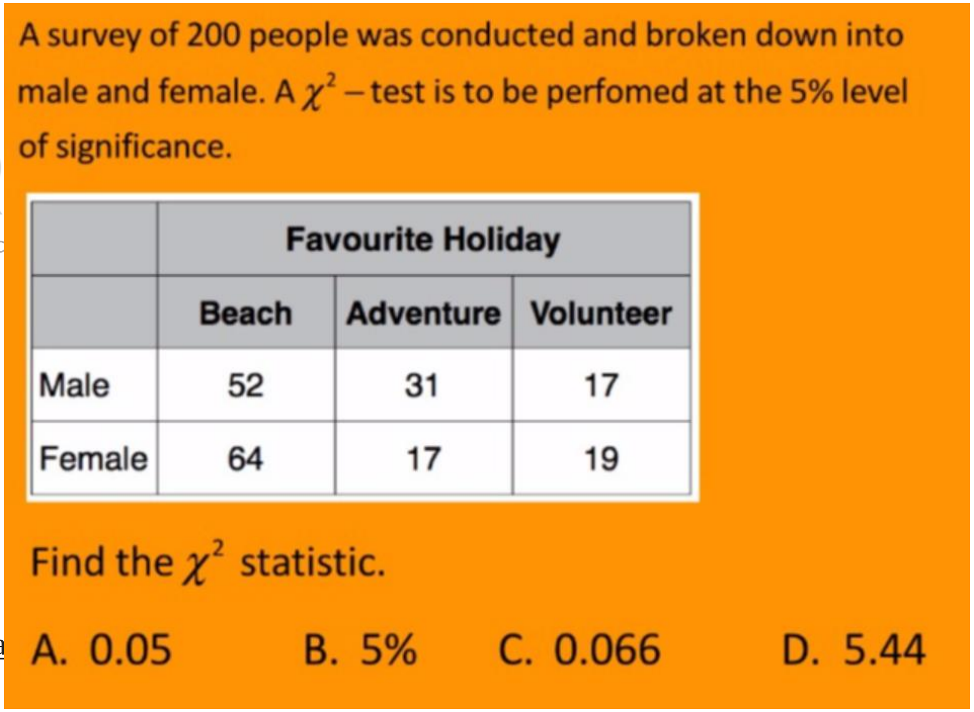
(c) $P=2600$ (d) $P=2600$

Find the equation of the straight line that passes through the point A, which has position vector $\begin{pmatrix}3\\-5\\4\end{pmatrix}$, and is parallel to the vector $\begin{pmatrix}7\\0\\-3\end{pmatrix}$.

The straight line l has vector equation:

$$\boldsymbol{r}=(3\boldsymbol{i}+2\boldsymbol{j}-5\boldsymbol{k})+t(\boldsymbol{i}-6\boldsymbol{j}-2\boldsymbol{k})$$

Given that the point $(a,b,0)$ lies on l, find the value of a and the value of b.

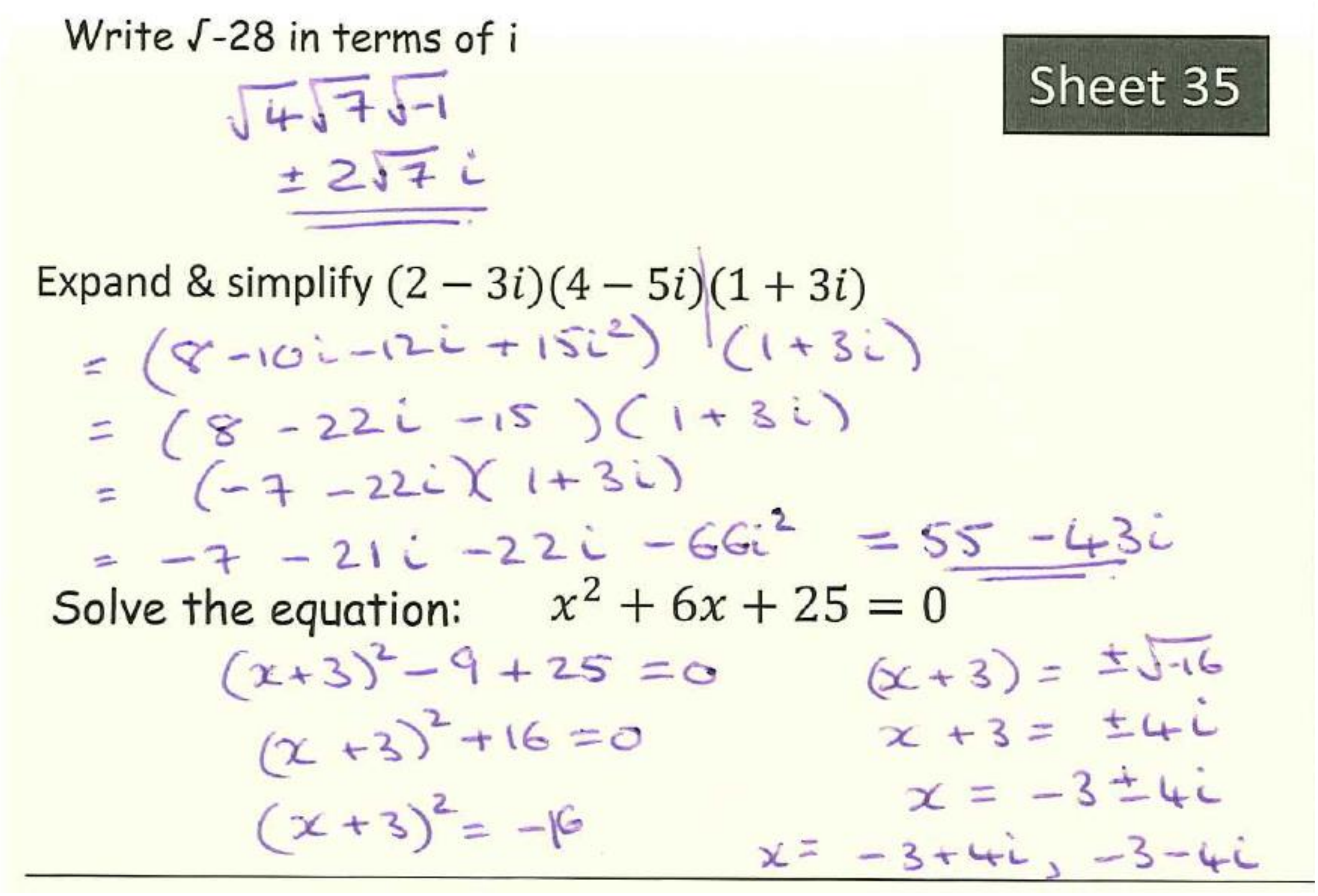


A survey of 200 people was conducted and broken down into male and female. A χ^2 – test is to be perfomed at the 5% level of significance.

	Favourite Holiday		
	Beach	Adventure	Volunteer
Male	52	31	17
Female	64	17	19

Find the χ^2 statistic.

A. 0.05 B. 5% C. 0.066 D. 5.44



Sheet 35

Write √-28 in terms of i

$\sqrt{4}\sqrt{7}\sqrt{-1}$

$\pm 2\sqrt{7}\,i$

Expand & simplify $(2-3i)(4-5i)(1+3i)$

$= (8 - 10i - 12i + 15i^2)(1+3i)$

$= (8 - 22i - 15)(1+3i)$

$= (-7 - 22i)(1+3i)$

$= -7 - 21i - 22i - 66i^2 = 55 - 43i$

Solve the equation: $x^2 + 6x + 25 = 0$

$(x+3)^2 - 9 + 25 = 0$

$(x+3)^2 + 16 = 0$

$(x+3)^2 = -16$

$(x+3) = \pm\sqrt{-16}$

$x + 3 = \pm 4i$

$x = -3 \pm 4i$

$x = -3+4i, \ -3-4i$

Find the equation of the straight line that passes through the point A,

which has position vector $\begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$, and

is parallel to the vector $\begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}$.

$$r = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 0 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -6 \\ -2 \end{pmatrix}$$

The straight line l has vector equation:

$$\boldsymbol{r} = (3\boldsymbol{i} + 2\boldsymbol{j} - 5\boldsymbol{k}) + t(\boldsymbol{i} - 6\boldsymbol{j} - 2\boldsymbol{k})$$

Given that the point $(a, b, 0)$ lies on l, find the value of a and the value of b.

$$-5 - 2t = 0$$
$$-5 = 2t$$
$$-\tfrac{5}{2} = t$$

$$a = 3 - \tfrac{5}{2} = \tfrac{1}{2}$$
$$b = 2 + 15 = 17$$

1. Solve the following LP:

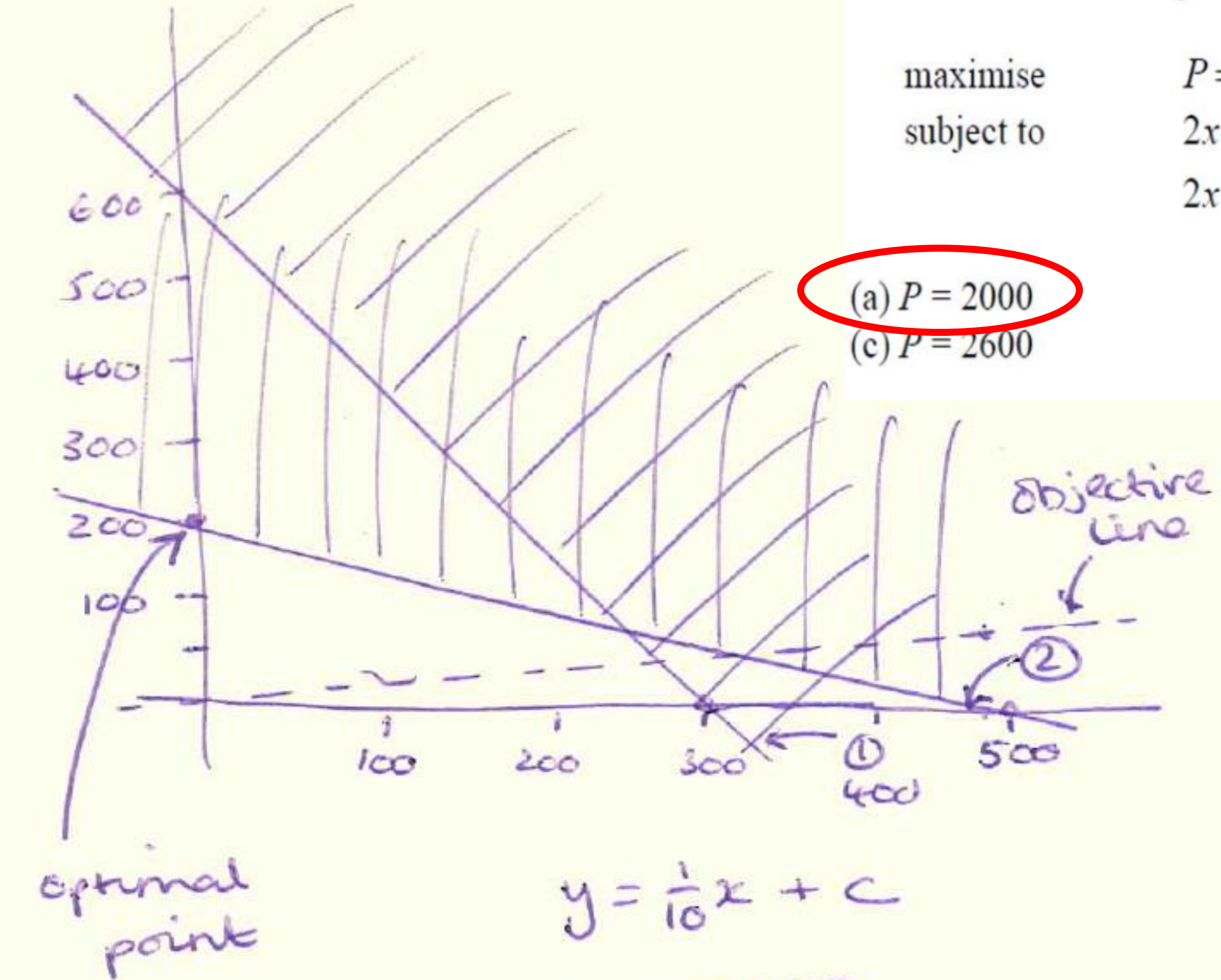
maximise $P = x + 10y$

subject to $2x + y \le 600$

$2x + 5y \le 1000$

[✓ (a) $P = 2000$] (b) $P = 1250$

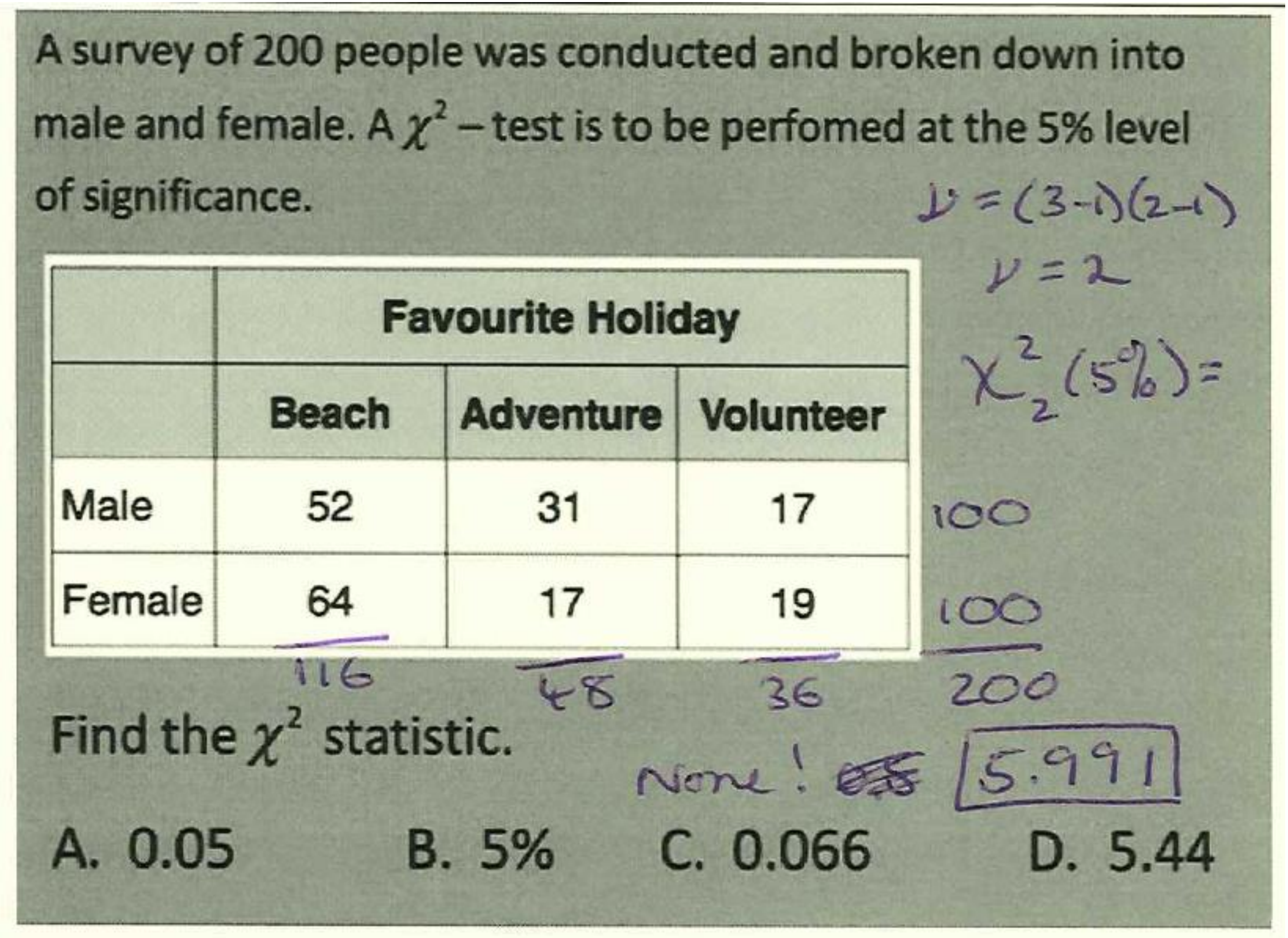
(c) $P = 2600$ (d) $P = 2600$



$y = \frac{1}{10}x + c$

Let $x = 0$, $y = 200$

$P = x + 10y = 10(200) = \underline{\underline{2000}}$.



A survey of 200 people was conducted and broken down into male and female. A χ^2 – test is to be perfomed at the 5% level of significance.

$\nu = (3-1)(2-1)$

$\nu = 2$

$\chi^2_2(5\%) =$

	Favourite Holiday		
	Beach	**Adventure**	**Volunteer**
Male	52	31	17
Female	64	17	19

116	48	36	200

100

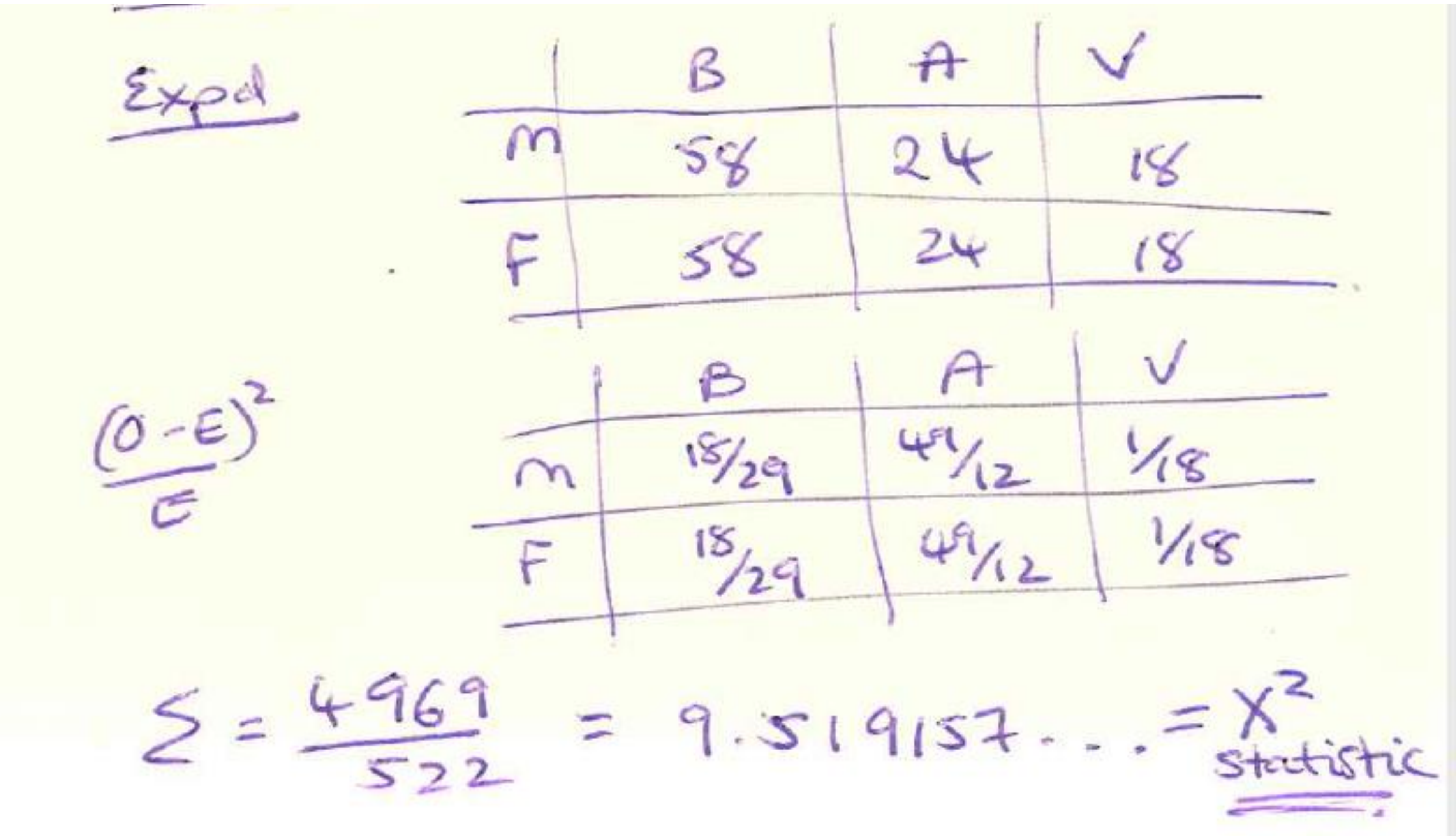
100

Find the χ^2 statistic.

None! $\boxed{5.991}$

A. 0.05 B. 5% C. 0.066 D. 5.44

In case they meant the X^2 test statistic ….



Exp'd

	B	A	V
m	58	24	18
F	58	24	18

$\frac{(O-E)^2}{E}$

	B	A	V
m	$^{18}/_{29}$	$^{49}/_{12}$	$^{1}/_{18}$
F	$^{18}/_{29}$	$^{49}/_{12}$	$^{1}/_{18}$

$$\Sigma = \frac{4969}{522} = 9.519157... = X^2$$

statistic

Sheet 36

The plane Π has equation:

$\boldsymbol{r}.(\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k})=5$

The point P has coordinates:

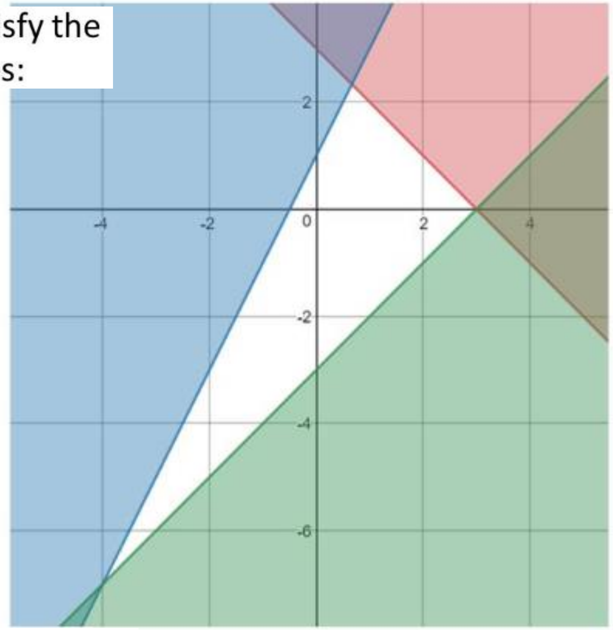
$$(1,3,-2)$$

a) Find the shortest distance between P and Π

b) The point Q is a reflection of P in Π. Find the coordinates of Q.

The numbers x and y satisfy the following inequalities:

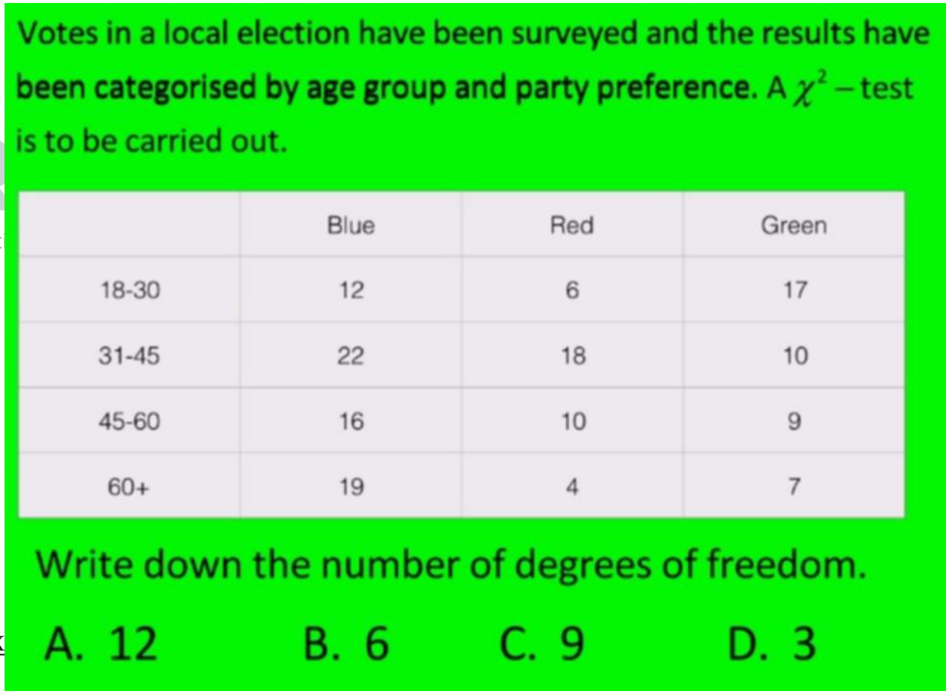
$$x+y\leq 3$$
$$y\leq 2x+1$$
$$y\geq x-3$$



find the maximum value of $\boldsymbol{2x+y}$

The matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are non-singular. Prove that $(\boldsymbol{PQ})^{-1}=\boldsymbol{Q}^{-1}\boldsymbol{P}^{-1}$.

HINT: Start by letting $\boldsymbol{C}=(\boldsymbol{PQ})^{-1}$



Votes in a local election have been surveyed and the results have been categorised by age group and party preference. A χ^2-test is to be carried out.

	Blue	Red	Green
18-30	12	6	17
31-45	22	18	10
45-60	16	10	9
60+	19	4	7

Write down the number of degrees of freedom.

A. 12 B. 6 C. 9 D. 3

Sheet 36

The plane Π has equation:

$\boldsymbol{r}.(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) = 5$

The point P has coordinates:

$$(1,3,-2)$$

a) Find the shortest distance between P and Π

b) The point Q is a reflection of P in Π. Find the coordinates of Q.

a) formula Book $\pi: \; x + 2y + 2k - 5 = 0$

$$\text{dist} = \frac{|1 + 6 - 4 - 5|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{2}{3}$$

b) $Q = P + 2\left(\frac{2}{3}\right) \times$ unit direction vector (Normal) $= \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \frac{4}{3} \times \frac{1}{3\cancel{9}} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\sqrt{1^2+2^2+2^2}$

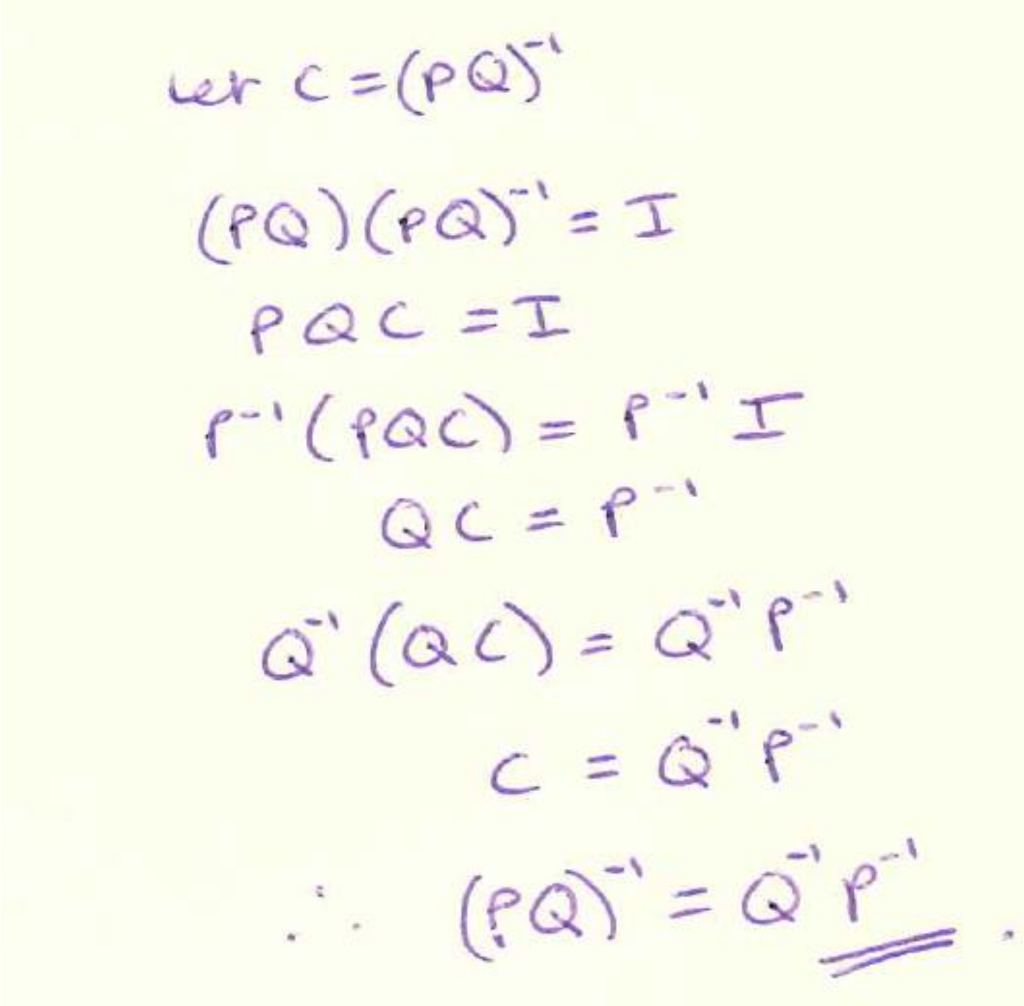
$$= \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \frac{4}{9} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 13/9 \\ 35/9 \\ -10/9 \end{pmatrix}$$

✓

The matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are non-singular. Prove that $(\boldsymbol{PQ})^{-1} = \boldsymbol{Q}^{-1}\boldsymbol{P}^{-1}$.

HINT: Start by letting $\boldsymbol{C} = (\boldsymbol{PQ})^{-1}$



let $C = (PQ)^{-1}$

$$(PQ)(PQ)^{-1} = I$$

$$PQC = I$$

$$P^{-1}(PQC) = P^{-1}I$$

$$QC = P^{-1}$$

$$Q^{-1}(QC) = Q^{-1}P^{-1}$$

$$C = Q^{-1}P^{-1}$$

$$\therefore \quad (PQ)^{-1} = \underline{\underline{Q^{-1}P^{-1}}} \,.$$

The numbers x and y satisfy the following inequalities:

✓ $x + y \leq 3$ ①

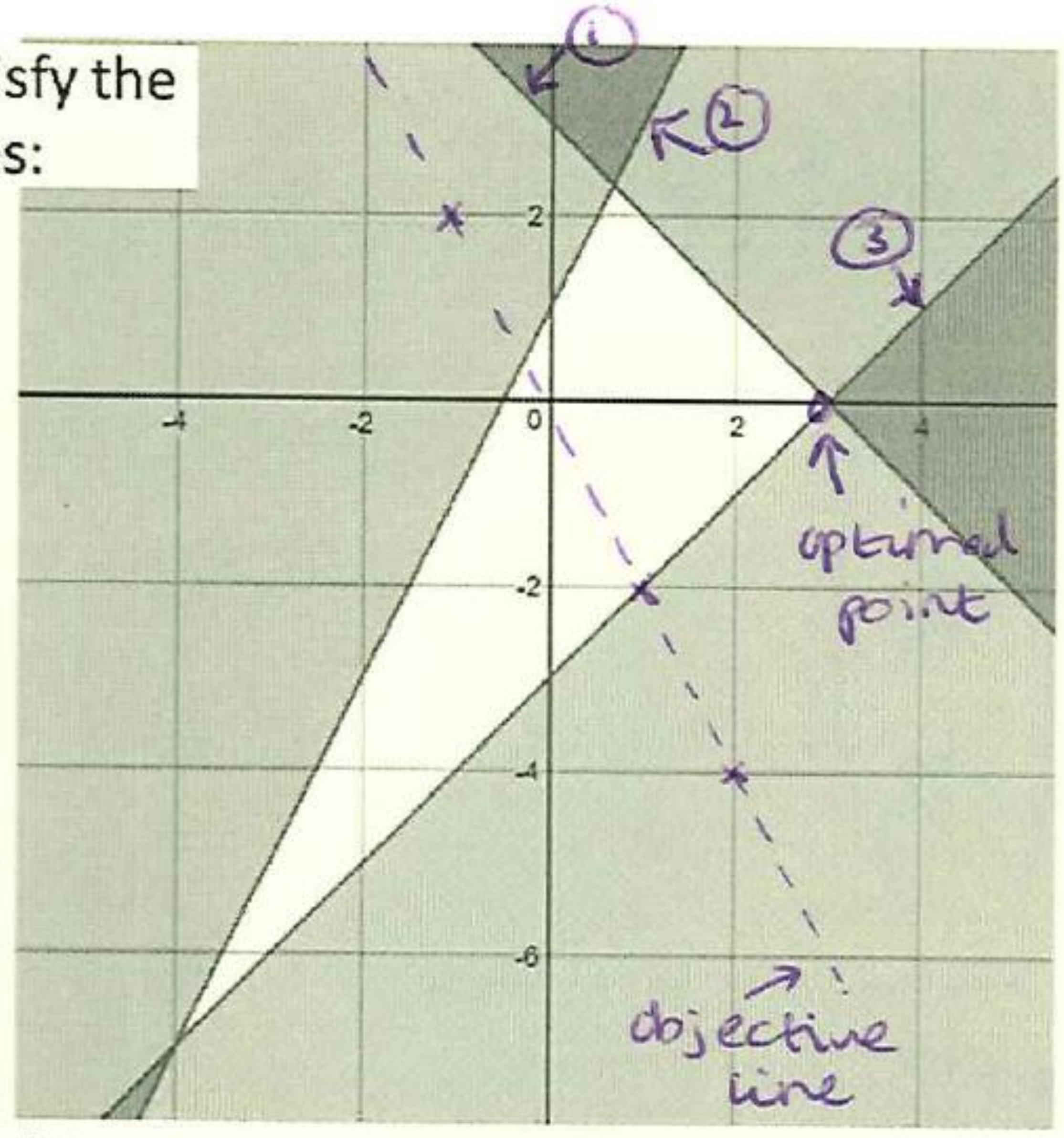
✓ $y \leq 2x + 1$ ②

✓ $y \geq x - 3$ ③

(3, 0)

$2x + \cancel{3}y$

$= \underline{\underline{6}}.$



find the maximum value of $\mathbf{2x + y}$

$y = -2x + c$

Votes in a local election have been surveyed and the results have been categorised by age group and party preference. A χ^2 – test is to be carried out.

	Blue	Red	Green
18-30	12	6	17
31-45	22	18	10
45-60	16	10	9
60+	19	4	7

Write down the number of degrees of freedom.

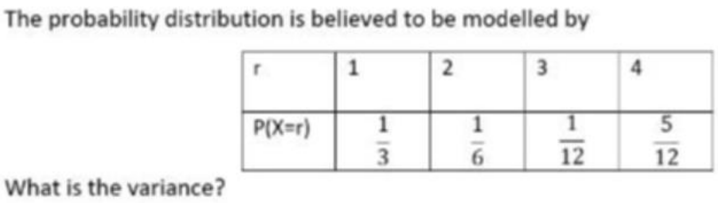
A. 12 [✓ B. 6] C. 9 D. 3

$\nu = (3-1)(4-1)$
$= 2(3)$
$= 6$

Given that the Matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{bmatrix}$

find A^{-1}

Sheet 37



The probability distribution is believed to be modelled by

r	1	2	3	4
P(X=r)	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{5}{12}$

What is the variance?

A: $Var(X) = \frac{101}{12}$

B: $Var(X) = 1.76$

C: $Var(X) = \frac{251}{144}$

D: $Var(X) = \frac{127}{144}$

Given that 3 + i is a root of the quartic equation:

$$2x^4 - 3x^3 - 39x^2 + 120x - 50 = 0$$

Solve the equation completely.

Prove by mathematical induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} (r^2) = \frac{1}{6} n(n+1)(2n+1)$$

Given that the Matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{bmatrix}$ find A^{-1}

Sheet 37

minors

Top $\begin{vmatrix} 4 & 1 \\ -1 & 0 \end{vmatrix} = 1$ $\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = -2$ $\begin{vmatrix} 0 & 4 \\ 2 & -1 \end{vmatrix} = -8$

Mid $\begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} = 1$ $\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2$ $\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -7$

bott $\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = -1$ $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$ $\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = 4$

Given that the Matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{bmatrix}$ find A^{-1}

Sheet 37

minors

Top $\begin{vmatrix} 4 & 1 \\ -1 & 0 \end{vmatrix} = 1$ $\quad \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = -2$ $\quad \begin{vmatrix} 0 & 4 \\ 2 & -1 \end{vmatrix} = -8$

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bot $\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = -1$ $\quad \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$ $\quad \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = 4$

Matrix of cofactors $= \begin{pmatrix} 1 & 2 & -8 \\ -1 & -2 & 7 \\ -1 & -1 & 4 \end{pmatrix} \xrightarrow[\text{Transpose}]{} \begin{pmatrix} 1 & -1 & -1 \\ 2 & -2 & -1 \\ -8 & 7 & 4 \end{pmatrix}$

Given that the Matrix $A = \begin{bmatrix}1 & 3 & 1\\0 & 4 & 1\\2 & -1 & 0\end{bmatrix}$ Sheet 37

find A^{-1}

minors

Top $\begin{vmatrix}4 & 1\\-1 & 0\end{vmatrix} = 1$ $\begin{vmatrix}0 & 1\\2 & 0\end{vmatrix} = -2$ $\begin{vmatrix}0 & 4\\2 & -1\end{vmatrix} = -8$

Mid $\begin{vmatrix}3 & 1\\-1 & 0\end{vmatrix} = 1$ $\begin{vmatrix}1 & 1\\2 & 0\end{vmatrix} = -2$ $\begin{vmatrix}1 & 3\\2 & -1\end{vmatrix} = -7$

bot $\begin{vmatrix}3 & 1\\4 & 1\end{vmatrix} = -1$ $\begin{vmatrix}1 & 1\\0 & 1\end{vmatrix} = 1$ $\begin{vmatrix}1 & 3\\0 & 4\end{vmatrix} = 4$

Matrix of cofactors $= \begin{pmatrix}1 & 2 & -8\\-1 & -2 & 7\\-1 & -1 & 4\end{pmatrix} \xrightarrow[\text{Transpose}]{} \begin{pmatrix}1 & -1 & -1\\2 & -2 & -1\\-8 & 7 & 4\end{pmatrix}$

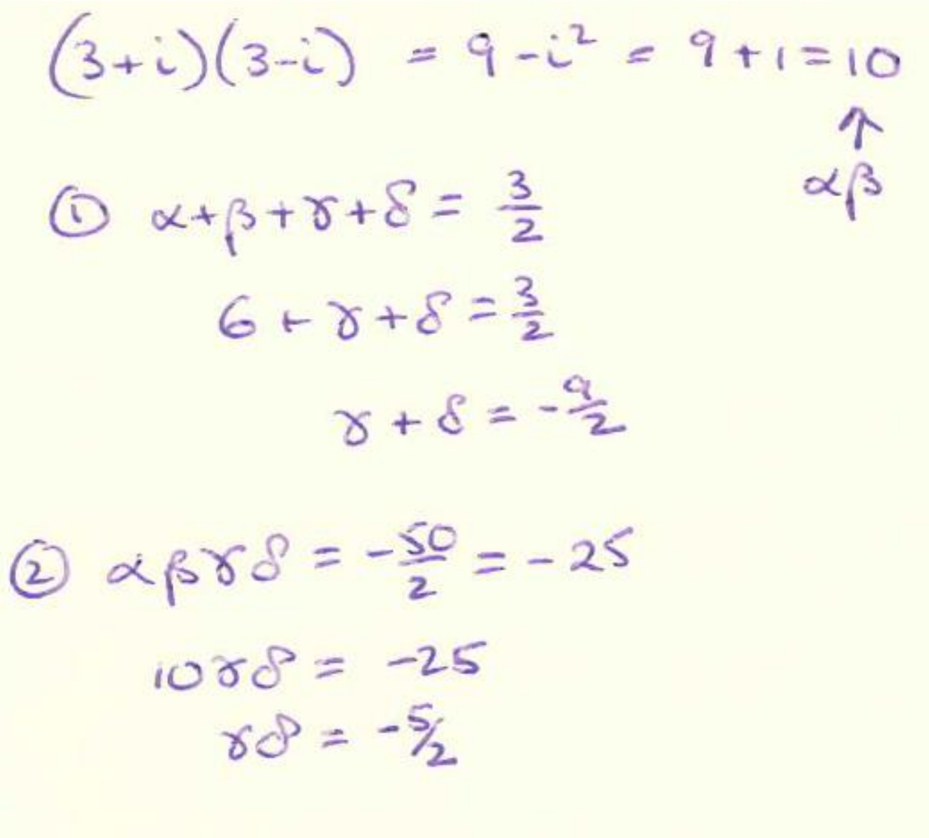
$$A^{-1} = -1\begin{pmatrix}1 & -1 & -1\\2 & -2 & -1\\-8 & 7 & 4\end{pmatrix}$$

$$= \begin{pmatrix}-1 & 1 & 1\\-2 & 2 & 1\\8 & -7 & 4\end{pmatrix}$$

Given that 3 + i is a root of the
quartic equation:

$$2x^4 - 3x^3 - 39x^2 + 120x - 50 = 0$$

Solve the equation completely.



$(3+i)(3-i) = 9 - i^2 = 9 + 1 = 10$

$\uparrow$

$\alpha\beta$

① $\alpha + \beta + \gamma + \delta = \frac{3}{2}$

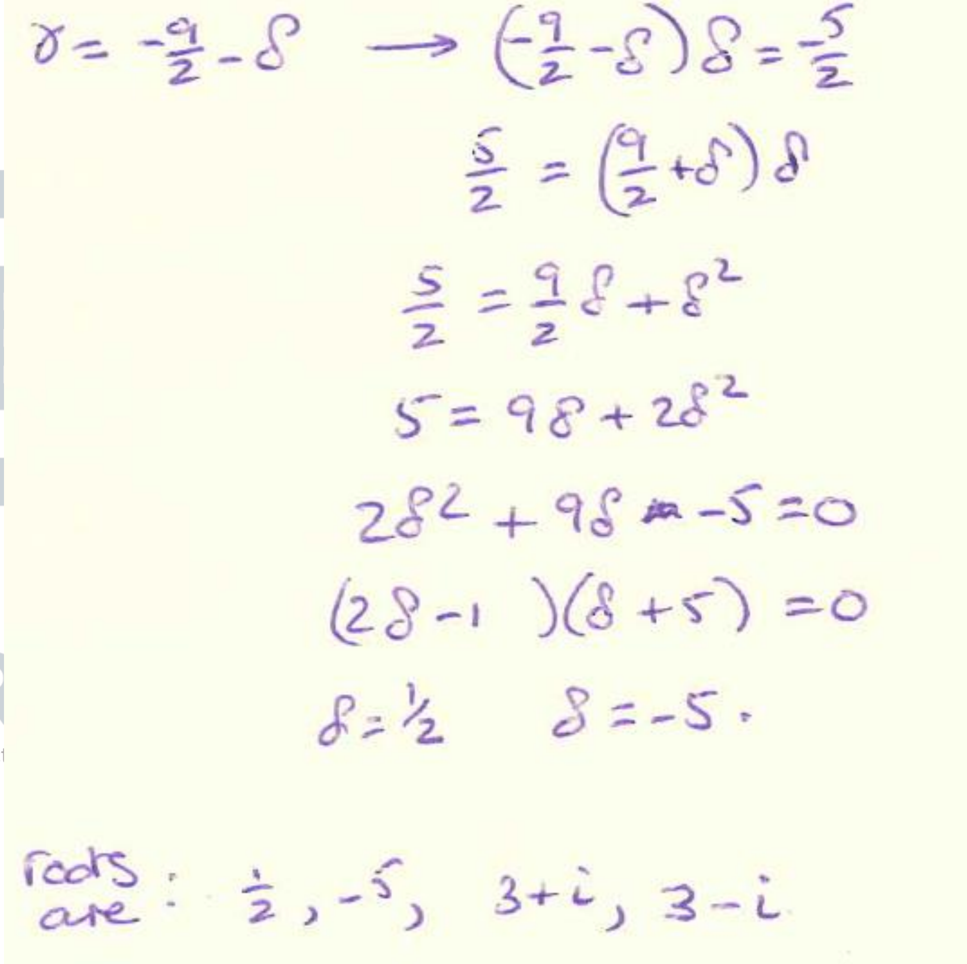
$6 + \gamma + \delta = \frac{3}{2}$

$\gamma + \delta = -\frac{9}{2}$

② $\alpha\beta\gamma\delta = -\frac{50}{2} = -25$

$10\gamma\delta = -25$

$\gamma\delta = -5/2$



$\gamma = -\frac{9}{2} - \delta \longrightarrow \left(-\frac{9}{2} - \delta\right)\delta = -\frac{5}{2}$

$\frac{5}{2} = \left(\frac{9}{2} + \delta\right)\delta$

$\frac{5}{2} = \frac{9}{2}\delta + \delta^2$

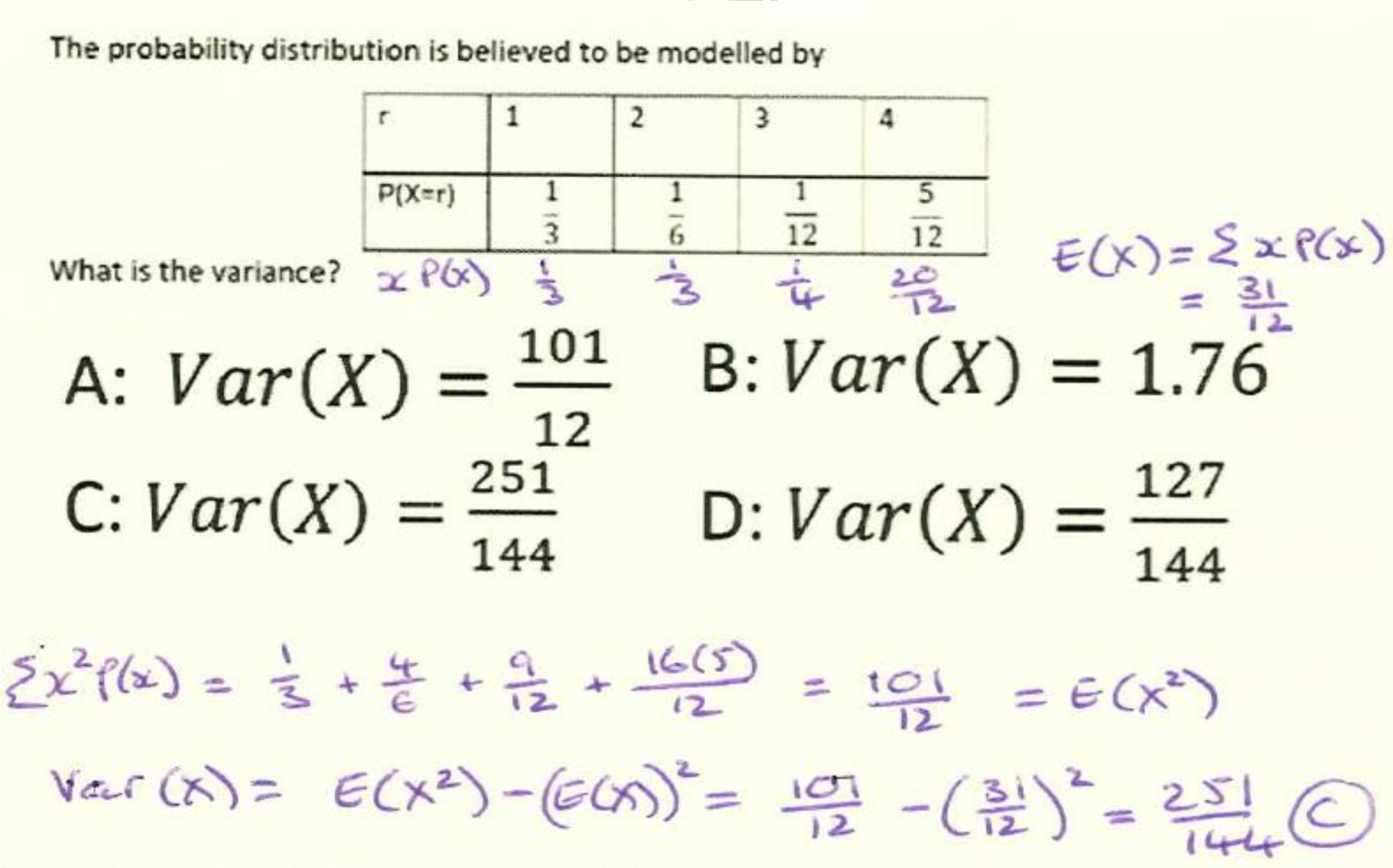
$5 = 9\delta + 2\delta^2$

$2\delta^2 + 9\delta - 5 = 0$

$(2\delta - 1)(\delta + 5) = 0$

$\delta = 1/2 \quad \delta = -5.$

roots are: $\frac{1}{2}, -5, 3+i, 3-i$



The probability distribution is believed to be modelled by

r	1	2	3	4
P(X=r)	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{5}{12}$
$xP(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{20}{12}$

What is the variance?

$E(X) = \sum x P(x)$

$= \frac{31}{12}$

A: $Var(X) = \frac{101}{12}$

B: $Var(X) = 1.76$

C: $Var(X) = \frac{251}{144}$

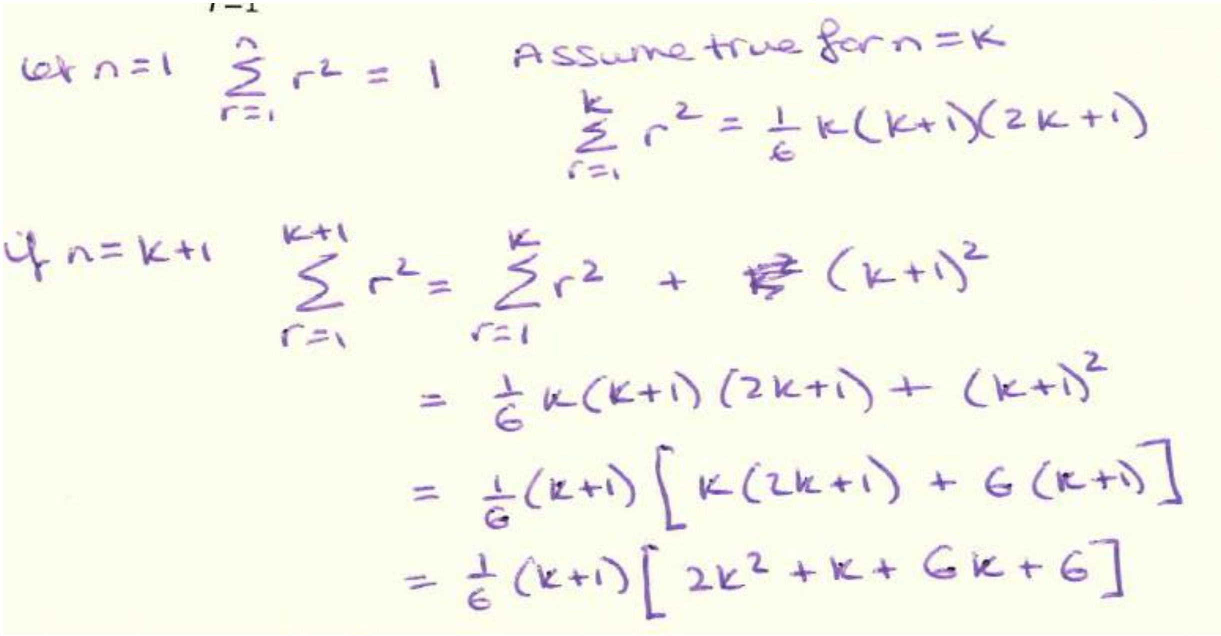
D: $Var(X) = \frac{127}{144}$

$\sum x^2 P(x) = \frac{1}{3} + \frac{4}{6} + \frac{9}{12} + \frac{16(5)}{12} = \frac{101}{12} = E(X^2)$

$Var(X) = E(X^2) - (E(X))^2 = \frac{101}{12} - \left(\frac{31}{12}\right)^2 = \frac{251}{144}$ Ⓒ

Prove by mathematical induction that,
for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n}(r^2) = \frac{1}{6}n(n+1)(2n+1)$$



let $n=1$ $\sum_{r=1}^{n} r^2 = 1$

Assume true for $n=k$

$$\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$$

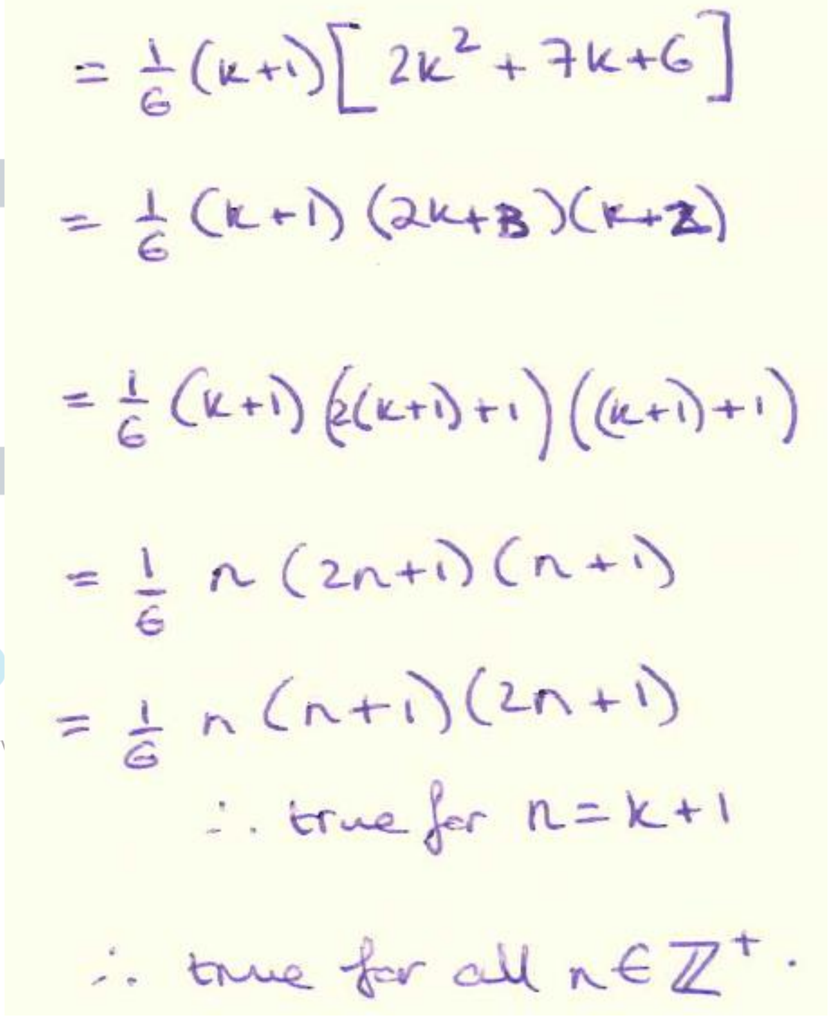
if $n=k+1$

$$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$$

$$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$

$$= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right]$$

$$= \frac{1}{6}(k+1)\left[2k^2 + k + 6k + 6\right]$$



$$= \frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right]$$

$$= \frac{1}{6}(k+1)(2k+3)(k+2)$$

$$= \frac{1}{6}(k+1)\left(2(k+1)+1\right)\left((k+1)+1\right)$$

$$= \frac{1}{6}n(2n+1)(n+1)$$

$$= \frac{1}{6}n(n+1)(2n+1)$$

∴ true for $n=k+1$

∴ true for all $n \in \mathbb{Z}^+$.

Given that -1 is a root of the equation:

$$x^3 - x^2 + 3x + k = 0$$

Find the other two roots of the equation.

Sheet 38

Prove, by induction, that $3^{2n} + 11$ is divisible by 4 for all positive integers $n \in \mathbb{Z}^+$

The matrix $\boldsymbol{A} = \begin{bmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ and the matrix $\boldsymbol{B}$ is such that $(\boldsymbol{AB})^{-1} = \begin{bmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{bmatrix}$

a) Show that $\boldsymbol{A}^{-1} = \boldsymbol{A}$

b) Find $\boldsymbol{B}^{-1}$

If:

$$|z - 5 - 3i| = 3$$

Sketch the locus of P(x,y) which is represented by z on an Argand diagram

Sheet 38

$(x+1)$ is a factor

↓

Given that -1 is a root of the equation:

let $\alpha = -1$

$$x^3 - x^2 + 3x + k = 0$$

$f(-1) = 0$

$-1 - 1 - 3 + k = 0$

$-5 + k = 0$

$\underline{k = 5}$.

Find the other two roots of the equation.

$\alpha + \beta + \gamma = -\frac{b}{a} = 1$

$\beta + \gamma = 2$ ①

$\beta = (2 - \gamma)$

$\gamma\beta\gamma = -\frac{d}{a} = -\frac{5}{1} = -5$

$-1\beta\gamma = -5$

$\beta\gamma = 5$ ②

$(2-\gamma)\gamma = 5$

$2\gamma - \gamma^2 = 5$

$\gamma^2 - 2\gamma + 5 = 0$

$(\gamma - 1)^2 - 1 + 5 = 0$

$(\gamma - 1)^2 = -4$

$\gamma - 1 = \pm 2i$

$\gamma = 1 \pm 2i$

The matrix $\boldsymbol{A} = \begin{bmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ and the matrix $\boldsymbol{B}$ is such that $(\boldsymbol{AB})^{-1} = \begin{bmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{bmatrix}$

a) Show that $\boldsymbol{A}^{-1} = \boldsymbol{A}$

b) Find $\boldsymbol{B}^{-1}$

$$\det A = -2\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

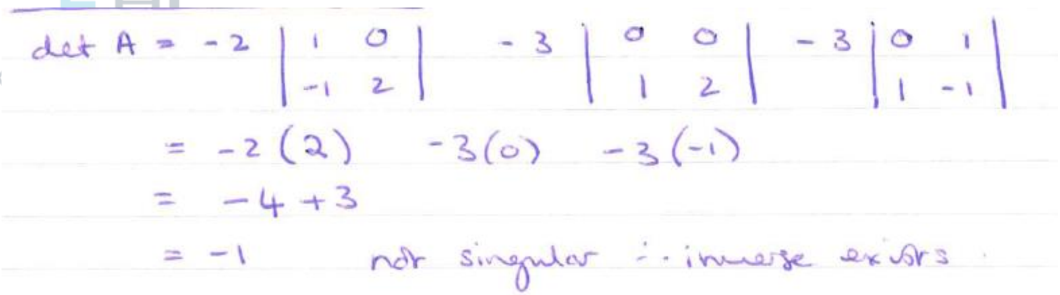
$$= -2(2) \quad -3(0) \quad -3(-1)$$

$$= -4 + 3$$

$= -1$ not singular $\therefore$ inverse exists

The matrix $\boldsymbol{A} = \begin{bmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ and the matrix

$\boldsymbol{B}$ is such that $(\boldsymbol{AB})^{-1} = \begin{bmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{bmatrix}$



$\det A = -2 \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} \quad -3 \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} \quad -3 \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$

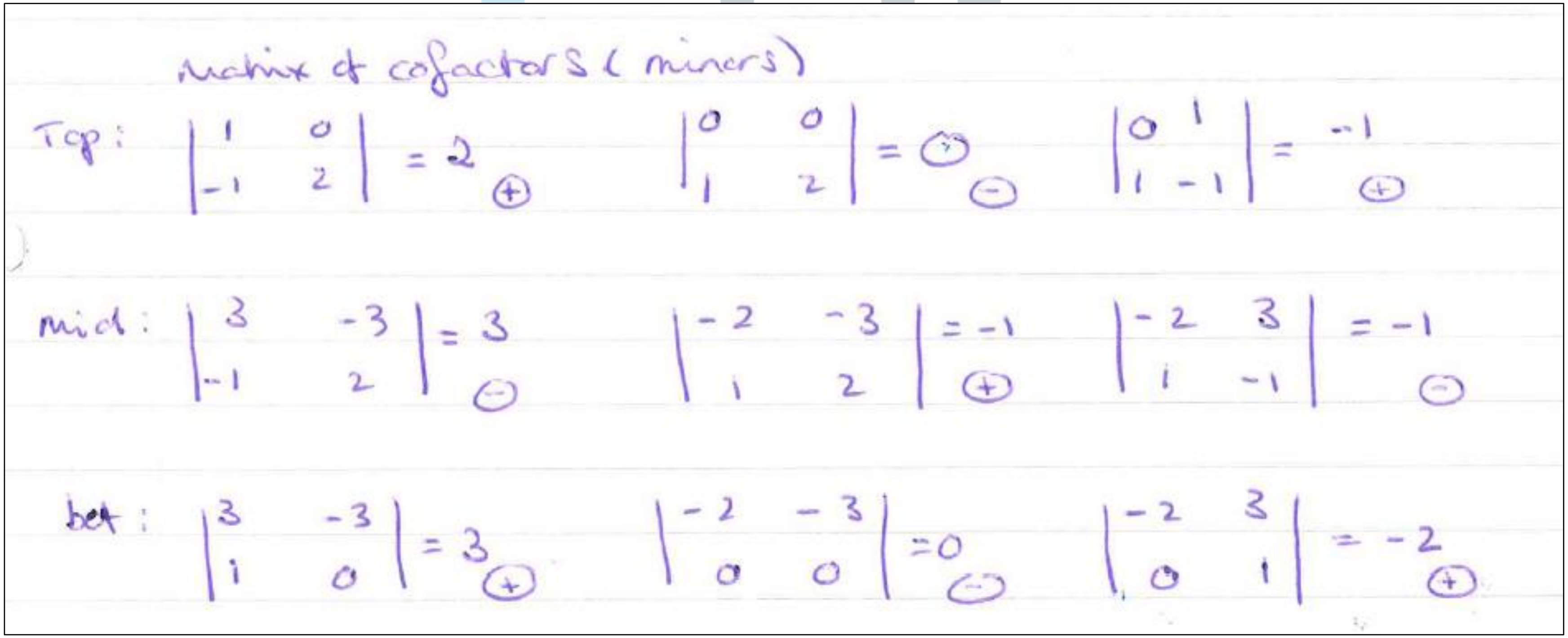
$= -2(2) \quad -3(0) \quad -3(-1)$

$= -4 + 3$

$= -1$ not singular $\therefore$ inverse exists

a) Show that $\boldsymbol{A}^{-1} = \boldsymbol{A}$

b) Find $\boldsymbol{B}^{-1}$



Matrix of cofactors (miners)

Top: $\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2 \; \oplus \qquad \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} = 0 \; \ominus \qquad \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1 \; \oplus$

mid: $\begin{vmatrix} 3 & -3 \\ -1 & 2 \end{vmatrix} = 3 \; \ominus \qquad \begin{vmatrix} -2 & -3 \\ 1 & 2 \end{vmatrix} = -1 \; \oplus \qquad \begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} = -1 \; \ominus$

bot: $\begin{vmatrix} 3 & -3 \\ 1 & 0 \end{vmatrix} = 3 \; \oplus \qquad \begin{vmatrix} -2 & -3 \\ 0 & 0 \end{vmatrix} = 0 \; \ominus \qquad \begin{vmatrix} -2 & 3 \\ 0 & 1 \end{vmatrix} = -2 \; \oplus$

The matrix $\boldsymbol{A} = \begin{bmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ and the matrix

$\boldsymbol{B}$ is such that $(\boldsymbol{AB})^{-1} = \begin{bmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{bmatrix}$

a) Show that $\boldsymbol{A}^{-1} = \boldsymbol{A}$

b) Find $\boldsymbol{B}^{-1}$

Matrix of cofactors (miners)

Top: $\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2 \;\oplus$ $\quad \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} = 0 \;\ominus$ $\quad \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1 \;\oplus$

mid: $\begin{vmatrix} 3 & -3 \\ -1 & 2 \end{vmatrix} = 3 \;\ominus$ $\quad \begin{vmatrix} -2 & -3 \\ 1 & 2 \end{vmatrix} = -1 \;\oplus$ $\quad \begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} = -1 \;\ominus$

bot: $\begin{vmatrix} 3 & -3 \\ 1 & 0 \end{vmatrix} = 3 \;\oplus$ $\quad \begin{vmatrix} -2 & -3 \\ 0 & 0 \end{vmatrix} = 0 \;\ominus$ $\quad \begin{vmatrix} -2 & 3 \\ 0 & 1 \end{vmatrix} = -2 \;\oplus$

$$\begin{pmatrix} 2 & 0 & -1 \\ -3 & -1 & 1 \\ 3 & 0 & -2 \end{pmatrix} \xrightarrow{\text{transpose}} \begin{pmatrix} 2 & -3 & 3 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{-1} \begin{pmatrix} 2 & -3 & 3 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix} = \underline{\underline{A}}.$$

The matrix $A = \begin{bmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ and the matrix

B is such that $(AB)^{-1} = \begin{bmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{bmatrix}$

a) Show that $A^{-1} = A$

b) Find B^{-1}

b) $(AB)^{-1}A = A^{-1}$

$$\begin{pmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{pmatrix}$$

$(AB)^{-1} = B^{-1}A^{-1} \xrightarrow{\times B} B(AB)^{-1} = BB^{-1}A^{-1}$

← Don't do this $= A^{-1} = A.$

$(AB)^{-1} = B^{-1}A^{-1}$ ← Do this

$(AB)^{-1}\underline{A} = (B^{-1})(A^{-1}\underline{A}) = B^{-1}$ $\times A$ (afterwards)

$$B^{-1} = \begin{pmatrix} 8 & -17 & 9 \\ -5 & 10 & -6 \\ -3 & 5 & -4 \end{pmatrix}\begin{pmatrix} -2 & 3 & -3 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -7 & -2 & -6 \\ 4 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

Prove, by induction, that $3^{2n} + 11$ is divisible by 4 for all positive integers $n \in \mathbb{Z}^+$

let $n = 1$ $\quad 3^{2n} + 11 = 3^2 + 11 = 9 + 11 = 20$
which is divisible by 4.

assume true for $n = k$ ie) $3^{2k} + 11$ is divisible by 4

if $n = k+1$

$$3^{2n} + 11 = 3^{2(k+1)} + 11$$
$$= 3^{2k+2} + 11$$
$$= 3^2 . 3^{2k} + 11$$
$$= 9.(3^{2k}) + 11$$
$$= 1(3^{2k}) + 8(3^{2k}) + 11$$
$$= \boxed{8(3^{2k})} + (3^{2k} + 11)$$

$\underline{4} \times 2 \times 3^{2k}$ is divisible by 4.

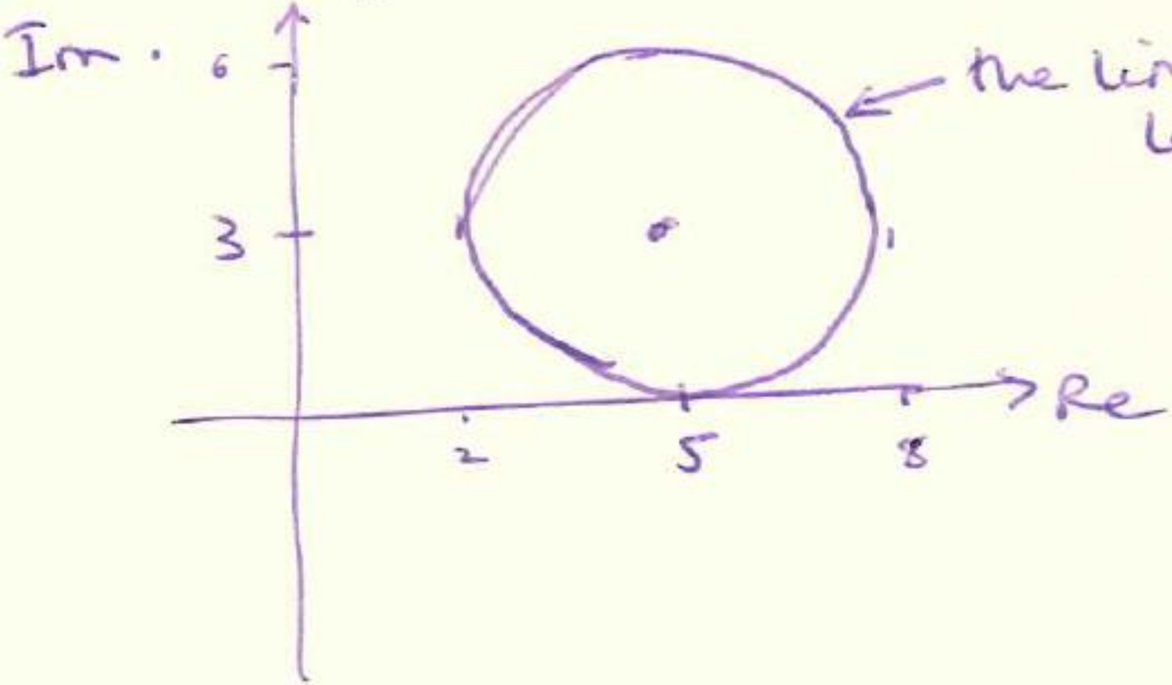
$(3^{2k} + 11)$ is divisible by 4

$\therefore$ true for $k+1$, and all $n \in \mathbb{Z}^+$

If: $|z-(5+3i)| = 3$

$$|z-5-3i| = 3$$

Sketch the locus of P(x,y) which is represented by z on an Argand diagram



the line is the locus.

Determine the number of solutions to this set of equations, and give a geometric interpretation:

$$x + y + z = 2$$

$$2x + 3y - z = 13$$

$$x - 2y + 3z = -11$$

Sheet 39

Use mathematical induction to prove that:

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^n = \begin{bmatrix} 1 & 1-2^n \\ 0 & 2^n \end{bmatrix} \quad for\ \ n \in \mathbb{Z}^+$$

Expand & Simplify $(2i)^5$

Write the following in the form $a + bi$ $\frac{(10+5i)}{(1+2i)}$

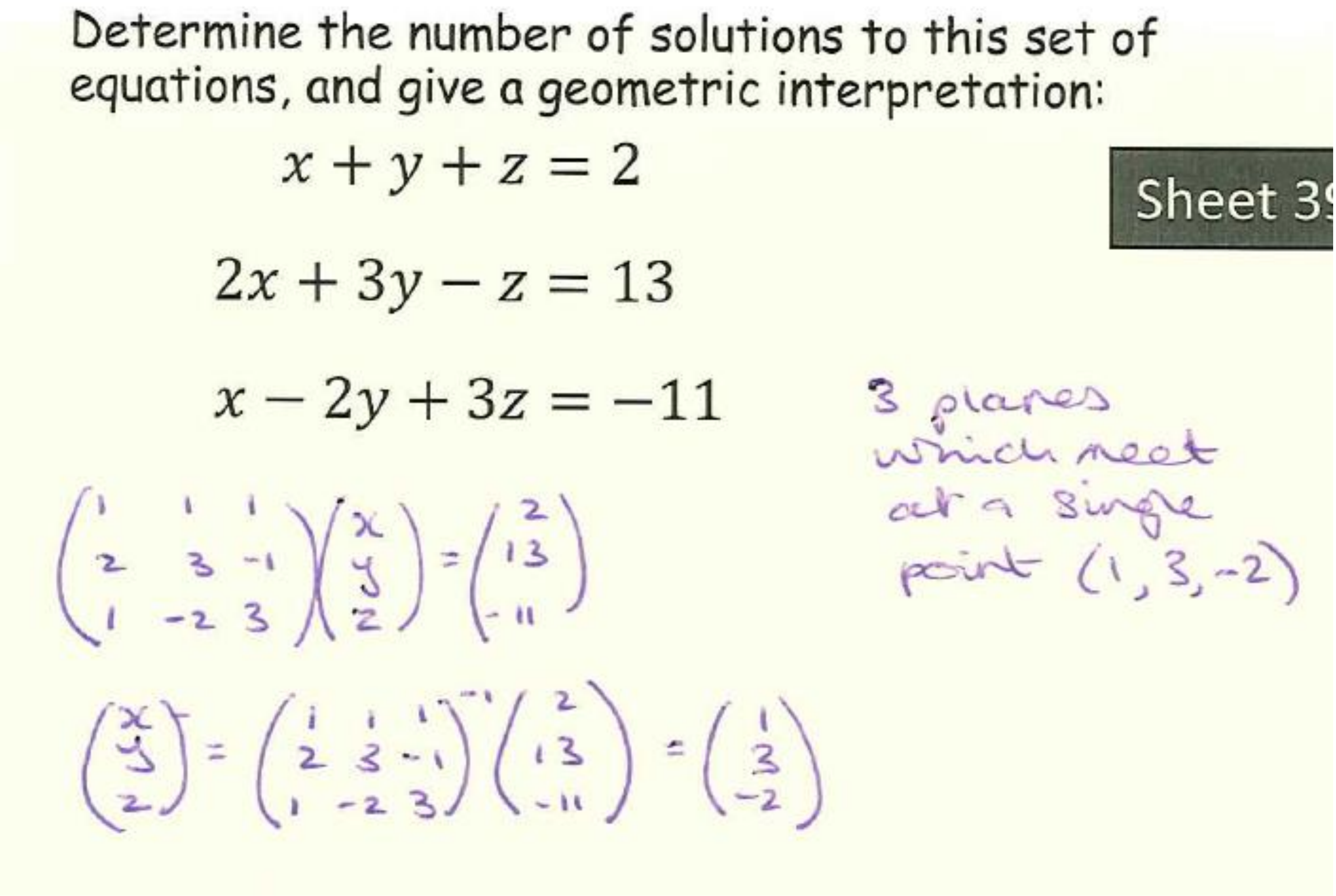
Find z + z*, and zz*, given that:

z = 2√2 + i√2

If:

$$|z - 5 - 3i| = 3$$

Find the maximum value of argz in the interval (-π,π)



Determine the number of solutions to this set of equations, and give a geometric interpretation:

Sheet 3

$$x + y + z = 2$$

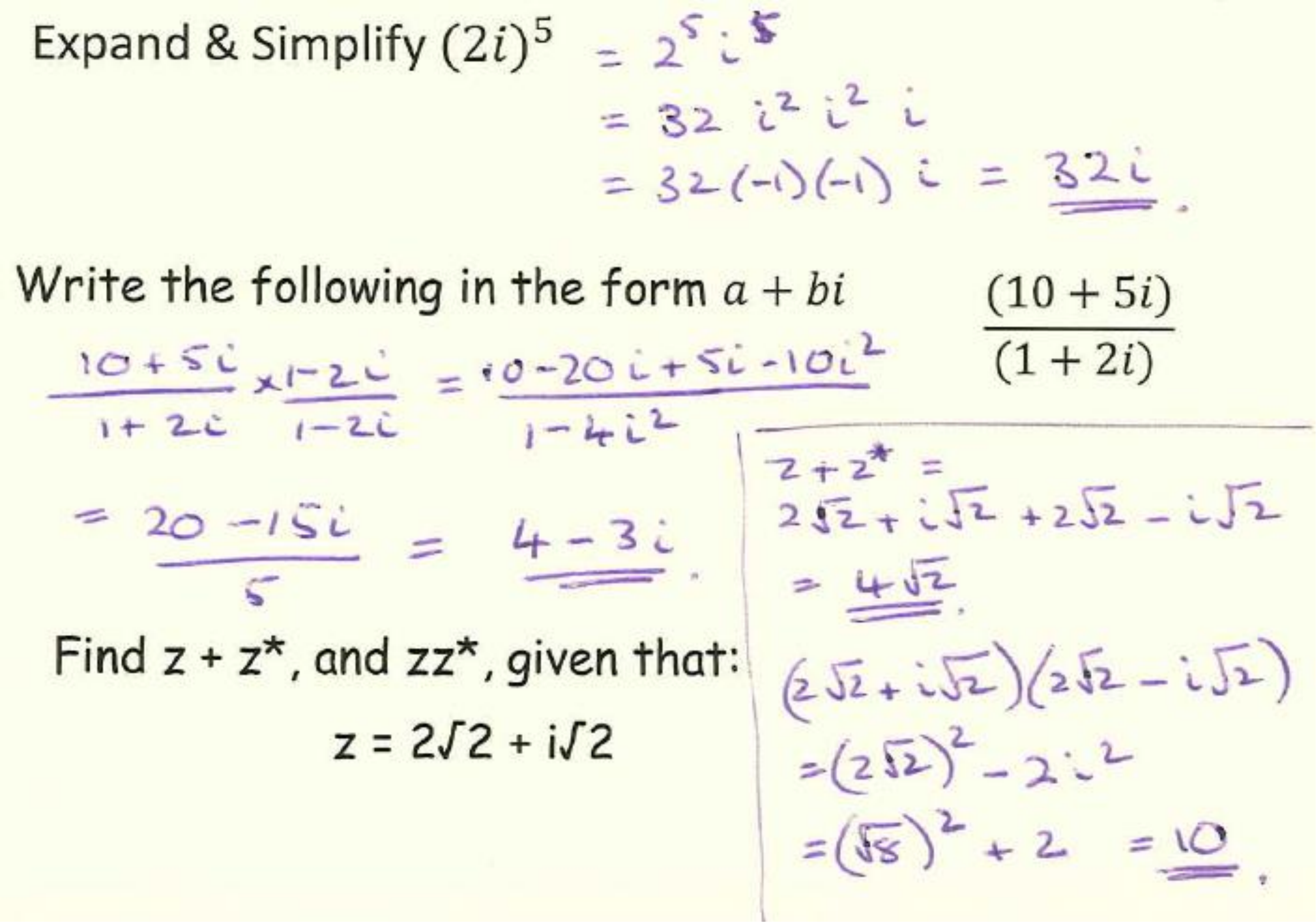
$$2x + 3y - z = 13$$

$$x - 2y + 3z = -11$$

3 planes which meet at a single point (1, 3, -2)

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 13 \\ -11 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 13 \\ -11 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$



Expand & Simplify $(2i)^5$ $= 2^5 i^5$

$= 32\, i^2\, i^2\, i$

$= 32(-1)(-1)\, i = \underline{\underline{32i}}$.

Write the following in the form $a + bi$ $\quad \frac{(10+5i)}{(1+2i)}$

$$\frac{10+5i}{1+2i} \times \frac{1-2i}{1-2i} = \frac{10-20i+5i-10i^2}{1-4i^2}$$

$$= \frac{20-15i}{5} = \underline{\underline{4-3i}}.$$

Find z + z*, and zz*, given that:

z = 2√2 + i√2

$z + z^* =$

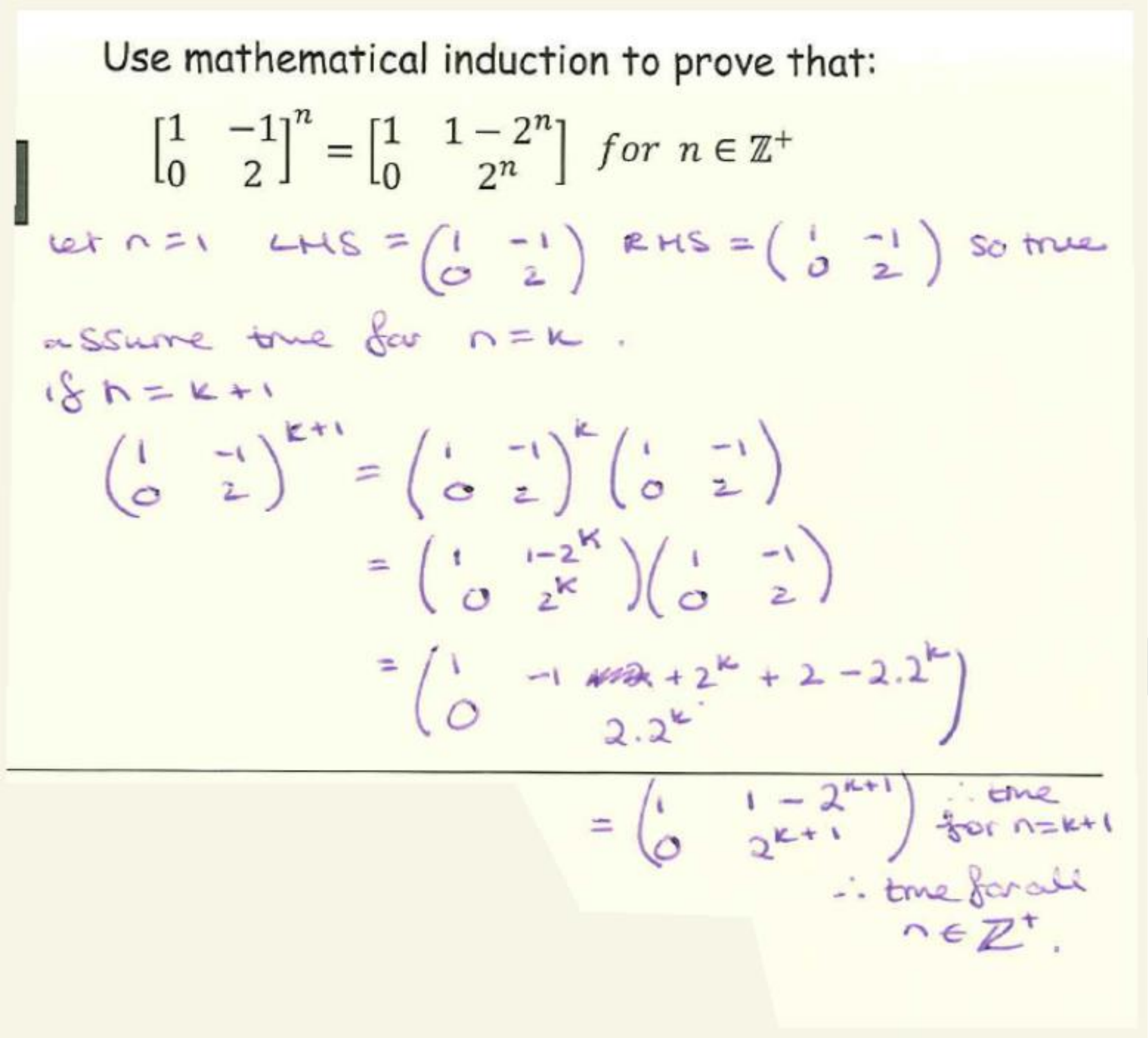
$2\sqrt{2} + i\sqrt{2} + 2\sqrt{2} - i\sqrt{2}$

$= \underline{\underline{4\sqrt{2}}}$.

$(2\sqrt{2} + i\sqrt{2})(2\sqrt{2} - i\sqrt{2})$

$= (2\sqrt{2})^2 - 2i^2$

$= (\sqrt{8})^2 + 2 \quad = \underline{\underline{10}}$.



Use mathematical induction to prove that:

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^n = \begin{bmatrix} 1 & 1-2^n \\ 0 & 2^n \end{bmatrix} \quad for\ n \in \mathbb{Z}^+$$

let $n=1$ LHS $= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ RHS $= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ so true

assume true for $n=k$.

if $n=k+1$

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{k} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1-2^k \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 + 2^k + 2 - 2.2^k \\ 0 & 2.2^k \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1-2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix}$$ $\therefore$ true for $n=k+1$

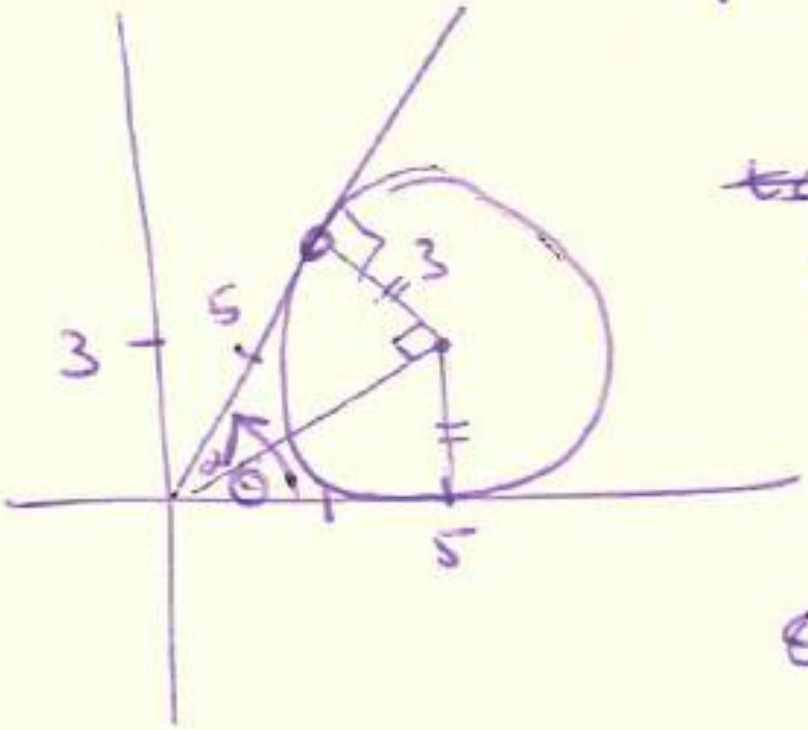
$\therefore$ true for all $n \in \mathbb{Z}^+$.

If:

$$|z - 5 - 3i| = 3$$

Find the maximum value of argz in the interval (-π,π)

$$|z-(5+3i)|=3$$



~~$\tan\alpha =$~~

$$\sin\alpha = \frac{3}{5}$$

$$\alpha = \sin^{-1}\left(\frac{3}{5}\right)$$

$$\theta = 2\alpha = 1.287 \text{ rad}$$

(3dp).

Sheet 40

Use an inverse matrix to solve the simultaneous equations:

$$-x + 6y - 2z = 21$$

$$6x - 2y - z = -16$$

$$-2x + 3y + 5z = 24$$

Prove by mathematical induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n}(r2^r) = 2[1 + (n-1)2^n]$$

Find the quadratic equation that has roots 3 + 5i and 3 - 5i

If:

$$|z - 5 - 3i| = 3$$

Use an algebraic method to find a Cartesian equation of the locus of z

Use an inverse matrix to solve the simultaneous equations:

Sheet 40

$$-x + 6y - 2z = 21$$

$$6x - 2y - z = -16$$

$$-2x + 3y + 5z = 24$$

$$\left.\begin{array}{l} x = -1 \\ y = 4 \\ z = 2 \end{array}\right\}$$

$$\begin{pmatrix} -1 & 6 & -2 \\ 6 & -2 & -1 \\ -2 & 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 21 \\ -16 \\ 24 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 6 & -2 \\ 6 & -2 & -1 \\ -2 & 3 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 21 \\ -16 \\ 24 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$$

$$(z-(3+5i))(z-(3-5i))=0$$

Repeat !

$$(z-3-5i)(z-3+5i)=0$$

$$z^2-3z\boxed{+5iz}-3z+9\boxed{-15i}\boxed{-5iz}\boxed{+15i}-25i^2=0$$

Find the quadratic equation that has roots 3 + 5i and 3 - 5i

$$z^2-6z+9+25=0$$

$$z^2-6z+34=0$$

Check on classwiz.

$x_1=3+5i$

$x_2=3-5i$

✓ o.k.

Prove by mathematical induction that,
for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n}(r2^r) = 2[1 + (n-1)2^n]$$

Let $n=1$ LHS $= 1.2^1 = 2$ RHS $= 2(1+(0)2^1) = 2$

$\therefore$ true for $n=1$, LHS = RHS.

assume true for $n=k$

if $n=k+1$

$$\sum_{r=1}^{k+1}(r2^r) = \sum_{r=1}^{k}(r2^r) + (r2^r)\Big|_{k+1}$$

$$= 2[1+(k-1)2^k] + (k+1)2^{k+1}$$

$$= 2 + (k-1)2^{k+1} + (k+1)2^{k+}$$

$$= 2 + 2^{k+1}(k-1+k+1)$$

$$= 2 + 2^{k+1}(2k)$$

$$= 2[1 + k.2^{k+1}]$$

$$= 2[1 + ((k+1)-1)2^{(k+1)}]$$

$$= 2\{1 + (n-1)2^n]$$

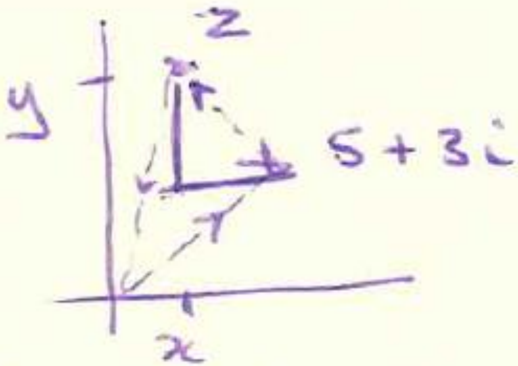
$\therefore$ true for $n=k+1$

$\therefore$ true for all $n \in \mathbb{Z}^+$.

If:

$$|z - 5 - 3i| = 3$$

Use an algebraic method to find a Cartesian equation of the locus of z



$z = x + iy$

$$|z - (5 + 3i)| = 3$$

$$|x + iy - (5 + 3i)| = 3$$

$$|(x - 5) + i(y - 3)| = 3$$

$$\sqrt{(x-5)^2 + (y-3)^2} = 3$$

$$(x-5)^2 + (y-3)^2 = 9.$$

Sheet 41

Determine the number of solutions to this set of equations, and give a geometric interpretation:

$$3x - y - 6z = 1$$

$$x + 3y + 3z = 2$$

$$-3x - y + 3z = -2$$

Prove, by induction, that the expression '$11^{n+1} + 12^{2n-1}$' is divisible by 133 for all positive integers $n \in \mathbb{Z}^+$

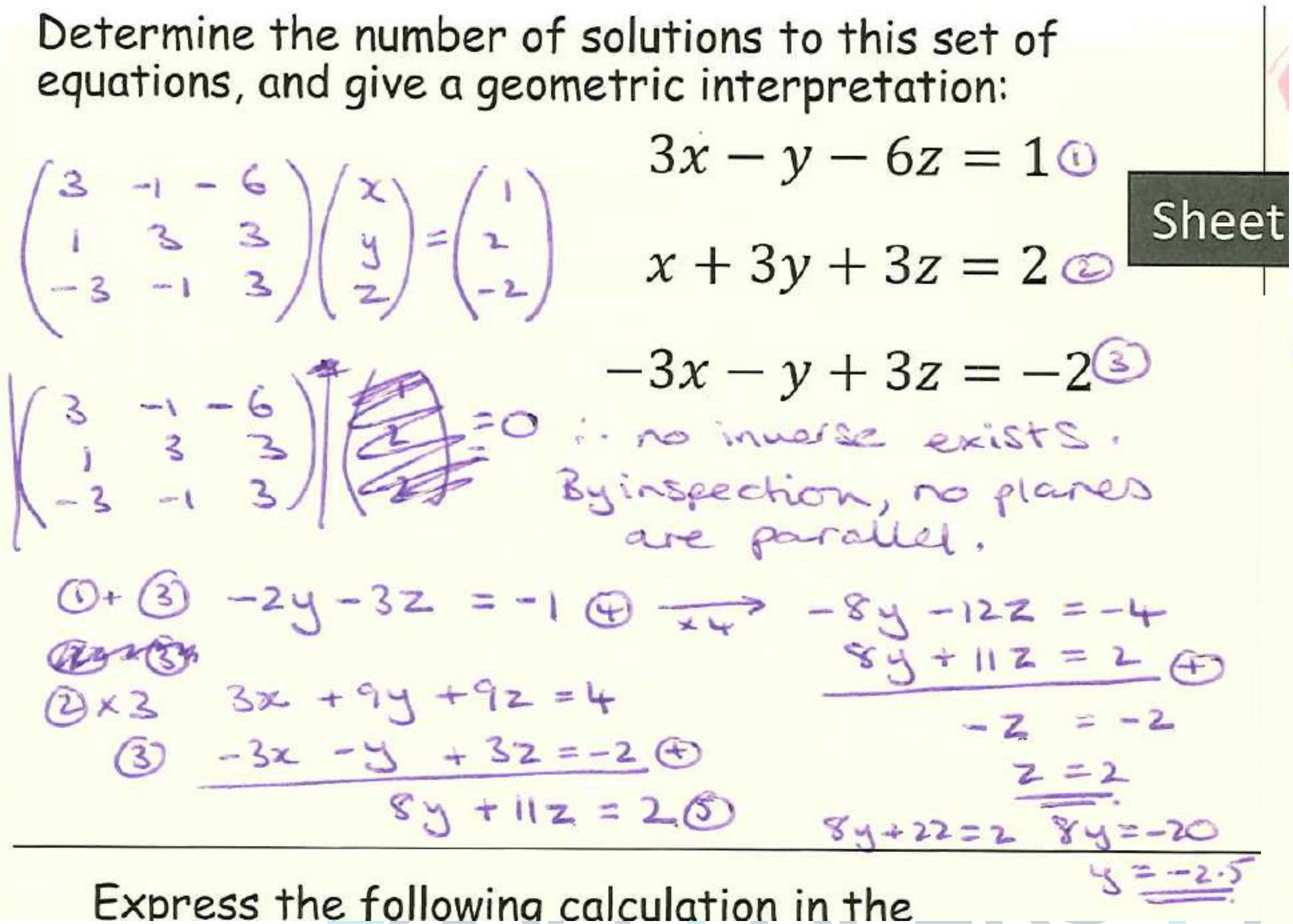
Express the following calculation in the form x + iy:

$$\frac{\sqrt{2}\left(cos\frac{\pi}{12} + isin\frac{\pi}{12}\right)}{2\left(cos\frac{5\pi}{6} + isin\frac{5\pi}{6}\right)}$$

HINT: $\frac{z_1}{z_2} = \frac{r_1}{r_2}(cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2))$

Sketch the locus of P(x,y) which is represented by z on an Argand diagram, if:

$$|z| = |z - 6i|$$



Determine the number of solutions to this set of equations, and give a geometric interpretation:

$$\begin{pmatrix} 3 & -1 & -6 \\ 1 & 3 & 3 \\ -3 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$3x - y - 6z = 1 \text{ ①}$$

$$x + 3y + 3z = 2 \text{ ②}$$

$$-3x - y + 3z = -2 \text{ ③}$$

$$\begin{vmatrix} 3 & -1 & -6 \\ 1 & 3 & 3 \\ -3 & -1 & 3 \end{vmatrix} = 0$$

∴ no inverse exists. By inspection, no planes are parallel.

① + ③ $-2y - 3z = -1$ ④ $\xrightarrow[\times 4]{}$ $-8y - 12z = -4$

② × 3 $3x + 9y + 9z = 4$

③ $-3x - y + 3z = -2$ (+)

$8y + 11z = 2$ ⑤

$8y + 11z = 2$ (+)

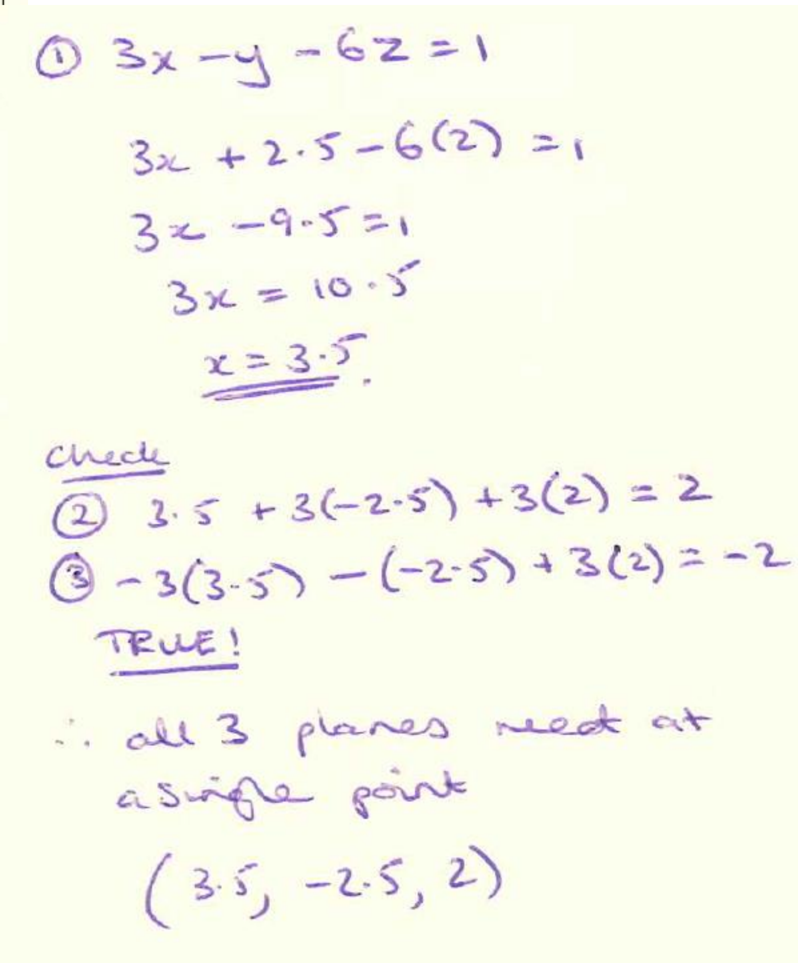
$-z = -2$

$z = 2$

$8y + 22 = 2$ $8y = -20$

$y = -2.5$

Express the following calculation in the



① $3x - y - 6z = 1$

$3x + 2.5 - 6(2) = 1$

$3x - 9.5 = 1$

$3x = 10.5$

$x = 3.5$

Check

② $3.5 + 3(-2.5) + 3(2) = 2$

③ $-3(3.5) - (-2.5) + 3(2) = -2$

TRUE!

∴ all 3 planes meet at a single point

$(3.5, -2.5, 2)$

$\underline{\underline{y = -2.5}}$

Express the following calculation in the form x + iy:

$$\frac{\sqrt{2}\left(cos\frac{\pi}{12} + isin\frac{\pi}{12}\right)}{2\left(cos\frac{5\pi}{6} + isin\frac{5\pi}{6}\right)}$$

÷ the lengths
subtract angles.

HINT: $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)\right)$

$\frac{\sqrt{2}}{2}$

$\frac{\pi}{12} - \frac{5\pi}{6} = -\frac{3\pi}{4}$

$$\text{Answer} = \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$

$$= -\frac{1}{2} - \frac{1}{2}i$$

✓ check classwiz

Prove, by induction, that the expression '$11^{n+1} + 12^{2n-1}$' is divisible by 133 for all positive integers $n \in \mathbb{Z}^+$

41

let $n=1$ $\quad 11^2 + 12^1 = 121 + 12 = 133$

which is divisible by 133, ∴ true

assume true for $n=k$

let $n=k+1$

$$11^{k+1+1} + 12^{2(k+1)-1}$$

$$= 11^{k+2} + 12^{2k+2-1}$$

$$= 11.11^{k+1} + 12^{2k+1}$$

$$= 11.11^{k+1} + 12^2.12^{2k-1}$$

$$= 11.11^{k+1} + 144.12^{2k-1}$$

$$= 11.11^{k+1} + 11.12^{2k-1} + 133.12^{2k-1}$$

$$= 11\underbrace{\left(11^{k+1} + 12^{2k-1}\right)}_{\text{when } n=k \text{ we said this was divisible by 133}} + \underbrace{133\left(12^{2k-1}\right)}_{\text{divisible by 133}}$$

So this is also divisible by 133

So this integer is also divisible by 133

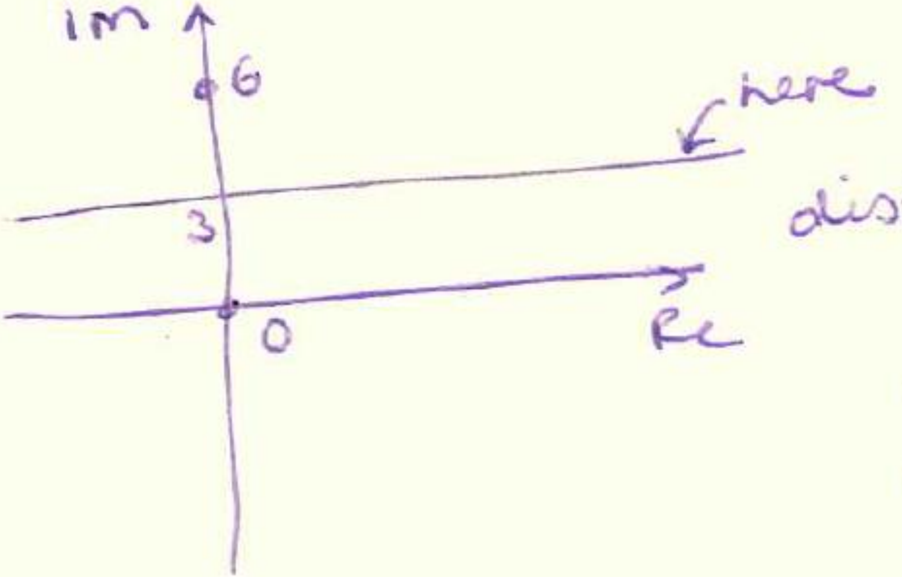
∴ true for $n=k+1$

true for $n=1$ and $n=k$

∴ true for all $n \in \mathbb{Z}^+$.

Sketch the locus of P(x,y) which is represented by z on an Argand diagram, if:

$$|z| = |z - 6i|$$



distance from 6i = distance from 0 anywhere on here

Shade on an Argand diagram the region indicated by:

$$|z-4| < |z-6|$$

Sheet 42

Use mathematical induction to prove that:

$$\begin{bmatrix} -2 & 9 \\ -1 & 4 \end{bmatrix}^n = \begin{bmatrix} -3n+1 & 9n \\ -n & 3n+1 \end{bmatrix} \quad for\ \ n \in \mathbb{Z}^+$$

Show that:

$$\sum_{r=1}^{n} r^2 + r - 2 = \frac{n}{3}(n+4)(n-1)$$

Use an algebraic method to find the Cartesian equation of the locus of z if:

$$|z-3| = |z+i|$$

Given that: $$\sum_{r=1}^{n} r^2 + r - 2 = \frac{n}{3}(n+4)(n-1)$$

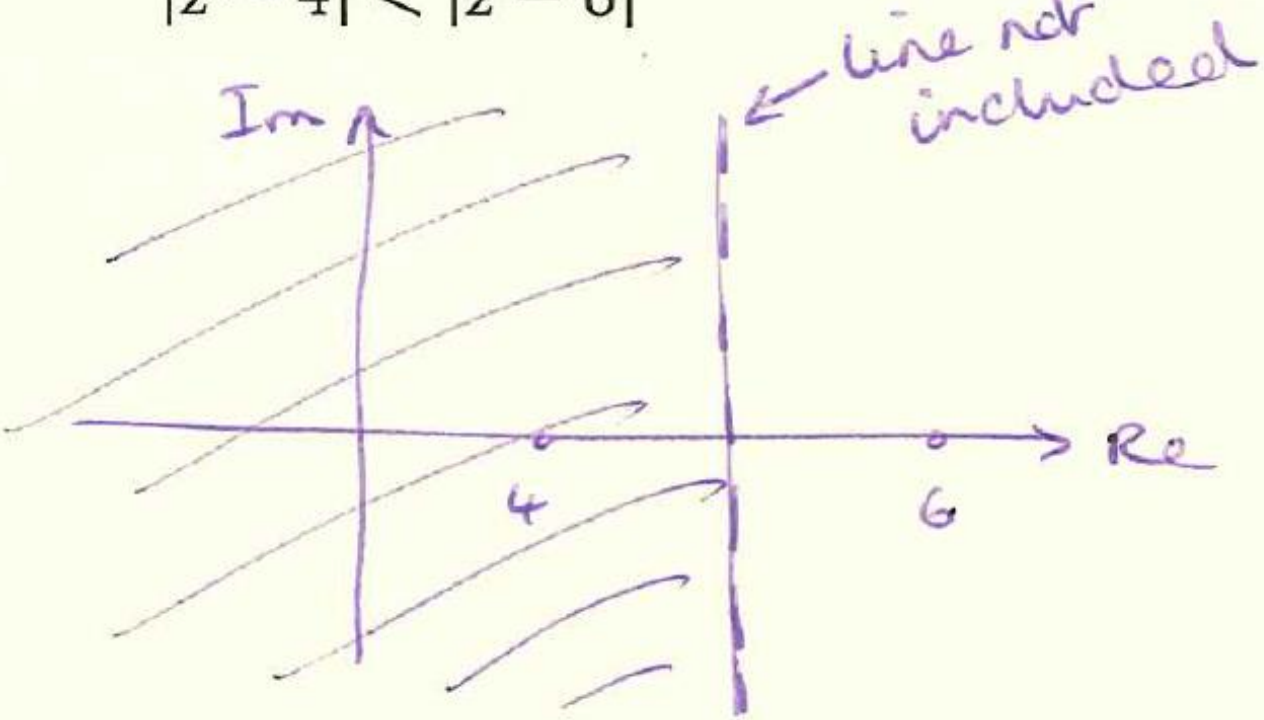
calculate the sum of the series:

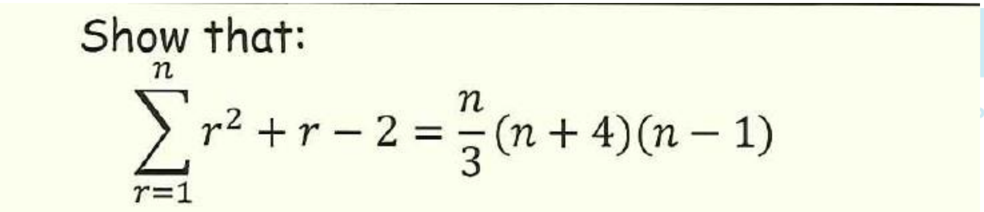
4 + 10 + 18 + 28 + 40 … … … + 418

Shade on an Argand diagram the region indicated by:

Sheet 42

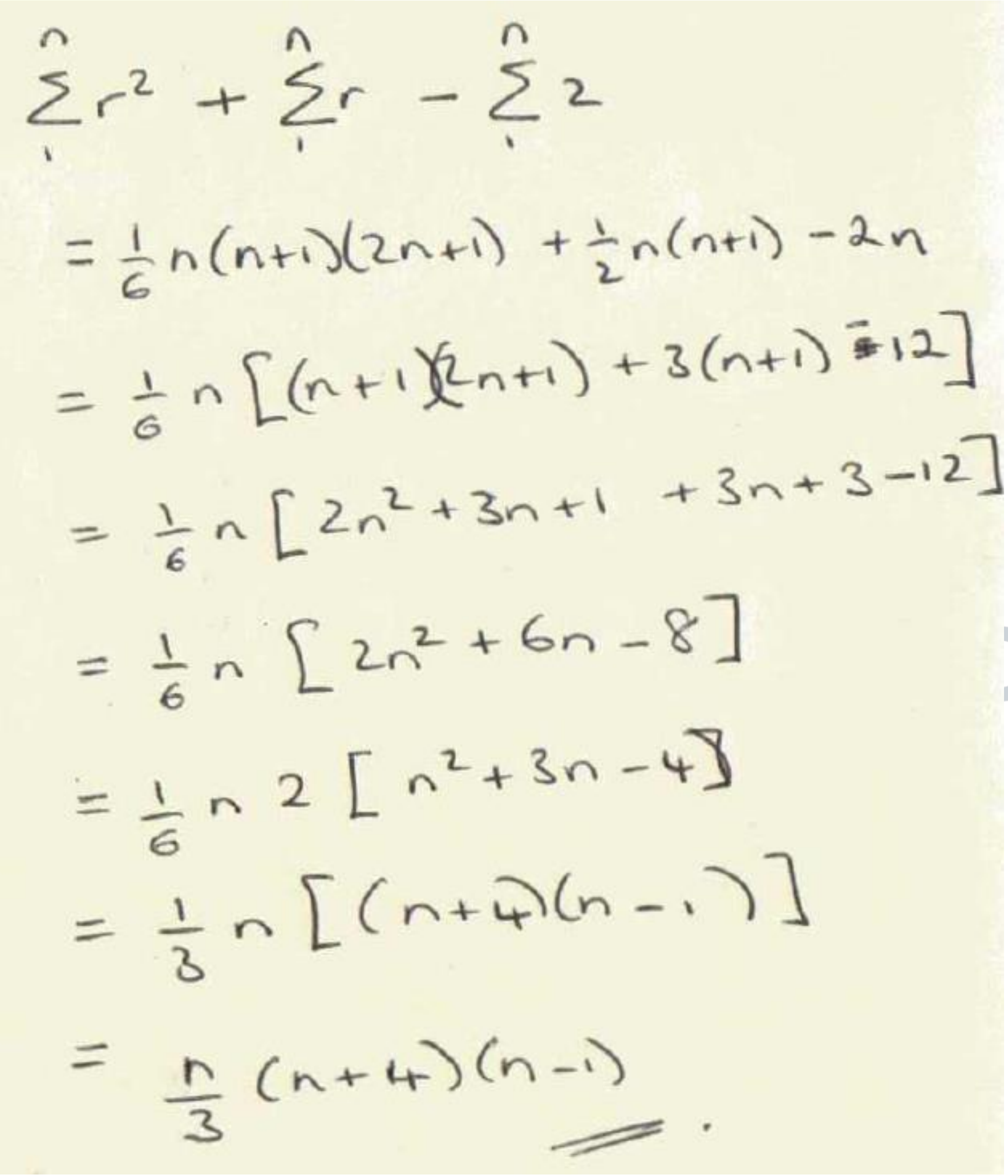
$$|z-4| < |z-6|$$





Show that:

$$\sum_{r=1}^{n} r^2 + r - 2 = \frac{n}{3}(n+4)(n-1)$$



$$\sum_{1}^{n} r^2 + \sum_{1}^{n} r - \sum_{1}^{n} 2$$

$$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - 2n$$

$$= \frac{1}{6}n\left[(n+1)(2n+1) + 3(n+1) - 12\right]$$

$$= \frac{1}{6}n\left[2n^2 + 3n + 1 + 3n + 3 - 12\right]$$

$$= \frac{1}{6}n\left[2n^2 + 6n - 8\right]$$

$$= \frac{1}{6}n\,2\left[n^2 + 3n - 4\right]$$

$$= \frac{1}{3}n\left[(n+4)(n-1)\right]$$

$$= \underline{\underline{\frac{n}{3}(n+4)(n-1)}}\,.$$

Show that:

$$\sum_{r=1}^{n} r^2 + r - 2 = \frac{n}{3}(n+4)(n-1)$$

Given that: $$\sum_{r=1}^{n} r^2 + r - 2 = \frac{n}{3}(n+4)(n-1)$$

calculate the sum of the series:

4 + 10 + 18 + 28 + 40 … … … + 418

$n=1$ $\quad 1^2+1-2 = 0 \leftarrow$ can include this.

$\boxed{n=2}$ $\quad 2^2+2-2 = 4$

$418 = r^2+r-2$ $\quad \boxed{n=20}$ $\quad \overset{20}{r}(\overset{21}{r+1}) = 420$

~~$r^2+r-420=0$~~

} work out limits for n.

Need

$$\sum_{1}^{20} r^2+r-2 = \frac{20}{3}(20+4)(20-1)$$

$$= \frac{20}{3} \times 24 \times 19$$

$$= \underline{\underline{3040}}.$$

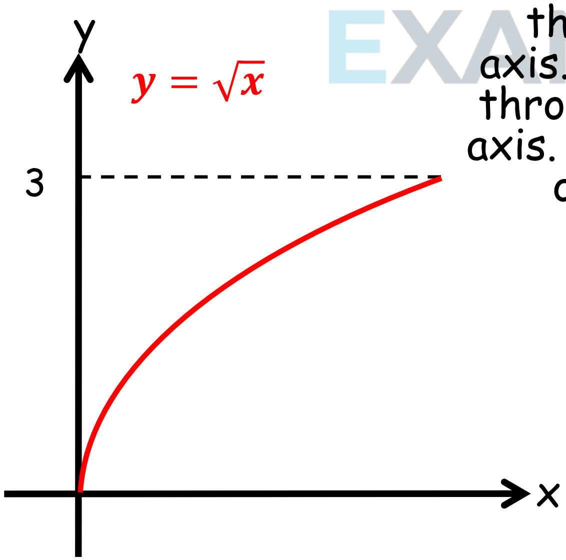
Check classwiz: $\sum_{1}^{20} x^2+x-2$ same ✓

Sheet 43

The three roots of a cubic equation are α, β and γ. Given that $\alpha\beta\gamma = 4$, $\alpha\beta + \beta\gamma + \gamma\alpha = -5$ and $\alpha + \beta + \gamma = 3$, find the value of $(\alpha + 3)(\beta + 3)(\gamma + 3)$.

If: $\arg(z - 2) = \frac{\pi}{3}$

Sketch the locus of P(x,y) which is represented by z on an Argand diagram. Then find the Cartesian equation of this locus algebraically.



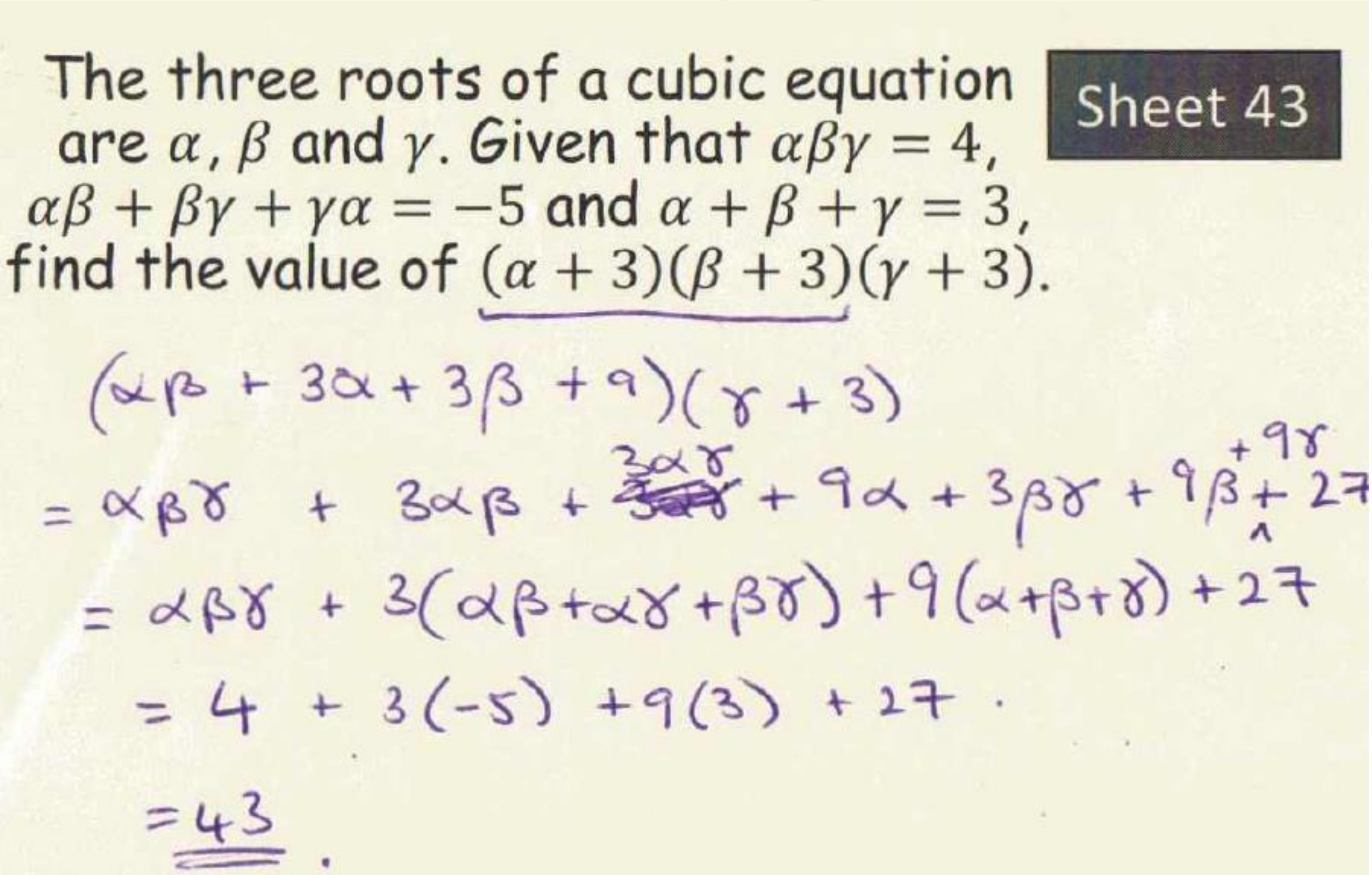
The diagram shows the region R bounded by the curve with equation $y = \sqrt{x}$, the line $y = 3$ and the y-axis. The region is rotated through 360° about the y-axis. Find the exact volume of the solid generated.

Determine the number of solutions to this set of equations, and give a geometric interpretation:

$$x + y + z = 8$$

$$2x + 2y + 2z = 14$$

$$3x - y - z = 10$$



Sheet 43

The three roots of a cubic equation are α, β and γ. Given that $\alpha\beta\gamma = 4$, $\alpha\beta + \beta\gamma + \gamma\alpha = -5$ and $\alpha + \beta + \gamma = 3$, find the value of $\underline{(\alpha + 3)(\beta + 3)}(\gamma + 3)$.

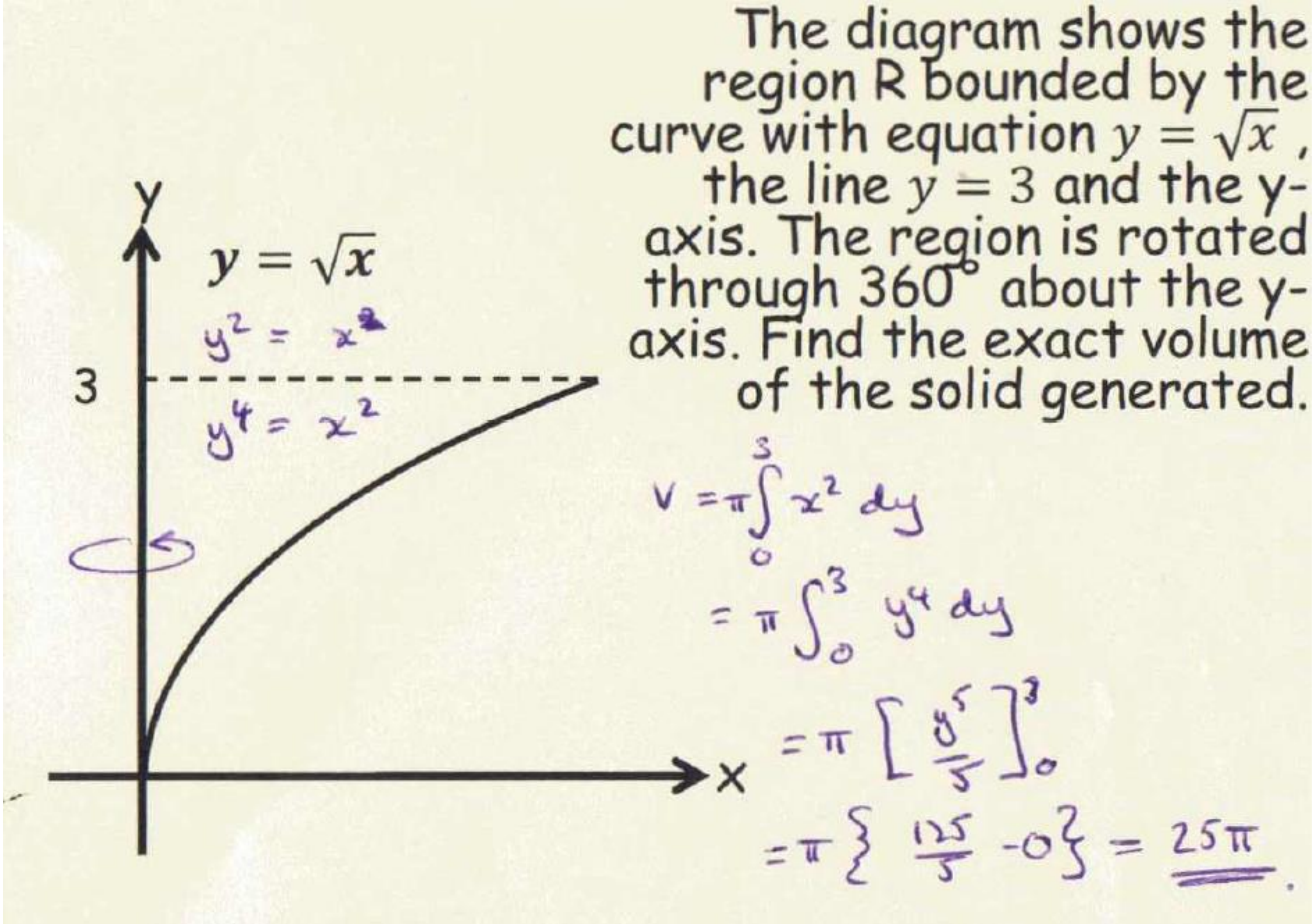
$$(\alpha\beta + 3\alpha + 3\beta + 9)(\gamma + 3)$$

$$= \alpha\beta\gamma + 3\alpha\beta + \overset{3\alpha\gamma}{\cancel{3\alpha\gamma}} + 9\alpha + 3\beta\gamma + 9\beta \overset{+9\gamma}{\wedge} + 27$$

$$= \alpha\beta\gamma + 3(\alpha\beta + \alpha\gamma + \beta\gamma) + 9(\alpha + \beta + \gamma) + 27$$

$$= 4 + 3(-5) + 9(3) + 27.$$

$$= \underline{\underline{43}}.$$



The diagram shows the region R bounded by the curve with equation $y = \sqrt{x}$, the line $y = 3$ and the y-axis. The region is rotated through 360° about the y-axis. Find the exact volume of the solid generated.

$y^2 = x$

$y^4 = x^2$

$$V = \pi \int_0^3 x^2 \, dy$$

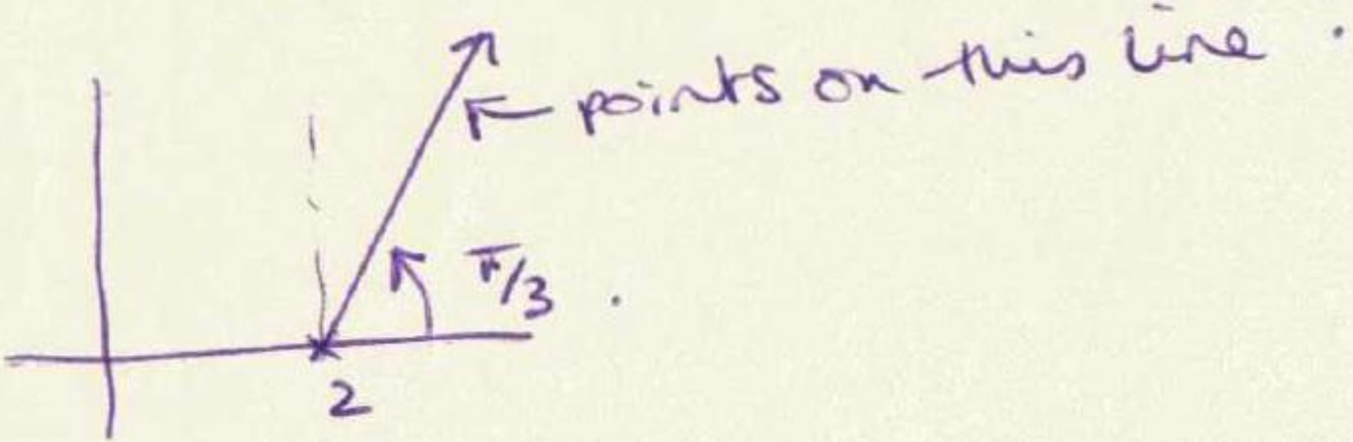
$$= \pi \int_0^3 y^4 \, dy$$

$$= \pi \left[\frac{y^5}{5} \right]_0^3$$

$$= \pi \left\{ \frac{125}{5} - 0 \right\} = \underline{\underline{25\pi}}.$$

If: $\arg(z-2)=\dfrac{\pi}{3}$

Sketch the locus of P(x,y) which is represented by z on an Argand diagram. Then find the Cartesian equation of this locus algebraically.



Determine the number of solutions to this set of equations, and give a geometric interpretation:

$$\begin{pmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 3 & -1 & -1\end{pmatrix}^{-1}\begin{pmatrix}8\\ 14\\ 10\end{pmatrix}.$$

$$\left.\begin{aligned} x+y+z &= 8 \text{①} \\ 2x+2y+2z &= 14 \text{②} \end{aligned}\right\} \text{2 parallel planes.}$$

$$x + y + z = 7$$

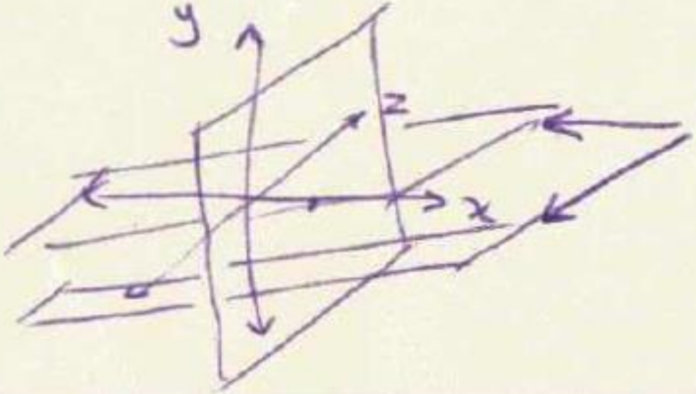
$$3x - y - z = 10 \text{③}$$

see if ② and ③ meet:

$4x = 17.$

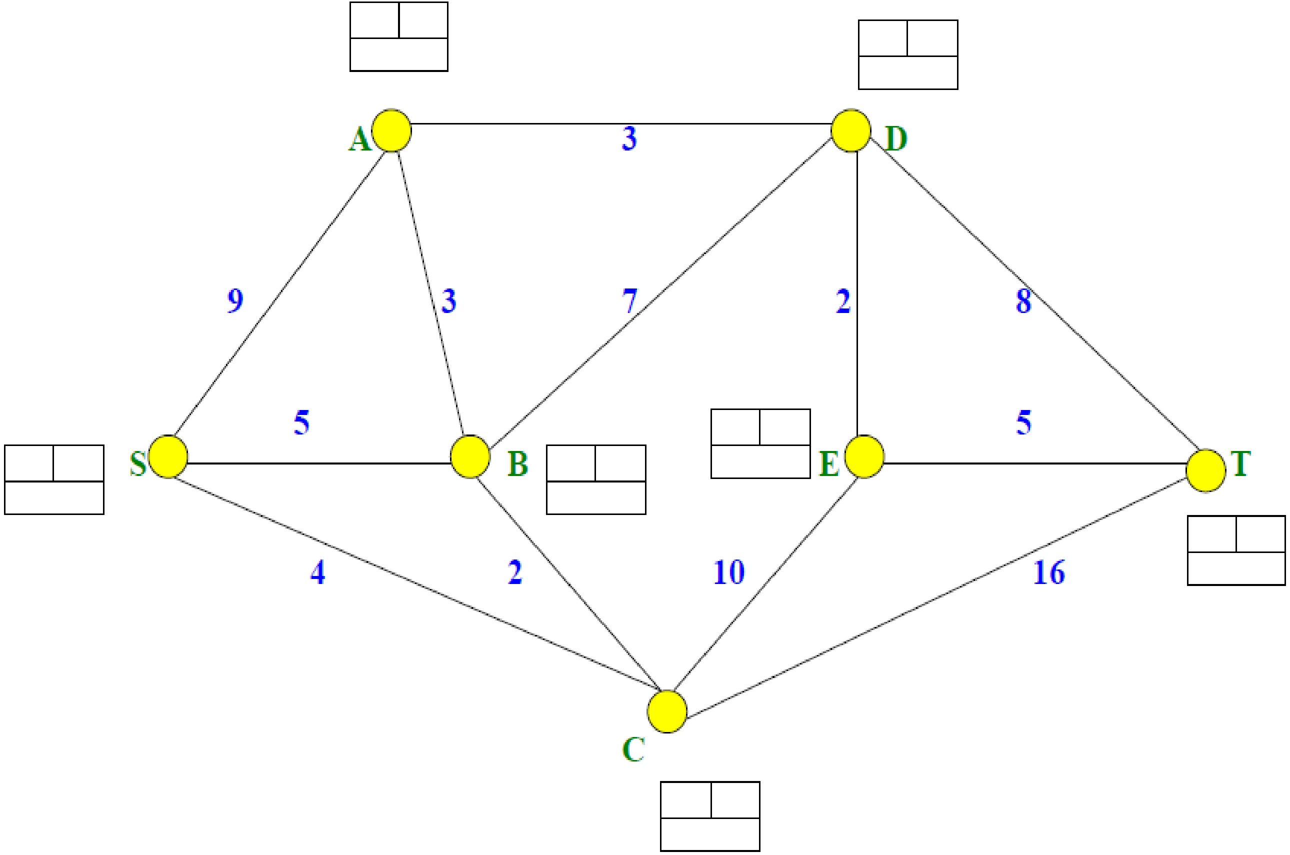
$x = 17/4 = 3\tfrac{3}{4} \nearrow$ ② $y + z = 7 - 3\tfrac{3}{4} = 3\tfrac{1}{4}$

infinitely many solutions

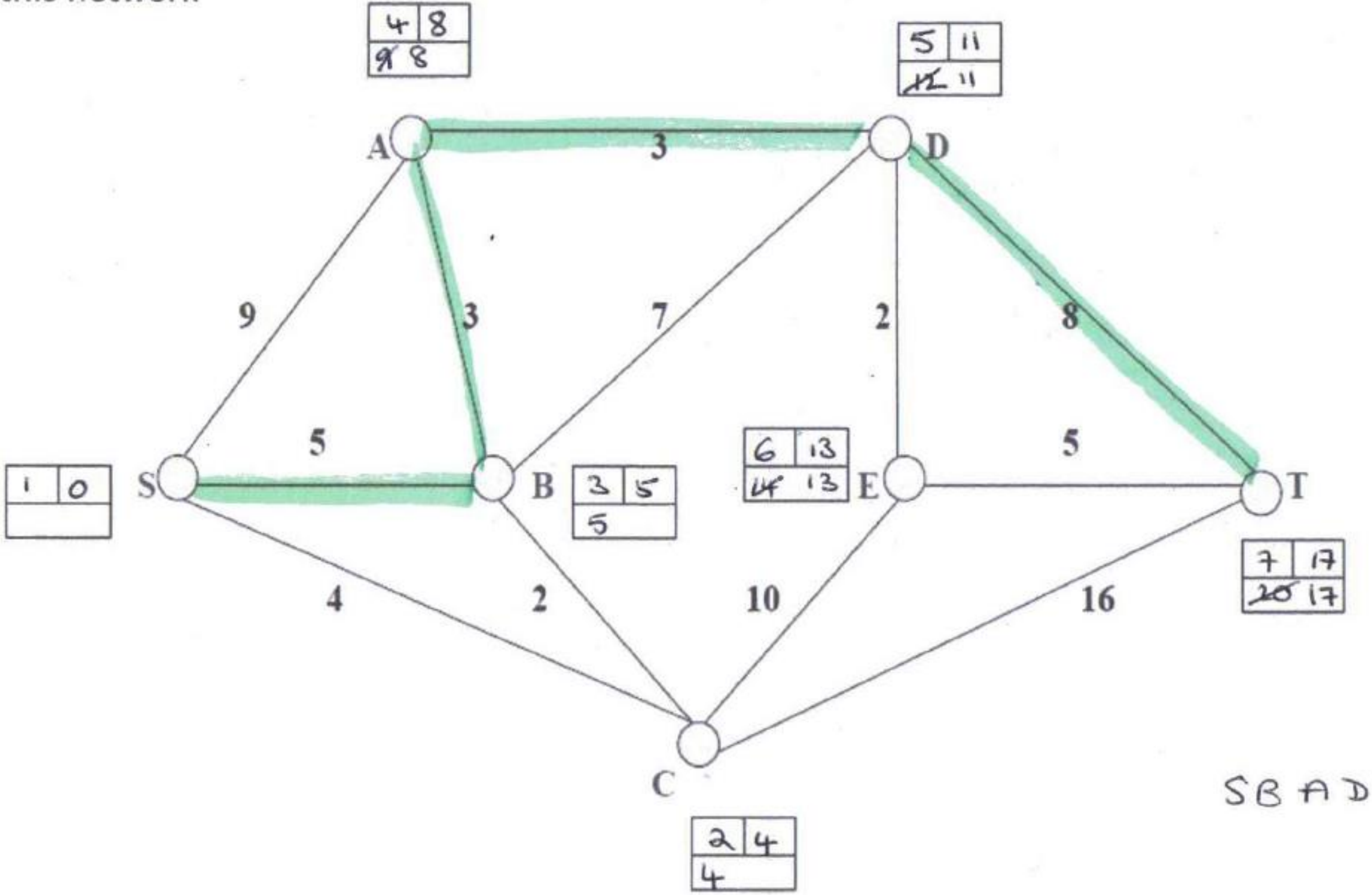


① and ② are parallel planes and ③ intersects them both.

Use Dijkstra’s Algorithm to find the shortest route from S to T in this network

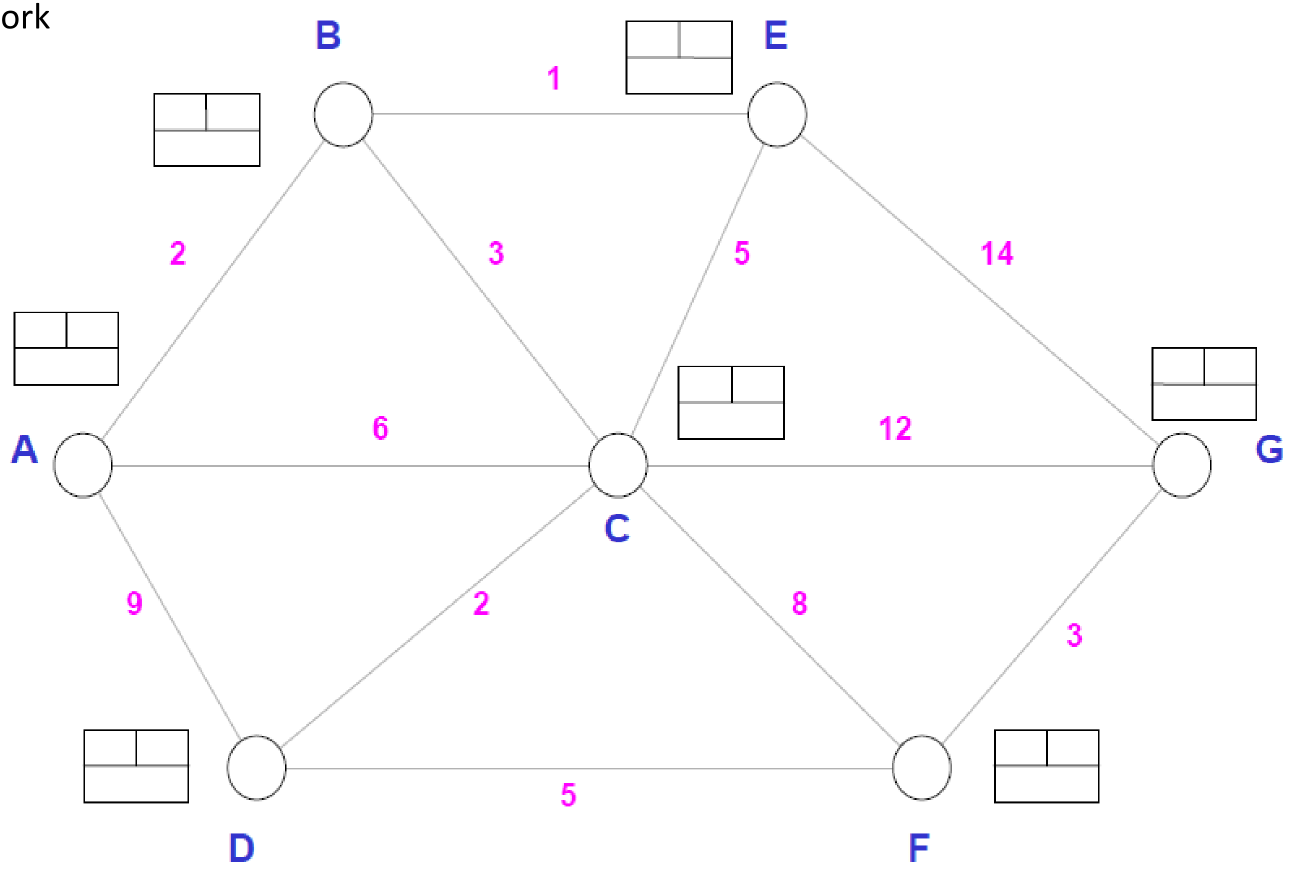


Use Dijkstra's Algorithm to find the shortest route from S to T in this network

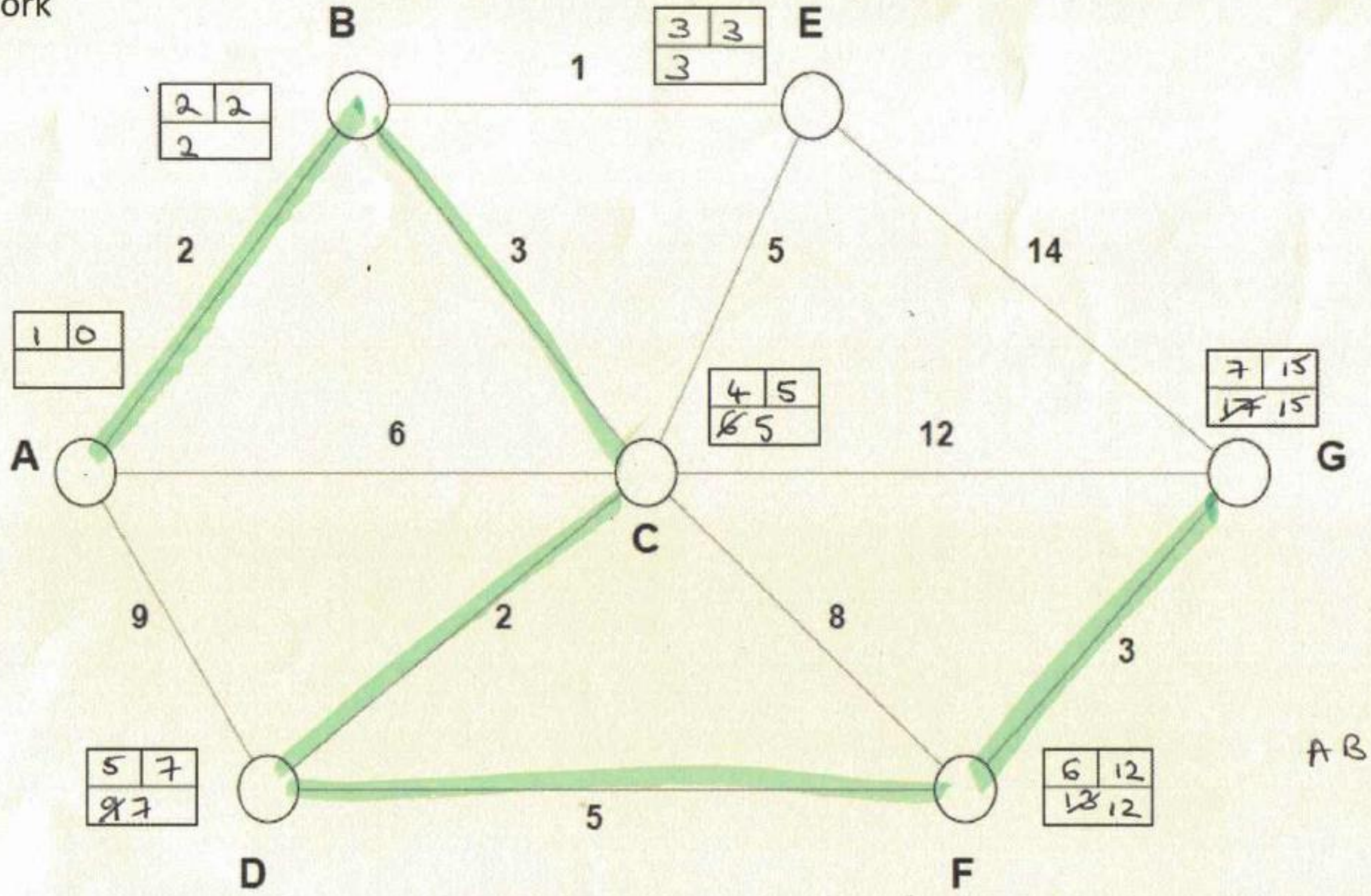


SBADT = 17

Use Dijkstra's Algorithm to find the shortest route from A to G in this network



Use Dijkstra's Algorithm to find the shortest route from A to G in this network



ABCDFG = 15