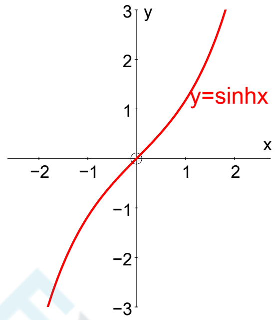
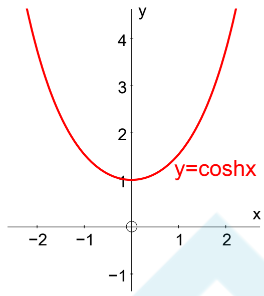
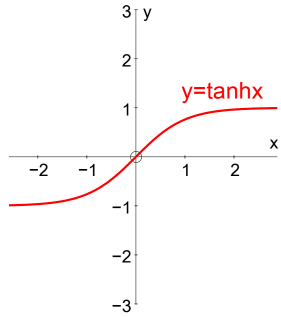
1 Hyperbolic functions

Definitions and graphs

$\boldsymbol{\sinh x = \frac{1}{2}(e^x - e^{-x})}$ $\boldsymbol{\cosh x = \frac{1}{2}(e^x + e^{-x})}$ $\boldsymbol{\tanh x = \frac{\sinh x}{\cosh x} = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}}$

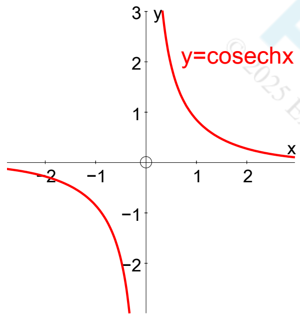


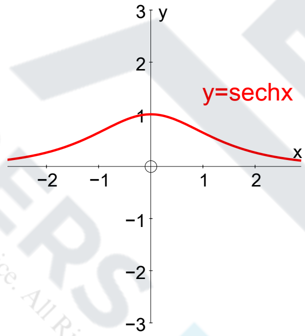


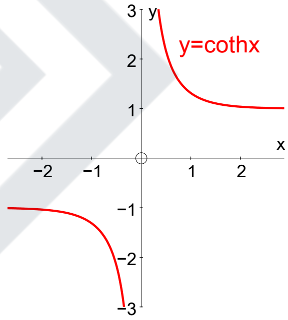


You should be able to draw the graphs of cosech x, sech x and coth x from the above:

cosech *x* **sech *x*** **coth *x***







Addition formulae, double angle formulae etc.

The standard trigonometric formulae are very similar to the hyperbolic formulae.

Osborne's rule

If a trigonometric identity involves the **product of two sines**, then we change the sign to write down the corresponding hyperbolic identity.

Examples:

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\Rightarrow \sinh(A + B) = \sinh A \cosh B + \cosh A \sinh B$ no change

but $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\Rightarrow \cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B$ product of two sines, so change sign

and $1 + \tan^2 A = \sec^2 A$

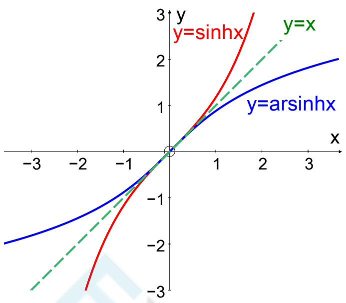
$\Rightarrow 1 - \tanh^2 A = \operatorname{sech}^2 A$ $\tan^2 A = \frac{\sin^2 A}{\cos^2 A}$, product of two sines, so change sign

Inverse hyperbolic functions

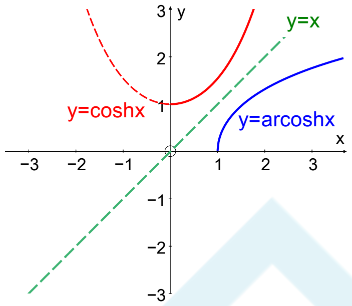
Graphs

Remember that the graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in $y = x$.

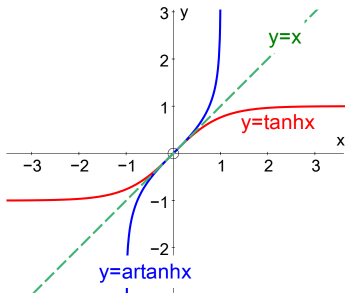
y* = arsinh *x



y* = arcosh *x



y* = artanh *x



Notice arcosh x is a function defined so that arcosh $x \geq 0$.

$\Rightarrow$ there is only **one** value of arcosh x.

However, the equation $\cosh z = 2$, has **two** solutions, +arcosh 2 and –arcosh 2.

Logarithmic form

1) ***y* = arsinh *x***

$\Rightarrow \quad \sinh y = \frac{1}{2}(e^y - e^{-y}) = x$

$\Rightarrow \quad e^{2y} - 2xe^y - 1 = 0$

$\Rightarrow \quad e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x + \sqrt{x^2+1}$ or $x - \sqrt{x^2+1}$

But $e^y > 0$ and $x - \sqrt{x^2+1} < 0 \quad \Rightarrow e^y = x + \sqrt{x^2+1}$ **only**

$\Rightarrow \quad y = \text{arsinh}\, x = \ln\left(x + \sqrt{x^2+1}\right)$

2) ***y* = arcosh *x***

$\Rightarrow \quad \cosh y = \frac{1}{2}(e^y + e^{-y}) = x$

$\Rightarrow \quad e^{2y} - 2xe^y + 1 = 0$

$\Rightarrow \quad e^y = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$ both roots are positive

$\Rightarrow \quad y = \text{arcosh}\, x = \ln\left(x + \sqrt{x^2-1}\right)$ or $\ln\left(x - \sqrt{x^2-1}\right)$

It can be shown that $\ln\left(x - \sqrt{x^2-1}\right) = -\ln\left(x + \sqrt{x^2-1}\right)$

$\Rightarrow \quad y = \text{arcosh}\, x = \pm\ln\left(x + \sqrt{x^2-1}\right)$

But arcosh x is a function and therefore has only one value (positive)

$\Rightarrow \quad y = \text{arcosh}\, x = \ln\left(x + \sqrt{x^2-1}\right) \quad (x \geq 1)$

3) Similarly $\text{artan}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1)$

Equations involving hyperbolic functions

It would be possible to solve $6\sinh x - 2\cosh x = 7$ using the $R\sinh(x-\alpha)$ technique from trigonometry, but it is easier to use the exponential form.

Example: Solve $6\sinh x - 2\cosh x = 7$

Solution: $6\sinh x - 2\cosh x = 7$

$\Rightarrow \quad 6 \times \frac{1}{2}(e^x - e^{-x}) - 2 \times \frac{1}{2}(e^x + e^{-x}) = 7$

$\Rightarrow \quad 2e^{2x} - 7e^x - 4 = 0$

$\Rightarrow \quad (2e^x + 1)(e^x - 4) = 0$

$\Rightarrow \quad e^x = -\frac{1}{2}$ (not possible) or 4

$\Rightarrow \quad x = \ln 4$

In other cases, the 'trigonometric' solution may be preferable

Example: Solve $\cosh 2x + 5\sinh x - 4 = 0$

Solution: $\cosh 2x + 5\sinh x - 4 = 0$

$\Rightarrow \quad 1 + 2\sinh^2 x + 5\sinh x - 4 = 0$ note use of Osborn's rule

$\Rightarrow \quad 2\sinh^2 x + 5\sinh x - 3 = 0$

$\Rightarrow \quad (2\sinh x - 1)(\sinh x + 3) = 0$

$\Rightarrow \quad \sinh x = \frac{1}{2}$ or -3

$\Rightarrow \quad x = \text{arsinh } 0.5$ or $\text{arsinh }(-3)$

$\Rightarrow \quad x = \ln\left(0.5 + \sqrt{0.5^2 + 1}\right)$ or $\ln\left((-3) + \sqrt{(-3)^2 + 1}\right)$ using log form of inverse

$\Rightarrow \quad x = \ln\left(\frac{1+\sqrt{5}}{2}\right)$ or $\ln\left(\sqrt{10} - 3\right)$

2 Further coordinate systems

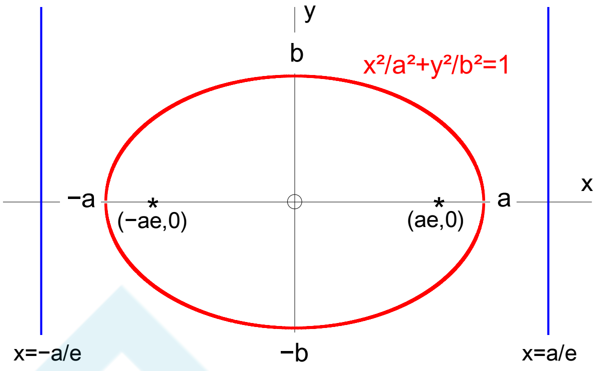
Ellipse

Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parametric equations

$$x = a\cos\theta\,, \quad y = b\sin\theta$$

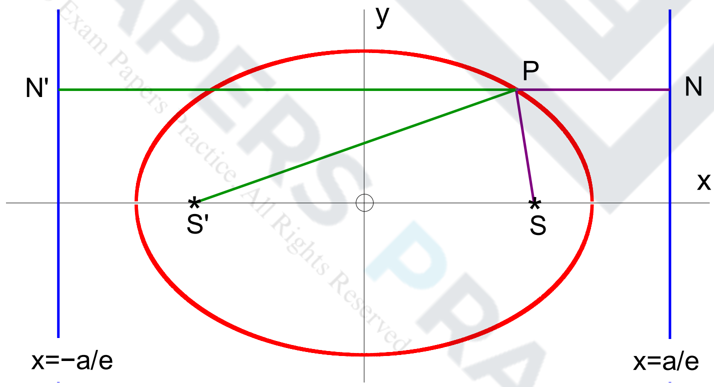


Foci at $S\,(ae, 0)$ and $S'\,(-ae, 0)$

Directrices at $x = \pm\dfrac{a}{e}$

Eccentricity $e < 1$, $b^2 = a^2(1 - e^2)$

An ellipse can be defined as the locus of a point P which moves so that $PS = e\,PN$, where S is one of the foci, $e < 1$ and N lies on the corresponding directrix.



This is true for either focus with the corresponding directrix.

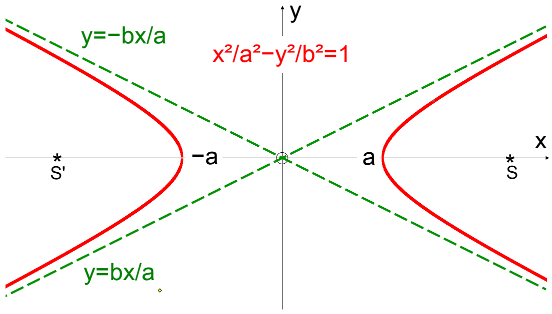
$\Rightarrow \quad PS = e\,PN$ and $PS' = e\,PN'$

$\Rightarrow \quad PS + PS' = e\,(PN + PN') = eNN'$

$\Rightarrow \quad PS + PS' = e\,\dfrac{2a}{e} = 2a.$

This justifies the 'string method' of drawing an ellipse.

Hyperbola



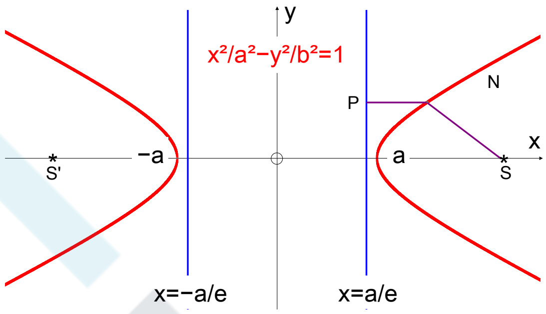
Cartesian equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Parametric equations

$$x = a\cosh\theta\,,\quad y = b\sinh\theta$$

$(x = a\sec\theta\,,\quad y = b\tan\theta$ also work)



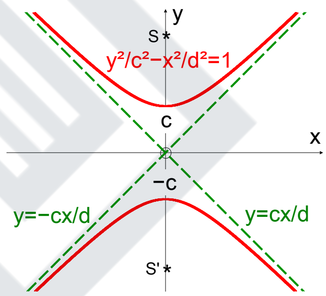
Asymptotes $\frac{x}{a} = \pm\frac{y}{b}$

Foci at $S\,(ae, 0)$ and $S'\,(-ae, 0)$

Directrices at $x = \pm\frac{a}{e}$

Eccentricity $e > 1,\quad b^2 = a^2(e^2 - 1)$

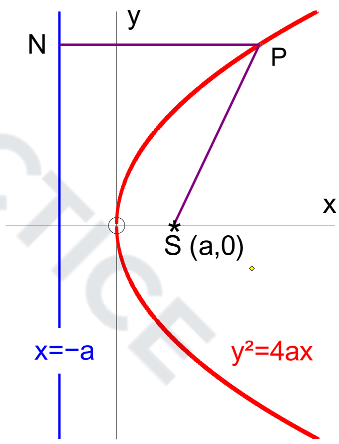
A hyperbola can be defined as the locus of a point P which moves so that $PS = e\,PN$, where S is the focus, $e > 1$ and N lies on the directrix.



$$\frac{y^2}{c^2} - \frac{x^2}{d^2} = 1$$

is a hyperbola with foci on the y-axis,

Parabola



Cartesian equation

$$y^2 = 4ax$$

Parametric equations

$$x = at^2,\quad y = 2at$$

Focus at $S\,(a, 0)$

Directrix at $x = -a$

A parabola can be defined as the locus of a point P which moves so that $PS = PN$, where S is the focus, N lies on the directrix and the eccentricity $e = 1$.

Parametric differentiation

From the chain rule $\frac{dy}{d\theta} = \frac{dy}{dx} \times \frac{dx}{d\theta}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ using any parameter.

Tangents and normals

It is now easy to find tangents and normals.

Example: Find the equation of the normal to the curve given by the parametric equations $x = 5\cos\theta$, $y = 8\sin\theta$ at the point where $\theta = \frac{\pi}{3}$

Solution: When $\theta = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad x = \frac{5}{2}, \quad y = 4\sqrt{3}$

and $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{8\cos\theta}{-5\sin\theta} = \frac{-8}{5\sqrt{3}}$ when $\theta = \frac{\pi}{3}$

$\Rightarrow$ gradient of normal is $\frac{5\sqrt{3}}{8}$

$\Rightarrow$ equation of normal is $y - 4\sqrt{3} = \frac{5\sqrt{3}}{8}\left(x - \frac{5}{2}\right)$

$\Rightarrow \quad 5\sqrt{3}\,x - 8y + \frac{39\sqrt{3}}{2} = 0$

Sometimes normal, or implicit, differentiation is (slightly) easier.

Example: Find the equation of the tangent to $xy = 36$, or $x = 6t$, $y = \frac{6}{t}$, at the point where $t = 3$.

Solution: When $t = 3$, $x = 18$ and $y = 2$.

$\frac{dy}{dx}$ can be found in two (or more!) ways:

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-6t^{-2}}{6}$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{t^2} = \frac{-1}{9}$, when $t = 3$

$xy = 36 \quad \Rightarrow \quad y = \frac{36}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{-36}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{-36}{18^2} = \frac{-1}{9}$, when $x = 18$

$\Rightarrow$ equation of tangent is $y - 2 = \frac{-1}{9}(x - 18)$

$\Rightarrow \quad x + 9y - 36 = 0$

Finding a locus

First find expressions for x and y coordinates in terms of a parameter, t or θ, then eliminate the parameter to give an expression involving **only** x and y, which will be the equation of the locus.

Example: The tangent to the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$, at the point P, $(3\cos\theta, 4\sin\theta)$, crosses the x-axis at A, and the y-axis at B.

Find an equation for the locus of the mid-point of AB as P moves round the ellipse, or as θ varies.

Solution: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4\cos\theta}{-3\sin\theta}$

$\Rightarrow$ equation of tangent is $y - 4\sin\theta = \frac{4\cos\theta}{-3\sin\theta}(x - 3\cos\theta)$

$\Rightarrow \quad 3y\sin\theta + 4x\cos\theta = 12\cos^2\theta + 12\sin^2\theta = 12$

Tangent crosses x-axis at A when $y = 0$, $\Rightarrow \quad x = \frac{3}{\cos\theta}$,

and crosses y-axis at B when $x = 0$, $\Rightarrow \quad y = \frac{4}{\sin\theta}$

$\Rightarrow$ mid-point of AB is $\left(\frac{3}{2\cos\theta}, \frac{4}{2\sin\theta}\right) \Leftrightarrow (X, Y)$

Here $X = \frac{3}{2\cos\theta}$ and $Y = \frac{2}{\sin\theta}$

$\Rightarrow \quad \cos\theta = \frac{3}{2X}$ and $\sin\theta = \frac{2}{Y}$

$\Rightarrow$ equation of the locus is $\frac{9}{4X^2} + \frac{4}{Y^2} = 1$ since $\cos^2\theta + \sin^2\theta = 1$

or $\quad \frac{9}{4x^2} + \frac{4}{y^2} = 1$

3 Differentiation

Derivatives of hyperbolic functions

$$y = \sinh x = \tfrac{1}{2}(e^x - e^{-x})$$

$$\Rightarrow \frac{dy}{dx} = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x$$

and, similarly, $\dfrac{d(\cosh x)}{dx} = \sinh x$

Also, $y = \tanh x = \dfrac{\sinh x}{\cosh x}$

$$\Rightarrow \frac{dy}{dx} = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \text{sech}^2\, x$$

In a similar way, all the derivatives of hyperbolic functions can be found.

$f(x)$	$f'(x)$	
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	**all positive**
$\tanh x$	$\text{sech}^2 x$	
$\coth x$	$-\text{cosech}^2 x$	
$\text{cosech}\, x$	$-\text{cosech}\, x \coth x$	**all negative**
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$	

Notice: these are similar to the results for $\sin x$, $\cos x$, $\tan x$ etc., **but** the **minus** signs do not always agree.

The minus signs are '*wrong*' only for $\cosh x$ and $\text{sech}\, x$ $\left(= \dfrac{1}{\cosh x}\right)$.

Derivatives of inverse functions

$$y = \text{arsinh}\, x$$

$$\Rightarrow \sinh y = x \quad \Rightarrow \quad \cosh y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}}$$

$$\Rightarrow \frac{d(\text{arsinh}\, x)}{dx} = \frac{1}{\sqrt{1 + x^2}}$$

The derivatives for other inverse hyperbolic functions can be found in a similar way.

You can also use integration by substitution to find the integrals of the $f'(x)$ column

$f(x)$	$f'(x)$	**substitution needed for integration**
arcsin x	$\frac{1}{\sqrt{1-x^2}}$	$1-\sin^2 u = \cos^2 u \quad \Rightarrow$ use $x = \sin u$
arccos x	$\frac{-1}{\sqrt{1-x^2}}$	$1-\cos^2 u = \sin^2 u \quad \Rightarrow$ use $x = \cos u$
arctan x	$\frac{1}{1+x^2}$	$1+\tan^2 u = \sec^2 u \quad \Rightarrow$ use $x = \tan u$
arsinh x	$\frac{1}{\sqrt{1+x^2}}$	$1+\sinh^2 u = \cosh^2 u \quad \Rightarrow$ use $x = \sinh u$
arcosh x	$\frac{1}{\sqrt{x^2-1}}$	$\cosh^2 u - 1 = \sinh^2 u \quad \Rightarrow$ use $x = \cosh u$
artanh x	$\frac{1}{1-x^2}$	$1-\tanh^2 u = \text{sech}^2 u \quad \Rightarrow$ use $x = \tanh u$
$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$\frac{1}{1-x^2}$	partial fractions, see example below

Note that $\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\int \frac{1}{1+x} + \frac{1}{1-x}\,dx = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + c$

With chain rule, product rule and quotient rule you should be able to handle a large variety of combinations of functions.

4 Integration

Standard techniques

Recognise a standard function

Examples: $\int \sec x \tan x \; dx = \sec x + c$

$\int \operatorname{sech} x \tanh x \; dx = -\operatorname{sech} x + c$

Using formulae to change the integrand

Examples: $\int \tan^2 x \; dx = \int \sec^2 x - 1 \, dx = \tan x - x + c$

$\int \cos^2 x \; dx = \frac{1}{2} \int 1 + \cos 2x \; dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + c$

$\int \sinh^2 x \; dx = \frac{1}{2} \int \cosh 2x - 1 \, dx = \frac{1}{2}\left(\frac{1}{2}\sinh 2x - x\right) + c$

Reverse chain rule

Notice the chain rule pattern, guess an answer and differentiate to find the constant.

Example: $\int \cos^2 x \sin x \; dx$ 'looks like' $u^2 \frac{du}{dx}$ so try $u^3 \Leftrightarrow \cos^3 x$

$\frac{d(\cos^3 x)}{dx} = 3\cos^2 x(-\sin x) = -3\cos^2 x \sin x$ so divide by -3

$\Rightarrow \int \cos^2 x \sin x \; dx = -\frac{1}{3}\cos^3 x + c$

Example: $\int x^2 (2x^3 - 7)^4 \; dx$ 'looks like' $u^4 \frac{du}{dx}$ so try $u^5 \Leftrightarrow (2x^3 - 7)^5$

$\frac{d(2x^3-7)^5}{dx} = 5(2x^3 - 7)^4 \times 6x^2 = 30x^2(2x^3 - 7)^4$ so divide by 30

$\Rightarrow \int x^2 (2x^3 - 7)^4 \; dx = \frac{1}{30}(2x^3 - 7)^5 + c$

Example: $\int \operatorname{sech}^4 x \tanh x \; dx$

$= \int \operatorname{sech}^3 x (\operatorname{sech} x \tanh x) \; dx$ 'looks like' $u^3 \frac{du}{dx}$ so try $u^4 = \operatorname{sech}^4 x$

$\frac{d(\operatorname{sech}^4 x)}{dx} = -4\operatorname{sech}^3 x \operatorname{sech} x \tanh x$ so divide by -4

$\Rightarrow \int \operatorname{sech}^4 x \tanh x \; dx = -\frac{1}{4}\operatorname{sech}^4 x + c$

Standard substitutions

$\int \frac{1}{a^2+b^2x^2}\ dx$ — $bx = a\tan u$ — better than $bx = a\sinh u$ when there is **no** $\sqrt{\ }$

$\int \frac{1}{\sqrt{a^2+b^2x^2}}\ dx$ — $bx = a\sinh u$ — better than $bx = a\tan u$ when there **is** $\sqrt{\ }$

$\int \frac{1}{a^2-b^2x^2}\ dx$ — $bx = a\tanh u$ — or use partial fractions

$\int \frac{1}{\sqrt{b^2x^2-a^2}}\ dx$ — $bx = a\cosh u$ — better than $bx = a\sec u$ when there **is** $\sqrt{\ }$

For more complicated integrals like

$$\int \frac{1}{px^2+qx+r}\ dx \quad \text{or} \quad \int \frac{1}{\sqrt{px^2+qx+r}}\ dx$$

complete the square to give $p(x+a)^2 + b$ and then use a substitution similar to one of the four above.

Example: $\int \frac{1}{\sqrt{4x^2-8x-5}}\ dx$ $\qquad 4x^2-8x-5 = 4(x^2-2x+1)-9 = 4(x-1)^2-9$

$$= \int \frac{1}{\sqrt{4(x-1)^2-9}}\ dx$$

Substitute $2(x-1) = 3\cosh u \Rightarrow 2\,dx = 3\sinh u\ du$

$$= \int \frac{1}{\sqrt{9(\cosh^2 u-1)}}\ \frac{3\sinh u}{2}\ du$$

$$= \frac{1}{2}\int du = u + c = \frac{1}{2}\text{arcosh}\left(\frac{2x-2}{3}\right) + c$$

Nice trick

Example: $$I = \int \frac{1}{\sqrt{(2x-3)^2 + 25}}\ dx$$

Solution: Substitute $u = 2x-3$, $\Rightarrow dx = \frac{1}{2}du$

$$\Rightarrow I = \int \frac{1}{\sqrt{u^2+5^2}}\ \frac{1}{2}du = \frac{1}{2}\text{arsinh}\left(\frac{u}{5}\right) + c,$$ using standard formula

$$= \frac{1}{2}\text{arsinh}\left(\frac{2x-3}{5}\right) + c$$

Important tip

$\int \frac{x^n}{\sqrt{a^2 \pm x^2}}\ dx$, etc., is best done with the substitution

(i) u (or u^2) $= a^2 \pm x^2$, when n is **odd,**

or (ii) a trigonometric or hyperbolic function when n is **even**.

EXAM PAPERS PRACTICE

Integration inverse functions and ln *x*

To integrate inverse trigonometric or hyperbolic functions and $\ln x$ we use integration by parts with $\frac{dv}{dx} = 0$

Example: Find $\int \arctan x \; dx$

Solution: $I = \int \arctan x \; dx$ take $u = \arctan x \Rightarrow \frac{du}{dx} = \frac{1}{1+x^2}$

and $\frac{dv}{dx} = 1 \Rightarrow v = x$

$\Rightarrow \quad I = x \arctan x - \int x \times \frac{1}{1+x^2} \; dx$

$\Rightarrow \quad I = \int \arctan x \; dx = x \arctan x - \frac{1}{2} \ln(1 + x^2) + c$

Example: Find $\int \text{arcosh}\, x \; dx$

Solution: $I = \int \text{arcosh}\, x \; dx$ take $u = \text{arcosh}\, x \Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{x^2-1}}$

and $\frac{dv}{dx} = 1 \Rightarrow v = x$

$\Rightarrow \quad I = x\, \text{arcosh}\, x - \int x \times \frac{1}{\sqrt{x^2-1}} \; dx$

$\Rightarrow \quad I = \int \text{arcosh}\, x \; dx = x\, \text{arcosh}\, x - \sqrt{x^2 - 1} + c$

Reduction formulae

The first step in finding a reduction formula is often (but not always) integration by parts (sometimes twice). The following examples show a variety of techniques.

Example 1: $I_n = \int x^n e^x \, dx$.

(*a*) Find a reduction formula,
(*b*) Find I_0,
(*c*) Find I_4

Solution:

(*a*) Integrating by parts

$$u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$$

and $\frac{dv}{dx} = e^x \Rightarrow v = e^x$

$$\Rightarrow \quad I_n = x^n e^x - \int nx^{n-1} e^x \, dx$$

$$\Rightarrow \quad I_n = x^n e^x - nI_{n-1}$$

(*b*) $I_0 = \int e^x \, dx = e^x + c$

(*c*) Using the reduction formula

$$I_4 = x^4 e^x - 4I_3 = x^4 e^x - 4(x^3 e^x - 3I_2)$$

$$= x^4 e^x - 4x^3 e^x + 12(x^2 e^x - 2I_1)$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24(xe^x - I_0)$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + c \qquad \text{since } I_0 = e^x + c$$

Example 2: Find a reduction formula for $I_n = \int \tan^n x \, dx$.

Solution: $I_n = \int \tan^n x \, dx = \int \tan^{n-2} x \tan^2 x \, dx$

$$\Rightarrow \quad I_n = \int \tan^{n-2} x \, (\sec^2 x - 1) \, dx$$

$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

$$\Rightarrow \quad I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}.$$

Example 3: (i) Find a reduction formula for $I_n = \int_0^{\pi/2} \sin^n x \; dx$.

(ii) Use the formula to find $I_6 = \int_0^{\pi/2} \sin^6 x \; dx$

Solution: (i) By splitting $\sin^n x = \sin^{n-1} x \sin x$, we can differentiate $\sin^{n-1} x$ reducing the power, and we can integrate $\sin x$

$$I_n = \int_0^{\pi/2} \sin^n x \; dx = \int_0^{\pi/2} \sin^{n-1} x \sin x \; dx$$

take $u = \sin^{n-1} x \Rightarrow \frac{du}{dx} = (n-1) \sin^{n-2} x \cos x$

and $\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$$\Rightarrow \quad I_n = \left[-\cos x \sin^{n-1} x\right]_0^{\pi/2} - \int_0^{\pi/2} -\cos x \; (n-1) \sin^{n-2} x \cos x \, dx$$

$$= 0 + (n-1) \int_0^{\pi/2} \cos^2 x \; \sin^{n-2} x \; dx$$

$$= (n-1) \int_0^{\pi/2} (1 - \sin^2 x) \; \sin^{n-2} x \; dx$$

$$= (n-1) \int_0^{\pi/2} \sin^{n-2} x \; dx - (n-1) \int_0^{\pi/2} \sin^n x \; dx$$

$$\Rightarrow \quad I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow \quad I_n = \frac{n-1}{n} I_{n-2}.$$

(ii) $$I_6 = \frac{5}{6} I_4 = \frac{5}{6} \times \frac{3}{4} I_2 = \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} I_0$$

$$\Rightarrow \quad I_6 = \frac{5}{16} \int_0^{\pi/2} 1 \; dx = \frac{5\pi}{32}.$$

Example 4: Find a reduction formula for $I_n = \int \sec^n x \; dx$.

Solution: By splitting $\sec^n x = \sec^{n-2} x \sec^2 x$, we can differentiate $\sec^{n-2} x$ reducing the power, and we can integrate $\sec^2 x$

$$I_n = \int \sec^n x \; dx = \int \sec^{n-2} x \; \sec^2 x \; dx$$

take $u = \sec^{n-2} x \Rightarrow \frac{du}{dx} = (n-2) \sec^{n-3} x \sec x \tan x$

and $\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$

$$\Rightarrow \quad I_n = \sec^{n-2} x \tan x - \int \tan x \, (n-2) \sec^{n-3} x \sec x \tan x \; dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \tan^2 x \; \sec^{n-2} x \; dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int (\sec^2 x - 1) \; \sec^{n-2} x \; dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$\Rightarrow \quad (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

Example 5: Find a reduction formula for $I_n = \int_{-1}^{0} x^n (1+x)^2\, dx$.

Solution: We can differentiate x^n reducing the power, and we can integrate $(1+x)^2$

$I_n = \int_{-1}^{0} x^n (1+x)^2\, dx$ take $u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$

and $\frac{dv}{dx} = (1+x)^2 \Rightarrow v = \frac{1}{3}(1+x)^3$

$\Rightarrow \quad I_n = \left[x^n \times \frac{1}{3}(1+x)^3\right]_{-1}^{0} - \int_{-1}^{0} nx^{n-1} \times \frac{1}{3}(1+x)^3\, dx.$

Writing $(1+x)^3 = (1+x)^2(1+x)$ allows us to write the integral in terms of I_{n-1} and I_n.

Many reduction formulae need a fiddle like this – e.g. $(1-x^2)^{\frac{3}{2}} = (1-x^2)^{\frac{1}{2}}(1-x^2)$

$\Rightarrow \quad I_n = 0 - \frac{n}{3}\int_{-1}^{0} x^{n-1}(1+x)^2(1+x)dx$

$\Rightarrow \quad I_n = -\frac{n}{3}\int_{-1}^{0} x^{n-1}(1+x)^2 dx - \frac{n}{3}\int_{-1}^{0} x^n (1+x)^2 dx$

$\Rightarrow \quad I_n = -\frac{n}{3} I_{n-1} - \frac{n}{3} I_n$

$\Rightarrow \quad \frac{n+3}{3} I_n = -\frac{n}{3} I_{n-1}$

$\Rightarrow \quad I_n = -\frac{n}{n+3} I_{n-1}$

Example 6: Find a reduction formula for $I_n = \int_{0}^{\pi/2} x^n \cos x\; dx$

Solution: We can differentiate x^n reducing the power, and we can integrate $\cos x$

$I_n = \int_{0}^{\pi/2} x^n \cos x\; dx$ take $u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$

and $\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$

$\Rightarrow \quad I_n = [x^n \sin x]_{0}^{\pi/2} - n\int_{0}^{\pi/2} \sin x \times x^{n-1}\; dx$

Integrating by parts again will change the sin x to cos x, and reduce the power further.

take $u = x^{n-1} \Rightarrow \frac{du}{dx} = (n-1)x^{n-2}$

and $\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$\Rightarrow \quad I_n = \left(\frac{\pi}{2}\right)^n - n\left\{[x^{n-1}(-\cos x)]_{0}^{\pi/2} - \int_{0}^{\pi/2} -\cos x \times (n-1)x^{n-2}\; dx\right\}$

$\Rightarrow \quad I_n = \left(\frac{\pi}{2}\right)^n - n\left\{0 + (n-1)\int_{0}^{\pi/2} x^{n-2}\cos x\; dx\right\}$

$\Rightarrow \quad I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}$

Example 7: Find a reduction formula for $I_n = \int \frac{\sin nx}{\sin x}\, dx$

Solution: $I_n = \int \frac{\sin[(n-2)x+2x]}{\sin x}\, dx$

$$= \int \frac{\sin(n-2)x\cos 2x + \cos(n-2)x\sin 2x}{\sin x}\, dx$$

$$= \int \frac{\sin(n-2)x\,(1-2\sin^2 x) + \cos(n-2)x \times 2\sin x\cos x}{\sin x}\, dx$$

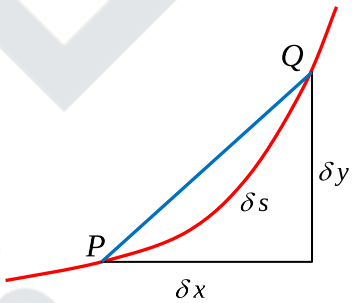
$$= \int \frac{\sin(n-2)x}{\sin x}\, dx + 2\int \cos(n-2)x\cos x - \sin(n-2)x\sin x \;\; dx$$

$$= I_{n-2} + 2\int \cos(n-1)x \;\; dx \qquad \text{using } \cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\Rightarrow \quad I_n = I_{n-2} + \frac{2}{n-1}\sin(n-1)x\,.$$

Arc length

All the formulae you need can be remembered from this diagram



arc $PQ \approx$ line segment PQ

$$\Rightarrow \quad (\delta s)^2 \approx (\delta x)^2 + (\delta y)^2$$

$$\Rightarrow \quad \left(\frac{\delta s}{\delta x}\right)^2 \approx 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

and as $\delta x \to 0$

$$\Rightarrow \quad \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \quad \Rightarrow \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \quad \text{arc length} = s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\;\; dx$$

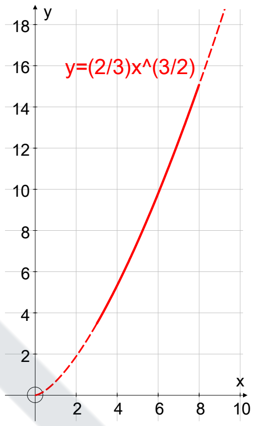
Similarly $\left(\frac{\delta s}{\delta y}\right)^2 \approx \left(\frac{\delta x}{\delta y}\right)^2 + 1 \quad \Rightarrow \quad s = \int \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\ dy$

and $\left(\frac{\delta s}{\delta t}\right)^2 \approx \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$

$$\Rightarrow \quad s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \quad \text{or} \quad s = \int \sqrt{\dot{x}^2 + \dot{y}^2}\ dt$$

for parametric equations.

Example 1: Find the length of the curve $y = \frac{2}{3}x^{3/2}$, from the point where $x = 3$ to the point where $x = 8$.



Solution: The equation of the curve is in Cartesian form so we use

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx\ .$$

$$y = \frac{2}{3}x^{3/2} \quad \Rightarrow \quad \frac{dy}{dx} = \sqrt{x}$$

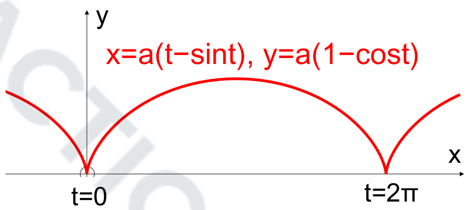
$$\Rightarrow \quad s = \int_3^8 \sqrt{1+x}\ dx$$

$$= \left[\frac{2}{3}(1+x)^{3/2}\right]_3^8 = \frac{2}{3} \times (9)^{3/2} - \frac{2}{3} \times (4)^{3/2}$$

$$\Rightarrow \quad s = 12\frac{2}{3}.$$

Example 2: Find the length of one arch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

Solution: The curve is given in parametric form so we use $s = \int \sqrt{\dot{x}^2 + \dot{y}^2}\ dt$.



$x = a(t - \sin t)$, $y = a(1 - \cos t)$

$$\Rightarrow \quad \frac{dx}{dt} = a(1 - \cos t), \text{ and } \frac{dy}{dt} = a \sin t$$

$$\Rightarrow \quad \dot{x}^2 + \dot{y}^2 = a^2(1 - 2\cos t + \cos^2 t + \sin^2 t) = 2a^2(1 - \cos t)$$

$$\Rightarrow \quad \sqrt{\dot{x}^2 + \dot{y}^2} = a\sqrt{2\left(1 - \left[1 - 2\sin^2\left(\frac{t}{2}\right)\right]\right)} = 2a\sin\left(\frac{t}{2}\right)$$

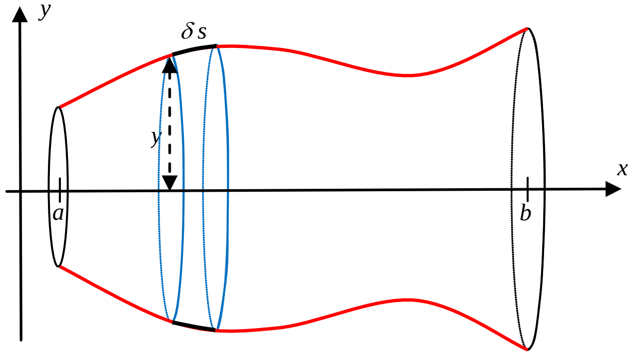
$$\Rightarrow \quad s = \int_0^{2\pi} 2a\sin\left(\frac{t}{2}\right)\ dt$$

$$\Rightarrow \quad s = \left[-4a\cos\left(\frac{t}{2}\right)\right]_0^{2\pi} = 4a - - 4a = 8a.$$

Area of a surface of revolution

A curve is rotated about the x-axis.

To find the area of the surface formed between $x = a$ and $x = b$, we consider a small section of the curve, δs, at a distance of y from the x-axis.
When this small section is rotated about the x-axis, the shape formed is approximately a cylinder of radius y and length δs.



The surface area of this (cylindrical) shape $\approx 2\pi\, y\delta s$

$\Rightarrow$ The total surface area $\approx \sum_a^b 2\pi\, y\delta s$

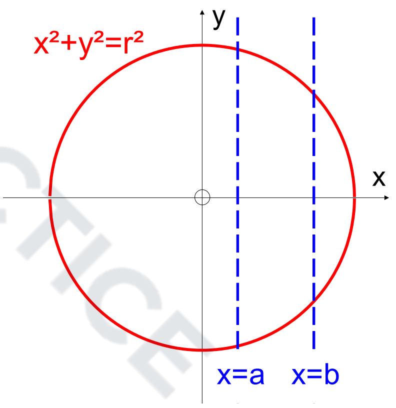
and, as $\delta s \to 0$, the area of the surface is $A = \int_a^b 2\pi\, y\ ds$.

And so $A = \int_a^b 2\pi\, y\, \frac{ds}{dx}\, dx$ or $A = \int_a^b 2\pi\, y\, \frac{ds}{dt}\, dt$

We can use $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ or $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$, as appropriate,

remembering that $(\delta s)^2 \approx (\delta x)^2 + (\delta y)^2$

Example 1: A sphere has radius r. Find the surface area of the sphere between the planes $x = a$ and $x = b$.



Solution: The Cartesian form is most suitable here.

$A = \int_a^b 2\pi\, y\, \frac{ds}{dx}\, dx$

$x^2 + y^2 = r^2$

$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-x}{y}$

and $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$\Rightarrow \quad A = \int_a^b 2\pi y \sqrt{1 + \frac{x^2}{y^2}}\, dx = \int_a^b 2\pi \sqrt{y^2 + x^2}\, dx$

$= \int_a^b 2\pi r\ dx$ since $x^2 + y^2 = r^2$

$\Rightarrow \quad A = [2\pi r x]_a^b = 2\pi r(b - a)$ since r is constant

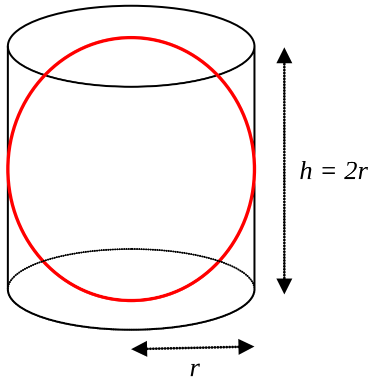
Notice that the area of the whole sphere is from $a = -r$ to $b = r$ giving surface area of a sphere is $4\pi r^2$.

Historical note.

Archimedes showed that the area of a sphere is equal to the area of the curved surface of the surrounding cylinder.

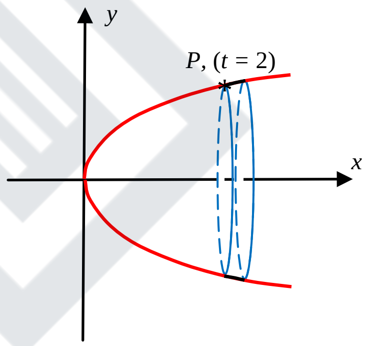
Thus the area of the sphere is

$A = 2\pi rh = 4\pi r^2$ since $h = 2r$.



Example 2: The parabola, $x = at^2$, $y = 2at$, between the origin ($t = 0$) and P ($t = 2$) is rotated about the x-axis.

Find the surface area of the shape formed.



Solution: The parametric form is suitable here.

$$A = \int_a^b 2\pi\, y\, \frac{ds}{dt}\, dt$$

and $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$$\frac{dx}{dt} = 2at \quad \text{and} \quad \frac{dy}{dt} = 2a$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{(2at)^2 + (2a)^2} = 2a\sqrt{t^2+1}$$

$$\Rightarrow \quad A = \int_0^2 2\pi\, 2at \times 2a\sqrt{t^2+1}\ dt$$

$$= 8\pi a^2 \times \tfrac{1}{3}\left[(t^2+1)^{3/2}\right]_0^2$$

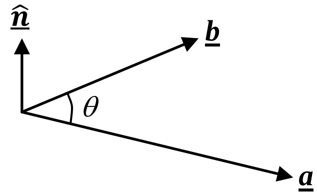
$$\Rightarrow \quad A = \frac{8\pi a^2}{3}\left(5^{3/2} - 1\right)$$

5 Vectors

Vector product

The vector, or cross, product of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ is

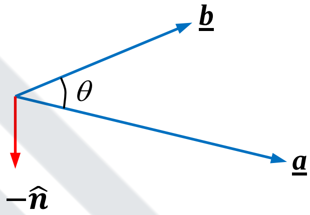
$$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = ab \sin\theta \; \underline{\hat{\mathbf{n}}}$$



where $\underline{\hat{\mathbf{n}}}$ is a *unit* (length 1) vector which is *perpendicular* to both $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$, and θ is the angle between $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$.

The direction of $\underline{\hat{\mathbf{n}}}$ is that in which a right hand corkscrew would move when turned through the angle θ from $\underline{\mathbf{a}}$ to $\underline{\mathbf{b}}$.

Notice that $\underline{\mathbf{b}} \times \underline{\mathbf{a}} = ab \sin\theta(-\underline{\hat{\mathbf{n}}})$, where $-\underline{\hat{\mathbf{n}}}$ is in the opposite direction to $\underline{\hat{\mathbf{n}}}$, since the corkscrew would move in the opposite direction when moving from $\underline{\mathbf{b}}$ to $\underline{\mathbf{a}}$.



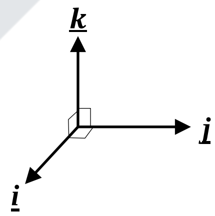
Thus $\underline{\mathbf{b}} \times \underline{\mathbf{a}} = -\underline{\mathbf{a}} \times \underline{\mathbf{b}}$.

The vectors $\underline{\mathbf{i}}, \underline{\mathbf{j}}$ and $\underline{\mathbf{k}}$

For unit vectors, $\underline{\mathbf{i}}, \underline{\mathbf{j}}$ and $\underline{\mathbf{k}}$, in the directions of the axes

$\underline{\mathbf{i}} \times \underline{\mathbf{j}} = \underline{\mathbf{k}}$, $\underline{\mathbf{j}} \times \underline{\mathbf{k}} = \underline{\mathbf{i}}$, $\underline{\mathbf{k}} \times \underline{\mathbf{i}} = \underline{\mathbf{j}}$,

$\underline{\mathbf{i}} \times \underline{\mathbf{k}} = -\underline{\mathbf{j}}$, $\underline{\mathbf{j}} \times \underline{\mathbf{i}} = -\underline{\mathbf{k}}$, $\underline{\mathbf{k}} \times \underline{\mathbf{j}} = -\underline{\mathbf{i}}$.



Properties

$\underline{\mathbf{a}} \times \underline{\mathbf{a}} = \underline{\mathbf{0}}$		since $\theta = 0$
$\underline{\mathbf{a}} \times \underline{\mathbf{b}} = \underline{\mathbf{0}} \quad \Rightarrow$	$\underline{\mathbf{a}}$ is parallel to $\underline{\mathbf{b}}$	since $\sin\theta = 0 \;\Rightarrow\; \theta = 0$ or π
	or $\underline{\mathbf{a}}$ or $\underline{\mathbf{b}} = \underline{\mathbf{0}}$	
$\underline{\mathbf{a}} \times (\underline{\mathbf{b}} + \underline{\mathbf{c}}) = \underline{\mathbf{a}} \times \underline{\mathbf{b}} + \underline{\mathbf{a}} \times \underline{\mathbf{c}}$		remember the brilliant demo with the straws!
$\underline{\mathbf{a}} \times \underline{\mathbf{b}}$ is perpendicular to both $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$		from the definition

Component form

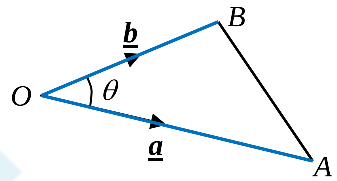
Using the above we can show that

$$\underline{a} \times \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Applications of the vector product

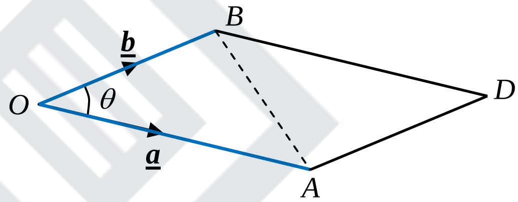
Area of triangle $OAB = \frac{1}{2}ab\sin\theta$

$\Rightarrow$ area of triangle $OAB = \frac{1}{2}\left|\underline{a} \times \underline{b}\right|$



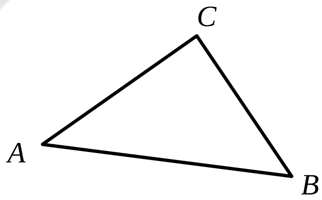
Area of parallelogram $OADB$ is twice the area of the triangle OAB

$\Rightarrow$ area of parallelogram $OADB = \left|\underline{a} \times \underline{b}\right|$



Example: A is $(-1, 2, 1)$, B is $(2, 3, 0)$ and C is $(3, 4, -2)$.

Find the area of the triangle ABC.



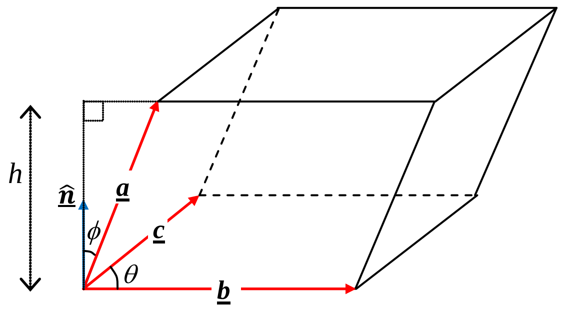
Solution: The area of the triangle $ABC = \left|\frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \text{ and } \overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 1 & -1 \\ 4 & 2 & -3 \end{vmatrix} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$$

$$\Rightarrow \text{area } ABC = \left|\frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{1^2 + 5^2 + 3^2} = \frac{1}{2}\sqrt{35}$$

Volume of a parallelepiped



In the parallelepiped,

the base is parallel to $\underline{\boldsymbol{b}}$ and $\underline{\boldsymbol{c}}$

$\underline{\hat{\boldsymbol{n}}}$ is a unit vector perpendicular to the base

and the height $\underline{\boldsymbol{h}} = h\,\underline{\hat{\boldsymbol{n}}}$,

where $h = \pm a \cos\phi = \pm\, \underline{\boldsymbol{a}} \,.\, \underline{\hat{\boldsymbol{n}}}$
$\pm$ because ϕ might be obtuse

The area of base $= bc \sin\theta$

$\Rightarrow$ volume $\quad V = \pm h \times bc \sin\theta$

$\Rightarrow \quad \pm V = a \cos\phi \times bc \sin\theta$

$\underline{\boldsymbol{a}} \,.\, (\underline{\boldsymbol{b}} \times \underline{\boldsymbol{c}}) = \underline{\boldsymbol{a}} \,.\, (bc \sin\theta\, \underline{\hat{\boldsymbol{n}}}) = \underline{\boldsymbol{a}} \,.\, \underline{\hat{\boldsymbol{n}}}\,(bc \sin\theta\,)$

$\Rightarrow \quad \underline{\boldsymbol{a}} \,.\, (\underline{\boldsymbol{b}} \times \underline{\boldsymbol{c}}) = a \cos\phi \times bc \sin\theta = \pm V$

$\Rightarrow$ volume of parallelepiped $= |\underline{\boldsymbol{a}} \,.\, (\underline{\boldsymbol{b}} \times \underline{\boldsymbol{c}})|$

Triple scalar product

$$|\underline{\boldsymbol{a}} \,.\, (\underline{\boldsymbol{b}} \times \underline{\boldsymbol{c}})| = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_2c_3 - b_3c_2 \\ -b_1c_3 + b_3c_1 \\ b_1c_2 - b_2c_1 \end{pmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(-b_1c_3 + b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

By expanding the determinants we can show that

$\underline{\boldsymbol{a}} \,.\, (\underline{\boldsymbol{b}} \times \underline{\boldsymbol{c}}) = (\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}}) \,.\, \underline{\boldsymbol{c}}$ keep the order of $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}$ but change the order of the $\times$ and **.**

For this reason the triple scalar product is written as $\{\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}\}$

$$\{\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}\} = \underline{\boldsymbol{a}} \,.\, (\underline{\boldsymbol{b}} \times \underline{\boldsymbol{c}}) = (\underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}}) \,.\, \underline{\boldsymbol{c}}$$

It can also be shown that a cyclic change of the order of $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}$ does not change the value, but interchanging two of the vectors multiplies the value by -1.

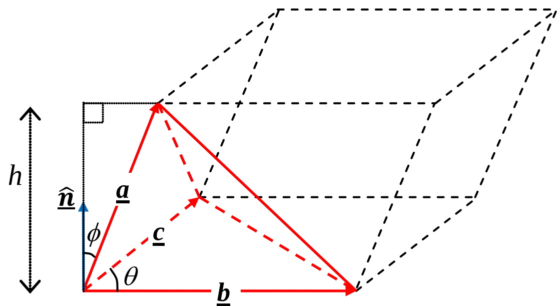
$$\Rightarrow \quad \{\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}\} = \{\underline{\boldsymbol{c}}, \underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}\} = \{\underline{\boldsymbol{b}}, \underline{\boldsymbol{c}}, \underline{\boldsymbol{a}}\} = -\{\underline{\boldsymbol{a}}, \underline{\boldsymbol{c}}, \underline{\boldsymbol{b}}\} = -\{\underline{\boldsymbol{c}}, \underline{\boldsymbol{b}}, \underline{\boldsymbol{a}}\} = -\{\underline{\boldsymbol{b}}, \underline{\boldsymbol{a}}, \underline{\boldsymbol{c}}\}$$

Volume of a tetrahedron

The volume of a tetrahedron is

$\frac{1}{3}$ Area of base $\times h$

The height of the tetrahedron is the same as the height of the parallelepiped, but its base has half the area



$\Rightarrow$ volume of tetrahedron $= \frac{1}{6}$ volume of parallelepiped

$\Rightarrow$ volume of tetrahedron $= \frac{1}{6}|\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}|$

Example: Find the volume of the tetrahedron $ABCD$,

given that A is (1, 0, 2), B is (–1, 2, 2), C is (1, 1, –3) and D is (4, 0, 3).

Solution: Volume $= \frac{1}{6}\left|\{\overrightarrow{AD}, \overrightarrow{AC}, \overrightarrow{AB}\}\right|$

$$\overrightarrow{AD} = \underline{\boldsymbol{d}} - \underline{\boldsymbol{a}} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}, \quad \overrightarrow{AB} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \{\overrightarrow{AD}, \overrightarrow{AC}, \overrightarrow{AB}\} = \begin{vmatrix} 3 & 0 & 1 \\ 0 & 1 & -5 \\ -2 & 2 & 0 \end{vmatrix} = 3 \times 10 + 2 = 32$$

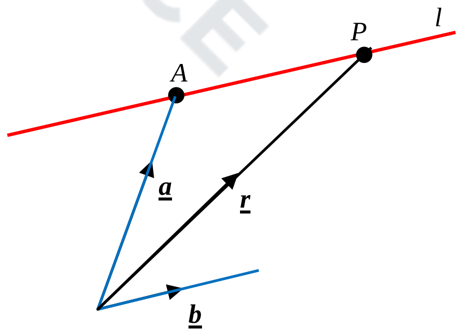
$\Rightarrow$ volume of tetrahedron is $\frac{1}{6} \times 32 = 5\frac{1}{3}$

Equations of straight lines

Vector equation of a line

$\underline{\boldsymbol{r}} = \underline{\boldsymbol{a}} + \lambda \underline{\boldsymbol{b}}$ is the equation of a line through the point A and parallel to the vector $\underline{\boldsymbol{b}}$,

or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} l \\ m \\ n \end{pmatrix} + \lambda \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$.



Cartesian equation of a line in 3-D

Eliminating λ from the above equation we obtain

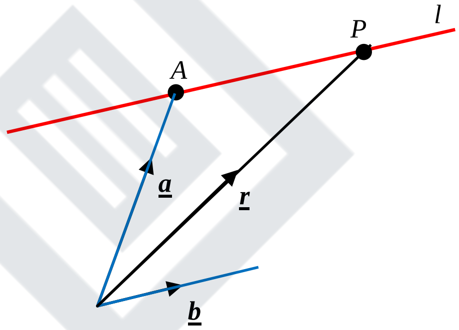
$$\frac{x-l}{\alpha} = \frac{y-m}{\beta} = \frac{z-n}{\gamma} \quad (= \lambda)$$

is the equation of a line through the point (l, m, n) and parallel to the vector $\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$.

This strange form of equation is really the intersection of the planes

$$\frac{x-l}{\alpha} = \frac{y-m}{\beta} \quad \text{and} \quad \frac{y-m}{\beta} = \frac{z-n}{\gamma} \qquad \left(\text{and } \frac{x-l}{\alpha} = \frac{z-n}{\gamma}\right).$$

Vector product equation of a line



$\overrightarrow{AP} = \underline{\mathbf{r}} - \underline{\mathbf{a}}$ and is parallel to the vector $\underline{\mathbf{b}}$

$\Rightarrow \quad \overrightarrow{AP} \times \underline{\mathbf{b}} = \underline{\mathbf{0}}$

$\Rightarrow \quad (\underline{\mathbf{r}} - \underline{\mathbf{a}}) \times \underline{\mathbf{b}} = \underline{\mathbf{0}}$ is the equation of a line through A and parallel to $\underline{\mathbf{b}}$.

or $\quad \underline{\mathbf{r}} \times \underline{\mathbf{b}} = \underline{\mathbf{a}} \times \underline{\mathbf{b}} = \underline{\mathbf{c}}$ is the equation of a line parallel to $\underline{\mathbf{b}}$.

Notice that all three forms of equation refer to *a line through the point A and parallel to the vector $\underline{\mathbf{b}}$.*

Example: A straight line has Cartesian equation

$$x = \frac{2y+4}{5} = \frac{3-z}{2}.$$

Find its equation (i) in the form $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda\underline{\mathbf{b}}$, (ii) in the form $\underline{\mathbf{r}} \times \underline{\mathbf{b}} = \underline{\mathbf{c}}$.

Solution:

First re-write the equation in the *standard* manner

$$\Rightarrow \quad \frac{x-0}{1} = \frac{y--2}{2.5} = \frac{z-3}{-2}$$

$\Rightarrow$ the line passes through A, $(0, -2, 3)$, and is parallel to $\underline{\mathbf{b}}$, $\begin{pmatrix}1\\ 2.5\\ -2\end{pmatrix}$ or $\begin{pmatrix}2\\ 5\\ -4\end{pmatrix}$

(i) $\underline{r} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$

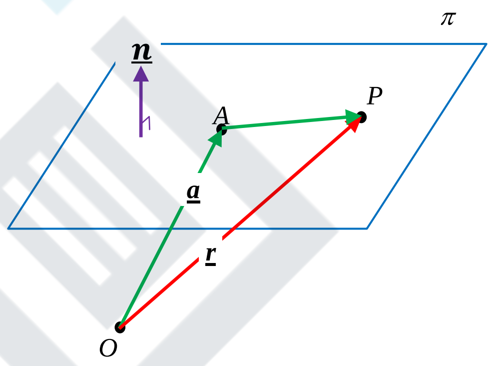
(ii) $\left(\underline{r} - \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}\right) \times \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} = \underline{\mathbf{0}}$

$$\Rightarrow \quad \underline{r} \times \begin{pmatrix} 1 \\ 2.5 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & -2 & 3 \\ 2 & 5 & -4 \end{vmatrix} = \begin{pmatrix} -7 \\ 6 \\ 4 \end{pmatrix}$$

$$\Rightarrow \quad \underline{r} \times \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} -7 \\ 6 \\ 4 \end{pmatrix}.$$

Equation of a plane

Scalar product form



Let $\underline{n}$ be a vector perpendicular to the plane π.

Let A be a fixed point in the plane, and P be a general point, (x, y, z), in the plane.

Then $\overrightarrow{AP}$ is parallel to the plane, and therefore perpendicular to $\underline{n}$

$\Rightarrow \quad \overrightarrow{AP} \cdot \underline{n} = 0 \quad \Rightarrow \quad (\underline{r} - \underline{a}) \cdot \underline{n} = 0$

$\Rightarrow \quad \underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n} =$ a constant, d

$\Rightarrow \quad \underline{r} \cdot \underline{n} = d$ is the equation of a plane **perpendicular to the vector** $\underline{n}$.

Cartesian form

If $\underline{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $\underline{r} \cdot \underline{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = ax + by + cz$

$\Rightarrow \quad ax + by + cz = d$ is the Cartesian equation of a plane **perpendicular to** $\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix}$.

Example: Find the scalar product form and the Cartesian equation of the plane through the points A, (3, 2, 5), B, (–1, 0, 3) and C, (2, 1, –2).

Solution: We first need a vector perpendicular to the plane.

A, (3, 2, 5), B, (–1, 0, 3) and C, (2, 1, –2) lie in the plane

$\Rightarrow \quad \overrightarrow{AB} = \begin{pmatrix} -4 \\ -2 \\ -2 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -1 \\ -1 \\ -7 \end{pmatrix}$ are parallel to the plane

$\Rightarrow \quad \overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to the plane

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ -4 & -2 & -2 \\ -1 & -1 & -7 \end{vmatrix} = \begin{pmatrix} 12 \\ -26 \\ 2 \end{pmatrix} = 2 \times \begin{pmatrix} 6 \\ -13 \\ 1 \end{pmatrix}$$ using smaller numbers

$\Rightarrow \quad 6x - 13y + z = d$

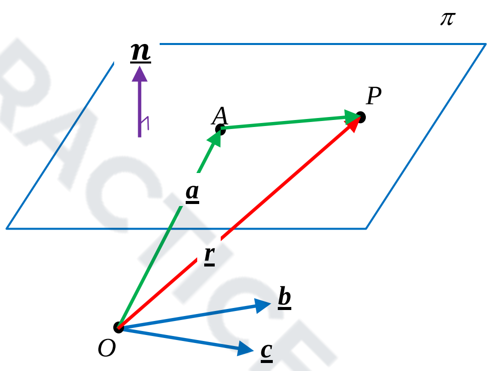
but A, (3, 2, 5) lies in the plane $\quad \Rightarrow \quad d = 6 \times 3 - 13 \times 2 + 5 = -3$

$\Rightarrow$ *Cartesian equation* is $6x - 13y + z = -3$

and *scalar product* equation is $\underline{\boldsymbol{r}} \cdot \begin{pmatrix} 6 \\ -13 \\ 1 \end{pmatrix} = -3$.

Vector equation of a plane

$\underline{\boldsymbol{r}} = \underline{\boldsymbol{a}} + \lambda \underline{\boldsymbol{b}} + \mu \underline{\boldsymbol{c}}$ is the equation of a plane, π, through A and parallel to the vectors $\underline{\boldsymbol{b}}$ and $\underline{\boldsymbol{c}}$.

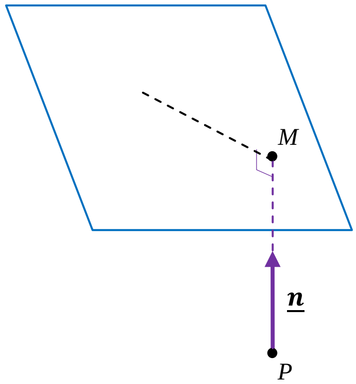


Example: Find the vector equation of the plane through the points A, (1, 4, –2), B, (1, 5, 3) and C, (4, 7, 2).

Solution: We want the plane through A, (1, 4, –2), parallel to $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$

$\Rightarrow$ vector equation is $\underline{\boldsymbol{r}} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$.

Distance from a point to a plane



Example: Find the distance from the point P (–2, 3, 5) to the plane $4x - 3y + 12z = 21$.

Solution: Let M be the foot of the perpendicular from P to the plane. The distance of the origin from the plane is PM.

We must first find the intersection of the line PM with the plane.

PM is perpendicular to the plane

and so is parallel to $\underline{\boldsymbol{n}} = \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix}$.

$\Rightarrow$ the line PM is $\underline{\boldsymbol{r}} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} = \begin{pmatrix} -2+4\lambda \\ 3-3\lambda \\ 5+12\lambda \end{pmatrix}$,

and the point of intersection of PM with the plane is given by

$$4(-2 + 4\lambda) - 3(3 - 3\lambda) + 12(5 + 12\lambda) = 21$$

$$\Rightarrow \quad -8 + 16\lambda - 9 + 9\lambda + 60 + 144\lambda = 21$$

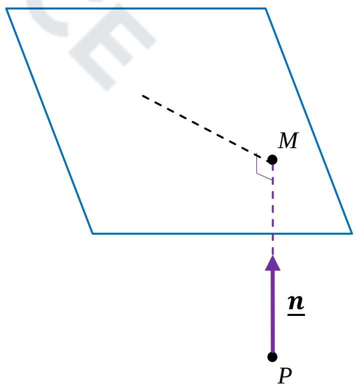
$$\Rightarrow \quad \lambda = \frac{-22}{169}$$

$$\Rightarrow \quad \overrightarrow{PM} = \frac{-22}{169}\begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix}$$

$$\Rightarrow \quad \text{distance} = \left|\overrightarrow{PM}\right| = \frac{22}{169}\sqrt{4^2 + 3^2 + 12^2} = \frac{22}{13}$$

The distance of the P from the plane is $\frac{22}{13}$.

Distance from any point to a plane



The above technique can be used to find the formula:-

distance, s, from the point P (α, β, γ) to the plane

$n_1x + n_2y + n_3z + d = 0$ is given by

$$s = \left|\frac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{n_1^{\,2} + n_2^{\,2} + n_3^{\,2}}}\right|$$

This formula is in your formula booklets, but **not** in your text books.

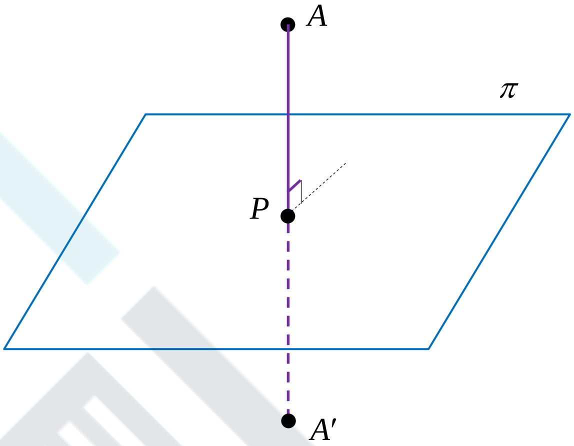
Reflection of a point in a plane

Example: Find the reflection of the point A (10, 1, 7) in the plane π, $\underline{\boldsymbol{r}} . \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 7$.

Solution: Find the point of intersection, P, of the line through A and perpendicular to π with the plane π. Then find $\overrightarrow{AP}$, to give $\overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AP}$.

Line through A perpendicular to π is

$$\underline{\boldsymbol{r}} = \begin{pmatrix} 10 \\ 1 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$



This meets the plane π when

$3(10+3\lambda) - 2(1-2\lambda) + (7+\lambda) = 7$

$\Rightarrow \quad 30 + 9\lambda - 2 + 4\lambda + 7 + \lambda = 7$

$\Rightarrow \quad \lambda = -2$

$\Rightarrow \quad \overrightarrow{OP} = \begin{pmatrix} 10 \\ 1 \\ 7 \end{pmatrix} + (-2)\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$\Rightarrow \quad \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (-2)\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ -2 \end{pmatrix}$

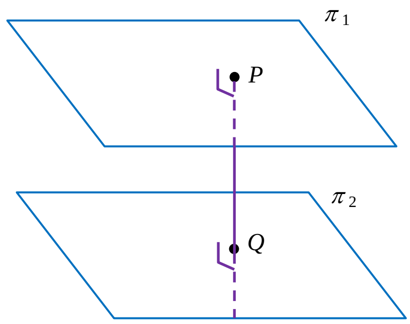
$\Rightarrow \quad \overrightarrow{OA'} = \overrightarrow{OA} + 2\overrightarrow{AP} = \begin{pmatrix} 10 \\ 1 \\ 7 \end{pmatrix} + 2\begin{pmatrix} -6 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 9 \\ 3 \end{pmatrix}$

$\Rightarrow$ the reflection of A is A', (–2, 9, 3)

Distance between parallel planes

Example: Find the distance between the parallel planes

π_1: $2x - 6y + 3z = 9$ and π_2: $2x - 6y + 3z = 5$



Solution: Take any point, P, on one of the planes, and then use the above formula for the shortest distance, PQ, between the planes.

By inspection the point P (0, 0, 3) lies on π_1

$\Rightarrow$ shortest distance s from P to the plane π_2 is $\left|\dfrac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{{n_1}^2 + {n_2}^2 + {n_3}^2}}\right|$

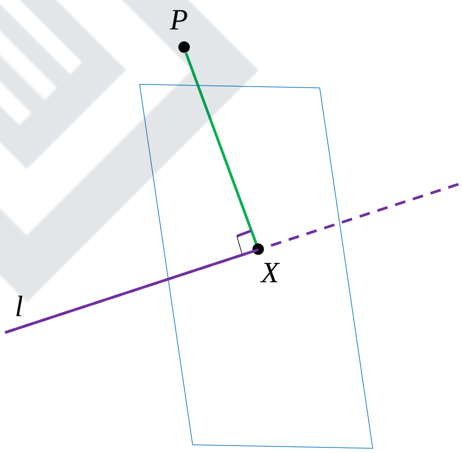
$$\Rightarrow \quad \text{shortest distance } s = \left|\frac{2 \times 0 - 6 \times 0 + 3 \times 3 - 5}{\sqrt{2^2 + 6^2 + 3^2}}\right| = \frac{4}{7}$$

The distance between the planes is $\frac{4}{7}$.

Shortest distance from a point to a line

Example: Find the shortest distance from the point

P (3, –2, 4) to the line l, $\underline{r} = \begin{pmatrix}-2\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\6\end{pmatrix}$



Solution: Any plane $2x - 3y + 6z = d$ must be perpendicular to the line l. If we make this plane pass through P and if it meets the line l in the point X, then PX must be perpendicular to the line l, and so PX is the shortest distance from P to the line l.

Plane passes through P (3, –2, 4)

$\Rightarrow$ $2x - 3y + 6z = 2 \times 3 - 3 \times (-2) + 6 \times 4 = 36$

$\Rightarrow$ $2x - 3y + 6z = 36$

l meets plane $\Rightarrow$ $2(-2 + 2\lambda) - 3(3 - 3\lambda) + 6(6\lambda) = 36$

$\Rightarrow$ $-4 + 4\lambda - 9 + 9\lambda + 36\lambda = 36$ $\Rightarrow$ $\lambda = 1$

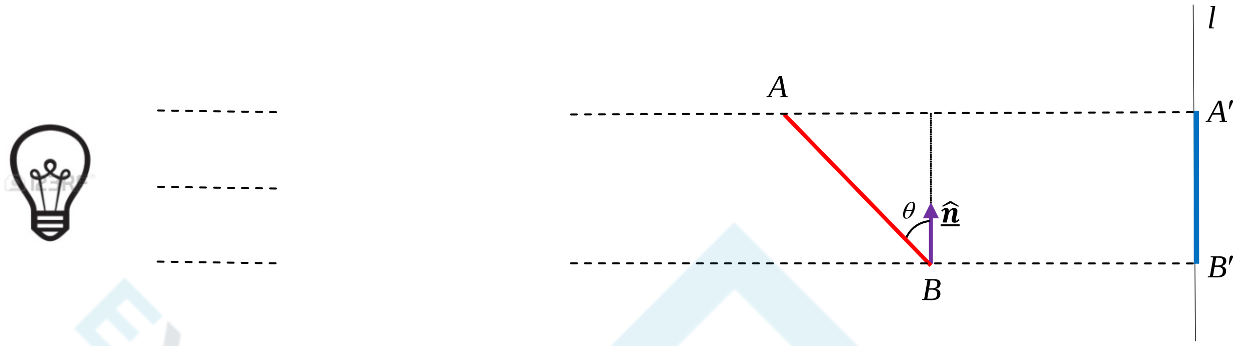
$\Rightarrow$ X is the point (–2, 0, 6)

$$\overrightarrow{PX} = \begin{pmatrix}0\\0\\6\end{pmatrix} - \begin{pmatrix}3\\-2\\4\end{pmatrix} = \begin{pmatrix}-3\\2\\2\end{pmatrix}$$

$\Rightarrow$ shortest distance is $PX = \sqrt{3^2 + 2^2 + 2^2} = \sqrt{17}$

Projections – an alternative approach

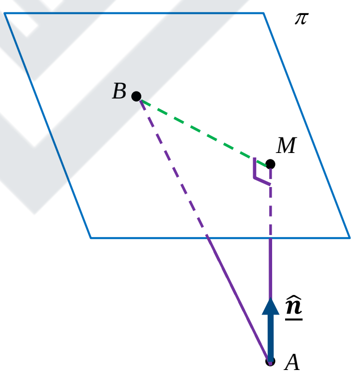
Imagine a light bulb causing a rod, AB, to make a shadow, $A'B'$, on the line l. If the light bulb is far enough away, we can think of all the light rays as parallel, and, if the rays are all perpendicular to the line l, the shadow is the *projection* of the rod onto l (strictly speaking an *orthogonal* projection).



The length of the shadow, $B'A'$, is $|BA\cos\theta| = |\overrightarrow{BA}\,.\,\hat{\mathbf{n}}|$, where $\hat{\mathbf{n}}$ is a unit vector parallel to the line l.

Modulus signs are needed in case $\hat{\mathbf{n}}$ is in the opposite direction.

Shortest distance from a point from a plane.



To find AM, the shortest distance from A to the plane π,

For any point, B, on π AM is the projection of AB onto the line AM

$$\Rightarrow \quad AM = |\overrightarrow{AB}\,.\,\hat{\mathbf{n}}|$$

Example: Find the shortest distance from the point A (–2, 3, 5)
to the plane $4x - 3y + 12z = 21$.

Solution: By inspection B (0, –7, 0) lies on the plane

$$\Rightarrow \quad \overrightarrow{AB} = \begin{pmatrix} 0 \\ -7 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \\ -5 \end{pmatrix}$$

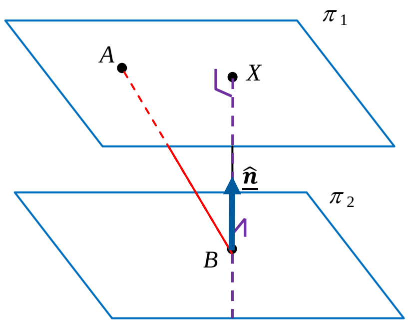
$$\mathbf{n} = \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \quad \Rightarrow \quad n = \sqrt{4^2 + 3^2 + 12^2} = 13$$

$$\Rightarrow \quad \text{shortest distance} = |\overrightarrow{AB}\,.\,\hat{\mathbf{n}}| = \left| \begin{pmatrix} 2 \\ -10 \\ -5 \end{pmatrix} . \frac{1}{13} \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \right| = \frac{22}{13}$$

Distance between parallel planes

Example: Find the distance between the parallel planes

π_1: $2x - 6y + 3z = 9$ and π_2: $2x - 6y + 3z = 5$



Solution: Take any point, B, on one of the planes, π_2, and then consider the line BX perpendicular to both planes; BX is then the shortest distance between the planes.

Then choose any point, A, on π_1, and BX is now the projection of AB onto BX

$\Rightarrow$ shortest distance $= BX = |\overrightarrow{AB} \cdot \hat{\boldsymbol{n}}|$

or shortest distance $= |(\boldsymbol{b} - \boldsymbol{a}) \cdot \hat{\boldsymbol{n}}|$, for any two points A and B, one on each plane, where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to both planes.

By inspection the point A (0, 0, 3) lies on π_1, and the point B (2·5, 0, 0) lies on π_2

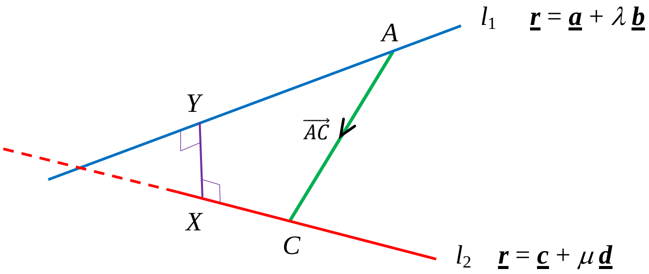
$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \cdot 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \cdot 5 \\ 0 \\ 3 \end{pmatrix}$$

$$\boldsymbol{n} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} \Rightarrow n = \sqrt{2^2 + 6^2 + 3^2} = 7$$

$$\Rightarrow \quad \text{shortest distance} = \left| \begin{pmatrix} -2 \cdot 5 \\ 0 \\ 3 \end{pmatrix} \cdot \frac{1}{7} \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} \right| = \frac{4}{7}$$

Shortest distance between two skew lines

It can be shown that there must be a line joining two skew lines which is perpendicular to both lines.



This line is XY and is the shortest distance between the lines.

The vector $\underline{n} = \underline{b} \times \underline{d}$ is perpendicular to both lines

$\Rightarrow$ the unit vector $\hat{\underline{n}} = \frac{\underline{b}\times\underline{d}}{|\underline{b}\times\underline{d}|}$

Now imagine two parallel planes π_1 and π_2, both perpendicular to $\hat{\underline{n}}$, one containing the line l_1 and the other containing the line l_2.

A and C are points on l_1 and l_2, and therefore on π_1 and π_2.

We now have two parallel planes with two points, A and C, one on each plane, and the planes are both perpendicular to $\hat{\underline{n}}$.

As in the example for the distance between parallel planes,

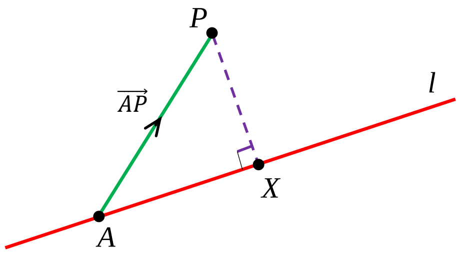
the shortest distance $d = |\overrightarrow{AC} \cdot \hat{\underline{n}}|$

$$\Rightarrow \quad d = \left|(\underline{c} - \underline{a}) \cdot \frac{\underline{b}\times\underline{d}}{|\underline{b}\times\underline{d}|}\right|$$

This result is not in your formula booklet, SO LEARN IT – please

Shortest distance from a point to a line

In trying to find the shortest distance from a point P to a line l, $\underline{\boldsymbol{r}} = \underline{\boldsymbol{a}} + \lambda\underline{\boldsymbol{b}}$, we do not know $\hat{\underline{\boldsymbol{n}}}$, the direction of the line through P perpendicular to l.



Some lateral thinking is needed.

We do know A, a point on the line, and $\hat{\underline{\boldsymbol{b}}}$, the direction of the line l

$\Rightarrow \quad \left|\overrightarrow{AP} \,.\, \hat{\underline{\boldsymbol{b}}}\right| = AX$, the projection of AP onto l

and we can now find $PX = \sqrt{AP^2 - AX^2}$, using Pythagoras

Example: Find the shortest distance from the point P (3, –2, 4)

to the line l, $\boldsymbol{r} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

Solution: If l is $\underline{\boldsymbol{r}} = \underline{\boldsymbol{a}} + \lambda\underline{\boldsymbol{b}}$, then $\underline{\boldsymbol{a}} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$ and $\underline{\boldsymbol{b}} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

$\Rightarrow \quad b = \sqrt{2^2 + 3^2 + 6^2} = 7, \quad \Rightarrow \quad \hat{\underline{\boldsymbol{b}}} = \frac{1}{7}\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

and $\overrightarrow{AP} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix}$

$\Rightarrow \quad AX = \left|\overrightarrow{AP} \,.\, \hat{\underline{\boldsymbol{b}}}\right| = \left|\begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix} . \frac{1}{7}\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}\right| = \frac{10+15+24}{7} = 7$

$\Rightarrow \quad PX = \sqrt{AP^2 - AX^2} = \sqrt{(5^2 + 5^2 + 4^2) - 7^2}$

$= \sqrt{17}$

Line of intersection of two planes

Example: Find an equation for the line of intersection of the planes

$$x + y + 2z = 4 \qquad \textbf{I}$$

and

$$2x - y + 3z = 4 \qquad \textbf{II}$$

Solution: Eliminate one variable –

$$\textbf{I} + \textbf{II} \quad \Rightarrow \quad 3x + 5z = 8$$

We are *not* expecting a unique solution, so put one variable, z say, equal to λ and find the other variables in terms of λ.

$$z = \lambda \quad \Rightarrow \quad x = \frac{8-5\lambda}{3}$$

$$\textbf{I} \Rightarrow \quad y = 4 - x - 2z = 4 - \frac{8-5\lambda}{3} - 2\lambda = \frac{4-\lambda}{3}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8/3 \\ 4/3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -5/3 \\ -1/3 \\ 1 \end{pmatrix}$$

or $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8/3 \\ 4/3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -1 \\ 3 \end{pmatrix}$$ making the numbers nicer in the **direction vector only**

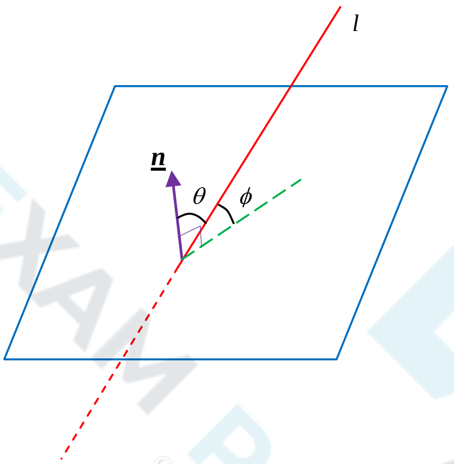
which is the equation of a line through $\left(\frac{8}{3}, \frac{4}{3}, 0\right)$ and parallel to $\begin{pmatrix} -5 \\ -1 \\ 3 \end{pmatrix}$.

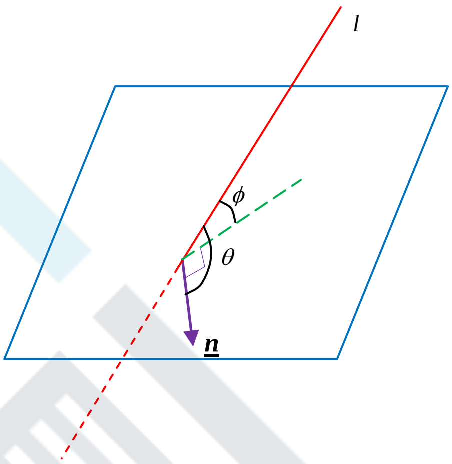
Angle between line and plane

Let the acute angle between the line and the plane be ϕ.

First find the angle between the line and the normal vector, θ.

There are two possibilities – as shown below:





(i) $\underline{\boldsymbol{n}}$ and the angle ϕ are on the same side of the plane

$\Rightarrow$ $\phi = 90 - \theta$

(ii) $\underline{\boldsymbol{n}}$ and the angle ϕ are on opposite sides of the plane

$\Rightarrow$ $\phi = \theta - 90$

Example: Find the angle between the line $\frac{x+1}{2} = \frac{y-2}{1} = \frac{z-3}{-2}$ and the plane $2x + 3y - 7z = 5$.

Solution: The line is parallel to $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, and the normal vector to the plane is $\begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix}$.

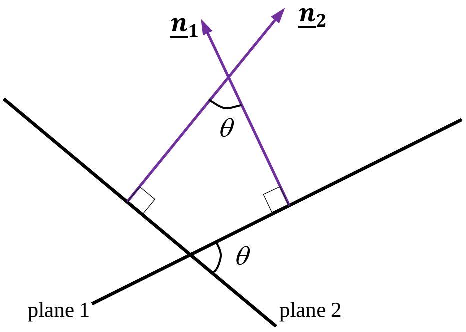
$$\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}} = ab\cos\theta \quad \Rightarrow \quad 21 = \sqrt{2^2 + 1^2 + 2^2}\sqrt{2^2 + 3^2 + 7^2}\ \cos\theta$$

$$\Rightarrow \cos\theta = \frac{7}{\sqrt{62}} \quad \Rightarrow \quad \theta = 27.3^{\circ}$$

$\Rightarrow$ the angle between the line and the plane, $\phi = 90 - 27.3 = 62.7^{\circ}$

Angle between two planes

If we look 'end-on' at the two planes, we can see that the angle between the planes, θ, equals the angle between the normal vectors.



Example: Find the angle between the planes

$2x + y + 3z = 5$ and $2x + 3y + z = 7$

Solution: The normal vectors are $\begin{pmatrix}2\\1\\3\end{pmatrix}$ and $\begin{pmatrix}2\\3\\1\end{pmatrix}$

$$\underline{a} \cdot \underline{b} = ab\cos\theta \Rightarrow 10 = \sqrt{2^2+1^2+3^2} \times \sqrt{2^2+1^2+3^2}\ \cos\theta$$

$$\Rightarrow \cos\theta = \frac{10}{14} \quad \Rightarrow \theta = 44.4^{\circ}$$

6 Matrices

Basic definitions

Dimension of a matrix

A matrix with r rows and c columns has *dimension* $r \times c$.

Transpose and symmetric matrices

The *transpose*, $\boldsymbol{A}^T$, of a matrix, $\boldsymbol{A}$, is found by interchanging rows and columns

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{A}^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

$(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$ - note the change of order of A and B.

A matrix, $\boldsymbol{S}$, is *symmetric* if the elements are symmetrically placed about the leading diagonal,

or if $\boldsymbol{S} = \boldsymbol{S}^T$.

Thus, $\boldsymbol{S} = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$ is a symmetric matrix.

Identity and zero matrices

The *identity* matrix $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

and the *zero* matrix is $\boldsymbol{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Determinant of a 3 × 3 matrix

The *determinant* of a 3×3 matrix, $\boldsymbol{A}$, is

$$\det(\boldsymbol{A}) = \Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\Rightarrow \Delta = aei - afh - bdi + bfg + cdh - ceg$$

Properties of the determinant

1) A determinant can be expanded by any row or column using $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

e.g $\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - f\begin{vmatrix} a & b \\ g & h \end{vmatrix}$ using the middle row and leaving the value unchanged

2) Interchanging two rows changes the sign of the determinant

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$ which can be shown by evaluating both determinants

3) A determinant with two identical rows (or columns) has value 0.

$\Delta = \begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix}$ interchanging the two identical rows gives $\Delta = -\Delta \Rightarrow \Delta = 0$

4) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A}) \times \det(\boldsymbol{B})$ this can be shown by multiplying out

Singular and non-singular matrices

A matrix, $\boldsymbol{A}$, is *singular* if its determinant is zero, $\det(\boldsymbol{A}) = 0$

A matrix, $\boldsymbol{A}$, is *non-singular* if its determinant is not zero, $\det(\boldsymbol{A}) \neq 0$

Inverse of a 3×3 matrix

This is tedious, but no reason to make a mistake if you are careful.

Cofactors

In $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ the cofactors of $a, b, c,$ etc. are A, B, C etc., where

$A = +\begin{vmatrix} e & f \\ h & i \end{vmatrix}$, $B = -\begin{vmatrix} d & f \\ g & i \end{vmatrix}$, $C = +\begin{vmatrix} d & e \\ g & h \end{vmatrix}$,

$D = -\begin{vmatrix} b & c \\ h & i \end{vmatrix}$, $E = +\begin{vmatrix} a & c \\ g & i \end{vmatrix}$, $F = -\begin{vmatrix} a & b \\ g & h \end{vmatrix}$,

$G = +\begin{vmatrix} b & c \\ e & f \end{vmatrix}$, $H = -\begin{vmatrix} a & c \\ d & f \end{vmatrix}$, $I = +\begin{vmatrix} a & b \\ d & e \end{vmatrix}$

These are the 2×2 matrices used in finding the determinant, together with the correct sign from $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Finding the inverse

1) Find the determinant, $\det(\boldsymbol{A})$.
If $\det(\boldsymbol{A}) = 0$, then $\boldsymbol{A}$ is singular and has no inverse.

2) Find the matrix of cofactors $\boldsymbol{C} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$

3) Find the transpose of $\boldsymbol{C}$, $\boldsymbol{C}^T = \begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$

4) Divide $\boldsymbol{C}^T$ by $\det(\boldsymbol{A})$ to give $\boldsymbol{A}^{-1} = \dfrac{1}{\det(\boldsymbol{A})}\begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$

See example 10 on page 148.

Properties of the inverse

1) $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$

2) $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ - note the change of order of A and B.

Proof $(\boldsymbol{AB})^{-1}\boldsymbol{AB} = \boldsymbol{I}$ from definition of inverse

$\Rightarrow \quad (\boldsymbol{AB})^{-1}\boldsymbol{AB}(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) = \boldsymbol{I}(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})$

$\Rightarrow \quad (\boldsymbol{AB})^{-1}\boldsymbol{A}(\boldsymbol{BB}^{-1})\boldsymbol{A}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ $\Rightarrow \quad (\boldsymbol{AB})^{-1}\boldsymbol{A}\,\boldsymbol{I}\,\boldsymbol{A}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

$\Rightarrow \quad (\boldsymbol{AB})^{-1}\boldsymbol{A}\,\boldsymbol{A}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ $\Rightarrow \quad (\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

3) $\det(\boldsymbol{A}^{-1}) = \dfrac{1}{\det(\boldsymbol{A})}$

Matrices and linear transformations

Linear transformations

$\boldsymbol{T}$ is a linear transformation on a set of vectors if

(i) $\boldsymbol{T}(\underline{x}_1 + \underline{x}_2) = \boldsymbol{T}(\underline{x}_1) + \boldsymbol{T}(\underline{x}_2)$ for all vectors $\underline{x}$ and $\underline{y}$

(ii) $\boldsymbol{T}(k\underline{x}) = k\boldsymbol{T}(\underline{x})$ for all vectors $\underline{x}$

Example: Show that $\boldsymbol{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ x+y \\ -z \end{pmatrix}$ is a linear transformation.

Solution: (i) $\boldsymbol{T}(\underline{x}_1 + \underline{x}_2) = \boldsymbol{T}\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = \boldsymbol{T}\left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}\right)$

$$= \begin{pmatrix} 2(x_1+x_2) \\ x_1+x_2+y_1+y_2 \\ -z_1-z_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_1+y_1 \\ -z_1 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ x_2+y_2 \\ -z_2 \end{pmatrix} = \boldsymbol{T}(\underline{x}_1) + \boldsymbol{T}(\underline{x}_2)$$

$\Rightarrow \quad \boldsymbol{T}(\underline{x}_1 + \underline{x}_2) = \boldsymbol{T}(\underline{x}_1) + \boldsymbol{T}(\underline{x}_2)$

(ii) $\boldsymbol{T}(k\underline{x}) = \boldsymbol{T}\left(k\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \boldsymbol{T}\begin{pmatrix} kx \\ ky \\ kz \end{pmatrix} = \begin{pmatrix} 2kx \\ kx+ky \\ -kz \end{pmatrix} = k\begin{pmatrix} 2x_1 \\ x_1+y_1 \\ -z_1 \end{pmatrix} = k\boldsymbol{T}(\underline{x})$

$\Rightarrow \quad \boldsymbol{T}(k\underline{x}) = k\boldsymbol{T}(\underline{x})$

Both (i) and (ii) are satisfied, and so $\boldsymbol{T}$ is a linear transformation.

All matrices can represent linear transformations.

Base vectors $\underline{i}$, $\underline{j}$, $\underline{k}$

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Under the transformation with matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} a \\ d \\ g \end{pmatrix}$ the first column of the matrix

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} b \\ e \\ h \end{pmatrix}$ the second column of the matrix

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} c \\ f \\ i \end{pmatrix}$ the third column of the matrix

This is an important result, as it allows us to find the matrix for given transformations.

Example: Find the matrix for a reflection in the plane $y = x$

Solution: The z-axis lies in the plane $y = x$ so $\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\Rightarrow$ the third column of the matrix is $\begin{pmatrix}0\\0\\1\end{pmatrix}$

Also $\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\Rightarrow$ the first column of the matrix is $\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\Rightarrow$ the second column of the matrix is $\begin{pmatrix}1\\0\\0\end{pmatrix}$

$\Rightarrow$ the matrix for a reflection in $y = x$ is $\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}$.

Example: Find the matrix of the linear transformation, $\boldsymbol{T}$, which maps $(1, 0, 0) \to (3, 4, 2)$, $(1, 1, 0) \to (6, 1, 5)$ and $(2, 1, -4) \to (1, 1, -1)$.

Solution:

Firstly $\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}3\\4\\2\end{pmatrix}$ $\Rightarrow$ first column is $\begin{pmatrix}3\\4\\2\end{pmatrix}$

Secondly $\begin{pmatrix}1\\1\\0\end{pmatrix} \to \begin{pmatrix}6\\1\\5\end{pmatrix}$ but $\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}3\\4\\2\end{pmatrix} + \boldsymbol{T}\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\Rightarrow$ $\boldsymbol{T}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}6\\1\\5\end{pmatrix} - \begin{pmatrix}3\\4\\2\end{pmatrix} = \begin{pmatrix}3\\-3\\3\end{pmatrix}$ $\Rightarrow$ second column is $\begin{pmatrix}3\\-3\\3\end{pmatrix}$

Thirdly $\begin{pmatrix}2\\1\\-4\end{pmatrix} \to \begin{pmatrix}1\\1\\-1\end{pmatrix}$

but $\begin{pmatrix}2\\1\\-4\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} - 4\begin{pmatrix}0\\0\\1\end{pmatrix} \to 2\begin{pmatrix}3\\4\\2\end{pmatrix} + \begin{pmatrix}3\\-3\\3\end{pmatrix} - 4\boldsymbol{T}\begin{pmatrix}0\\0\\1\end{pmatrix}$

$\Rightarrow$ $2\begin{pmatrix}3\\4\\2\end{pmatrix} + \begin{pmatrix}3\\-3\\3\end{pmatrix} - 4\boldsymbol{T}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$

$\Rightarrow$ $\boldsymbol{T}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\2\end{pmatrix}$ $\Rightarrow$ third column is $\begin{pmatrix}2\\1\\2\end{pmatrix}$

$\Rightarrow$ $\boldsymbol{T} = \begin{pmatrix}3 & 3 & 2\\4 & -3 & 1\\2 & 3 & 2\end{pmatrix}$.

Image of a line

Example: Find the image of the line $\underline{\boldsymbol{r}} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ under $\boldsymbol{T}$,

where $\boldsymbol{T} = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 3 & 4 \\ 2 & -1 & 1 \end{pmatrix}$.

Solution: As $\boldsymbol{T}$ is a linear transformation, we can find

$$\boldsymbol{T}(\underline{\boldsymbol{r}}) = \boldsymbol{T}\left(\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}\right) = \boldsymbol{T}\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \boldsymbol{T}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{T}(\underline{\boldsymbol{r}}) = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 3 & 4 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 & -2 & 1 \\ 1 & 3 & 4 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$\Rightarrow \quad \boldsymbol{T}(\underline{\boldsymbol{r}}) = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 14 \\ 1 \\ 9 \end{pmatrix}$ and so a vector equation of the new line is

$$\underline{\boldsymbol{r}} = \begin{pmatrix} 3 \\ -10 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 14 \\ 1 \\ 9 \end{pmatrix}.$$

Image of a plane 1

Similarly the image of a plane $\underline{\boldsymbol{r}} = \underline{\boldsymbol{a}} + \lambda \underline{\boldsymbol{b}} + \mu \underline{\boldsymbol{c}}$, under a linear transformation, $\boldsymbol{T}$, is

$$\boldsymbol{T}(\underline{\boldsymbol{r}}) = \boldsymbol{T}(\underline{\boldsymbol{a}} + \lambda \underline{\boldsymbol{b}} + \mu \underline{\boldsymbol{c}}) = \boldsymbol{T}(\underline{\boldsymbol{a}}) + \lambda \boldsymbol{T}(\underline{\boldsymbol{b}}) + \mu \boldsymbol{T}(\underline{\boldsymbol{c}}).$$

Image of a plane 2

To find the image of a plane with equation of the form $ax + by + cz = d$, first construct a vector equation.

Method 1

Example 1: Find the image of the plane $3x - 2y + 4z = 6$ under a linear transformation, $\boldsymbol{T}$.

Solution: First construct a vector equation,

(i) Put $x = z = 0 \Rightarrow y = -3 \Rightarrow (0, -3, 0)$ is a point on the plane

(ii) To find vectors parallel to the plane, they must be perpendicular to $\underline{\boldsymbol{n}} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$. By inspection, using the top two coordinates, $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$, and, using the bottom two coordinates, $\begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ must be $\perp$ to $\underline{\boldsymbol{n}}$ (look at the scalar products), and so are parallel to the plane.

$\Rightarrow$ The vector equation of the plane is $\underline{\boldsymbol{r}} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$\Rightarrow$ The image under the matrix $\boldsymbol{M}$ is $\boldsymbol{M}\,\underline{\boldsymbol{r}} = \boldsymbol{M} \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} + \lambda \boldsymbol{M} \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \mu\, \boldsymbol{M} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

Example 2: Find the image of the plane $\underline{\boldsymbol{r}} \cdot \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix} = 8$ under a linear transformation $\boldsymbol{T}$.

Solution: The equation can be written as $2x - 5y = 8$

(i) Put $y = 0 \Rightarrow x = 4 \Rightarrow (4, 0, 0)$ is a point on the plane *z* could be anything

(ii) To find vectors parallel to the plane, they must be perpendicular to $\underline{\boldsymbol{n}} = \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$. By inspection, using the top two coordinates, $\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$, and, using the 0 *z*-coordinate, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ must be $\perp$ to $\underline{\boldsymbol{n}}$ (look at the scalar products), and so are parallel to the plane.

Continue as in Example 1.

Method 2, as in the book

Fine until the vector $\underline{\boldsymbol{n}}$ has a zero coordinate, then life is a bit more complicated.

Example: Find the image of the plane $3x - 2y + 4z = 6$ under a linear transformation, $\boldsymbol{T}$.

Solution: To construct a vector equation, put $x = \lambda$, $y = \mu$ and find z in terms of λ and μ.

$$\Rightarrow \quad 3\lambda - 2\mu + 4z = 6 \quad \Rightarrow \quad z = \frac{6-3\lambda+2\mu}{4}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda \\ \mu \\ \frac{6-3\lambda+2\mu}{4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6/4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -3/4 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1/2 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6/4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$ making the numbers nicer in the 'parallel' vectors

and now continue as in method 1.

NOTE that $\boldsymbol{M}\,(\underline{\boldsymbol{b}} \times \underline{\boldsymbol{c}})$ is **not** equal to $\boldsymbol{M}\,(\underline{\boldsymbol{b}}) \times \boldsymbol{M}\,(\underline{\boldsymbol{c}})$, since this does not follow the conditions of a *linear* transformation, so you must use one of the methods above.

7 Eigenvalues and eigenvectors

Definitions

1) An *eigenvector* of a linear transformation, $\boldsymbol{T}$, is a non-zero vector whose *direction* is unchanged by $\boldsymbol{T}$.

So, if $\underline{e}$ is an eigenvector of $\boldsymbol{T}$ then its image $\underline{e}'$ is parallel to $\underline{e}$, or $\underline{e}' = \lambda\,\underline{e}$

$\Rightarrow \quad \underline{e}' = \boldsymbol{T}(\underline{e}) = \lambda\,\underline{e}$.

$\underline{e}$ defines a line which maps onto itself and so is invariant *as a whole line*.

If $\lambda = 1$ each point on the line remains in the same place, and we have a line of *invariant points*.

2) The *characteristic equation* of a matrix $\boldsymbol{A}$ is $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$

$\boldsymbol{A}\underline{e} = \lambda\,\underline{e}$

$\Rightarrow \quad (\boldsymbol{A} - \lambda\boldsymbol{I})\,\underline{e} = \underline{\boldsymbol{0}}$ has non-zero solutions — eigenvectors are non-zero

$\Rightarrow \quad \boldsymbol{A} - \lambda\boldsymbol{I}$ is a singular matrix

$\Rightarrow \quad \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$

$\Rightarrow$ the solutions of the characteristic equation are the eigenvalues.

2 × 2 matrices

Example: Find the eigenvalues and eigenvectors for the transformation with matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}.$$

Solution: The characteristic equation is $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$

$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (1-\lambda)(4-\lambda) + 2 = 0$

$\Rightarrow \quad \lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda_1 = 2$ and $\lambda_2 = 3$

For $\lambda_1 = 2$

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow \quad x + y = 2x \quad \Rightarrow \quad x = y$

and $\quad -2x + 4y = 2y \quad \Rightarrow \quad x = y$

$\Rightarrow \quad$ eigenvector $\underline{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ — we could use $\begin{pmatrix} 3.7 \\ 3.7 \end{pmatrix}$, but why make things nasty

For $\lambda_2 = 3$

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow \quad x + y = 3x \quad \Rightarrow \quad 2x = y$

and $\quad -2x + 4y = 3y \quad \Rightarrow \quad 2x = y$

$\Rightarrow$ eigenvector $\underline{e}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ choosing easy numbers.

Orthogonal matrices

Normalised eigenvectors

A normalised eigenvector is an eigenvector of length 1.

In the above example, the normalized eigenvectors are $\underline{e}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, and $\underline{e}_2 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$.

Orthogonal vectors

A posh way of saying perpendicular, scalar product will be zero.

Orthogonal matrices

If the columns of a matrix form vectors which are

(i) mutually orthogonal (or perpendicular)
(ii) each of length 1

then the matrix is an *orthogonal* matrix.

Example:

$\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$ and $\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$ are both unit vectors, and

$\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \bullet \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \frac{-2}{5} + \frac{2}{5} = 0, \Rightarrow$ the vectors are orthogonal

$\Rightarrow \boldsymbol{M} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$ is an orthogonal matrix

Notice that

$$M^TM = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and so ***the transpose of an orthogonal matrix is also its inverse***.

This is true for **all** orthogonal matrices — think of any set of perpendicular unit vectors

Another definition of an orthogonal matrix is

$\boldsymbol{M}$ is orthogonal $\Leftrightarrow$ $\boldsymbol{M}^T\boldsymbol{M} = \boldsymbol{I}$ $\Leftrightarrow$ $\boldsymbol{M}^{-1} = \boldsymbol{M}^T$

Diagonalising a 2 × 2 matrix

Let $\boldsymbol{A}$ be a 2×2 matrix with eigenvalues λ_1 and λ_2,
and eigenvectors $\underline{\boldsymbol{e}}_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $\underline{\boldsymbol{e}}_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$

then $\boldsymbol{A}\,\underline{\boldsymbol{e}}_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 u_1 \\ \lambda_1 v_1 \end{pmatrix}$ and $\boldsymbol{A}\,\underline{\boldsymbol{e}}_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 u_2 \\ \lambda_2 v_2 \end{pmatrix}$

$$\Rightarrow \quad \boldsymbol{A}\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 u_1 & \lambda_2 u_2 \\ \lambda_1 v_1 & \lambda_2 v_2 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \ldots\ldots\ldots \quad \mathbf{I}$$

Define $\boldsymbol{P}$ as the matrix whose columns are eigenvectors of $\boldsymbol{A}$, and $\boldsymbol{D}$ as the diagonal matrix, whose entries are the eigenvalues of $\boldsymbol{A}$

$$\mathbf{I} \quad \Rightarrow \quad \boldsymbol{P} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \text{ and } \boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{AP} = \boldsymbol{PD} \quad \Rightarrow \quad \boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{D}$$

The above is the general case for diagonalising **any** matrix.

In this course we consider only diagonalising symmetric matrices.

Diagonalising 2 × 2 symmetric matrices

Eigenvectors of symmetric matrices

Preliminary result:

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

The scalar product $\underline{x} \cdot \underline{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 + x_2 y_2$

but $(x_1 \ \ x_2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 + x_2 y_2$

$\Rightarrow \quad \underline{x}^T \underline{y} = \underline{x} \cdot \underline{y}$

This result allows us to use matrix multiplication for the scalar product.

Theorem: Eigenvectors, for different eigenvalues, of a symmetric matrix are orthogonal.

Proof: Let $\boldsymbol{A}$ be a symmetric matrix, then $\boldsymbol{A}^T = \boldsymbol{A}$

Let $\boldsymbol{A}\,\underline{\boldsymbol{e}_1} = \lambda_1 \underline{\boldsymbol{e}_1}$, and $\boldsymbol{A}\,\underline{\boldsymbol{e}_2} = \lambda_2 \underline{\boldsymbol{e}_2}$, $\quad \lambda_1 \neq \lambda_2$.

$\lambda_1 \underline{\boldsymbol{e}_1}^T = (\lambda_1 \underline{\boldsymbol{e}_1})^T = (\boldsymbol{A}\,\underline{\boldsymbol{e}_1})^T = \underline{\boldsymbol{e}_1}^T \boldsymbol{A}^T = \underline{\boldsymbol{e}_1}^T \boldsymbol{A}$ $\quad$ since $\boldsymbol{A}^T = \boldsymbol{A}$, and $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$

$\Rightarrow \quad \lambda_1 \underline{\boldsymbol{e}_1}^T = \underline{\boldsymbol{e}_1}^T \boldsymbol{A}$

$\Rightarrow \quad \lambda_1 \underline{\boldsymbol{e}_1}^T \underline{\boldsymbol{e}_2} = \underline{\boldsymbol{e}_1}^T \boldsymbol{A}\,\underline{\boldsymbol{e}_2} = \underline{\boldsymbol{e}_1}^T \lambda_2 \underline{\boldsymbol{e}_2} = \lambda_2 \underline{\boldsymbol{e}_1}^T \underline{\boldsymbol{e}_2}$

$\Rightarrow \quad \lambda_1 \underline{\boldsymbol{e}_1}^T \underline{\boldsymbol{e}_2} = \lambda_2 \underline{\boldsymbol{e}_1}^T \underline{\boldsymbol{e}_2}$

$\Rightarrow \quad (\lambda_1 - \lambda_2)\, \underline{\boldsymbol{e}_1}^T \underline{\boldsymbol{e}_2} = \underline{\boldsymbol{0}}$

But $\quad \lambda_1 - \lambda_2 \neq 0 \quad \Rightarrow \quad \underline{\boldsymbol{e}_1}^T \underline{\boldsymbol{e}_2} = \underline{\boldsymbol{0}} \quad \Leftrightarrow \quad \underline{\boldsymbol{e}_1} \cdot \underline{\boldsymbol{e}_2} = 0$

$\Rightarrow$ the eigenvectors are orthogonal $\quad$ or perpendicular

Diagonalising a symmetric matrix

The above theorem makes diagonalising a symmetric matrix, $\boldsymbol{A}$, easy.

1) Find eigenvalues, λ_1 and λ_2, and eigenvectors, $\underline{\boldsymbol{e}_1}$ and $\underline{\boldsymbol{e}_2}$

2) Normalise the eigenvectors, to give $\underline{\hat{\boldsymbol{e}}_1}$ and $\underline{\hat{\boldsymbol{e}}_2}$.

3) Write down the matrix $\boldsymbol{P}$ with $\underline{\hat{\boldsymbol{e}}_1}$ and $\underline{\hat{\boldsymbol{e}}_2}$ as columns.
$\boldsymbol{P}$ will now be an orthogonal matrix since $\underline{\hat{\boldsymbol{e}}_1}$ and $\underline{\hat{\boldsymbol{e}}_2}$ are orthogonal
$\Rightarrow \quad \boldsymbol{P}^{-1} = \boldsymbol{P}^T$

4) $\boldsymbol{P}^T\boldsymbol{A}\,\boldsymbol{P}$ will be the diagonal matrix $\boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Example: Diagonalise the symmetric matrix $\boldsymbol{A} = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}$.

Solution: The characteristic equation is $\begin{vmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (6-\lambda)(9-\lambda) - 4 = 0$

$\Rightarrow \quad \lambda^2 - 15\lambda + 50 = 0 \quad \Rightarrow \quad (\lambda-5)(\lambda-10) = 0$

$\Rightarrow \quad \lambda = 5 \text{ or } 10$

For $\lambda_1 = 5$

$$\begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow \quad 6x - 2y = 5x \quad \Rightarrow \quad x = 2y$

and $\quad -2x + 9y = 5y \quad \Rightarrow \quad x = 2y$

$\Rightarrow \quad \underline{\boldsymbol{e}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

and normalising $\Rightarrow \quad \underline{\hat{\boldsymbol{e}}}_1 = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

For $\lambda_2 = 10$

$$\begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 10\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow \quad 6x - 2y = 10x \quad \Rightarrow \quad -2x = y$

and $\quad -2x + 9y = 10y \quad \Rightarrow \quad -2x = y$

$\Rightarrow \quad \underline{\boldsymbol{e}}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

and normalising $\Rightarrow \quad \underline{\hat{\boldsymbol{e}}}_2 = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$

Notice that the eigenvectors are orthogonal

$$\Rightarrow \quad \boldsymbol{P} = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{D} = \boldsymbol{P}^T\boldsymbol{A}\,\boldsymbol{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}.$$

3×3 matrices

All the results for 2×2 matrices are also true for 3×3 matrices (or $n \times n$ matrices). The proofs are either the same, or similar in a higher number of dimensions.

Finding eigenvectors for 3×3 matrices.

Example: Given that $\lambda = 5$ is an eigenvector of the matrix

$M = \begin{pmatrix} 3 & -1 & 2 \\ -2 & 1 & -1 \\ 4 & -1 & -2 \end{pmatrix}$, find the corresponding eigenvector.

Solution: Consider $M\underline{e} = 5\underline{e}$

$$\Rightarrow \begin{pmatrix} 3 & -1 & 2 \\ -2 & 1 & -1 \\ 4 & -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Rightarrow \quad 3x - y + 2z = 5x \quad \Rightarrow \quad -2x - y + 2z = 0$ **I**

$-2x + y - z = 5y \quad \Rightarrow \quad -2x - 4y - z = 0$ **II**

$4x - y - 2z = 5z \quad \Rightarrow \quad 4x - y - 7z = 0$ **III**

Now eliminate one variable, say x:

I – II $\Rightarrow 3y + 3z = 0 \quad \Rightarrow \quad y = -z$

We are not expecting to find *unique* solutions, so put $z = 1$, and then find x and y.

$\Rightarrow \quad y = -1$, and,

from **I**, $\quad 2x = 2z - y = 2 + 1 = 3$

$\Rightarrow \quad x = 1{\cdot}5$

$\Rightarrow \quad \underline{e} = \begin{pmatrix} 1 \cdot 5 \\ -1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ as any multiple will also be an eigenvector

check in **II** and **III**, **O.K.**

Diagonalising 3 × 3 symmetric matrices

Example: $\boldsymbol{A} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}$.

Find an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^T\boldsymbol{AP}$ is a diagonal matrix.

Solution:

1) Find eigenvalues

The characteristic equation is $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & -2 & 0 \\ -2 & 1-\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)[-\lambda(1-\lambda) - 4] + 2 \times [2\lambda - 0] + 0 = 0$$

$$\Rightarrow \quad \lambda^3 - 3\lambda^2 - 6\lambda + 8 = 0$$

By inspection $\lambda = -2$ is a root $\Rightarrow$ $(\lambda + 2)$ is a factor

$$\Rightarrow \quad (\lambda+2)(\lambda^2 - 5\lambda + 4) = 0$$

$$\Rightarrow \quad (\lambda+2)(\lambda-1)(\lambda-4) = 0$$

$$\Rightarrow \quad \lambda = -2, 1 \text{ or } 4.$$

2) Find normalized eigenvectors

$$\lambda_1 = -2 \Rightarrow \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Rightarrow$
$2x - 2y = -2x$ **I**
$-2x + y + 2z = -2y$ **II**
$2y = -2z$ **III**

I $\Rightarrow y = 2x$, and **III** $\Rightarrow y = -z$ choose $x = 1$ and find y and z

$$\Rightarrow \quad \boldsymbol{e_1} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad \text{and} \quad |\boldsymbol{e_1}| = e_1 = \sqrt{9} = 3 \Rightarrow \hat{\boldsymbol{e}}_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

$$\lambda_2 = 1 \Rightarrow \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Rightarrow$
$2x - 2y = x$ **I**
$-2x + y + 2z = y$ **II**
$2y = z$ **III**

I $\Rightarrow$ $x = 2y$, and **II** $\Rightarrow$ $z = 2y$ choose $y = 1$ and find x and z

$$\Rightarrow \quad \underline{\boldsymbol{e}}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad |\underline{\boldsymbol{e}}_2| = e_2 = \sqrt{9} = 3$$

$$\Rightarrow \quad \underline{\hat{\boldsymbol{e}}}_2 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

$$\lambda_3 = 4 \quad \Rightarrow \quad \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2x - 2y &= 4x && \textbf{I} \\ -2x + y + 2z &= 4y && \textbf{II} \\ 2y &= 4z && \textbf{III} \end{aligned}$$

I $\Rightarrow x = -y$, and **III** $\Rightarrow$ $y = 2z$ choose $z = 1$ and find x and y

$$\Rightarrow \quad \underline{\boldsymbol{e}}_3 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad |\underline{\boldsymbol{e}}_3| = e_3 = \sqrt{9} = 3$$

$$\Rightarrow \quad \underline{\hat{\boldsymbol{e}}}_3 = \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

3) Find orthogonal matrix, *P*

$$\Rightarrow \quad \boldsymbol{P} = (\underline{\hat{\boldsymbol{e}}}_1 \quad \underline{\hat{\boldsymbol{e}}}_2 \quad \underline{\hat{\boldsymbol{e}}}_3)$$

$$\Rightarrow \quad \boldsymbol{P} = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{pmatrix}$$ is required orthogonal matrix

4) Find diagonal matrix, *D*

$$\Rightarrow \quad \boldsymbol{P}^T\boldsymbol{AP} = \boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

A nice long question! But, although you will not be asked to do a complete problem, the examiners can test every step above!

Index