Edexcel Core 2 Mathematics: Differentiation and integration
Section 1: Differentiation

Mark Scheme

1. (i) $y = x^3 + 6x^2 + 9x$

$$\frac{dy}{dx} = 3x^2 + 12x + 9$$

(ii) $\frac{dy}{dx} = 0$

$3x^2 + 12x + 9 = 0$

$x^2 + 4x + 3 = 0$

$(x+1)(x+3) = 0$

$x = -1$ or $x = -3$

When $x = -1$, $y = (-1)^3 + 6(-1)^2 + 9 \times -1 = -1 + 6 - 9 = -4$

When $x = -3$, $y = (-3)^3 + 6(-3)^2 + 9 \times -3 = -27 + 54 - 27 = 0$

The stationary points are (-1, -4) and (-3, 0)

(iii)

x	x< -3	x = -3	-3 < x < -1	x = -1	x > -1
$\frac{dy}{dx}$	+ve /	0 —	-ve \	0 —	+ve /

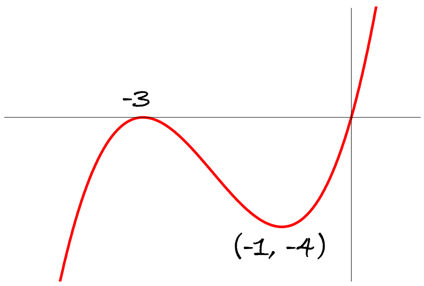
The point (-3, 0) is a maximum point.
The point (-1, -4) is a minimum point.

(iv) $y = x^3 + 6x^2 + 9x$

$= x(x^2 + 6x + 9)$

$= x(x+3)^2$

The graph cuts the x-axis at x = 0 and x = -3 (repeated).
The graph cuts the y-axis at y = 0.



2. (i) $y = 2x + x^2 - 4x^3$

$\frac{dy}{dx} = 2 + 2x - 12x^2$

At stationary points, $\frac{dy}{dx} = 0$

$2 + 2x - 12x^2 = 0$

$1 + x - 6x^2 = 0$

$6x^2 - x - 1 = 0$

$(3x + 1)(2x - 1) = 0$

$x = -\frac{1}{3}$ or $x = \frac{1}{2}$

When $x = -\frac{1}{3}$, $y = 2\left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)^2 - 4\left(-\frac{1}{3}\right)^3 = -\frac{2}{3} + \frac{1}{9} + \frac{4}{27} = \frac{-18+3+4}{27} = -\frac{11}{27}$

When $x = \frac{1}{2}$, $y = 2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right)^3 = 1 + \frac{1}{4} - \frac{1}{2} = \frac{3}{4}$

The turning points are $\left(-\frac{1}{3}, -\frac{11}{27}\right)$ and $\left(\frac{1}{2}, \frac{3}{4}\right)$.

x	$x < -\frac{1}{3}$	$x = -\frac{1}{3}$	$-\frac{1}{3} < x < \frac{1}{2}$	$x = \frac{1}{2}$	$x > \frac{1}{2}$
$\frac{dy}{dx}$	-ve	0	+ve	0	-ve
	\	—	/	—	\

$\left(-\frac{1}{3}, -\frac{11}{27}\right)$ is a minimum point.

$\left(\frac{1}{2}, \frac{3}{4}\right)$ is a maximum point.

(ii) $y = 2x + x^2 - 4x^3$

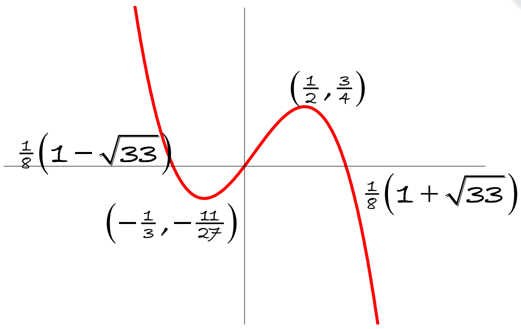
$= x(2 + x - 4x^2)$

$= -x(4x^2 - x - 2)$

The curve cuts the x-axis at $x = 0$ and at the points satisfying $4x^2 - x - 2 = 0$.

For this quadratic equation, $a = 4, b = -1, c = -2$

Using the quadratic formula, $x = \frac{1 \pm \sqrt{1 - 4 \times 4 \times -2}}{8} = \frac{1 \pm \sqrt{33}}{8}$



3. $y = x^3 - 3x^2 + 6$

$\frac{dy}{dx} = 3x^2 - 6x$

At turning points, $\frac{dy}{dx} = 0$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0$ or $x = 2$

When $x = 0$, $y = 6$

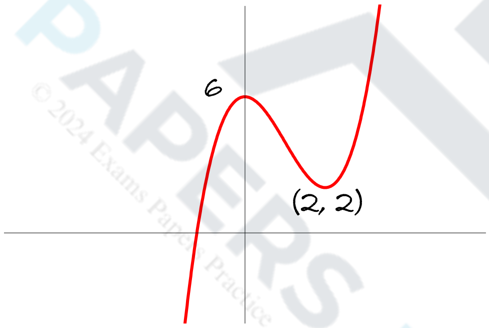
When $x = 2$, $y = 2^3 - 3 \times 2^2 + 6 = 8 - 12 + 6 = 2$

The turning points are $(0, 6)$ and $(2, 2)$.

$\frac{d^2y}{dx^2} = 6x - 6$

When $x = 0$, $\frac{d^2y}{dx^2} = 0 - 6 < 0$, so $(0, 6)$ is a maximum point.

When $x = 2$, $\frac{d^2y}{dx^2} = 12 - 6 > 0$, so $(2, 2)$ is a minimum point.



4. (i) $y = (x+1)(x-3)^2$

$= (x+1)(x^2 - 6x + 9)$

$= x^3 - 6x^2 + 9x + x^2 - 6x + 9$

$= x^3 - 5x^2 + 3x + 9$

(ii) $\frac{dy}{dx} = 3x^2 - 10x + 3$

$\frac{d^2y}{dx^2} = 6x - 10$

(iii) At turning points, $\frac{dy}{dx} = 0$

$$3x^2 - 10x + 3 = 0$$
$$(3x-1)(x-3) = 0$$
$$x = \tfrac{1}{3} \text{ or } x = 3$$

When $x = \frac{1}{3}$, $y = (\frac{1}{3}+1)(\frac{1}{3}-3)^2 = \frac{4}{3} \times (-\frac{8}{3})^2 = \frac{256}{27}$

When $x = 3$, $y = 0$

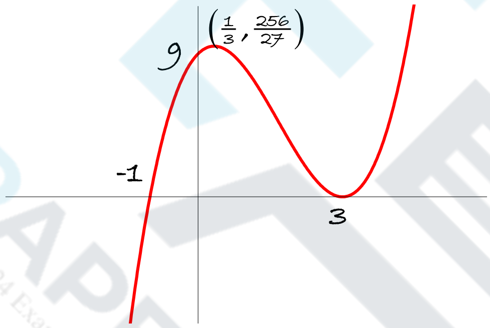
The turning points are $\left(\frac{1}{3}, \frac{256}{27}\right)$ and $(3, 0)$.

(iv) When $x = \frac{1}{3}$, $\frac{d^2y}{dx^2} = 6 \times \frac{1}{3} - 10 = 2 - 10 < 0$, so $\left(\frac{1}{3}, \frac{256}{27}\right)$ is a maximum.

When $x = 3$, $\frac{d^2y}{dx^2} = 18 - 10 > 0$, so $(3, 0)$ is a minimum.

(v) When $y = 0$, $x = -1$ or $x = 3$

When $x = 0$, $y = 9$



5. $y = x^4 - 2x^3$

$$\frac{dy}{dx} = 4x^3 - 6x^2$$

At stationary points, $4x^3 - 6x^2 = 0$

$$2x^3 - 3x^2 = 0$$
$$x^2(2x-3) = 0$$
$$x = 0 \text{ or } x = \tfrac{3}{2}$$

When $x = 0$, $y = 0$

When $x = \frac{3}{2}$, $y = \left(\frac{3}{2}\right)^4 - 2 \times \left(\frac{3}{2}\right)^3 = \frac{81}{16} - \frac{27}{4} = -\frac{27}{16}$

So stationary points are $(0, 0)$ and $\left(\frac{3}{2}, -\frac{27}{16}\right)$

$$\frac{dy}{dx} = 4x^3 - 6x^2 \quad \Rightarrow \frac{d^2y}{dx^2} = 12x^2 - 12x$$

When $x = 0$, $\frac{d^2y}{dx^2} = 0$, so need to test gradient.

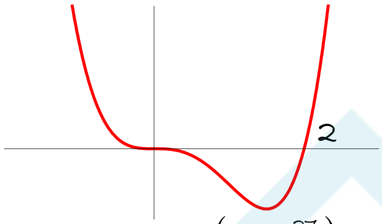
When x = -1, $\frac{dy}{dx} = -4 - 6 < 0$

When x = 1, $\frac{dy}{dx} = 4 - 6 < 0$

so (0, 0) is a point of inflection.

When $x = \frac{3}{2}$, $\frac{d^2y}{dx^2} = 12 \times \left(\frac{3}{2}\right)^2 - 12 \times \frac{3}{2} = 27 - 18 > 0$

so $\left(\frac{3}{2}, -\frac{27}{16}\right)$ is a local minimum point.



6. (i) $y = x - \frac{4}{x^2} = x - 4x^{-2}$

$\frac{dy}{dx} = 1 - 4 \times -2x^{-3} = 1 + \frac{8}{x^3}$

At stationary points, $1 + \frac{8}{x^3} = 0$

$1 = -\frac{8}{x^3}$

$x^3 = -8$

$x = -2$

When x = -2, $y = -2 - \frac{4}{4} = -3$

so (-2, -3) is a stationary point.

$\frac{dy}{dx} = 1 + 8x^{-3} \quad \Rightarrow \frac{d^2y}{dx^2} = -24x^{-4} < 0$

so (-2, -3) is a maximum point.

(ii) $y = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$

At stationary points, $\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x^3}} = 0$

$$\frac{1}{\sqrt{x}}\left(1 - \frac{1}{x}\right) = 0$$

$$x = 1$$

When $x = 1$, $y = 1 + \frac{1}{1} = 2$

so (1, 2) is a stationary point.

$$\frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}} - \tfrac{1}{2}x^{-\frac{3}{2}} \quad \Rightarrow \frac{d^2y}{dx^2} = -\tfrac{1}{4}x^{-\frac{3}{2}} + \tfrac{3}{4}x^{-\frac{5}{2}}$$

When $x = 1$, $\frac{d^2y}{dx^2} = -\tfrac{1}{4} + \tfrac{3}{4} > 0$

so (1, 2) is a minimum point.

7. $y = x^3 + px^2 + q$

$$\frac{dy}{dx} = 3x^2 + 2px$$

At turning points, $\frac{dy}{dx} = 0$

$$3x^2 + 2px = 0$$

$$x(3x + 2p) = 0$$

$$x = 0 \text{ or } x = -\frac{2p}{3}$$

Since there is a minimum point at $x = 4$, $-\frac{2p}{3} = 4 \quad \Rightarrow p = -6$

The curve is therefore $y = x^3 - 6x^2 + q$.

The point (4, -11) lies on the curve, so $-11 = 4^3 - 6 \times 4^2 + q$

$$-11 = 64 - 96 + q$$

$$q = 21$$

The equation of the curve is $y = x^3 - 6x^2 + 21$.

The other turning point is at $x = 0$, so the maximum point is (0, 21).

8. (i) $y = x^3 + ax^2 + bx + c$

The graph passes through the point (1, 1)

so $1 = 1 + a + b + c$

$$a + b + c = 0$$

(ii) $\frac{dy}{dx} = 3x^2 + 2ax + b$

Turning points are when $3x^2 + 2ax + b = 0$

There is a turning point when $x = -1$, so $3(-1)^2 + 2a \times -1 + b = 0$

$$3 - 2a + b = 0$$

$$2a - b = 3$$

There is a turning point when $x = 3$, so $3 \times 3^2 + 2a \times 3 + b = 0$

$$27 + 6a + b = 0$$

$$6a + b = -27$$

(iii) $a + b + c = 0$ (1)

$2a - b = 3$ (2)

$6a + b = -27$ (3)

Adding (2) and (3): $8a = -24 \Rightarrow a = -3$

Substituting into (2) gives: $b = 2a - 3 = -6 - 3 = -9$

Substituting into (1) gives: $c = -a - b = 9 + 3 = 12$

$a = -3, b = -9, c = 12$

9. Let the length of the sides be x and y.

Considering the perimeter: $2(x + y) = 20 \Rightarrow x + y = 10$

Let the area be A: $A = xy$

$$= x(10 - x)$$

$$= 10x - x^2$$

$$\frac{dA}{dx} = 10 - 2x$$

At turning point, $10 - 2x = 0$

$$2x = 10$$

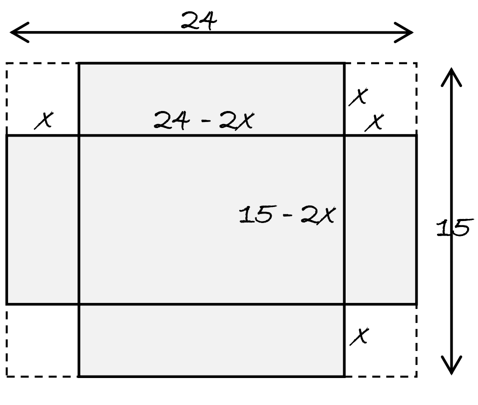
$$x = 5$$

When $x = 5$, $y = 10 - 5 = 5$.

$\frac{d^2A}{dx^2} = -2$ so turning point is a maximum.

The area is a maximum when the lengths of the sides are 5 cm (i.e. the rectangle is a square).

10. (i)



Height of box is x cm
Length of box is (24 – 2x) cm
Width of box is (15 – 2x) cm

$$\text{Volume } V = x(15-2x)(24-2x)$$
$$= x(360-78x+4x^2)$$
$$= 4x^3-78x^2+360x$$

(ii) $\dfrac{dV}{dx} = 12x^2 - 156x + 360$

At turning points, $12x^2 - 156x + 360 = 0$

$$x^2 - 13x + 30 = 0$$
$$(x-3)(x-10) = 0$$
$$x = 3 \text{ or } x = 10$$

x = 10 is not possible since this would mean that the width would be negative.

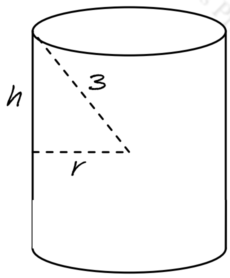
$\dfrac{d^2V}{dx^2} = 24x - 156$

When x = 3, $\dfrac{d^2V}{dx^2} = 72 - 156 < 0$, so x = 1 is a maximum point.

The volume of the box is maximised when x = 3.

(iii) Volume when x = 3 is $V = 3 \times 9 \times 18 = 486$ cm³

11. (i)



$$r^2 + h^2 = 3^2$$
$$r = \sqrt{9-h^2}$$

(ii) Volume $V = \pi r^2 h$

$$= \pi(9-h^2) \times 2h$$
$$= 2\pi h(9-h^2)$$

(iii) $V = 18\pi h - 2\pi h^3$

$\frac{dV}{dh} = 18\pi - 6\pi h^2$

At turning points, $18\pi - 6\pi h^2 = 0$

$3 - h^2 = 0$

$h = \sqrt{3}$

$\frac{d^2V}{dh^2} = -12\pi h$ so $h = \sqrt{3}$ is a maximum point.

Maximum volume $V = 2\pi\sqrt{3}(9-3) = 12\pi\sqrt{3}$ cm³.

12. (i) Volume of cylinder $= \pi r^2 h$

$2 = \pi r^2 h$

$h = \frac{2}{\pi r^2}$

(ii) Surface area $= 2\pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\pi r \times \frac{2}{\pi r^2}$

$= 2\pi r^2 + \frac{4}{r}$

(iii) $S = 2\pi r^2 + 4r^{-1}$

$\frac{dS}{dr} = 4\pi r - 4r^{-2}$

At stationary point, $4\pi r - 4r^{-2} = 0$

$4\pi r = \frac{4}{r^2}$

$r^3 = \frac{1}{\pi}$

$r = \frac{1}{\sqrt[3]{\pi}} = 0.683$ (3 s.f.)

$\frac{d^2S}{dr^2} = 4\pi + 8r^{-3} > 0$, so the stationary point is a minimum point.

The surface area is minimised when the radius is 0.683 m (3 s.f.)