Question 1

Consider the first-order differential equation

$$\frac{dy}{dx} - 5x^4 = 3$$

Solve the equation given that $y = 40$ when $x = 2$, giving your answer in the form $y = f(x)$.

[5]

$\frac{dy}{dx} - 5x^4 = 3$

$\frac{dy}{dx} = 5x^4 + 3$

$y = \int \frac{dy}{dx} = \int (5x^4 + 3)\,dx = x^5 + 3x + c$

$y = 40$ when $x = 2 \longrightarrow (2, 40)$

$40 = (2)^5 + 3(2) + c = 32 + 6 + c = 38 + c \quad \therefore c = 2$

$\boxed{y = x^5 + 3x + 2}$

Question 2

Use separation of variables to solve each of the following differential equations for y:

(a)

$$\frac{dy}{dx} = \frac{4x^2}{y^4}$$

[4]

(b)

$$\frac{dy}{dx} = (x^2 + 1)e^{-y}$$

[4]

a) $\frac{dy}{dx} = \frac{4x^2}{y^4}$

$\int y^4\,dy = \int 4x^2\,dx$

$y^5 = \frac{20}{3}x^3 + c$

$\boxed{y = \sqrt[5]{\frac{20}{3}x^3 + c}}$

Note: if the power was an even number we would need a $\pm$ infront of the root

Use separation of variables to solve each of the following differential equations for y:

(a)

$$\frac{dy}{dx} = \frac{4x^2}{y^4}$$

[4]

(b)

$$\frac{dy}{dx} = (x^2 + 1)e^{-y}$$

[4]

b) $\frac{dy}{dx} = (x^2 + 1)e^{-y} \longrightarrow \frac{dy}{dx} = (x^2 + 1) \times \frac{1}{e^y}$

$e^y \frac{dy}{dx} = x^2 + 1$

$\int e^y\,dy = \int (x^2 + 1)\,dx \longrightarrow e^y = \frac{1}{3}x^3 + x + c$

$\boxed{y = \ln\left(\frac{1}{3}x^3 + x + c\right)}$

Question 3

Use separation of variables to solve each of the following differential equations for y which satisfies the given boundary condition:

(a)

$$\frac{dy}{dx} = xy^2; \quad y(2) = 1$$

[5]

(b)

$$(x+3)\frac{dy}{dx} = \sec y; \quad y(-2) = \frac{3\pi}{2}$$

[5]

a) $\frac{dy}{dx} = xy^2 \longrightarrow y^{-2}\frac{dy}{dx} = x$

$\int y^{-2}\,dy = \int x\,dx \longrightarrow -\frac{1}{y} = \frac{1}{2}x^2 + c$

$y(2) = 1 \rightarrow y = 1$ when $x = 2$

$-\frac{1}{(1)} = \frac{1}{2}(2)^2 + c \rightarrow -1 = 2 + c \quad \therefore c = -3$

$-\frac{1}{y} = \frac{1}{2}x^2 - 3 = \frac{x^2 - 6}{2}$

$$\boxed{y = \frac{2}{6 - x^2}}$$

Use separation of variables to solve each of the following differential equations for y which satisfies the given boundary condition:

(a)

$$\frac{dy}{dx} = xy^2; \quad y(2) = 1$$

[5]

(b)

$$(x+3)\frac{dy}{dx} = \sec y; \quad y(-2) = \frac{3\pi}{2}$$

[5]

b) $(x+3)\frac{dy}{dx} = \sec y$

$\cos y \frac{dy}{dx} = \frac{1}{(x+3)}$

$\int \cos y\,dy = \int \frac{1}{(x+3)}\,dx$

$\sin y = \ln|x+3| + c$

$y(-2) = \frac{3\pi}{2} \longrightarrow y = \frac{3\pi}{2}$ when $x = -2$

$\sin\left(\frac{3\pi}{2}\right) = \ln|(-2)+3| + c$

$-1 = \ln 1 + c = 0 + c \quad \therefore c = -1$

$$\boxed{\sin y = \ln|x+3| - 1}$$

Question 4

At any point in time, the rate of growth of a colony of bacteria is proportional to the current population size P. At time $t = 0$ hours, the population size is 5000.

(a) Write a differential equation to model the size of the population of bacteria.

[1]

After 1 hour, the population has grown to 7000.

(b) By first solving the differential equation from part (a), determine the constant of proportionality.

[6]

(c) (i) Show that, according to the model, it will take exactly $\frac{\ln 20}{\ln 7 - \ln 5}$ hours (from $t = 0$) for the population of bacteria to grow to 100 000.

(ii) Confirm your answer to part (c)(i) graphically.

[5]

a) $\boxed{\frac{dP}{dt} = kP, \text{ for some constant } k.}$

At any point in time, the rate of growth of a colony of bacteria is proportional to the current population size P. At time $t = 0$ hours, the population size is 5000.

(a) Write a differential equation to model the size of the population of bacteria.

$\boxed{\frac{dP}{dt} = kP, \text{ for some constant } k.}$

[1]

After 1 hour, the population has grown to 7000.

(b) By first solving the differential equation from part (a), determine the constant of proportionality.

[6]

(c) (i) Show that, according to the model, it will take exactly $\frac{\ln 20}{\ln 7 - \ln 5}$ hours (from $t = 0$) for the population of bacteria to grow to 100 000.

(ii) Confirm your answer to part (c)(i) graphically.

[5]

b) This is separation of variables

$\frac{dP}{dt} = kP \longrightarrow \frac{1}{P}\frac{dP}{dt} = k$

$\int \frac{1}{P}\, dP = \int k\, dt \longrightarrow \ln P = kt + c$

Note: we can drop the modulus since P is positive.

$P = e^{kt+c} = (e^{kt})(e^{c}) = Ae^{kt}$, where $A = e^{c}$

$P(0) = 5000$ and $P(1) = 7000$

$P(0) = 5000 = Ae^{k(0)} = A(1) = A \quad \therefore A = e^{c} = 5000$

$P(1) = 7000 = 5000e^{k(1)} = 5000e^{k}$

$\therefore k = \ln\frac{7}{5} = 0.3364...$

$\boxed{P = 5000e^{(\ln\frac{7}{5})t}}$

At any point in time, the rate of growth of a colony of bacteria is proportional to the current population size P. At time $t = 0$ hours, the population size is 5000.

(a) Write a differential equation to model the size of the population of bacteria.

[1]

After 1 hour, the population has grown to 7000.

(b) By first solving the differential equation from part (a), determine the constant of proportionality.

$$\boxed{P = 5000e^{\left(\ln\frac{7}{5}\right)t}}$$

[6]

(c) (i) Show that, according to the model, it will take exactly $\frac{\ln 20}{\ln 7 - \ln 5}$ hours (from $t = 0$) for the population of bacteria to grow to 100 000.

(ii) Confirm your answer to part (c)(i) graphically.

[5]

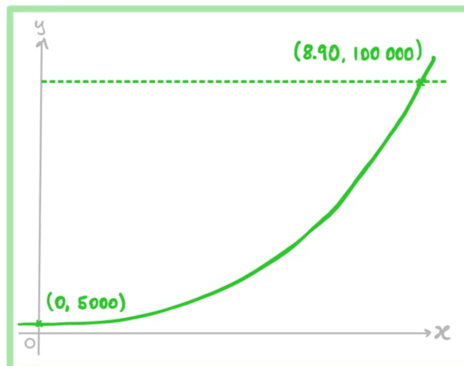
c) i) $100\,000 = 5000e^{\left(\ln\frac{7}{5}\right)t} \longrightarrow e^{\left(\ln\frac{7}{5}\right)t} = 20$

$\left(\ln\frac{7}{5}\right)t = \ln 20$

$(\ln 7 - \ln 5)t = \ln 20$ $\qquad \log_a \frac{x}{y} = \log_a x - \log_a y$

$$\boxed{t = \frac{\ln 20}{\ln 7 - \ln 5} = 8.9033\ldots = 8.90 \ (3\text{s.f})}$$

ii) Use your GDC to plot the graph of $P(t)$ and $P = 100\,000$.



Question 5

After clearing a large forest of malign influences, a wizard introduces a population of 100 unicorns to the forest. According to the wizard's mathemagicians, the population P of unicorns in the forest may be modelled by the logistic equation

$$\frac{dP}{dt} = 0.0006P(250 - P)$$

where t is the time in years after the unicorns were introduced to the forest.

(a) Show that the population of unicorns at time t years is given by

$$P(t) = \frac{500e^{0.15t}}{3 + 2e^{0.15t}}$$

[8]

(b) Find the length of time predicted by the model for the population of unicorns to double in size.

[3]

(c) Determine the maximum size that the model predicts the population of unicorns can grow to.

[2]

a) This is separation of variables plus partial fractions.

$$\frac{dP}{dt} = 0.0006P(250 - P)$$

$$\int \frac{1}{P(250-P)}\, dP = \int 0.0006\, dt$$

$$\frac{1}{250}\int\left(\frac{1}{P} + \frac{1}{250-P}\right) dP = 0.0006 \int dt$$

$\times 250$

$$\int\left(\frac{1}{P} + \frac{1}{250-P}\right) dP = 0.15 \int dt$$

$$\ln P - \ln(250 - P) = \ln\left(\frac{P}{250-P}\right) = 0.15t + c$$

Note: we can drop the modulus since P and $(250 - P)$ are positive.

$$\frac{P}{250-P} = e^{0.15t + c} = (e^c)(e^{0.15t}) = Ae^{0.15t}, \quad \text{where } A = e^c$$

$P(0) = 100$

$$\frac{(100)}{250-(100)} = Ae^{0.15(0)} \longrightarrow A = \frac{2}{3}$$

$$\frac{P}{250-P} = \frac{2}{3}e^{0.15t}$$

$\times (250 - P)$

$$P = \frac{500}{3}e^{0.15t} - \frac{2P}{3}e^{0.15t}$$

$$P + \frac{2P}{3}e^{0.15t} = P\left(1 + \frac{2}{3}e^{0.15t}\right) = \frac{500}{3}e^{0.15t}$$

$$P = \frac{\frac{500}{3}e^{0.15t}}{1 + \frac{2}{3}e^{0.15t}} = \frac{\left(\frac{500e^{0.15t}}{\cancel{3}}\right)}{\left(\frac{3 + 2e^{0.15t}}{\cancel{3}}\right)}$$

$$\boxed{P = \frac{500e^{0.15t}}{3 + 2e^{0.15t}}}$$

After clearing a large forest of malign influences, a wizard introduces a population of 100 unicorns to the forest. According to the wizard's mathemagicians, the population P of unicorns in the forest may be modelled by the logistic equation

$$\frac{dP}{dt} = 0.0006P(250 - P)$$

where t is the time in years after the unicorns were introduced to the forest.

(a) Show that the population of unicorns at time t years is given by

$$P(t) = \frac{500e^{0.15t}}{3 + 2e^{0.15t}}$$

[8]

(b) Find the length of time predicted by the model for the population of unicorns to double in size.

[3]

(c) Determine the maximum size that the model predicts the population of unicorns can grow to.

[2]

b) Set $P = 200$ and solve for t with your GDC.

$$200 = \frac{500e^{0.15t}}{3 + 2e^{0.15t}} \longrightarrow t = \frac{\ln 6}{0.15} = 11.9450\ldots$$

$$\boxed{t = 11.9 \text{ years (3 s.f.)}}$$

After clearing a large forest of malign influences, a wizard introduces a population of 100 unicorns to the forest. According to the wizard's mathemagicians, the population P of unicorns in the forest may be modelled by the logistic equation

$$\frac{dP}{dt} = 0.0006P(250 - P)$$

where t is the time in years after the unicorns were introduced to the forest.

(a) Show that the population of unicorns at time t years is given by

$$P(t) = \frac{500e^{0.15t}}{3 + 2e^{0.15t}}$$

[8]

(b) Find the length of time predicted by the model for the population of unicorns to double in size.

[3]

(c) Determine the maximum size that the model predicts the population of unicorns can grow to.

[2]

c) Need to consider limit as $t \to \infty$.

$$\frac{500e^{0.15t}}{3 + 2e^{0.15t}} \times \underbrace{\frac{e^{-0.15t}}{e^{-0.15t}}}_{=1} = \frac{500}{3e^{-0.15t} + 2}$$

$$P(\infty) = \lim_{t \to \infty} \frac{500e^{0.15t}}{3 + 2e^{0.15t}} = \lim_{t \to \infty} \frac{500}{3e^{-0.15t} + 2} = \frac{500}{0 + 2}$$

$$\boxed{P(\infty) = \lim_{t \to \infty} \frac{500e^{0.15t}}{3 + 2e^{0.15t}} = 250}$$

Question 6

(a) Show that

$$x^2\frac{dy}{dx} = xy + 2x^2$$

is a homogeneous differential equation.

[2]

(b) Using the substitution $v = \frac{y}{x}$, show that the solution to the differential equation in part (a) is

$$y = 2x\ln|x| + cx$$

where c is a constant of integration.

[4]

a) A homogeneous differential equation can be written in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ and solved using the substitution $v = \frac{y}{x}$, where v is a function in x.

$$x^2\frac{dy}{dx} = xy + 2x^2$$

$$\boxed{\frac{dy}{dx} = \frac{y}{x} + 2 = f\left(\frac{y}{x}\right) \longrightarrow \text{homogeneous}}$$

(a) Show that

$$x^2\frac{dy}{dx} = xy + 2x^2$$

is a homogeneous differential equation.

[2]

(b) Using the substitution $v = \frac{y}{x}$, show that the solution to the differential equation in part (a) is

$$y = 2x\ln|x| + cx$$

where c is a constant of integration.

[4]

b) Rearrange and use implicit differentiation plus product rule.

$$v = \frac{y}{x} \longrightarrow y = vx \qquad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{y}{x} + 2 = v + 2$$

$$\therefore x\frac{dv}{dx} = 2 \longrightarrow \frac{dv}{dx} = \frac{2}{x} \longrightarrow v = 2\ln|x| + c$$

$$\frac{y}{x} = 2\ln|x| + c$$

$$\boxed{y = 2x\ln|x| + cx}$$

Question 7

(a) Use the substitution $v = \frac{y}{x}$ to show that the differential equation

$$y' = \frac{y^2}{x^2} - \frac{y}{x} + 1$$

may be rewritten in the form

$$v' = \frac{(v-1)^2}{x}$$

[3]

(b) Hence use separation of variables to solve the differential equation in part (a) for y which satisfies the boundary condition $y(1) = \frac{2}{3}$. Give your answer in the form $y = f(x)$.

[5]

a) $v = \frac{y}{x} \longrightarrow y' = v + xv'$

$y' = \frac{y^2}{x^2} - \frac{y}{x} + 1 = v^2 - v + 1 = v + xv'$

$\therefore v' = \frac{v^2 - 2v + 1}{x}$

$$\boxed{v' = \frac{(v-1)^2}{x}}$$

Note: The tick mark (v') is used to indicate the specifically derivative with respect to x.

(a) Use the substitution $v = \frac{y}{x}$ to show that the differential equation

$$y' = \frac{y^2}{x^2} - \frac{y}{x} + 1$$

may be rewritten in the form

$$v' = \frac{(v-1)^2}{x}$$

[3]

(b) Hence use separation of variables to solve the differential equation in part (a) for y which satisfies the boundary condition $y(1) = \frac{2}{3}$. Give your answer in the form $y = f(x)$.

[5]

b) $v' = \frac{(v-1)^2}{x} \longrightarrow \frac{1}{(v-1)^2} v' = \frac{1}{x}$

$\int \frac{1}{(v-1)^2} dv = \int \frac{1}{x} dx \longrightarrow -\frac{1}{v-1} = \ln|x| + c$

$v = \frac{y}{x} = 1 - \frac{1}{\ln|x| + c} \longrightarrow y = x - \frac{x}{\ln|x| + c}$

sub $y(1) = \frac{2}{3}$

$\frac{2}{3} = 1 - \frac{1}{\ln|1| + c} = 1 - \frac{1}{c} \quad \therefore c = 3$

$$\boxed{y = x - \frac{x}{\ln|x| + 3}}$$

Question 8

Consider the differential equation

$$y' + 2xy = (4x+2)e^x$$

(a) Explain why it would be appropriate to use an integrating factor in attempting to solve the differential equation.

[2]

(b) Show that the integrating factor for this differential equation is e^{x^2}.

[2]

(c) Hence solve the differential equation.

[5]

a) The equation is in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$, and so multiplying it by the appropriate integrating factor will convert it into an exact differential equation of the form $\frac{d}{dx}(y(x)u(x)) = v(x)$.

Consider the differential equation

$$y' + 2xy = (4x+2)e^x$$

(a) Explain why it would be appropriate to use an integrating factor in attempting to solve the differential equation.

[2]

(b) Show that the integrating factor for this differential equation is e^{x^2}.

[2]

(c) Hence solve the differential equation.

[5]

b) Referring to 'standard form', $P(x) = 2x$, so

$$\boxed{e^{\int P(x)dx} = e^{\int 2x dx} = e^{x^2}}$$

Consider the differential equation

$$y' + 2xy = (4x+2)e^x$$

(a) Explain why it would be appropriate to use an integrating factor in attempting to solve the differential equation.

[2]

(b) Show that the integrating factor for this differential equation is e^{x^2}.

[2]

(c) Hence solve the differential equation.

[5]

c) $y' + 2xy = (4x+2)e^x$

multiply by integrating factor, e^{x^2}

$e^{x^2}y' + 2xe^{x^2}y = e^{x^2}((4x+2)e^x)$

$e^{x^2}y' + 2xe^{x^2}y = (4x+2)e^{x^2+x}$

reverse product rule

$\frac{d}{dx}(e^{x^2}y) = (4x+2)e^{x^2+x}$

$\int \frac{d}{dx}(e^{x^2}y)dx = \int (4x+2)e^{x^2+x}dx$

$e^{x^2}y = 2e^{x^2+x} + c$

$$\boxed{y = 2e^x + ce^{-x^2}}$$

Question 9

Use an integrating factor to solve the differential equation

$$(x+3)\frac{dy}{dx} - 4y = (x+3)^6$$

for y which satisfies the boundary condition $y(-2) = 0$.

[7]

Start by rearranging into standard form.

$$(x+3)\frac{dy}{dx} - 4y = (x+3)^6$$

$$\frac{dy}{dx} - \left(\frac{4}{x+3}\right)y = (x+3)^5$$

So the integrating factor is

$$e^{\int\left(\frac{-4}{x+3}\right)dx} = e^{-4\ln|x+3|} = e^{\ln(|x+3|)^{-4}} = (x+3)^{-4}$$

$$\frac{dy}{dx} - \left(\frac{4}{x+3}\right)y = (x+3)^5$$

$\times (x+3)^{-4}$

$$\left(\frac{dy}{dx} - \left(\frac{4}{x+3}\right)y\right)(x+3)^{-4} = (x+3)^5(x+3)^{-4}$$

$$(x+3)^{-4}\frac{dy}{dx} - 4(x+3)^{-5}y = x+3$$

$$\frac{d}{dx}\left((x+3)^{-4}y\right) = x+3$$

$$\int\frac{d}{dx}\left((x+3)^{-4}y\right)dx = \int(x+3)\,dx$$

$$\frac{y}{(x+3)^4} = \frac{1}{2}x^2 + 3x + c$$

$$y = (x+3)^4\left(\frac{1}{2}x^2 + 3x + c\right)$$

$y(-2) = 0$

$$0 = (-2+3)^4\left(\frac{1}{2}(-2)^2 + 3(-2) + c\right)$$

$$\therefore 0 = \frac{1}{2}(-2)^2 + 3(-2) + c = 2 - 6 + c \quad \therefore c = 4$$

$$\boxed{y = (x+3)^4\left(\frac{1}{2}x^2 + 3x + 4\right)}$$

Question 10

Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + 1$$

with the boundary condition $y(1) = 0$.

(a) Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 2$.

[3]

(b) (i) Explain what method you could use to solve the above differential equation analytically (i.e., exactly).

(ii) The exact solution to the differential equation with the given boundary condition is $y = x\ln x$. Compare your approximation from part (a) to the exact value of the solution at $x = 2$.

[4]

(c) Explain how the accuracy of the approximation in part (a) could be improved.

[1]

a) $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$h = 0.2$, $f(x,y) = \frac{y}{x} + 1$

$\therefore y_n = y_{n-1} + 0.2\left(\frac{y_{n-1}}{x_{n-1}} + 1\right)$

n	x_n	y_n
0	1	0
1	1.2	0.2
2	1.4	0.4333
3	1.6	0.6952
4	1.8	0.9821
5	2	1.2912

$\boxed{y(2) = 1.2912... = 1.29 \text{ (3 s.f.)}}$

This table has the exact values.

n	x_n	y_n	$f(x_n, y_n) = \frac{y_n}{x_n} + 1$
0	1	0	$\frac{0}{1} + 1 = 1$
1	1.2	$0 + (0.2)(1) = 0.2$	$\frac{0.2}{1.2} + 1 = \frac{7}{6}$
2	1.4	$0.2 + (0.2)\left(\frac{7}{6}\right) = \frac{13}{30}$	$\frac{\frac{13}{30}}{1.4} + 1 = \frac{55}{42}$
3	1.6	$\frac{13}{30} + (0.2)\left(\frac{55}{42}\right) = \frac{73}{105}$	$\frac{\frac{73}{105}}{1.6} + 1 = \frac{241}{168}$
4	1.8	$\frac{73}{105} + (0.2)\left(\frac{241}{168}\right) = \frac{55}{56}$	$\frac{\frac{55}{56}}{1.8} + 1 = \frac{779}{504}$
5	2.0	$\frac{55}{56} + (0.2)\left(\frac{779}{504}\right) = \frac{1627}{1260}$	not needed.

Note, these are the exact values if you worked them out 'by hand'. You wouldn't want to do this in an exam because doing them on your GDC is much faster.

Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + 1$$

with the boundary condition $y(1) = 0$.

(a) Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 2$.

$\boxed{y(2) \approx \frac{1627}{1260} = 1.2912\ldots = 1.29 \text{ (3 s.f.)}}$

[5]

(b) (i) Explain what method you could use to solve the above differential equation analytically (i.e., exactly).

(ii) The exact solution to the differential equation with the given boundary condition is $y = x \ln x$. Compare your approximation from part (a) to the exact value of the solution at $x = 2$.

[4]

(c) Explain how the accuracy of the approximation in part (a) could be improved.

[1]

b) i)

$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, so the differential equation is homogeneous.

$\therefore$ solve using substitution $v = \frac{y}{x}$

ii) when $x = 2$, $y = 2\ln 2$, so percentage error, $\mathcal{E}$, is

$$\mathcal{E} = \left| \frac{2\ln 2 - \frac{1627}{1260}}{2\ln 2} \right| \times 100\% = 6.8545\ldots$$

$\boxed{\mathcal{E} = 6.85\% \text{ (3 s.f.)}}$

Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + 1$$

with the boundary condition $y(1) = 0$.

(a) Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 2$.

[5]

(b) (i) Explain what method you could use to solve the above differential equation analytically (i.e., exactly).

(ii) The exact solution to the differential equation with the given boundary condition is $y = x \ln x$. Compare your approximation from part (a) to the exact value of the solution at $x = 2$.

[4]

(c) Explain how the accuracy of the approximation in part (a) could be improved.

[1]

c) Use a smaller value of h, and thereby increase the number of incremental steps between $x = 1$ and $x = 2$.

EXAM PAPERS PRACTICE

Question 11

A particle moves in a straight line, such that its displacement x at time t is described by the differential equation

$$\frac{dx}{dt} = \frac{te^{3t^2} + 1}{4x^2}, \quad t \geq 0$$

At time $t = 0,\ x = \frac{1}{2}$.

(a) By using Euler's method with a step length of 0.1, find an approximate value for x at time $t = 0.3$.

[3]

(b) (i) Solve the differential equation with the given boundary condition to show that

$$x = \frac{1}{2}\sqrt[3]{e^{3t^2} + 6t}$$

(ii) Hence find the percentage error in your approximation for x at time $t = 0.3$.

[5]

a)

Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$	h is a constant (step length)
Euler's method for coupled systems	$x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$ $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$ $t_{n+1} = t_n + h$	h is a constant (step length)

$x_n = x_{n-1} + hf(x_{n-1}, t_{n-1}),\ h = 0.1,\quad f(x,t) = \frac{te^{3t^2}+1}{4x^2}$

$\therefore\ x_n = x_{n-1} + 0.1\left(\frac{t_{n-1}e^{3t_{n-1}^2} + 1}{4x_{n-1}^2}\right)$

Use recursion function on your GDC.

n	t_n	x_n
0	0	0.5
1	0.1	0.6
2	0.2	0.6766
3	0.3	0.7435

$\boxed{x(0.3) = x_3 = 0.7435... = 0.744 \quad (3sf)}$

A particle moves in a straight line, such that its displacement x at time t is described by the differential equation

$$\frac{dx}{dt} = \frac{te^{3t^2} + 1}{4x^2}, \quad t \geq 0$$

At time $t = 0,\ x = \frac{1}{2}$.

(a) By using Euler's method with a step length of 0.1, find an approximate value for x at time $t = 0.3$.

[3]

(b) (i) Solve the differential equation with the given boundary condition to show that

$$x = \frac{1}{2}\sqrt[3]{e^{3t^2} + 6t}$$

(ii) Hence find the percentage error in your approximation for x at time $t = 0.3$.

[5]

b) i) Separation of variables with reverse chain rule

$\frac{dx}{dt} = \frac{te^{3t^2}+1}{4x^2} \longrightarrow \int 4x^2\,dx = \int (te^{3t^2} + 1)\,dt$

$\frac{4}{3}x^3 = \frac{1}{6}e^{3t^2} + t + c$

$x^3 = \frac{1}{8}e^{3t^2} + \frac{3}{4}t + \frac{3}{4}c \qquad$ sub in $x = \frac{1}{2},\ t = 0$

$\left(\frac{1}{2}\right)^3 = \frac{1}{8}e^{3(0)^2} + \frac{3}{4}(0) + \frac{3}{4}c \qquad \therefore\ c = 0$

$\therefore\ x^3 = \frac{1}{8}e^{3t^2} + \frac{3}{4}t = \frac{1}{8}\left(e^{3t^2} + 6t\right)$

$\boxed{x = \sqrt[3]{\frac{1}{8}\left(e^{3t^2} + 6t\right)} = \frac{1}{2}\sqrt[3]{e^{3t^2} + 6t}}$

ii) Exact value of x when $t = 0.3$

$x = \frac{1}{2}\sqrt[3]{e^{3(0.3)^2} + 6(0.3)} = 0.7298...$

Percentage error	$\varepsilon = \left\|\frac{v_A - v_E}{v_E}\right\| \times 100\%$	v_E is the exact value and v_A is the approximate value of v

$\boxed{\varepsilon = \left|\frac{0.7435... - 0.7298...}{0.7298...}\right| \times 100\% = 1.841... = 1.84\%}$