Question 1

Consider the first-order differential equation

$$\frac{dy}{dx} - 5x^4 = 3$$

(a) Find the general solution to the differential equation, giving your answer in the form $y = f(x)$.

[3]

(b) Find the specific solution to the equation given that $y = 40$ when $x = 2$.

[2]

a) $\frac{dy}{dx} - 5x^4 = 3$

$\frac{dy}{dx} = 5x^4 + 3$

$y = \int \frac{dy}{dx} = \int (5x^4 + 3)\,dx$

$\boxed{y = x^5 + 3x + c}$

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[3]

(b) Find the specific solution to the equation given that $y = 40$ when $x = 2$.

[2]

b) $y = 40$ when $x = 2 \longrightarrow (2, 40)$

$40 = (2)^5 + 3(2) + C = 32 + 6 + c = 38 + c \quad \therefore c = 2$

$\boxed{y = x^5 + 3x + 2}$

Question 2

Use separation of variables to find the general solution of each of the following differential equations, giving your answers in the form $y = f(x)$:

(a)

$$\frac{dy}{dx} = \frac{4x^2}{y^4}$$

[4]

(b)

$$\frac{dy}{dx} = (x^2 + 1)e^{-y}$$

[4]

a) $\frac{dy}{dx} = \frac{4x^2}{y^4}$

$\int y^4\,dy = \int 4x^2\,dx$

$y^5 = \frac{20}{3}x^3 + c$

$\boxed{y = \sqrt[5]{\frac{20}{3}x^3 + c}}$

Note: if the power was an even number we would need a ± infront of the root

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[4]

(b)

$$\frac{dy}{dx} = (x^2 + 1)e^{-y}$$

[4]

b) $\frac{dy}{dx} = (x^2+1)e^{-y} \longrightarrow \frac{dy}{dx} = (x^2+1) \times \frac{1}{e^y}$

$e^y \frac{dy}{dx} = x^2 + 1$

$\int e^y dy = \int (x^2+1)\,dx \longrightarrow e^y = \frac{1}{3}x^3 + x + c$

$\boxed{y = \ln\left(\frac{1}{3}x^3 + x + c\right)}$

Question 3

Use separation of variables to solve each of the following differential equations for y which satisfies the given boundary condition:

(a)

$$\frac{dy}{dx} = xy^2; \quad y(2) = 1$$

[5]

(b)

$$(x+3)\frac{dy}{dx} = \sec y; \quad y(-2) = \frac{3\pi}{2}$$

[5]

a) $\frac{dy}{dx} = xy^2 \longrightarrow y^{-2}\frac{dy}{dx} = x$

$\int y^{-2} dy = \int x\,dx \longrightarrow -\frac{1}{y} = \frac{1}{2}x^2 + c$

$y(2) = 1 \longrightarrow y = 1$ when $x = 2$

$-\frac{1}{(1)} = \frac{1}{2}(2)^2 + c \longrightarrow -1 = 2 + c \quad \therefore c = -3$

$-\frac{1}{y} = \frac{1}{2}x^2 - 3 = \frac{x^2-6}{2}$

$\boxed{y = \frac{2}{6-x^2}}$

EXAM PAPERS PRACTICE

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[5]

(b)

$$(x+3)\frac{\mathrm{d}y}{\mathrm{d}x} = \sec y; \quad y(-2) = \frac{3\pi}{2}$$

[5]

b) $(x+3)\dfrac{dy}{dx} = \sec y$

$\cos y \dfrac{dy}{dx} = \dfrac{1}{(x+3)}$

$\int \cos y \, dy = \int \dfrac{1}{(x+3)} \, dx$

$\sin y = \ln|x+3| + c$

$y(-2) = \dfrac{3\pi}{2} \rightarrow y = \dfrac{3\pi}{2}$ when $x = -2$

$\sin\left(\dfrac{3\pi}{2}\right) = \ln|(-2)+3| + c$

$-1 = \ln 1 + c = 0 + c \quad \therefore c = -1$

$\boxed{\sin y = \ln|x+3| - 1}$

Question 4

Scientists are studying a large pond where an invasive plant has been observed growing, and they have begun measuring the area, A m^2, of the pond's surface that is covered by the plant. According to the scientists' model, the rate of change of the area of the pond covered by the plant at any time, t, is proportional to the square root of the area already covered.

$\sqrt{A}$

(a) Write down a differential equation to represent the scientists' model.

[2]

a) Separation of variables

$\boxed{\dfrac{dA}{dt} = k\sqrt{A}}$

(b) Solve the differential equation to show that

$$A = \left(\frac{kt+c}{2}\right)^2$$

where k is the constant of proportionality and c is a constant of integration.

[4]

At the time when the scientists begin studying the pond the invasive plant covers an area of 100 m^2. One week later the area has increased to 225 m^2.

(c) Use this information to determine the values of k and c.

[3]

The pond has a total area of 250 000 m^2.

(d) Determine how long it will take, according to the scientists' model, for the invasive plant to cover the entire surface of the pond.

[2]

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[2]

b) Separation of variables

$\frac{dA}{dt} = k\sqrt{A} \longrightarrow \int A^{-\frac{1}{2}}\, dA = \int k\, dt$

$2A^{\frac{1}{2}} = kt + c$

$\sqrt{A} = \frac{kt + c}{2}$

$$\boxed{A = \left(\frac{kt + c}{2}\right)^2}$$

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[2]

c) when $t = 0$, $A = 100$

$\therefore \sqrt{100} = 10 = \frac{k(0) + c}{2}$ $\quad \boxed{c = 20}$

You can use weeks or day as the unit for t.

Method 1: t in weeks

when $t = 1$, $A = 225$

$\therefore \sqrt{225} = \frac{k(1) + 20}{2}$ $\quad \boxed{k = 10}$

Method 2: t in days

when $t = 7$, $A = 225$

$\therefore \sqrt{225} = \frac{k(7) + 20}{2}$ $\quad \boxed{k = \frac{10}{7}}$

EXAM PAPERS PRACTICE

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d) For t in weeks

$$A = \left(\frac{10t + 20}{2}\right)^2 = (5t + 10)^2 \qquad \text{(from part c)}$$

For t in days

$$A = \left(\frac{5}{7}t + 10\right)^2 \qquad \text{(from part c)}$$

Method 1: t in weeks

$$\sqrt{250\,000} = 5t + 10 \qquad \boxed{t = 98 \text{ weeks}}$$

Method 2: t in days

$$\sqrt{250\,000} = \frac{5}{7}t + 10 \qquad \boxed{t = 686 \text{ days}}$$

Question 5

At any point in time, the rate of growth of a colony of bacteria is proportional to the current population size P. At time $t = 0$ hours, the population size is 5000.

(a) Write a differential equation to model the size of the population of bacteria.

[1]

After 1 hour, the population has grown to 7000.

(b) By first solving the differential equation from part (a), determine the constant of proportionality.

[6]

(c) (i) Show that, according to the model, it will take exactly $\frac{\ln 20}{\ln 7 - \ln 5}$ hours (from $t = 0$) for the population of bacteria to grow to 100 000.

(ii) Confirm your answer to part (c)(i) graphically.

[5]

a) $\boxed{\frac{dP}{dt} = kP, \text{ for some constant } k.}$

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[5]

b) This is separation of variables

$\frac{dP}{dt} = kP \longrightarrow \frac{1}{P}\frac{dP}{dt} = k$

$\int \frac{1}{P}\, dP = \int k\, dt \longrightarrow \ln P = kt + c$

Note: we can drop the modulus since P is positive.

$P = e^{kt+c} = (e^{kt})(e^{c}) = Ae^{kt}$, where $A = e^{c}$

$P(0) = 5000$ and $P(1) = 7000$

$P(0) = 5000 = Ae^{k(0)} = A(1) = A \quad \therefore A = e^{c} = 5000$

$P(1) = 7000 = 5000e^{k(1)} = 5000e^{k}$

$\therefore k = \ln\frac{7}{5} = 0.3364\ldots$

$\boxed{P = 5000e^{(\ln\frac{7}{5})t}}$

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[5]

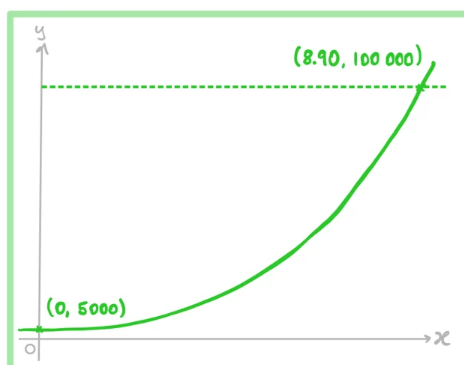
c) i) $100\,000 = 5000e^{(\ln\frac{7}{5})t} \longrightarrow e^{(\ln\frac{7}{5})t} = 20$

$\left(\ln\frac{7}{5}\right)t = \ln 20$

$(\ln 7 - \ln 5)t = \ln 20 \qquad \log_a \frac{x}{y} = \log_a x - \log_a y$

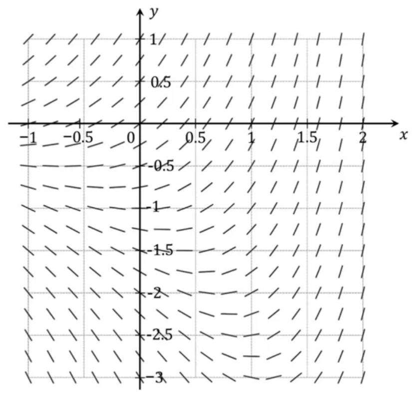
$\boxed{t = \frac{\ln 20}{\ln 7 - \ln 5} = 8.9033\ldots = 8.90 \text{ (3s.f)}}$

ii) Use your GDC to plot the graph of $P(t)$ and $P = 100\,000$.



Question 6

The graph below shows the slope field for the differential equation $\frac{dy}{dx} = e^x + y$ in the intervals $-1 \le x \le 2$ and $-3 \le y \le 1$.



(a) Calculate the value of $\frac{dy}{dx}$ at the point $(0, -3)$.

[1]

(b) On the graph above sketch:

(i) a curve that represents the points where $\frac{dy}{dx} = 0$

(ii) the solution curve that passes through the point $(0, -1)$

(iii) the solution curve that passes through the point $(0, -2)$

[6]

(a) $x = 0,\ y = -3$

$\frac{dy}{dx} = e^0 + (-3)$

$= 1 - 3$

$\boxed{\frac{dy}{dx} = -2 \text{ at } (0, -3)}$

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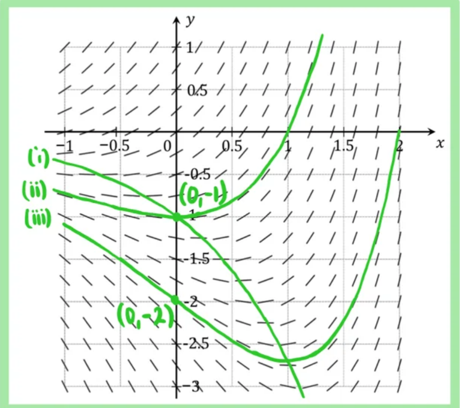

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(iii) the solution curve that passes through the point $(0, -2)$

[6]

(b) The turning points occur when $\frac{dy}{dx} = 0$

$\therefore$ The turning points of the curves in (ii) and (iii) should lie on the curve given in (i)

Question 7

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + 1$$

with the boundary condition $y(1) = 0$.

(a) Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 2$.

[3]

(b) It can be shown that the exact solution to the differential equation with the given boundary condition is $y = x \ln x$. Compare your approximation from part (a) to the exact value of the solution at $x = 2$.

[3]

(c) Explain how the accuracy of the approximation in part (a) could be improved.

[1]

a) $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$h = 0.2, \qquad f(x, y) = \frac{y}{x} + 1$

$\therefore y_n = y_{n-1} + 0.2\left(\frac{y_{n-1}}{x_{n-1}} + 1\right)$

n	x_n	y_n
0	1	0
1	1.2	0.2
2	1.4	0.4333
3	1.6	0.6952
4	1.8	0.9821
5	2	1.2912

$\boxed{y(2) = 1.2912\ldots = 1.29 \text{ (3 s.f.)}}$

This table has the exact values.

n	x_n	y_n	$f(x_n, y_n) = \frac{y_n}{x_n} + 1$
0	1	0	$\frac{0}{1} + 1 = 1$
1	1.2	$0 + (0.2)(1) = 0.2$	$\frac{0.2}{1.2} + 1 = \frac{7}{6}$
2	1.4	$0.2 + (0.2)\left(\frac{7}{6}\right) = \frac{13}{30}$	$\frac{\frac{13}{30}}{1.4} + 1 = \frac{55}{42}$
3	1.6	$\frac{13}{30} + (0.2)\left(\frac{55}{42}\right) = \frac{73}{105}$	$\frac{\frac{73}{105}}{1.6} + 1 = \frac{241}{168}$
4	1.8	$\frac{73}{105} + (0.2)\left(\frac{241}{168}\right) = \frac{55}{56}$	$\frac{\frac{55}{56}}{1.8} + 1 = \frac{779}{504}$
5	2.0	$\frac{55}{56} + (0.2)\left(\frac{779}{504}\right) = \frac{1627}{1260}$	not needed.

Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + 1$$

with the boundary condition $y(1) = 0$.

(a) Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 2$.

$\boxed{y(2) = 1.2912... = 1.29 \text{ (3 s.f.)}}$

[3]

(b) It can be shown that the exact solution to the differential equation with the given boundary condition is $y = x\ln x$. Compare your approximation from part (a) to the exact value of the solution at $x = 2$.

[3]

(c) Explain how the accuracy of the approximation in part (a) could be improved.

[1]

b) when $x = 2$, $y = 2\ln 2$, so percentage error, $\mathcal{E}$, is

$$\mathcal{E} = \left| \frac{2\ln 2 - 1.2912...}{2\ln 2} \right| \times 100\% = 6.8545...$$

$\boxed{\mathcal{E} = 6.85\% \text{ (3 s.f.)}}$

Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + 1$$

with the boundary condition $y(1) = 0$.

(a) Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 2$.

[3]

(b) It can be shown that the exact solution to the differential equation with the given boundary condition is $y = x\ln x$. Compare your approximation from part (a) to the exact value of the solution at $x = 2$.

[3]

(c) Explain how the accuracy of the approximation in part (a) could be improved.

[1]

c) Use a smaller value of h, and thereby increase the number of incremental steps between $x = 1$ and $x = 2$.

Question 8

A particle moves in a straight line, such that its displacement x at time t is described by the differential equation

$$\dot{x} = \frac{te^{3t^2}+1}{4x^2}, \quad t \geq 0$$

At time $t = 0,\ x = \frac{1}{2}$.

(a) By using Euler's method with a step length of 0.1, find an approximate value for x at time $t = 0.3$.

[3]

(b) (i) Solve the differential equation with the given boundary condition to show that

$$x = \frac{1}{2}\sqrt[3]{e^{3t^2}+6t}$$

(ii) Hence find the percentage error in your approximation for x at time $t = 0.3$.

[5]

a)

Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$	h is a constant (step length)
Euler's method for coupled systems	$x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$ $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$ $t_{n+1} = t_n + h$	h is a constant (step length)

$x_n = x_{n-1} + hf(x_{n-1}, t_{n-1}),\ h = 0.1,\quad f(x,t) = \frac{te^{3t^2}+1}{4x^2}$

$\therefore\ x_n = x_{n-1} + 0.1\left(\frac{t_{n-1}e^{3t_{n-1}^2}+1}{4x_{n-1}^2}\right)$

Use recursion function on your GDC.

n	t_n	x_n
0	0	0.5
1	0.1	0.6
2	0.2	0.6766
3	0.3	0.7435

$\boxed{x(0.3) = x_3 = 0.7435\ldots = 0.744\ (3sf)}$

A particle moves in a straight line, such that its displacement x at time t is described by the differential equation

$$\dot{x} = \frac{te^{3t^2}+1}{4x^2}, \quad t \geq 0$$

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$$x = \frac{1}{2}\sqrt[3]{e^{3t^2}+6t}$$

(ii) Hence find the percentage error in your approximation for x at time $t = 0.3$.

[5]

b) i) Separation of variables with reverse chain rule

$\frac{dx}{dt} = \frac{te^{3t^2}+1}{4x^2} \longrightarrow \int 4x^2\,dx = \int (te^{3t^2}+1)\,dt$

$\frac{4}{3}x^3 = \frac{1}{6}e^{3t^2} + t + c$

$x^3 = \frac{1}{8}e^{3t^2} + \frac{3}{4}t + \frac{3}{4}c$ sub in $x = \frac{1}{2},\ t = 0$

$\left(\frac{1}{2}\right)^3 = \frac{1}{8}e^{3(0)^2} + \frac{3}{4}(0) + \frac{3}{4}c \qquad \therefore c = 0$

$\therefore\ x^3 = \frac{1}{8}e^{3t^2} + \frac{3}{4}t = \frac{1}{8}\left(e^{3t^2}+6t\right)$

$\boxed{x = \sqrt[3]{\frac{1}{8}\left(e^{3t^2}+6t\right)} = \frac{1}{2}\sqrt[3]{e^{3t^2}+6t}}$

ii) Exact value of x when $t = 0.3$

$x = \frac{1}{2}\sqrt[3]{e^{3(0.3)^2}+6(0.3)} = 0.7298\ldots$

Percentage error	$\varepsilon = \left\lvert\frac{v_A - v_E}{v_E}\right\rvert \times 100\%$	v_E is the exact value and v_A is the approximate value of v

$\boxed{\varepsilon = \left\lvert\frac{0.7435\ldots - 0.7298\ldots}{0.7298\ldots}\right\rvert \times 100\% = 1.841\ldots = 1.84\%}$