EXAM PAPERS PRACTICE

- Boost your performance and confidence with these topic-based exam questions
- Practice questions created by actual examiners and assessment experts
- Detailed mark scheme
- Suitable for all boards
- Designed to test your ability and thoroughly prepare you

1. Number & Algebra

1.1 Number Toolkit

MATH

IB AI HL

IB Maths DP

1.Number & Algebra

CONTENTS

1.1 Number Toolkit

1.1.1 Standard Form

Standard Form

Standard form (sometimes called **scientific notation** or **standard index form**) gives us a way of writing very big and very small numbers using powers of 10.

Why use standard form?

- Some numbers are too big or too small to write easily or for your calculator to display at all
  - Imagine the number 50^{50}, the answer would take 84 digits to write out
  - Try typing 50^{50} into your calculator, you will see it displayed in **standard form**
- Writing very big or very small numbers in standard form allows us to:
  - Write them more neatly
  - Compare them more easily
  - Carry out calculations more easily
- Exam questions could ask for your answer to be written in standard form

How is standard form written?

- In standard form numbers are always written in the form $a \times 10^k$ where a and k satisfy the following conditions:
  - $1 \le a < 10$
    - So there is one non – zero digit before the decimal point
  - $k \in \mathbb{Z}$
    - So k must be an integer
  - $k > 0$ for large numbers
    - How many times a is multiplied by 10
  - $k < 0$ for small numbers
    - How many times a is divided by 10

How are calculations carried out with standard form?

- Your GDC will display large and small numbers in standard form when it is in normal mode
  - Your GDC may display standard form as *a*E*n*
    - For example, 2.1×10^{-5} will be displayed as $2.1\text{E}-5$
    - If so, be careful to **rewrite the answer given in the correct form**, you will not get marks for copying directly from your GDC
- Your GDC will be able to carry out calculations in standard form
  - If you put your GDC into scientific mode it will automatically convert numbers into standard form
    - Beware that your GDC may have more than one mode when in scientific mode
    - This relates to the number of significant figures the answer will be displayed in
    - Your GDC may add extra zeros to fill spaces if working with a high number of significant figures, you do not need to write these in your answer
- To add or subtract numbers written in the form $a \times 10^k$ without your GDC you will need to write them in full form first
  - Alternatively you can use 'matching powers of 10', because if the powers of 10 are the same, then the 'number parts' at the start can just be added or subtracted normally
    - For example $(6.3\times10^{14}) + (4.9\times10^{13}) = (6.3\times10^{14}) + (0.49\times10^{14}) = 6.79\times10^{14}$
    - Or $(7.93\times10^{-11}) - (5.2\times10^{-12}) = (7.93\times10^{-11}) - (0.52\times10^{-11}) = 7.41\times10^{-11}$
- To multiply or divide numbers written in the form $a \times 10^k$ without your GDC you can either write them in full form first or use the laws of indices

Exam Tip

- Your GDC will give very big or very small answers in standard form and will have a setting which will allow you to carry out calculations in scientific notation
- Make sure you are familiar with the form that your GDC gives answers in as it may be different to the form you are required to use in the exam

Worked Example

Calculate the following, giving your answer in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

i)
3780×200

Using GDC: Choose scientific mode.

Input directly into GDC as ordinary numbers.

$3780 \times 200 = 7.56 \times 10^5$

GDC will automatically give answer in standard form.

Without GDC:

Calculate the value:

$3780 \times 200 = 756000$

Convert to standard form:

$756000 = 7.56 \times 10^5$

7.56×10^5

ii) $(7 \times 10^5) - (5 \times 10^4)$

Using GDC: Choose scientific mode.

Input directly into GDC

$7 \times 10^5 - 5 \times 10^4 = 6.5 \times 10^5$

This may be displayed as 6.5E5

Without GDC:

Convert to ordinary numbers:

$7 \times 10^5 = 700000$

$5 \times 10^4 = 50000$

Carry out the calculation:

$700000 - 50000 = 650000$

Convert to standard form:

$650000 = 6.5 \times 10^5$

6.5×10^5

iii)

$(3.6 \times 10^{-3})(1.1 \times 10^{-5})$

Input directly into GDC:

(Choose scientific mode).

$(3.6 \times 10^{-3})(1.1 \times 10^{-5}) = 3.96 \times 10^{-8}$

3.96×10^{-8}

Note:

$10^{-3} \times 10^{-5} = 10^{-8}$

$(3.6 \times 10^{-3})(1.1 \times 10^{-5}) = 3.96 \times 10^{-8}$

$3.6 \times 1.1 = 3.96$

1.1.2 Approximation & Estimation

Upper & Lower Bounds

What are bounds?

- Bounds are the smallest **(lower bound, LB)** and largest **(upper bound, UB)** numbers that a **rounded number** can lie between
  - It simply means how low or high the number could have been before it was rounded
- The bounds for a number, x, can be written as $\text{LB} \le x < \text{UB}$
  - Note that the lower bound is included in the range of values x could have taken but the upper bound is not

How do we find bounds?

- The basic rule is "half up, half down"
  - To find the upper bound add on half the degree of accuracy
  - To find the lower bound take off half the degree of accuracy
- Remember that the upper bound is the cut off point for the greatest value that the number could have been rounded from but will not actually round to the number itself

How do we calculate using bounds?

- Find bounds before carrying out the calculation and then use the rules:
  - To add **UB = UB + UB** and **LB = LB + LB**
  - To multiply **UB = UB × UB** and **LB = LB × LB**
  - To divide **UB = UB / LB** and **LB = LB / UB**
  - To subtract **UB = UB – LB** and **LB = LB – UB**
- Use logic to decide which bound to use within the calculation
  - For example if you are finding the **maximum** volume of a sphere with the radius given correct to 1 decimal place substitute the **upper bound** of the radius into your calculation for the volume

Exam Tip

- When in an exam environment it can be easy to make silly errors in questions like this, read the question carefully to determine which parts bounds need to be found for
  - This will normally be any part in the question that has been rounded

Worked Example

A rectangular field has length, L, of 14.3 m correct to 1 decimal place and width, W, of 9.61 m correct to 2 decimal places.

a)
Calculate the lower and upper bound for L and W.

$L = 14.3$ m (1d.p.) the degree of accuracy is 1 d.p (0.1)

Find half the degree of accuracy:

$$\frac{0.1}{2} = 0.05$$

The upper bound is

$$14.3 + 0.05 = 14.35$$

The lower bound is

$$14.3 - 0.05 = 14.25$$

$$14.25 \leq L < 14.35$$

$W = 9.61$ m (2d.p.) the degree of accuracy is 2d.p (0.01)

Find half the degree of accuracy:

$$\frac{0.01}{2} = 0.005$$

The upper bound is

$$9.61 + 0.005 = 9.615$$

The lower bound is

$$9.61 - 0.005 = 9.605$$

$$9.605 \leq W < 9.615$$

b)
Calculate the lower and upper bound for the perimeter, P, and area, A, of the field.

For the lower bound use:

$L = 14.25 \quad W = 9.605$

$P = 2(14.25) + 2(9.605)$

$P = 47.71$ m

$A = (14.25)(9.605)$

$A = 136.87125$

For the upper bound use:

$L = 14.35 \quad W = 9.615$

$P = 2(14.35) + 2(9.615)$

$P = 47.93$ m

$A = (14.35)(9.615)$

$A = 137.97525$

$47.7\text{ m} \leq P < 47.9\text{ m}$ (3 s.f.)

$137\text{ m}^2 \leq A < 138\text{ m}^2$ (3 s.f.)

Approximating Values

How do I know what to round my answer to?

- Unless otherwise told, always round your answers to **3 significant figures** (3 s.f.)
  - The first non-zero digit is the first **significant** digit
  - The first digit after the third significant digit determines whether to 'round up' (≥ 5) or 'leave it alone' (<5)
    - where the ‘it’ we are rounding up or leaving alone is the third significant figure
  - Your final answer will have three **significant digits** and the rest will be zero
    - Any zero **after** the first significant digit is still significant
    - For large numbers be careful not to change the **place value** of the significant digits, you will have to fill in any zeros after the third significant figure
    - If your GDC is in **scientific mode** it may display unnecessary zeros after the decimal point, you do not need to copy these
- Look out for any questions that ask you to round your answer in a different way
  - Questions often ask for **2 decimal places** (2 d.p.)
    - Your final answer will only have 2 digits after the decimal point
    - For 2 d.p. it is the third digit after the decimal place that determines whether to 'round up' (5) or 'leave it alone' (<5)
- If you are working with a **currency** you must choose the appropriate degree of accuracy
  - For most this will be a **whole number**
    - E.g. yen, yuan, peso
  - For others this will be to **2 decimal places**
    - E.g. dollars, euro, pounds
  - It will be clear from the question which currency you are using and how you should round your answer
    - The question will state the name of the currency and the symbol you should use as a unit
    - E.g. YEN, ¥

Are there cases when I always have to round up?

- Yes - there are cases when it makes sense to always round up (or down)
- These normally involve finding the **minimum** or **maximum number** of objects
  - For example consider the scenario: There are 26 people and 5 people can fit in a single vehicle, how many vehicles are needed?
    - $\frac{26}{5} = 5.2$ and normally we'd round to 5
    - However 5 vehicles wouldn't be enough as there would only be room for 25 people
    - In this case we would round up to find the **minimum** number needed
- These kind of problems can be solved by inequalities
  - For $x > k$ take the **smallest value** of x at the appropriate degree of accuracy that is **greater than k**
    - For example: Using 3sf the smallest solution to $x > 2.5731...$ is $x = 2.58$

- For $x < k$ take the **biggest value** of x at the appropriate degree of accuracy that is **less than *k***
  - For example: The biggest integer solution to $x < 10.901...$ is $x = 10$

Exam Tip

- In the exam you should always give non exact answers correct to 3 significant figures unless otherwise told
  - This means you must round using a higher degree of accuracy within your working to ensure that your final answer is rounded correctly
  - Where possible always use exact values within your working rather than rounding mid way through a question

Worked Example

Let $T = \dfrac{b\sin(3a)}{5}$, where $a = 15°$ and $b = 20$.

a)

Calculate the exact value of T.

Substitute a and b into T:

$$T = \frac{20\sin(3 \times 15)}{5}$$

$$T = 2\sqrt{2}$$

$$2\sqrt{2} = 2.82|842...$$

≥ 5 so round up

2nd digit after decimal point

$$T = 2.83 \text{ (2d.p.)}$$

c)

Give your answer from part a) correct to two significant figures.

first significant figure

$$2\sqrt{2} = 2.8|2842...$$

< 5 so don't round up

2nd significant figure

$$T = 2.8 \text{ (2 s.f.)}$$

Percentage Error

What is percentage error?

- Percentage error is how far away from the actual value an estimated or rounded answer is
  - Percentage error can be calculated using the formula

$$\varepsilon = \left| \frac{v_A - v_E}{v_E} \right| \times 100\%$$

    - where v_E is the exact value and v_A is the approximate value of v
    - The | | is the **absolute value** meaning if you get a negative value within these straight brackets, you should take the **positive** value
  - This formula is **in the formula booklet** so you do not need to remember it
- The further away the estimated answer is from the true answer the greater the percentage error
- If the exact value is given as a surd or a multiple of π make sure you enter it into the formula exactly as you see it
- Percentage error should always be a positive number

Exam Tip

- In the exam percentage error will usually be a part of a bigger question on another topic, make sure you know how to find the formula for it in the formula book so that you are prepared to answer these questions

Worked Example

Let $P = x\cos(2y)$, where $y = 15°$ and $x = 4$.

a)
Calculate the exact value of P.

$P = x\cos 2y = 4\cos(2 \times 15°)$ ($x = 4$, $y = 15°$)

$= 4\cos(30°)$

$= 2\sqrt{3}$ ← leave answer as exact value

$P = 2\sqrt{3}$

b)
Calculate the percentage error if an estimate for P was 3.5.

Percentage error formula:

$$\varepsilon = \left|\frac{V_A - V_E}{V_E}\right| \times 100\%$$

$V_A = 3.5$ (approximated value)

$V_E = 2\sqrt{3}$ (exact value)

Sub V_A and V_E into the formula:

$$\varepsilon = \left|\frac{3.5 - 2\sqrt{3}}{2\sqrt{3}}\right| \times 100\%$$

$= 1.03629...\%$

$\varepsilon = 1.04\%$ (3s.f.)

Accuracy & Estimation

What are exact values?

- Exact values are forms that represent the full and precise value of a number
  - For example, π is an exact value and 3.14 is an approximation using 3 significant figures
- If a number has an infinite number of non-zero digits after the decimal point then you can use three dots to signal that the decimal representation goes on for example
  - For example, $\sqrt{2} = 1.414...$
- Exact values can involve
  - Fractions: $\frac{2}{7}$
  - Roots: $\sqrt{3}$, $\sqrt[5]{7}$
  - Logarithms: $\ln 2$, $\log_{10} 5$
  - Mathematical constants: π, e
- Your GDC might automatically give your answer as an exact answer
- If your GDC does not do this then you may need to evaluate parts of the expression separately and use algebra
  - For example: If $f(x) = e^x(2 + \sqrt{x})$ then your GDC will probably not give you the exact value of $f(2)$
  - You would insist evaluate it without a GDC to get the exact value: $f(2) = e^2(2 + \sqrt{2})$

Why use estimation?

- We **estimate** to find approximate answers to difficult sums
- Or to check our answers are about the right size (order of magnitude)
    - For example, if the question is to find a length the answer cannot be negative
    - or if we are looking for the mean age of some people an answer of 150 must be incorrect
- Estimating an answer before carrying out a calculation will help you know what you are looking for and determine if your answer is likely to be correct or not
- In real life estimation skills are used every day in many activities

How do I choose the correct answer?

- Sometimes a mathematical argument will lead to more than one answer
    - This is common with problems involving quadratics, you will usually have two solutions
    - If you have more than one solution after you have solved a problem, **always** check to see if they are both valid
- Most of the time you can simply use logic to choose the correct answer
    - If the problem involves length or area and one of the answers is negative, the true solution will be the positive answer
- Occasionally you will need to see if an answer can be valid
    - If one of your answers is $\cos x > 1$ for example, x will not give a true solution

Exam Tip

- Be aware that your GDC will not always give you an answer as an exact value, this means that you will need to find the exact value by hand

Worked Example

A rectangular floor has an area of 40 m^2 to the nearest square metre. It is going to be tiled using square tiles with side length 39.8 cm.

a)
Use estimation to find the number of tiles needed to cover the whole area.

Each tile is approximately:

$40 \times 40 \text{ cm} = 1600 \text{ cm}^2$

Area of the rectangle is approximately:

$40 \text{m}^2 = 400\,000 \text{ cm}^2$

$400\,000 \text{ cm}^2 \div 1600 \text{ cm}^2 = \frac{400000}{1600}$

$= \frac{4000}{16}$

$= 250$ tiles

≈ 250 tiles

b)
Given that there are 15 more tiles places length-wise than width-wise, find the approximate length and width of the floor.

Let the number of tiles covering the width of the floor be x, then the number of tiles covering the length will be $x + 15$.

number of tiles placed widthways ↘ number of tiles placed lengthways ↙

$$x(x + 15) = 250$$

$$x^2 + 15x - 250 = 0$$

$$x = 10 \quad \text{or} \quad x = -25$$

↑ not possible as x cannot be a negative number

Width of floor $\approx 10 \times 40$ cm

$= 400$ cm $= 4$ m

Length of floor $\approx 25 \times 40$ cm

$= 1000$ cm $= 10$ m

Length $\approx$ 10 m, Width $\approx$ 4 m

1.1.3 GDC: Solving Equations

Systems of Linear Equations

What are systems of linear equations?

- A linear equation is an equation of the first order (degree 1)
  - It is usually written in the form $ax + by + c = 0$ where a, b, and c are constants
- A system of linear equations is where two or more linear equations work together
  - Usually there will be two equations with the variables x and y
  - The **variables** x and y will satisfy all equations
  - They are usually known as **simultaneous equations**
  - Occasionally there may be three equations with the variables x, y and z
- They can be complicated to solve but your GDC has a function allowing you to solve them
  - The question may say 'using technology, solve…'
    - This means you do not need to show a method of solving the system of equations, you can use your GDC

How do I use my GDC to solve a system of linear equations?

- Your GDC will have a function within the algebra menu to solve a system of linear equations
- You will need to choose the number of equations
  - For two equations the variables will be x and y
  - For three equations the variables will be x, y and z
- Enter the equations into your calculator as you see them written
- Your GDC will display the values of x and y (or x, y, and z)

How do I set up a system of linear equations?

- Not all questions will have the equations written out for you
- There will be two bits of information given about two variables
  - Look out for clues such as 'assuming a linear relationship'
- Choose to assign x to one of the given variables and y to the other
  - Or you can choose to use more meaningful variables if you prefer
  - Such as c for cats and d for dogs
- Write your system of equations in the form

$$ax + by = e$$
$$cx + dy = f$$

- Use your GDC to solve the system of equations
- This function on the GDC can also be used to find the points of intersection of two straight line graphs
  - You may wish to use the graphing section on your GDC to see the points of intersection

Exam Tip

- Be sure to write down what you are putting into your GDC
  - If you have had to set up the system of equations as well make sure you write them down clearly before typing into your GD

Worked Example

A theme park has set ticket prices for adults and children. A group of three adults and nine children costs $153 and a group of five adults and eleven children costs $211.

i)
Set up a system of linear equations for the cost of adult and child tickets.

Set up variables:

Let the cost of an adult ticket be 'a'

Let the cost of a child ticket be 'c'

Set up equations: $3a + 9c = 153$

$5a + 11c = 211$

$$3a + 9c = 153$$
$$5a + 11c = 211$$

ii)
Find the price of one adult and one child ticket.

Enter into GDC:

Let a be x and c be y, then GDC gives

$x = 18$

$y = 11$

$$a = \$18, \quad c = \$11$$

Polynomial Equations

What is a polynomial equation?

- A polynomial is an algebraic expression consisting of a finite number of terms, with non-negative integer indices only
  - It is in the form $ax^n + bx^{n-1} + cx^{n-2} + \ldots, n \in \mathbb{N}$
- A **polynomial equation** is an equation where a polynomial is equal to zero
- The number of **solutions (roots or zeros)** depend on the **order** of the polynomial equation
  - A polynomial equation of order two can have up to two solutions
  - A polynomial equation of order five can have up to five solutions
- A polynomial equation of an odd degree will always have at least one solution
- A polynomial equation of an even degree could have no solutions

How do I use my GDC to solve polynomial equations?

- You should use your GDC's graphing mode to look at the shape of the polynomial
  - You will be able to see the number of solutions
  - This will be the number of times the graph cuts through or touches the x-axis
  - When entering a function into the graphing section you may need to adjust your zoom settings to be able to see the full graph on your display
  - Whilst in this mode you can then choose to **analyse** the graph
  - This will give you the option to see the solutions of the polynomial equation
    - This may be written as the **zeros** (points where the graph meets the x-axis)
- Your GDC will also have a function within the algebra menu to solve polynomial equations
  - You will need to enter the **order (highest degree)** of the polynomial
  - This is the highest power (or exponent) in the equation
  - Enter the equation into your calculator
  - Your GDC will display the solutions (roots) of the equation
    - Be aware that your GDC may either show all solutions or only the first solution, it is always worth plotting a graph of the function to check how many solutions there should be

Exam Tip

- Be sure to write down what you are putting into your GDC
- If you are using a graphical method it is often a good idea to sketch the graph that your GDC display shows
  - Don't spend too much time on this, a very quick sketch is fine

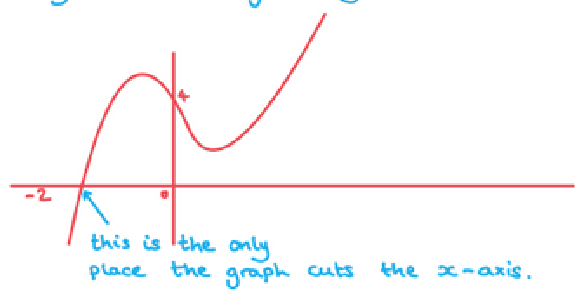
Worked Example

For the polynomial equation $2x^3 - 2x^2 - 3x + 4 = 0$:

i)

Use your GDC's graphing function to sketch the graph of $y = 2x^3 - 2x^2 - 3x + 4$ and determine the number of solutions to the polynomial equation.

Enter the equation $y = 2x^3 - 2x^2 - 3x + 4$ into your GDC's graphing software :



The polynomial equation $2x^3 - 2x^2 - 3x + 4 = 0$ has 1 solution

ii)

Use your GDC to find the solution(s) of the polynomial equation.

Use your GDC's graph analysis tool to find the 'zeros'.

$x = -1.3101...$

$x = -1.31$ (3sf)

Alternative method:

Enter the equation $2x^3 - 2x^2 - 3x + 4 = 0$ into your GDC's equation solving mode.

1.2 Exponentials & Logs

1.2.1 Exponents

Laws of Indices

What are the laws of indices?

- Laws of indices (or index laws) allow you to simplify and manipulate expressions involving exponents
  - An exponent is a power that a number (called the base) is raised to
  - Laws of indices can be used when the numbers are written with the same base
- The index laws you need to know are:
  - $(xy)^m = x^m y^m$
  - $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$
  - $x^m \times x^n = x^{m+n}$
  - $x^m \div x^n = x^{m-n}$
  - $(x^m)^n = x^{mn}$
  - $x^1 = x$
  - $x^0 = 1$
  - $\frac{1}{x^m} = x^{-m}$
  - $x^{\frac{1}{n}} = \sqrt[n]{x}$
  - $x^{\frac{m}{n}} = \sqrt[n]{x^m}$
- These laws are **not in the formula booklet** so you must remember them

How are laws of indices used?

- You will need to be able to carry out multiple calculations with the laws of indices
  - Take your time and apply each law individually
  - Work with numbers first and then with algebra
- Index laws only work with terms that have the same base, make sure you **change the base** of the term before using any of the index laws
  - Changing the base means rewriting the number as an exponent with the base you need
  - For example, $9^4 = (3^2)^4 = 3^{2 \times 4} = 3^8$
  - Using the above can them help with problems like $9^4 \div 3^7 = 3^8 \div 3^7 = 3^1 = 3$

Exam Tip

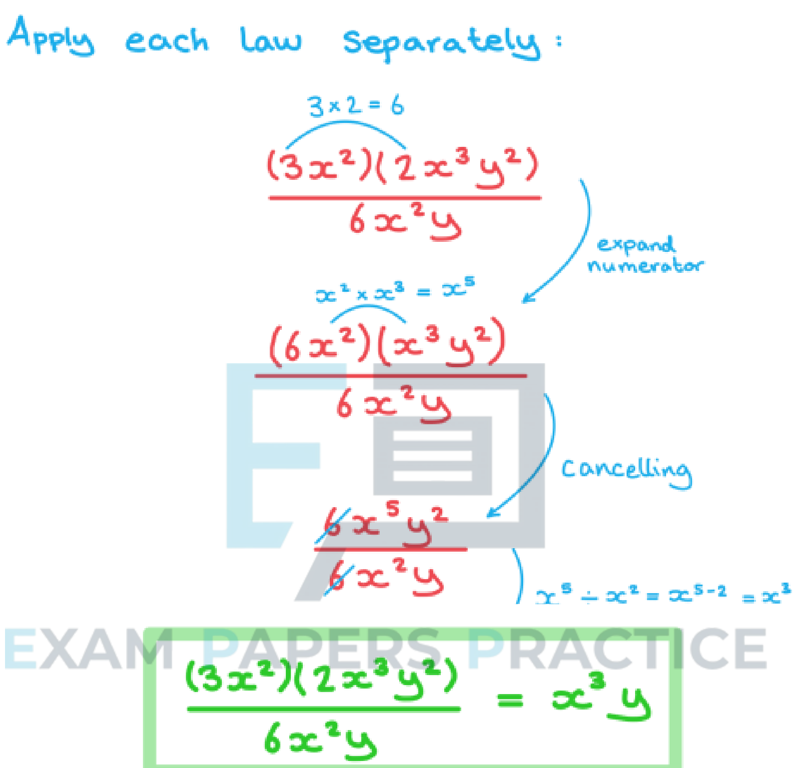
- Index laws are rarely a question on their own in the exam but are often needed to help you solve other problems, especially when working with logarithms or polynomials
- Look out for times when the laws of indices can be applied to help you solve a problem algebraically

Worked Example

Simplify the following equations:

i)

$$\frac{(3x^2)(2x^3y^2)}{(6x^2y)}.$$



ii)

$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$.

$$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$$

Rewrite as a fraction

$$\frac{(4x^2y^{-4})^3}{(2x^3y^{-1})^2}$$

expand numerator and denominator

$$\frac{64x^6y^{-12}}{4x^6y^{-2}}$$

cancelling

$$\frac{64x^6y^2}{4x^6y^{12}}$$

The negative exponents can be rewritten as their reciprocals

$$16y^{-10}$$

$$\frac{16}{y^{10}}$$

1.2.2 Logarithms

Introduction to Logarithms

What are logarithms?

- A logarithm is the inverse of an exponent
  - If $a^x = b$ then $\log_a(b) = x$ where $a > 0, b > 0, a \neq 1$
    - This is in the formula booklet
    - The number *a* is called the **base** of the logarithm
    - Your GDC will be able to use this function to solve equations involving exponents
- Try to get used to 'reading' logarithm statements to yourself
  - $\log_a(b) = x$ would be read as "the power that you raise a to, to get b, is x"
  - So $\log_5 125 = 3$ would be read as "the power that you raise 5 to, to get 125, is 3"
- Two important cases are:
  - $\ln x = \log_e(x)$
    - Where e is the mathematical constant 2.718…
    - This is called the **natural logarithm** and will have its own button on your GDC
  - $\log x = \log_{10}(x)$
    - Logarithms of **base 10** are used often and so abbreviated to **log *x***

Why use logarithms?

- Logarithms allow us to solve equations where the exponent is the unknown value
  - We can solve some of these by inspection
    - For example, for the equation $2^x = 8$ we know that *x* must be 3
  - Logarithms allow use to solve more complicated problems
    - For example, the equation $2^x = 10$ does not have a clear answer
    - Instead, we can use our GDCs to find the value of $\log_2 10$

Exam Tip

- Before going into the exam, make sure you are completely familiar with your GDC and know how to use its logarithm functions

Worked Example

Solve the following equations:

i)

$x = \log_3 27$,

$x = \log_3 27 \iff 3^x = 27$

We can see from inspection:

$3^3 = 27 \iff x = 3$

$x = 3$

OR: use GDC to find answer directly.

ii)

$2^x = 21.4$, giving your answer to 3 s.f.

$2^x = 21.4$ This cannot be seen from inspection:

$2^x = 21.4 \iff x = \log_2 21.4$

use GDC to find answer directly.

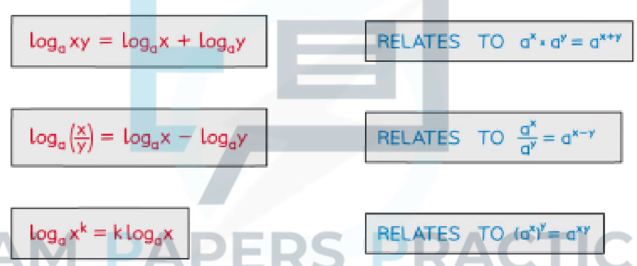
$\log_2 21.4 = 4.4195...$

$x = 4.42$ (3 s.f.)

Laws of Logarithms

What are the laws of logarithms?

- Laws of logarithms allow you to simplify and manipulate expressions involving logarithms
  - The laws of logarithms are equivalent to the **laws of indices**
- The laws you need to know are, given $a,\ x,\ y > 0$:
  - $\log_a xy = \log_a x + \log_a y$
    - This relates to $a^x \times a^y = a^{x+y}$
  - $\log_a \frac{x}{y} = \log_a x - \log_a y$
    - This relates to $a^x \div a^y = a^{x-y}$
  - $\log_a x^m = m\log_a x$
    - This relates to $(a^x)^y = a^{xy}$
- These laws are **in the formula booklet** so you do not need to remember them
  - You must make sure you know how to use them



Useful results from the laws of logarithms

- Given $a > 0,\ a \neq 1$
  - $\log_a 1 = 0$
    - This is equivalent to $a^0 = 1$
- If we substitute *b* for *a* into the given identity in the formula booklet
  - $a^x = b \Leftrightarrow \log_a b = x$ where $a > 0,\ b > 0,\ a \neq 1$
  - $a^x = a \Leftrightarrow \log_a a = x$ gives $a^1 = a \Leftrightarrow \log_a a = 1$
    - This is an important and useful result
- Substituting this into the third law gives the result
  - $\log_a a^k = k$
- Taking the inverse of its operation gives the result
  - $a^{\log_a x} = x$
- From the third law we can also conclude that
  - $\log_a \frac{1}{x} = -\log_a x$

$\log_a a = 1$	"THE POWER YOU RAISE a TO, TO GET a, IS 1"
$\log_a a^x = x$	$\log_a a^x = x \log_a a$ $= x$
$a^{\log_a x} = x$	AN OPERATION AND ITS INVERSE
$\log_a 1 = 0$	$a^0 = 1$
$\log_a \frac{1}{x} = -\log_a x$	$\log_a \frac{1}{x} = \log_a x^{-1}$ $= -\log_a x$

- These useful results are **not in the formula booklet** but can be deduced from the laws that are
- Beware...
  - ...$\log_a(x + y) \neq \log_a x + \log_a y$
- These results apply to $\ln x$ $(\log_e x)$ too
  - Two particularly useful results are
    - $\ln e^x = x$
    - $e^{\ln x} = x$
- Laws of logarithms can be used to ...
  - simplify expressions
  - solve logarithmic equations
  - solve exponential equations

Exam Tip

- Remember to check whether your solutions are valid
  - log (x+k) is only defined if x > -k
  - You will lose marks if you forget to reject invalid solutions

Worked Example

a)

Write the expression $2\log 4 - \log 2$ in the form $\log k$, where $k \in \mathbb{Z}$.

Using the law $\log_a x^m = m\log_a x$

$2\log 4 = \log 4^2 = \log 16$

$2\log 4 - \log 2 = \log 4^2 - \log 2$

$= \log 16 - \log 2$

Using the law $\log_a \frac{x}{y} = \log_a x - \log_a y$

$\log 16 - \log 2 = \log \frac{16}{2} = \log 8$

$2\log 4 - \log 2 = \log 8$

b) Hence, or otherwise, solve $2\log 4 - \log 2 = -\log \frac{1}{x}$.

To solve $2\log 4 - \log 2 = \log \frac{1}{x}$ rewrite as

$\log 8 = -\log \frac{1}{x}$

from part (a)

Use the index law $\frac{1}{x} = x^{-1}$

$\log 8 = -\log x^{-1}$

$\log 8 = \log x$ ← $\log_a x^m = m\log_a x$

$8 = x$

$x = 8$

1.3 Sequences & Series

1.3.1 Language of Sequences & Series

Language of Sequences & Series

What is a sequence?

- A **sequence** is an ordered set of numbers with a well-defined rule for getting from one number to the next
  - For example 1, 3, 5, 7, 9, ... is a sequence with the rule *'start at one and add two to get each subsequent number'*
- The numbers in a sequence are often called **terms**
- The terms of a sequence are often referred to by letters with a subscript
  - In IB this will be the letter u
  - So in the sequence above, $u_1 = 1$, $u_2 = 3$, $u_3 = 5$ and so on
- Each term in a sequence can be found by **substituting** the term number into the **formula for the nth term**

What is a series?

- You get a **series** by summing up the terms in a sequence
  - E.g. For the sequence 1, 3, 5, 7, ... the associated series is 1 + 3 + 5 + 7 + ...
- We use the notation S_n to refer to the sum of the first n terms in the series
  - $S_n = u_1 + u_2 + u_3 + ... + u_n$
  - So for the series above $S_5 = 1 + 3 + 5 + 7 + 9 = 25$

Worked Example

Determine the first five terms and the value of S_5 in the sequence with terms defined by $u_n = 5 - 2n$.

$u_n = 5 - 2n$

find the term you want by replacing n with it's value. (u_n) ; term number (n)

first term:

$u_1 = 5 - 2(1) = 3$

$u_2 = 5 - 2(2) = 1$

$u_3 = 5 - 2(3) = -1$

$u_4 = 5 - 2(4) = -3$

$u_5 = 5 - 2(5) = -5$

recognise the pattern. −2, −2 ← rule is subtract 2

'start with 3 and subtract 2 from each number'.

$S_5 = 3 + 1 + (-1) + (-3) + (-5) = -5$

(S_5:) the sum of the first 5 terms

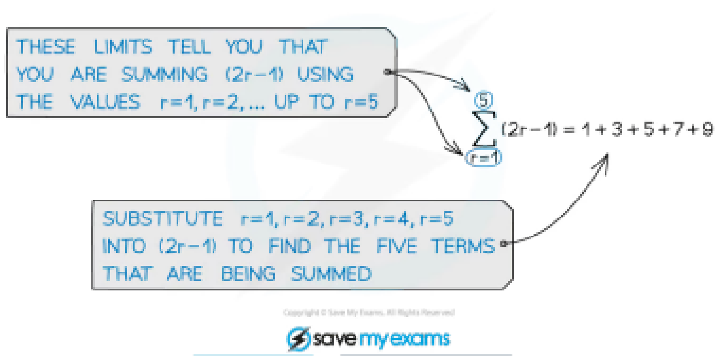
3, 1, −1, −3, −5

$S_5 = -5$

Sigma Notation

What is sigma notation?

- Sigma notation is used to show the sum of a certain number of terms in a sequence
- The symbol Σ is the capital Greek letter sigma
- Σ stands for 'sum'
    - The expression to the right of the Σ tells you what is being summed, and the limits above and below tell you which terms you are summing



- Be careful, the limits don't have to start with 1
    - For example $\sum_{k=0}^{4}(2k+1)$ or $\sum_{k=7}^{14}(2k-13)$
    - r and k are commonly used variables within sigma notation

Exam Tip

- Your GDC will be able to use sigma notation, familiarise yourself with it and practice using it to check your work

Worked Example

A sequence can be defined by $u_n = 2 \times 3^{n-1}$ for $n \in \mathbb{Z}^+$.

a)

Write an expression for $u_1 + u_2 + u_3 + \ldots + u_6$ using sigma notation.

$u_n = 2 \times 3^{n-1}, \; n \in \mathbb{Z}^+$ ← n is the set of all positive integers

Using sigma notation

$$u_1 + u_2 + \ldots + u_6 = \sum_{k=1}^{6} u_k$$

$$\sum_{k=1}^{6} (2 \times 3^{k-1})$$

b)

Write an expression for $u_7 + u_8 + u_9 + \ldots + u_{12}$ using sigma notation.

$u_n = 2 \times 3^{n-1}, \; n \in \mathbb{Z}^+$ ← n is the set of all positive integers

Using sigma notation

$$u_7 + u_8 + \ldots + u_{12} = \sum_{k=7}^{12} u_k$$

$$\sum_{k=7}^{12} (2 \times 3^{k-1})$$

1.3.2 Arithmetic Sequences & Series

Arithmetic Sequences

What is an arithmetic sequence?

- In an **arithmetic sequence,** the difference between consecutive terms in the sequence is constant
- This **constant difference** is known as the **common difference, *d*,** of the sequence
  - For example, 1, 4, 7, 10, … is an arithmetic sequence with the rule '*start at one and add three to each number*'
    - The **first term, u_1,** is 1
    - The **common difference, *d*,** is 3
  - An arithmetic sequence can be **increasing** (positive common difference) or **decreasing** (negative common difference)
  - Each term of an arithmetic sequence is referred to by the letter *u* with a subscript determining its place in the sequence

How do I find a term in an arithmetic sequence?

- The n^{th} term formula for an arithmetic sequence is given as

$$u_n = u_1 + (n-1)d$$

  - Where u_1 is the first term, and d is the common difference
  - This is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two terms and asked to find both the first term and the common difference
  - Substitute the information into the formula and set up a **system of linear equations**
  - Solve the simultaneous equations
    - You could use your GDC for this

Exam Tip

- Simultaneous equations are often needed within arithmetic sequence questions, make sure you are confident solving them with and without the GDC

Worked Example

The fourth term of an arithmetic sequence is 10 and the ninth term is 25, find the first term and the common difference of the sequence.

$u_4 = 10$, $u_9 = 25$

Formula for n^{th} term of an arithmetic series:

$$u_n = u_1 + (n-1)d$$

Sub in $u_4 = 10$ and $u_9 = 25$

$$u_4 = u_1 + (4-1)d = u_1 + 3d = 10$$

$$u_9 = u_1 + (9-1)d = u_1 + 8d = 25$$

Solve using GDC:

let $u_1 = x$ and $d = y$

$$x + 3y = 10$$
$$x + 8y = 25$$

$x = 1$, $y = 3$

$$u_1 = 1$$
$$d = 3$$

Arithmetic Series

How do I find the sum of an arithmetic series?

- An **arithmetic series** is the sum of the terms in an **arithmetic sequence**
  - For the arithmetic sequence 1, 4, 7, 10, ... the arithmetic series is 1 + 4 + 7 + 10 + ...
- Use the following formulae to find the sum of the first *n* terms of the arithmetic series:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) \quad ; \quad S_n = \frac{n}{2}(u_1 + u_n)$$

    - u_1 is the first term
    - d is the common difference
    - u_n is the last term
  - Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - If you know the first term and common difference use the first version
  - If you know the first and last term then the second version is easier to use
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this

Exam Tip

- The formulae you need for arithmetic series are in the formula book, you do not need to remember them
  - Practice finding the formulae so that you can quickly locate them in the exam

Worked Example

The sum of the first 10 terms of an arithmetic sequence is 630.

a)

Find the common difference, *d*, of the sequence if the first term is 18.

$S_{10} = 630$

Formula for the sum of an arithmetic series:

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

Sub in $S_{10} = 630$, $u_1 = 18$

$S_{10} = \frac{10}{2}(2(18) + (10-1)d) = 630$

$5(36 + 9d) = 630$

Solve: $36 + 9d = 126$

$9d = 90$

$d = 10$

$d = 10$

b)

Find the first term of the sequence if the common difference, *d*, is 11.

Sub in $S_{10} = 630$, $d = 11$

$$S_{10} = \frac{10}{2}(2u_1 + (10-1)(11)) = 630$$

$$5(2u_1 + 99) = 630$$

Solve: $2u_1 + 99 = 126$

$$2u_1 = 27$$

$$u_1 = 13.5$$

1.3.3 Geometric Sequences & Series

Geometric Sequences

What is a geometric sequence?

- In a **geometric sequence**, there is a **common ratio, *r*,** between consecutive terms in the sequence
  - For example, 2, 6, 18, 54, 162, ... is a sequence with the rule *'start at two and multiply each number by three'*
    - The **first term, u_1,** is 2
    - The **common ratio, *r*,** is 3
- A geometric sequence can be **increasing** ($r > 1$) or **decreasing** ($0 < r < 1$)
- If the common ratio is a **negative number** the terms will alternate between positive and negative values
  - For example, 1, -4, 16, -64, 256, ... is a sequence with the rule *'start at one and multiply each number by negative four'*
    - The **first term, u_1,** is 1
    - The **common ratio, *r*,** is -4
- Each term of a geometric sequence is referred to by the letter *u* with a subscript determining its place in the sequence

How do I find a term in a geometric sequence?

- The n^{th} term formula for a geometric sequence is given as

$$u_n = u_1 r^{n-1}$$

  - Where u_1 is the first term, and r is the common ratio
  - This formula allows you to find **any term** in the geometric sequence
  - It is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common ratio
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two or more consecutive terms and asked to find both the first term and the common ratio
  - Find the common ratio by dividing a term by the one before it
  - Substitute this and one of the terms into the formula to find the first term
- Sometimes you may be given a term and the formula for the n^{th} term and asked to find the value of *n*
  - You can solve these using **logarithms** on your GDC

Exam Tip

- You will sometimes need to use logarithms to answer geometric sequences questions
  - Make sure you are confident doing this
  - Practice using your GDC for different types of questions

Worked Example

The sixth term, u_6, of a geometric sequence is 486 and the seventh term, u_7, is 1458.

Find,

i)
the common ratio, r, of the sequence,

$u_6 = 486$, $u_7 = 1458$

The common ratio, r, is given by

$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \dots = \frac{u_{n+1}}{u_n}$$

Sub in $u_6 = 486$, $u_7 = 1458$

$$r = \frac{u_7}{u_6} = \frac{1458}{486} = 3$$

$r = 3$

ii)
the first term of the sequence, u_1.

Formula for n^{th} term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in $r = 3$ and either $u_6 = 486$ or $u_7 = 1458$

$$u_6 = u_1(3)^{6-1} = 486$$

Solve: $243 u_1 = 486$

$u_1 = 2$

$u_1 = 2$

Geometric Series

How do I find the sum of a geometric series?

- A **geometric series** is the sum of a certain number of terms in a **geometric sequence**
  - For the geometric sequence 2, 6, 18, 54, ... the geometric series is 2 + 6 + 18 + 54 + ...
- The following formulae will let you find the sum of the first n terms of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

    - u_1 is the first term
    - r is the common ratio
  - Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - The first version of the formula is more convenient if $r > 1$ and the second is more convenient if $r < 1$
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term, the common ratio, or the number of terms within the sequence
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this

Exam Tip

- The geometric series formulae are in the formulae booklet, you don't need to memorise them
  - Make sure you can locate them quickly in the formula booklet

Worked Example

A geometric sequence has $u_1 = 25$ and $r = 0.8$. Find the value of u_5 and S_5.

$u_1 = 25, \quad r = 0.8$

Formula for n^{th} term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in $u_1 = 25, \quad r = 0.8$

$$u_5 = 25(0.8)^4 = 10.24$$

Formula for the sum of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

$r < 1$ so this version is easier to use.

Sub in $u_1 = 25, \quad r = 0.8$

$$S_5 = \frac{u_1(1 - r^5)}{1 - r} = \frac{25(1 - 0.8^5)}{1 - 0.8} = 84.04$$

$u_5 = 10.24$

$S_5 = 84.04$

Sum to Infinity

What is the sum to infinity of a geometric series?

- A geometric sequence will either increase or decrease away from zero or the terms will get progressively closer to zero
  - Terms will get closer to zero if the common ratio, *r*, is between 1 and -1
- If the terms are getting closer to zero then the series is said to **converge**
  - This means that the sum of the series will approach a limiting value
  - As the number of terms increase, the sum of the terms will get closer to the limiting value

How do we calculate the sum to infinity?

- If asked to find out if a geometric sequence converges find the value of *r*
  - If $|r| < 1$ then the sequence converges
  - If $|r| \geq 1$ then the sequence does not converge and the sum to infinity cannot be calculated
  - $|r| < 1$ means $-1 < r < 1$
- If $|r| < 1$, then the geometric series **converges** to a finite value given by the formula

$$S_\infty = \frac{u_1}{1-r}, \quad |r| < 1$$

  - u_1 is the first term
  - r is the common ratio
  - This is **in the formula book**, you do not need to remember it

Exam Tip

- Learn and remember the conditions for when a sum to infinity can be calculated

Worked Example

The first three terms of a geometric sequence are 6, 2, $\frac{2}{3}$. Explain why the series converges and find the sum to infinity.

$u_1 = 6, \quad u_2 = 2, \quad u_3 = \frac{2}{3}$

Find the value of r: $r = \frac{u_2}{u_1}$

$$r = \frac{u_2}{u_1} = \frac{2}{6} = \frac{1}{3}$$

$|r| < 1$ so the series converges

Find the sum to infinity: $S_\infty = \frac{u_1}{1-r}$

$$S_\infty = \frac{u_1}{1-r} = \frac{6}{1-\frac{1}{3}} = \frac{6}{2/3} = 9$$

$S_\infty = 9$

1.3.4 Applications of Sequences & Series

Applications of Arithmetic Sequences & Series

Many real-life situations can be modelled using sequences and series, including but not limited to: patterns made when tiling floors; seating people around a table; the rate of change of a population; the spread of a virus and many more.

What do I need to know about applications of arithmetic sequences and series?

- If a quantity is changing repeatedly by having a fixed amount **added to** or **subtracted from** it then the use of **arithmetic sequences** and **arithmetic series** is appropriate to **model** the situation
  - If a sequence seems to fit the pattern of an arithmetic sequence it can be said to be **modelled** by an arithmetic sequence
  - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of arithmetic sequences and series is **simple interest**
  - Simple interest is when an initial investment is made and then a percentage of the initial investment is added to this amount on a regular basis (usually per year)
- Arithmetic sequences can be used to make estimations about how something will change in the future

Exam Tip

- Exam questions won't always tell you to use sequences and series methods, practice spotting them by looking for clues in the question
- If a given amount is repeated periodically then it is likely the question is on arithmetic sequences or series

Worked Example

Jasper is saving for a new car. He puts USD $100 into his savings account and then each month he puts in USD $10 more than the month before. Jasper needs USD $1200 for the car. Assuming no interest is added, find,

i)
the amount Jasper has saved after four months,

Identify the arithmetic sequence:

$u_1 = 100$, $d = 10$

After 4 months Jasper will have saved:

$u_1 + u_2 + u_3 + u_4 = S_4$

Formula for the sum of an arithmetic series:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$S_4 = \frac{4}{2}(2u_1 + (4-1)d)$$

Sub in $u_1 = 100$ and $d = 10$

$$S_4 = \frac{4}{2}(2(100) + (4-1)(10))$$

$$= 2(200 + 30)$$

$$= 2(230)$$

$$S_4 = \$460$$

ii)
the month in which Jasper reaches his goal of USD $1200.

Sub $S_n = 1200$, $u_1 = 100$, $d = 10$ into formula:

$$1200 = \frac{n}{2}(2(100) + (n-1)(10))$$

Solve using algebraic solver on GDC:

$n = 8.67...$ or $n = -27.67...$

↑ disregard as n cannot be negative.

$\therefore$ $S_8 < 1200$

$S_9 > 1200$ reaches total in 9^{th} month

Jasper will reach USD $1200 in the 9^{th} month.

Applications of Geometric Sequences & Series

What do I need to know about applications of geometric sequences and series?

- If a quantity is changing repeatedly by a fixed **percentage,** or by being **multiplied** repeatedly by a fixed amount, then the use of **geometric sequences** and **geometric series** is appropriate to **model** the situation
  - If a sequence seems to fit the pattern of a geometric sequence it can be said to be **modelled** by a geometric sequence
  - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of geometric sequences and series is **compound interest**
  - Compound interest is when an initial investment is made and then interest is paid on the initial amount **and on the interest already earned** on a regular basis (usually every year)
- Geometric sequences can be used to make estimations about how something will change in the future
- The questions won't always tell you to use sequences and series methods, so be prepared to spot 'hidden' sequences and series questions
  - Look out for questions on savings accounts, salaries, sales commissions, profits, population growth and decay, spread of bacteria etc

Exam Tip

- Exam questions won't always tell you to use sequences and series methods, practice spotting them by looking for clues in the question
- If a given amount is changing by a percentage or multiple then it is likely the question is on geometric sequences or series

Worked Example

A new virus is circulating on a remote island. On day one there were 10 people infected, with the number of new infections increasing at a rate of 40% per day.

a)

Find the expected number of people newly infected on the 7th day.

Identify the geometric sequence:

$u_1 = 10$, $r = 1.4$ ← 40% increase so 140% of the day before

New infections : u_7

Formula for nth term of a geometric series :

$$u_n = u_1 r^{n-1}$$

Sub in $u_1 = 10$, $r = 1.4$

$$u_7 = 10(1.4)^6 = 75.29...$$

Expected number of new infections = 75

b)

Find the expected number of infected people after one week (7 days), assuming no one has recovered yet.

Total infections : S_7

Formula for the sum of a geometric series :

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$

← $r > 1$ so this version is easier to use.

Sub in $u_1 = 10$, $r = 1.4$

$$S_7 = \frac{10(1.4^7 - 1)}{1.4 - 1} = 238.53...$$

Expected number of total infections = 239

1.4 Financial Applications

1.4.1 Compound Interest & Depreciation

Compound Interest

What is compound interest?

- Interest is a small percentage paid by a bank or company that is added on to an initial investment
    - Interest can also refer to an amount paid on a loan or debt, however IB compound interest questions will always refer to interest on **investments**
- **Compound interest** is where interest is paid on **both the initial investment** and any interest that has **already been paid**
    - Make sure you know the difference between compound interest and simple interest
        - Simple interest pays interest only on the initial investment
- The interest paid each time will increase as it is a percentage of a higher number
- Compound interest will be paid in instalments in a given timeframe
    - The interest rate, *r*, will be per annum (per year)
        - This could be written *r* % p.a.
    - Look out for phrases such as **compounding annually** (interest paid yearly) **or compounding monthly** (interest paid monthly)
        - If $\alpha\%$ p.a. (per annum) is paid compounding monthly, then $\frac{\alpha}{12}\%$ will be paid each month
        - The formula for compound interest allows for this so you do not have to compensate separately

How is compound interest calculated?

- The formula for calculating compound interest is:

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

  - Where
    - FV is the future value
    - PV is the present value
    - n is the number of years
    - k is the number of compounding periods per year
    - r % is the nominal annual rate of interest
  - This formula is **given in the formula booklet**, you do not have to remember it
- Be careful with the k value
  - Compounding annually means $k = 1$
  - Compounding half-yearly means $k = 2$
  - Compounding quarterly means $k = 4$
  - Compounding monthly means $k = 12$
- Your GDC will have a finance solver app on it which you can use to find the future value
  - This may also be called the TVM (time value of money) solver
  - You will have to enter the information from the question into your calculator
- Be aware that many questions will be set up such that you will have to use the formula
  - So for compound interest questions it is better to use the formula from your formula booklet than your GDC

Exam Tip

- Your GDC will be able to solve some compound interest problems so it is a good idea to make sure you are confident using it, however you must also familiarise yourself with the formula and make sure you can find it in the formula booklet

Worked Example

Kim invests MYR 2000 (Malaysian Ringgit) in an account that pays a nominal annual interest rate of 2.5% **compounded monthly**. Calculate the amount that Kim will have in her account after 5 years.

Compound interest formula:

$$FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$$

- FV – future value
- PV – present value
- r – interest rate
- k – compounding periods
- n – number of years

Substitute values in:

$PV = 2000$ (initial investment)

$k = 12$ (compounding monthly)

$r = 2.5\%$

$n = 5$ (number of years)

$$FV = 2000\left(1 + \frac{2.5}{(100)(12)}\right)^{(12 \times 5)}$$

$$= 2266.002\ldots$$

$FV \approx$ MYR 2270 (3sf)

Depreciation

What is depreciation?

- Depreciation is when the **value** of something **falls** over time
- The most common examples of depreciation are the value of cars and technology
- If the depreciation is occurring at a **constant rate** then it is **compound depreciation**

How is compound depreciation calculated?

- The formula for calculating compound depreciation is:

$$FV = PV \times \left(1 - \frac{r}{100}\right)^n$$

  - Where
    - *FV* is the future value
    - *PV* is the present value
    - *n* is the number of years
    - *r*% is the rate of depreciation
  - This formula is **not** given in the formula booklet, however it is almost the same as the formula for compound interest but
    - with a **subtraction** instead of an addition
    - the value of *k* will always be 1
- Your GDC **could** again be used to solve some compound depreciation questions, but watch out for those which are set up such that you will have to use the formula

Exam Tip

- You can use your GDC's "Finance Solver" (TI) or "Compound Interest" (Casio) feature to solve most depreciation questions, by entering the interest rate as a negative value

Worked Example

Kyle buys a new car for AUD $14 999. The value of the car depreciates by 15% each year.

a)

Find the value of the car after 5 years.

Depreciation formula:

$$FV = PV\left(1 - \frac{r}{100}\right)^n$$

FV ← future value; PV ← present value; r ← rate of depreciation; n ← number of years

Substitute values in:

$PV = 14999$ (initial cost)

$r = 15\%$

$n = 5$ (number of years)

$$FV = 14999\left(1 - \frac{15}{100}\right)^5$$

$$= 6655.13\ldots$$

$FV \approx$ AUD $6660 (3sf)

b)

Find the number of years and months it will take for the value of the car to be approximately AUD $9999.

$$FV = PV\left(1 - \frac{r}{100}\right)^n$$

$FV \approx 9999$

$PV = 14999$

$r = 15\%$

Substitute values in:

$$9999 \approx 14999\left(1 - \frac{15}{100}\right)^n$$

Use GDC to solve:

$n = 2.495...$

2 years — 0.495th of a year

Convert to years and months:

2 years + 0.495... × 12 months

≈ 2 years and 6 months

1.4.2 Amortisation & Annuities

Amortisation

What is amortisation?

- Amortisation is the process of repaying a loan over a fixed period of time
  - Most commonly questions will be about mortgages (loans taken out to buy a home) or loans taken out for a large purchase
- Interest will be paid on the original amount
  - Each repayment that is made will partly repay the original loan and partly pay the interest on the loan
  - As payments are made the amount owed will decrease and so the interest paid will decrease
    - As you continue to repay a loan more of the repayment goes on the loan and less on the interest

How can the GDC be used to make calculations involving loans?

- Your GDC should be used to solve questions involving loans
  - Use the **finance solver mode** (sometimes called the TVM (time value of money) solver)
    - *N* will be the number of **repayment periods** (remember to include months and years if necessary)
    - *I* (%) is the interest rate
    - *PV* is the amount that was borrowed at the start – as this has been received it will be entered as a **positive** number
    - *PMT* is the payments made per period – this is repaying the loan so will be a **negative** number
    - *FV* is the future value (this will be zero as the loan will be paid off at the end of the period)

        - *P/Y* is the number of payments per year, usually 12 as payments are made monthly
        - *C/Y* is the **compounding periods** per year
        - *PMT@* is the time of the year or month the payment is made (assume this is the end unless told otherwise)
    - Leave the section that you need to find out blank and fill in all other sections
    - Your GDC will fill in the last part for you
- It is sensible to check your final answer, you can do this by finding the total amount paid back overall and comparing it to the original loan
    - The total amount repaid will be **a little more** than the original loan plus *I* % of the original loan

Exam Tip

- Be sure to write down the values that you put into the financial solver on your GDC, don't just write down the final answer as if it is incorrect you won't get any marks if there is no working shown!
- Make sure that you are clear on what the signage of any monetary value is, if it's positive then money is coming in to you, if it's negative then you are paying money out

Worked Example

Olivia takes a mortgage of EUR €280 000 to purchase a house at a nominal annual interest rate of 3.2%, **compounded monthly**. She agrees to pay the bank EUR €1500 at the end of every month to amortise the loan. Find

i)

the number of years and months it will take Olivia to pay back the loan,

Use the finance / TVM solver on your GDC:

N	I%	PV	PMT	FV	P/Y	C/Y	PMT@
	3.2	280 000	−1500	0	12	12	END

N: GDC will fill this in for you.
PMT: negative because paying this back each month
P/Y: paid monthly
C/Y: compounding monthly
PMT@: paid at the end of each month

$N = 258.61$

Convert to years:

$$\text{Number of years} = \frac{258.61}{12} = 21.55$$

21 years and 7 months

ii)

the total amount Olivia will pay to purchase the house.

$$\text{Total amount paid} = N \times PMT$$

$$\text{Total amount paid} = 258.61 \times 1500$$

Total amount paid = €387 915

Annuities

What is an annuity?

- An annuity is a fixed sum of money paid to someone at specified intervals over a fixed period of time
  - Most commonly this will be because of an initial lump sum investment which will be returned at fixed intervals of time with a fixed interest rate
  - Either from personal savings or from receiving an inheritance

How are annuities calculated?

- Your GDC should be used to solve questions involving annuities
  - Use the **finance solver mode** (sometimes called the TVM (time value of money) solver)
    - *N* will be the number of **payment periods** (remember to include months and years if necessary)
    - *I* (%) is the interest rate
    - *PV* is the amount that was invested – as this has been invested it will be entered as a **negative number**
    - *PMT* is the amount paid per period – as this is being received it will be a **positive number**
    - *FV* is the future value (for an annuity this will be **zero** as the balance at the end of the payment period will be zero)
    - *P/Y* is the number of payments per year
    - *C/Y* is the **compounding periods** per year
    - *PMT@* is the time of the year or month the payment is made (usually the start)
  - Leave the section that you need to find out blank and fill in all other sections
  - Your GDC will fill in the last part for you
- Although you are unlikely to need to use it, the formula for calculating an annuity is:

$$FV = A\frac{(1+r)^n - 1}{r}$$

  - Where
    - *FV* is the future value
    - *A* is the amount invested
    - *n* is the number of years
    - *r%* is the interest rate as a decimal (e.g. at 6%, $r = 0.06$)
  - This formula is **not** given in the formula booklet, however your GDC will work out annuities for you so you do not need to remember it

Exam Tip

- Be sure to write down the values that you put into the financial solver on your GDC, don't just write down the final answer as if it is incorrect you won't get any marks if there is no working shown!
- Try to remember the difference between amortization and annuities:
  - with **amortization** you are **paying** money out
  - with **annuities** you are **receiving** money

Worked Example

Janni invests 2 million DKK (Danish krone) into an annuity for her retirement. The annuity returns 3% compounded annually. Find the monthly payments Janni will receive if she wants the annuity to last for 25 years.

Use the finance / TVM solver on your GDC:

N	I%	PV	PMT	FV	P/Y	C/Y	PMT@
300	3	-2 000 000		0	12	1	START

- N: 25 years x 12
- PV: negative because this was invested
- PMT: GDC will fill this in for you.
- P/Y: paid monthly
- C/Y: compounding annualy
- PMT@: paid at the start of each month

PMT = 9418.95

Janni receives DKK 9419 each month

1.5 Complex Numbers

1.5.1 Intro to Complex Numbers

Cartesian Form

What is an imaginary number?

- Up until now, when we have encountered an equation such as $x^2 = -1$ we would have stated that there are "no real solutions"
  - The solutions are $x = \pm\sqrt{-1}$ which are not real numbers
- To solve this issue, mathematicians have defined one of the square roots of negative one as i; an imaginary number
  - $\sqrt{-1} = \mathrm{i}$
  - $\mathrm{i}^2 = -1$
- The square roots of other negative numbers can be found by rewriting them as a multiple of $\sqrt{-1}$
  - using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

What is a complex number?

- Complex numbers have both a real part and an imaginary part
  - For example: $3 + 4\mathrm{i}$
  - The real part is 3 and the imaginary part is 4i
- Complex numbers are often denoted by z
  - We refer to the real and imaginary parts respectively using $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$
- Two complex numbers are equal if, and only if, both the real and imaginary parts are identical.
  - For example, $3 + 2\mathrm{i}$ and $3 + 3\mathrm{i}$ are **not equal**
- The set of all complex numbers is given the symbol $\mathbb{C}$

What is Cartesian Form?

- There are a number of different forms that complex numbers can be written in
- The form $z = a + bi$ is known as **Cartesian Form**
  - $a, b \in \mathbb{R}$
  - This is the first form given in the formula booklet
- In general, for $z = a + bi$
  - $\text{Re}(z) = a$
  - $\text{Im}(z) = b$
- A complex number can be easily represented geometrically when it is in Cartesian Form
- Your GDC may call this **rectangular form**
  - When your GDC is set in rectangular settings it will give answers in Cartesian Form
  - If your GDC is **not** set in a complex mode it will not give any output in complex number form
  - Make sure you can find the settings for using complex numbers in Cartesian Form and practice inputting problems
- Cartesian form is the easiest form for adding and subtracting complex numbers

Exam Tip

- Remember that complex numbers have both a real part and an imaginary part
  - 1 is purely real (its imaginary part is zero)
  - i is purely imaginary (its real part is zero)
  - $1 + i$ is a complex number (both the real and imaginary parts are equal to 1)

Worked Example

a)

Solve the equation $x^2 = -9$

$x^2 = -9$

$x = \pm\sqrt{-9}$

Using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ $\quad x = \pm\sqrt{9}\sqrt{-1}$

$x = \pm 3i$

b)

Solve the equation $(x+7)^2 = -16$, giving your answers in Cartesian form.

$(x+7)^2 = -16$

$x + 7 = \pm\sqrt{-16}$

$x + 7 = \pm\sqrt{16}\sqrt{-1}$ ← Using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

$x + 7 = \pm 4i$

Rearrange answer into Cartesian form:

$x = -7 \pm 4i$

Complex Addition, Subtraction & Multiplication

How do I add and subtract complex numbers in Cartesian Form?

- Adding and subtracting complex numbers should be done when they are in **Cartesian form**
- When adding and subtracting complex numbers, simplify the real and imaginary parts separately
  - Just like you would when collecting like terms in algebra and surds, or dealing with different components in vectors
  - $(a+bi)+(c+di)=(a+c)+(b+d)i$
  - $(a+bi)-(c+di)=(a-c)+(b-d)i$

How do I multiply complex numbers in Cartesian Form?

- Complex numbers can be multiplied by a constant in the same way as algebraic expressions:
  - $k(a+bi)=ka+kbi$
- Multiplying two complex numbers in Cartesian form is done in the same way as multiplying two linear expressions:
  - $(a+bi)(c+di)=ac+(ad+bc)i+bdi^2= ac+(ad+bc)i-bd$
  - This is a complex number with real part $ac-bd$ and imaginary part $ad+bc$
  - The most important thing when multiplying complex numbers is that
    - $i^2=-1$
- Your GDC will be able to multiply complex numbers in Cartesian form
  - Practise doing this and use it to check your answers
- It is easy to see that multiplying more than two complex numbers together in Cartesian form becomes a lengthy process prone to errors
  - It is easier to multiply complex numbers when they are in different forms and usually it makes sense to convert them from Cartesian form to either Polar form or Euler's form first
- Sometimes when a question describes multiple complex numbers, the notation z_1, z_2, ... is used to represent each complex number

How do I deal with higher powers of i?

- Because $i^2=-1$ this can lead to some interesting results for higher powers of i
  - $i^3=i^2\times i= -i$
  - $i^4=(i^2)^2=(-1)^2=1$
  - $i^5=(i^2)^2\times i=i$
  - $i^6=(i^2)^3=(-1)^3= -1$
- We can use this same approach of using i^2 to deal with much higher powers
  - $i^{23}=(i^2)^{11}\times i=(-1)^{11}\times i= -i$
  - Just remember that -1 raised to an even power is 1 and raised to an odd power is -1

Exam Tip

- When revising for your exams, practice using your GDC to check any calculations you do with complex numbers by hand
  - This will speed up using your GDC in rectangular form whilst also giving you lots of practice of carrying out calculations by hand

Worked Example

a)

Simplify the expression $2(8-6\text{i})-5(3+4\text{i})$.

Expand the brackets

$2(8-6i)-5(3+4i) = 16-12i-15-20i$

Collect the real and imaginary parts

$16-15-12i-20i$

Simplify

$1-32i$

b)

Given two complex numbers $z_1=3+4\text{i}$ and $z_2=6+7\text{i}$, find $z_1 \times z_2$.

Expand the brackets

$(3+4i)(6+7i) = 18+21i+24i+28i^2$

$= 18+21i+24i+(28)(-1)$ ← Using $i^2=-1$

Collect the real and imaginary parts

$18+21i+24i-28 = 18-28+(21+24)i$

Simplify

$-10+45i$

Complex Conjugation & Division

When **dividing** complex numbers, the **complex conjugate** is used to change the denominator to a real number.

What is a complex conjugate?

- For a given complex number $z = a + b\mathrm{i}$, **the complex conjugate of z is denoted as z^***, where $z^* = a - b\mathrm{i}$
- If $z = a - b\mathrm{i}$ then $z^* = a + b\mathrm{i}$
- You will find that:
  - $z + z^*$ is always real because $(a + b\mathrm{i}) + (a - b\mathrm{i}) = 2a$
    - For example: $(6 + 5\mathrm{i}) + (6 - 5\mathrm{i}) = 6 + 6 + 5\mathrm{i} - 5\mathrm{i} = 12$
  - $z - z^*$ is always imaginary because $(a + b\mathrm{i}) - (a - b\mathrm{i}) = 2b\mathrm{i}$
    - For example: $(6 + 5\mathrm{i}) - (6 - 5\mathrm{i}) = 6 - 6 + 5\mathrm{i} - (-5\mathrm{i}) = 10\mathrm{i}$
  - $z \times z^*$ is always real because $(a + b\mathrm{i})(a - b\mathrm{i}) = a^2 + ab\mathrm{i} - ab\mathrm{i} - b^2\mathrm{i}^2 = a^2 + b^2$ (as $\mathrm{i}^2 = -1$)
    - For example: $(6 + 5\mathrm{i})(6 - 5\mathrm{i}) = 36 + 30\mathrm{i} - 30\mathrm{i} - 25\mathrm{i}^2 = 36 - 25(-1) = 61$

How do I divide complex numbers?

- To divide two complex numbers:
  - STEP 1: Express the calculation in the form of a fraction
  - STEP 2: Multiply **the top and bottom by the conjugate of the denominator**:
    - $\dfrac{a + b\mathrm{i}}{c + d\mathrm{i}} = \dfrac{a + b\mathrm{i}}{c + d\mathrm{i}} \times \dfrac{c - d\mathrm{i}}{c - d\mathrm{i}}$
    - This ensures we are multiplying by 1; so not affecting the overall value
  - STEP 3: Multiply out and simplify your answer
    - This should have a real number as the denominator
  - STEP 4: Write your answer in Cartesian form as two terms, simplifying each term if needed
    - OR convert into the required form if needed
- Your GDC will be able to divide two complex numbers in Cartesian form
  - Practise doing this and use it to check your answers if you can

Exam Tip

- We can speed up the process for finding zz^* by using the basic pattern of $(x + a)(x - a) = x^2 - a^2$
- We can apply this to complex numbers: $(a + b\mathrm{i})(a - b\mathrm{i}) = a^2 - b^2\mathrm{i}^2 = a^2 + b^2$ (using the fact that $\mathrm{i}^2 = -1$)
  - So $3 + 4\mathrm{i}$ multiplied by its conjugate would be $3^2 + 4^2 = 25$

Worked Example

Find the value of $(1+7\text{i}) \div (3-\text{i})$.

Rewrite as a fraction: $\frac{1+7i}{3-i}$ complex conjugate of $3-i$ is $3+i$

Multiply top and bottom of the fraction by the complex conjugate of the denominator.

$$\frac{1+7i}{3-i} \times \frac{3+i}{3+i} = \frac{(1+7i)(3+i)}{(3-i)(3+i)}$$

$$= \frac{3+i+21i+7i^2}{9+3i-3i-i^2}$$

$i^2 = -1$

the imaginary parts eliminate each other

$$= \frac{3+22i+(-7)}{9-(-1)}$$

Simplify $= \frac{-4+22i}{10}$

Write in Cartesian form $= -\frac{4}{10} + \frac{22}{10}i$

$-\frac{2}{5} + \frac{11}{5}i$ Simplify final answer.

1.5.2 Modulus & Argument

Modulus & Argument

How do I find the modulus of a complex number?

- The modulus of a complex number is its **distance** from the origin when plotted on an Argand diagram
- The modulus of z is written $|z|$
- If $z = x + \mathrm{i}y$, then we can use **Pythagoras** to show...
  - $|z| = \sqrt{x^2 + y^2}$
- A modulus is **never negative**

What features should I know about the modulus of a complex number?

- the modulus is related to the complex **conjugate** by...
  - $zz^* = z^*z = |z|^2$
  - This is because $zz^* = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2$
- In general, $|z_1 + z_2| \neq |z_1| + |z_2|$
  - e.g. both $z_1 = 3 + 4\mathrm{i}$ and $z_2 = -3 + 4\mathrm{i}$ have a modulus of 5, but $z_1 + z_2$ simplifies to 8**i** which has a modulus of 8

How do I find the argument of a complex number?

- The argument of a complex number is the **angle** that it makes on an **Argand diagram**
  - The angle must be taken from the **positive real axis**
  - The angle must be in a **counter-clockwise** direction
- Arguments are measured in **radians**
  - They can be given exact in terms of π
- The argument of z is written $\arg z$
- Arguments can be calculated using right-angled **trigonometry**
  - This involves using the tan ratio plus a sketch to decide whether it is positive/negative and acute/obtuse

What features should I know about the argument of a complex number?

- Arguments are usually given in the range $-\pi < \arg z \leq \pi$
  - Negative arguments are for complex numbers in the third and fourth quadrants
  - Occasionally you could be asked to give arguments in the range $0 < \arg z \leq 2\pi$
    - The question will make it clear which range to use
- The argument of zero, $\arg 0$ is undefined (no angle can be drawn)

What are the rules for moduli and arguments under multiplication and division?

- When two complex numbers, z_1 and z_2, are **multiplied** to give $z_1 z_2$, their **moduli** are also **multiplied**
  - $|z_1 z_2| = |z_1||z_2|$
- When two complex numbers, z_1 and z_2, are **divided** to give $\frac{z_1}{z_2}$, their **moduli** are also **divided**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- When two complex numbers, z_1 and z_2, are **multiplied** to give $z_1 z_2$, their **arguments** are **added**
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- When two complex numbers, z_1 and z_2, are **divided** to give $\frac{z_1}{z_2}$, their **arguments** are **subtracted**

Exam Tip

- Always draw a quick sketch to help you see what quadrant the complex number lies in when working out an argument
- Look for the range of values within which you should give your argument
  - If it is $-\pi < \arg z \leq \pi$ then you may need to measure it in the negative direction
  - If it is $0 < \arg z \leq 2\pi$ then you will always measure in the positive direction (counter - clockwise)

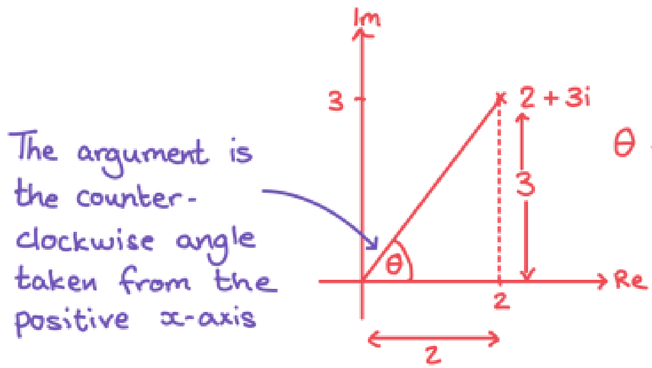
Worked Example

a)

Find the modulus and argument of $z = 2 + 3\text{i}$

$|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$

Draw a sketch to help find the argument:



$\theta = \tan^{-1}\left(\frac{3}{2}\right)$

$= 0.9827\ldots$

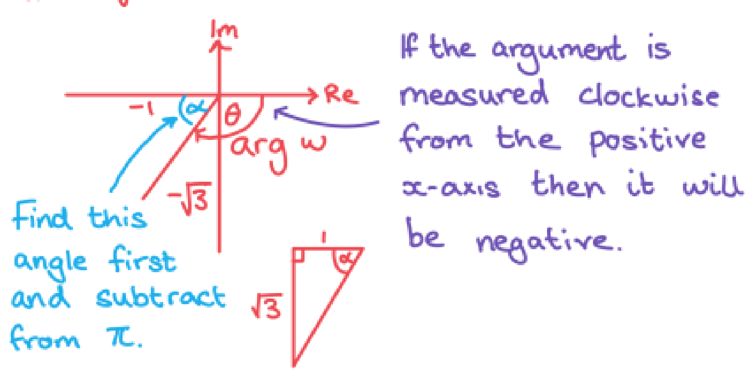
$\text{Mod } z = |z| = \sqrt{13}$

$\arg z = \theta = 0.983 \text{ (3sf)}$

b)

Find the modulus and argument of $w = -1 - \sqrt{3}\text{i}$

$|w| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{4}$



$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\text{Mod } z = |z| = 2$

$\arg z = -\theta = -\frac{2\pi}{3}$

1.5.3 Introduction to Argand Diagrams

Argand Diagrams

What is the complex plane?

- The complex plane, sometimes also known as the Argand plane, is a two-dimensional plane on which complex numbers can be represented geometrically
- It is similar to a two-dimensional Cartesian coordinate grid
  - The x-axis is known as the **real** axis (Re)
  - The y-axis is known as the **imaginary** axis (Im)
- The complex plane emphasises the fact that a complex number is two dimensional
  - i.e it has two parts, a real and imaginary part
  - Whereas a real number only has one dimension represented on a number line (the x-axis only)

What is an Argand diagram?

- An Argand diagram is a geometrical representation of complex numbers on a **complex plane**
  - A complex number can be represented as either a point or a vector
- The complex number $x + yi$ is represented by the point with cartesian coordinate (x, y)
  - The **real** part is represented by the point on the **real** (x-) axis
  - The **imaginary** part is represented by the point on the **imaginary** (y-) axis
- Complex numbers are often represented as **vectors**
  - A line segment is drawn from the origin to the cartesian coordinate point
  - An arrow is added in the direction away from the origin
  - This allows for geometrical representations of complex numbers

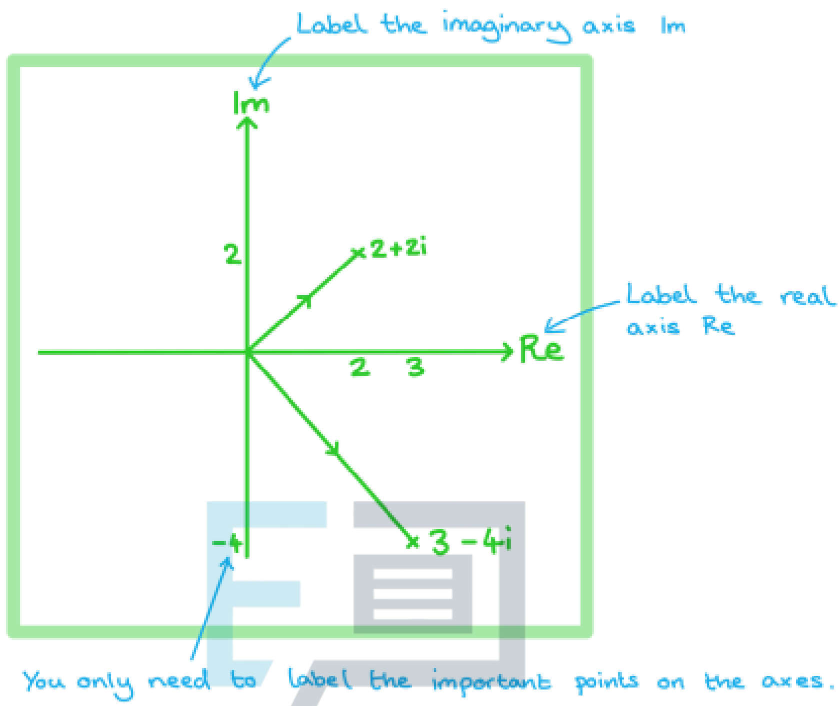
Exam Tip

- When setting up an Argand diagram you do not need to draw a fully scaled axes, you only need the essential information for the points you want to show, this will save a lot of time

Worked Example

a)

Plot the complex numbers $z_1 = 2 + 2i$ and $z_2 = 3 - 4i$ as points on an Argand diagram.



b)

Write down the complex numbers represented by the points A and B on the Argand diagram below.

A: $1 + 3i$
B: $-2 - i$

Complex Roots of Quadratics

What are complex roots?

- A quadratic equation can either have two real roots (zeros), a repeated real root or no real roots
  - This depends on the location of the graph of the quadratic with respect to the x-axis
- If a quadratic equation has no real roots we would previously have stated that it has **no real solutions**
  - The quadratic equation will have a **negative discriminant**
  - This means taking the square root of a negative number
- Complex numbers provide solutions for quadratic equations that have **no real roots**

How do we solve a quadratic equation when it has complex roots?

- If a quadratic equation takes the form $ax^2 + bx + c = 0$ it can be solved by either using the quadratic formula or completing the square
- If a quadratic equation takes the form $ax^2 + b = 0$ it can be solved by rearranging
- The property $i = \sqrt{-1}$ is used
  - $\sqrt{-a} = \sqrt{a \times -1} = \sqrt{a} \times \sqrt{-1}$
- If the coefficients of the quadratic are real then the complex roots will occur in complex conjugate pairs
  - If $\boldsymbol{z = m + ni\ (n \neq 0)}$ is a root of a quadratic with real coefficients then $\boldsymbol{z^* = m - ni}$ is also a root
- The **real part** of the solutions will have the same value as the x coordinate of the turning point on the graph of the quadratic

- When the coefficients of the quadratic equation are **non-real**, the solutions will **not** be complex conjugates
  - To solve these you can use the quadratic formula

How do we factorise a quadratic equation if it has complex roots?

- If we are given a quadratic equation in the form $az^2 + bz + c = 0$, where a, b, and $c \in \mathbb{R}$, $a \neq 0$ we can use its complex roots to write it in **factorised form**
  - Use the quadratic formula to find the two roots, $z = p + qi$ and $z^* = p - qi$
  - This means that $z - (p + qi)$ and $z - (p - qi)$ must both be factors of the quadratic equation
  - Therefore we can write $az^2 + bz + c = (z - (p + qi))(z - (p - qi))$
  - This can be rearranged into the form $(z - p - qi)(z - p + qi)$

Exam Tip

- Once you have your final answers you can check your roots are correct by substituting your solutions back into the original equation
  - You should get 0 if correct! [Note: 0 is equivalent to $0 + 0i$]

Worked Example

Solve the quadratic equation $z^2 - 2z + 5 = 0$ and hence, factorise $z^2 - 2z + 5$.

Use the quadratic formula or completing the square to find the solutions.

Solutions of a quadratic equation	$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ a \neq 0$

$a = 1$
$b = -2$
$c = 5$

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm \sqrt{16}\sqrt{-1}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$z_1 = 1 + 2i \qquad z_2 = 1 - 2i$$

If the solutions are $z_1 = 1 + 2i$ and $z_2 = 1 - 2i$ then the factors must be $z - (1+2i)$ and $z - (1-2i)$

$$z^2 - 2z + 5 = (z - (1+2i))(z - (1-2i))$$

$$(z - 1 - 2i)(z - 1 + 2i)$$

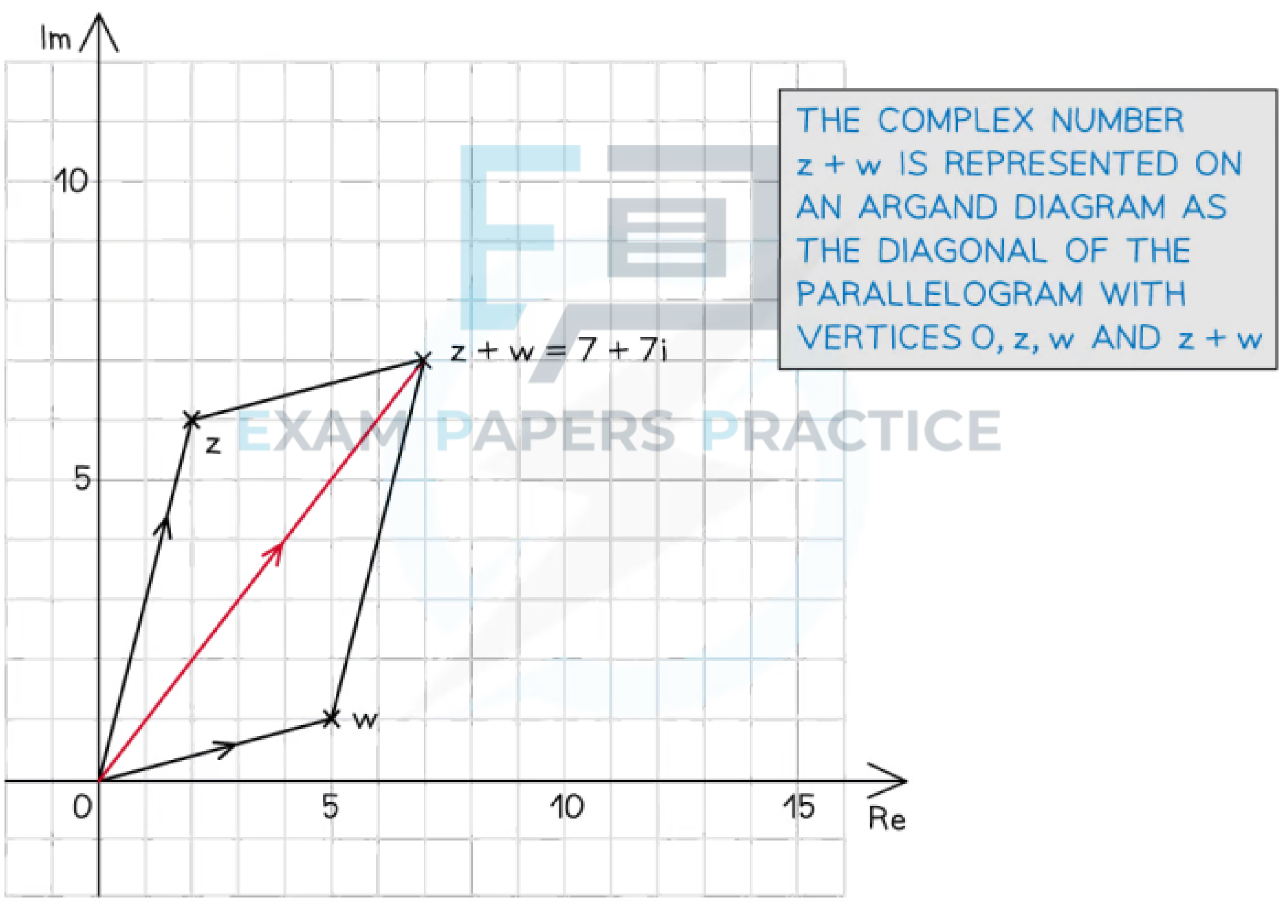
1.6 Further Complex Numbers

1.6.1 Geometry of Complex Numbers

Geometry of Complex Addition & Subtraction

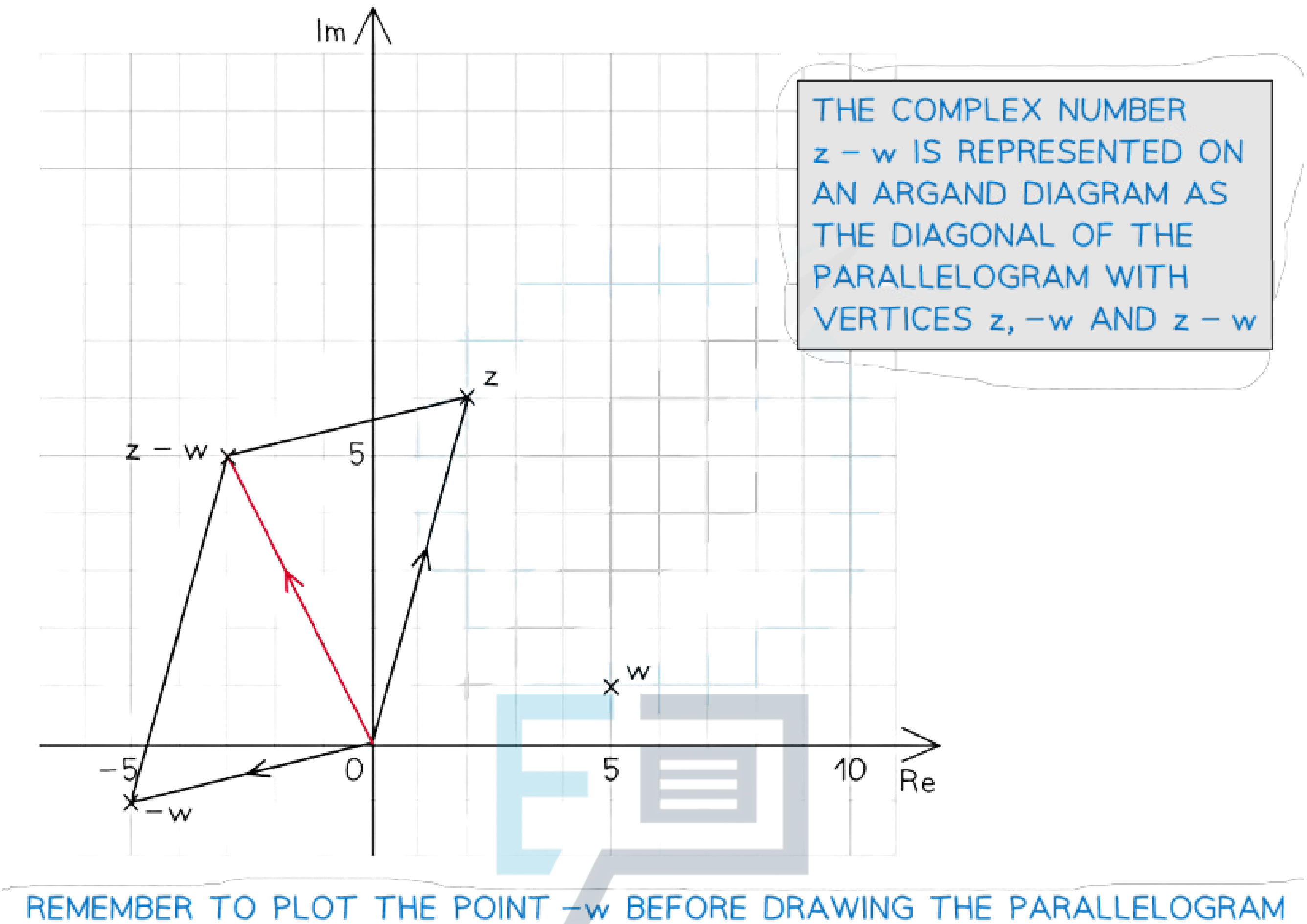
What does addition look like on an Argand diagram?

- In Cartesian form two complex numbers are added by adding the real and imaginary parts
- When plotted on an Argand diagram the complex number $z_1 + z_2$ is the longer diagonal of the parallelogram with vertices at the origin, z_1, z_2 and $z_1 + z_2$



What does subtraction look like on an Argand diagram?

- In Cartesian form the difference of two complex numbers is found by subtracting the real and imaginary parts
- When plotted on an Argand diagram the complex number $z_1 - z_2$ is the shorter diagonal of the parallelogram with vertices at the origin, z_1, $-z_2$ and $z_1 - z_2$



What are the geometrical representations of complex addition and subtraction?

- Let *w* be a given complex number with **real part *a*** and **imaginary part *b***
  - $w = a + bi$
- Let *z* be any complex number represented on an Argand diagram
- **Adding *w* to *z*** results in *z* being:
  - **Translated** by vector $\begin{pmatrix} a \\ b \end{pmatrix}$
- **Subtracting *w* from *z*** results in *z* being:
  - **Translated** by vector $\begin{pmatrix} -a \\ -b \end{pmatrix}$

Exam Tip

- Take extra care when representing a subtraction of a complex number geometrically
  - Remember that your answer will be a translation of the shorter diagonal of the parallelogram made up by the two complex numbers

Worked Example

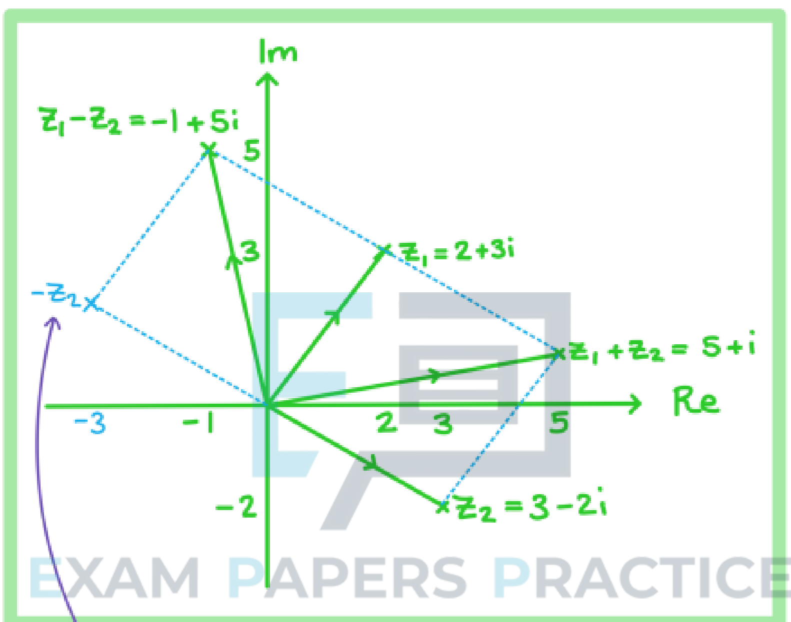
Consider the complex numbers $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$.

On an Argand diagram represent the complex numbers $z_1, z_2, z_1 + z_2$ and $z_1 - z_2$.

First find $z_1 + z_2$ and $z_1 - z_2$:

$z_1 + z_2 = (2 + 3i) + (3 - 2i) = 5 + i$

$z_1 - z_2 = (2 + 3i) - (3 - 2i) = -1 + 5i$



The geometrical properties can be seen by adding in $-z_2 = -3 + 2i$

Geometry of Complex Multiplication & Division

What do multiplication and division look like on an Argand diagram?

- The geometrical effect of multiplying a complex number by a real number, a, will be an enlargement of the vector by scale factor a
  - For positive values of a the direction of the vector will not change but the distance of the point from the origin will increase by scale factor a
  - For negative values of a the direction of the vector will change and the distance of the point from the origin will increase by scale factor a
- The geometrical effect of dividing a complex number by a real number, a, will be an enlargement of the vector by scale factor $1/a$
  - For positive values of a the direction of the vector will not change but the distance of the point from the origin will increase by scale factor $1/a$
  - For negative values of a the direction of the vector will change and the distance of the point from the origin will increase by scale factor $1/a$
- The geometrical effect of multiplying a complex number by i will be a rotation of the vector 90° counter-clockwise
  - $\mathrm{i}(x + y\mathrm{i}) = -y + x\mathrm{i}$
- The geometrical effect of multiplying a complex number by an imaginary number, ai, will be a rotation 90° counter-clockwise and an enlargement by scale factor a
  - $a\mathrm{i}(x + y\mathrm{i}) = -ay + ax\mathrm{i}$
- The geometrical effect of multiplying or dividing a complex number by a complex number will be an enlargement and a rotation
  - The direction of the vector will change
    - The angle of rotation is the **argument**
  - The distance of the point from the origin will change
    - The scale factor is the **modulus**

What does complex conjugation look like on an Argand diagram?

- The geometrical effect of plotting a **complex conjugate** on an Argand diagram is a reflection in the real axis
  - The **real** part of the complex number will stay the same and the **imaginary** part will change sign

Exam Tip

- Make sure you remember the transformations that different operations have on complex numbers, this could help you check your calculations in an exam

EXAM PAPERS PRACTICE

Worked Example

Consider the complex number $z = 2 - i$.

On an Argand diagram represent the complex numbers z, 3z, iz, z* and zz*.

First find 3z, iz and z*

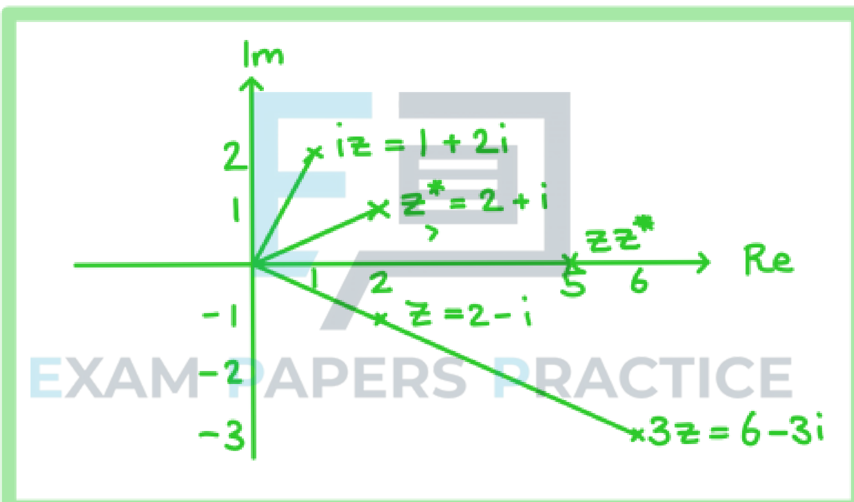
$z = 2 - i$

$3z = 3(2-i) = 6 - 3i$

$iz = i(2-i) = 2i - i^2 = 2i - (-1) = 1 + 2i$

$z^* = 2 + i$

$zz^* = (2-i)(2+i) = 4 - i^2 = 4 - (-1) = 5$



1.6.2 Forms of Complex Numbers

Modulus-Argument (Polar) Form

How do I write a complex number in modulus-argument (polar) form?

- The **Cartesian form** of a complex number, $z = x + \mathrm{i}y$, is written in terms of its real part, x, and its imaginary part, y
- If we let $r = |z|$ and $\theta = \arg z$, then it is possible to write a complex number in terms of its modulus, r, and its argument, θ, called the **modulus-argument (polar) form**, given by...
  - $z = r(\cos\theta + \mathrm{i}\sin\theta)$
  - This is often written as $z = r\,\mathrm{cis}\,\theta$
  - This is given in the formula book under Modulus-argument (polar) form and exponential (Euler) form
- It is usual to give arguments in the range $-\pi < \theta \le \pi$ or $0 \le \theta < 2\pi$
  - Negative arguments should be shown clearly
  - e.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right) = 2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$
    - without simplifying $\cos\left(-\frac{\pi}{3}\right)$ to either $\cos\left(\frac{\pi}{3}\right)$ or $\frac{1}{2}$
- The **complex conjugate** of $r\,\mathrm{cis}\,\theta$ is $r\,\mathrm{cis}\,(-\theta)$
- If a complex number is given in the form $z = r(\cos\theta - \mathrm{i}\sin\theta)$, then it is not in modulus-argument (polar) form due to the minus sign
  - It can be converted by considering transformations of trigonometric functions
    - $-\sin\theta = \sin(-\theta)$ and $\cos\theta = \cos(-\theta)$
  - So $z = r(\cos\theta - \mathrm{i}\sin\theta) = z = r(\cos(-\theta) + \mathrm{i}\sin(-\theta)) = r\,\mathrm{cis}\,(-\theta)$
- To convert from modulus-argument (polar) form back to Cartesian form, evaluate the real and imaginary parts
  - E.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ becomes $z = 2\left(\frac{1}{2} + \mathrm{i}\left(-\frac{\sqrt{3}}{2}\right)\right) = 1 - \sqrt{3}\,\mathrm{i}$

How do I multiply complex numbers in modulus-argument (polar) form?

- The main benefit of writing complex numbers in modulus-argument (polar) form is that they multiply and divide very easily
- To **multiply** two complex numbers in modulus-argument (polar) form we **multiply their moduli** and **add their arguments**
  - $|z_1 z_2| = |z_1||z_2|$
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- So if $z_1 = r_1\,\mathrm{cis}\,(\theta_1)$ and $z_2 = r_2\,\mathrm{cis}\,(\theta_2)$
  - $z_1 z_2 = r_1 r_2\,\mathrm{cis}\,(\theta_1 + \theta_2)$
- Sometimes the new argument, $\theta_1 + \theta_2$, does not lie in the range $-\pi < \theta \le \pi$ (or $0 \le \theta < 2\pi$ if this is being used)
  - An out-of-range argument can be adjusted by either **adding or subtracting** 2π
  - E.g. If $\theta_1 = \frac{2\pi}{3}$ and $\theta_2 = \frac{\pi}{2}$ then $\theta_1 + \theta_2 = \frac{7\pi}{6}$

- This is currently not in the range $-\pi < \theta \leq \pi$
- Subtracting 2π from $\frac{7\pi}{6}$ to give $-\frac{5\pi}{6}$, a new argument is formed
  - This lies in the correct range and represents the same angle on an Argand diagram
- The rules of **multiplying the moduli** and **adding the arguments** can also be applied when…
  - …multiplying three complex numbers together, $z_1 z_2 z_3$, or more
  - …finding powers of a complex number (e.g. z^2 can be written as zz)
- The rules for multiplication can be proved algebraically by multiplying $z_1 = r_1 \text{cis}(\theta_1)$ by $z_2 = r_2 \text{cis}(\theta_2)$, expanding the brackets and using compound angle formulae

How do I divide complex numbers in modulus-argument (polar) form?

- To **divide** two complex numbers in modulus-argument (polar) form, we **divide their moduli** and **subtract their arguments**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
  - $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
- So if $z_1 = r_1 \text{cis}(\theta_1)$ and $z_2 = r_2 \text{cis}(\theta_2)$ then
  - $\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$

- Sometimes the new argument, $\theta_1 - \theta_2$, can lie out of the range $-\pi < \theta \leq \pi$ (or the range $0 < \theta \leq 2\pi$ if this is being used)
  - You can **add or subtract** 2π to bring out-of-range arguments back in range
- The rules for division can be proved algebraically by dividing $z_1 = r_1 \text{cis}(\theta_1)$ by $z_2 = r_2 \text{cis}(\theta_2)$ using **complex division** and the compound angle formulae

Exam Tip

- Remember that $r \text{cis}\, \theta$ only refers to $r(\cos\theta + \text{i}\sin\theta)$
  - If you see a complex number written in the form $z = r(\cos\theta - \text{i}\sin\theta)$ then you will need to convert it to the correct form first
  - Make sure you are confident with basic trig identities to help you do this

Worked Example

Let $z_1 = 4\sqrt{2}\ \text{cis}\ \frac{3\pi}{4}$ and $z_2 = \sqrt{8}\left(\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\right)$

a)

Find $z_1 z_2$, giving your answer in the form $r(\cos\theta + i\sin\theta)$ where $0 \le \theta < 2\pi$

$z_1 = 4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$, $z_2 = \sqrt{8}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = 2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)$

For $z_1 z_2$, multiply the moduli and add the arguments.

$z_1 z_2 = \left(4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)\right)\left(2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)\right)$

$= (4\sqrt{2})(2\sqrt{2})\ \text{cis}\left(\frac{3\pi}{4} + \left(-\frac{\pi}{2}\right)\right)$

$= 16\ \text{cis}\left(\frac{\pi}{4}\right)$

$$z_1 z_2 = 16\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

b)

Find $\frac{z_1}{z_2}$, giving your answer in the form $r(\cos\theta + i\sin\theta)$ where $-\pi \le \theta < \pi$

For $\frac{z_1}{z_2}$, divide the moduli and subtract the arguments

$\frac{z_1}{z_2} = \frac{4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)}{2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)} = \frac{4\sqrt{2}}{2\sqrt{2}}\ \text{cis}\left(\frac{3\pi}{4} - \left(-\frac{\pi}{2}\right)\right)$

$= 2\ \text{cis}\left(\frac{5\pi}{4}\right)$

$= 2\ \text{cis}\left(\frac{5\pi}{4} - 2\pi\right)$

$\frac{5\pi}{4}$ is not in the range $-\pi \le \theta \le \pi$ so subtract 2π to bring it into range

$$\frac{z_1}{z_2} = 2\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$$

Exponential (Euler's) Form

How do we write a complex number in Euler's (exponential) form?

- A complex number can be written in Euler's form as $z = re^{i\theta}$
  - This relates to the modulus-argument (polar) form as $z = re^{i\theta} = r \operatorname{cis} \theta$
  - This shows a clear link between exponential functions and trigonometric functions
  - This is given in the formula booklet under 'Modulus-argument (polar) form and exponential (Euler) form'
- The argument is normally given in the range $0 \le \theta < 2\pi$
  - However in exponential form other arguments can be used and the same convention of adding or subtracting 2π can be applied

How do we multiply and divide complex numbers in Euler's form?

- Euler's form allows for quick and easy multiplication and division of complex numbers
- If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then
  - $z_1 \times z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
    - Multiply the moduli and add the arguments
  - $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$
    - Divide the moduli and subtract the arguments
- Using these rules makes multiplying and dividing more than two complex numbers much easier than in Cartesian form
- When a complex number is written in Euler's form it is easy to raise that complex number to a power
  - If $z = re^{i\theta}$, $z^2 = r^2 e^{2i\theta}$ and $z^n = r^n e^{ni\theta}$

What are some common numbers in exponential form?

- As $\cos(2\pi) = 1$ and $\sin(2\pi) = 0$ you can write:
  - $1 = e^{2\pi i}$
- Using the same idea you can write:
  - $1 = e^0 = e^{2\pi i} = e^{4\pi i} = e^{6\pi i} = e^{2k\pi i}$
  - where k is any integer
- As $\cos(\pi) = -1$ and $\sin(\pi) = 0$ you can write:
  - $e^{\pi i} = -1$
  - Or more commonly written as $e^{i\pi} + 1 = 0$
    - This is known as Euler's identity and is considered by some mathematicians as the most beautiful equation
- As $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$ you can write:
  - $i = e^{\frac{\pi}{2} i}$

Exam Tip

- Euler's form allows for easy manipulation of complex numbers, in an exam it is often worth the time converting a complex number into Euler's form if further calculations need to be carried out
  - Familiarise yourself with which calculations are easier in which form, for example multiplication and division are easiest in Euler's form but adding and subtracting are easiest in Cartesian form

Worked Example

Consider the complex number $z = 2e^{\frac{\pi}{3}i}$. Calculate z^2 giving your answer in the form $re^{i\theta}$.

$$z^2 = \left(2e^{\frac{\pi}{3}i}\right)^2 = \left(2e^{\frac{\pi}{3}i}\right)\left(2e^{\frac{\pi}{3}i}\right) = 4e^{2\left(\frac{\pi}{3}i\right)}$$

multiply the moduli

add the arguments

$$z^2 = 4e^{\frac{2\pi}{3}i}$$

Conversion of Forms

Converting from Cartesian form to modulus-argument (polar) form or exponential (Euler's) form.

- To convert from Cartesian form to modulus-argument (polar) form or exponential (Euler) form use
  - $r = |z| = \sqrt{x^2 + y^2}$
- and
  - $\theta = \arg z$

Converting from modulus-argument (polar) form or exponential (Euler's) form to Cartesian form.

- To convert from modulus-argument (polar) form to Cartesian form
  - Write $z = r(\cos\theta + i\sin\theta)$ as $z = r\cos\theta + (r\sin\theta)i$
  - Find the values of the trigonometric ratios $r\sin\theta$ and $r\cos\theta$
    - You may need to use your knowledge of trig exact values
  - Rewrite as $z = a + bi$ where
    - $a = r\cos\theta$ and $b = r\sin\theta$
- To convert from exponential (Euler's) form to Cartesian form first rewrite $z = re^{i\theta}$ in the form $z = r\cos\theta + (r\sin\theta)i$ and then follow the steps above

Exam Tip

- When converting from Cartesian form into Polar or Euler's form, always leave your modulus and argument as an exact value
  - Rounding values too early may result in inaccuracies later on

Worked Example

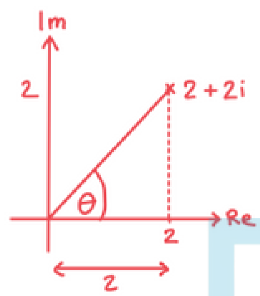
Two complex numbers are given by $z_1 = 2 + 2\text{i}$ and $z_2 = 3\text{e}^{\frac{2\pi}{3}\text{i}}$.

a)
Write z_1 in the form $r\text{e}^{\text{i}\theta}$.

$z_1 = 2 + 2i$

Find the modulus: $|z_1| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Draw a sketch to help find the argument:



$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1)$
$= \frac{\pi}{4}$

$r = 2\sqrt{2}, \quad \theta = \frac{\pi}{4}$

$z_1 = 2\sqrt{2}\, e^{\frac{\pi}{4}i}$

b)
Write z_2 in the form $r(\cos\theta + \text{i}\sin\theta)$ and then convert it to Cartesian form.

$z_2 = 3e^{\frac{2\pi}{3}i} = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
$= 3\left(-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right)$

$z_2 = \frac{3}{2}\left(-1 + \sqrt{3}i\right)$

1.6.3 Applications of Complex Numbers

Frequency & Phase of Trig Functions

How are complex numbers and trig functions related?

- A sinusoidal function is of the form $a\sin(bx+c)$
  - a represents the **amplitude**
  - b represents the **period** (also known as frequency)
  - c represents the **phase shift**
    - The function may be written $a\sin(bx+bc) = a\sin b(x+c)$ where the phase shift is represented by bc
    - This will be made clear in the exam
- When written in **modulus-argument** form the **imaginary part** of a complex number relates only to the **sin** part and the **real** part relates to the **cos** part
  - This means that the complex number can be rewritten in Euler's form and relates to the sinusoidal functions as follows:
  - $a\sin(bx+c) = \text{Im}\left(ae^{i(bx+c)}\right)$
  - $a\cos(bx+c) = \text{Re}\left(ae^{i(bx+c)}\right)$
- Complex numbers are particularly useful when working with electrical currents or voltages as these follow sinusoidal wave patterns
  - AC voltages may be given in the form $V = a\sin(bt+c)$ or $V = a\cos(bt+c)$

How are complex numbers used to add two sinusoidal functions?

- Complex numbers can help to add two **sinusoidal** functions if they have the same **frequency** but different **amplitudes** and **phase shifts**
  - e.g. $2\sin(\mathbf{3x}+1)$ can be added to $3\sin(\mathbf{3x}-5)$ but **not** $2\sin(\mathbf{5x}+1)$
- To add $a\sin(bx+c)$ to $d\sin(bx+e)$
  - or $a\cos(bx+c)$ to $d\cos(bx+e)$
- STEP 1: Consider the complex numbers $z_1 = ae^{i(bx+c)}$ and $z_2 = de^{i(bx+e)}$
  - Then $a\sin(bx+c) + d\sin(bx+e) = \text{Im}(z_1+z_2)$
  - Or $a\cos(bx+c) + d\cos(bx+e) = \text{Re}(z_1+z_2)$
- STEP 2: Factorise $z_1+z_2 = ae^{i(bx+c)} + de^{i(bx+e)} = e^{ibx}\left(ae^{ci} + de^{ei}\right)$
- STEP 3: Convert $ae^{ci} + de^{ei}$ into a single complex number in exponential form
  - You may need to convert it into Cartesian form first, simplify and then convert back into exponential form
  - Your GDC will be able to do this quickly
- STEP 4: Simplify the whole term and use the rules of indices to collect the powers
- STEP 5: Convert into polar form and take...
  - only the imaginary part for sin
  - or only the real part for cos

Exam Tip

- An exam question involving applications of complex numbers will often be made up of various parts which build on each other
  - Remember to look back at your answers from previous question parts to see if they can help you, especially when looking to convert from Euler's form to a sinusoidal graph form

Worked Example

Two AC voltage sources are connected in a circuit. If $V_1 = 20\sin(30t)$ and $V_2 = 30\sin(30t+5)$ find an expression for the total voltage in the form $V = A\sin(30t+B)$.

$20\sin(30t) + 30\sin(30t+5)$

Frequencies are the same so they can be added

STEP 1: Let $z_1 = 20e^{i(30t)}$ and $z_2 = 30e^{i(30t+5)}$

In polar form the imaginary parts are the sinusoidal functions we want to add.

STEP 2: Find $z_1 + z_2$

$$z_1 + z_2 = 20e^{i(30t)} + 30e^{i(30t+5)}$$

$$= 10e^{30ti}\left(2 + 3e^{5i}\right)$$

$2 + (3(\cos(5) + i\sin(5)))$
$= 2.850... - 2.876...i$

STEP 3: Use GDC to find $2 + 3e^{5i}$ in Euler's form

$$= 10e^{30ti}\left(4.05...e^{-0.789...i}\right)$$

STEP 4: Use index laws to simplify.

$$= 40.5e^{i(30t - 0.789...)}$$

STEP 5: Convert to Polar form

$$= 40.5\left(\cos(30t - 0.789...) + i\sin(30t - 0.789...)\right)$$

The imaginary part is the solution

$$V = 40.5\sin(30t - 0.790)$$

1.7 Matrices

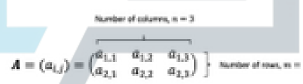
1.7.1 Introduction to Matrices

Introduction to Matrices

Matrices are a useful way to represent and manipulate data in order to model situations. The elements in a matrix can represent data, equations or systems and have many real-life applications.

What are matrices?

- A matrix is a **rectangular array** of **elements** (numerical or algebraic) that are arranged in **rows** and **columns**
- The **order** of a matrix is defined by the **number** of rows and columns that it has
  - The order of a matrix with m rows and n columns is $m \times n$
- A matrix $\boldsymbol{A}$ can be defined by $\boldsymbol{A} = (a_{ij})$ where $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$ and a_{ij} refers to the element in row i, column j



What type of matrices are there?

- A **column matrix** (or column vector) is a matrix with a **single column**, $n = 1$
- A **row matrix** is a matrix with a **single row**, $m = 1$
- A **square matrix** is one in which the number of rows is **equal** to the number of columns, $m = n$
- Two matrices are **equal** when they are of the **same order** and their **corresponding elements** are **equal**, i.e. $a_{ij} = b_{ij}$ for all elements
- A **zero matrix**, $\boldsymbol{O}$, is a matrix in which all the elements are 0, e.g. $\boldsymbol{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
- The identity matrix, $\boldsymbol{I}$, is a **square** matrix in which all elements along the **leading diagonal** are 1 and the rest are 0, e.g. $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

> **Exam Tip**
>
> - Make sure that you know how to enter and store a matrix on your GDC

Worked Example

Let the matrix $\boldsymbol{A} = \begin{pmatrix} 5 & -3 & 7 \\ -1 & 2 & 4 \end{pmatrix}$

a)
Write down the order of $\boldsymbol{A}$.

A is a 2x3 Matrix

b)
State the value of $a_{2,3}$.

$a_{23} = 4$

1.7.2 Operations with Matrices

Matrix Addition & Subtraction

Just as with ordinary numbers, **matrices** can be **added** together and **subtracted** from one another, provided that they meet certain conditions.

How is addition and subtraction performed with matrices?

- Two matrices of the **same order** can be added or subtracted
- Only **corresponding elements** of the two matrices are added or subtracted
  - $\boldsymbol{A} \pm \boldsymbol{B} = (a_{ij}) \pm (b_{ij}) = (a_{ij} \pm b_{ij})$
- The **resultant** matrix is of the **same order** as the original matrices being added or subtracted

What are the properties of matrix addition and subtraction?

- $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ (commutative)
- $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$ (associative)
- $\boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$
- $\boldsymbol{O} - \boldsymbol{A} = -\boldsymbol{A}$
- $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$

Exam Tip

- Make sure that you know how to add and subtract matrices on your GDC for speed or for checking work in an exam!

Worked Example

Consider the matrices $\boldsymbol{A}=\begin{pmatrix}-4 & 2\\ 7 & 3\\ 1 & -5\end{pmatrix}$, $\boldsymbol{B}=\begin{pmatrix}2 & 6\\ 5 & -9\\ -2 & -3\end{pmatrix}$.

a)
Find $\boldsymbol{A}+\boldsymbol{B}$.

$$A + B = \begin{pmatrix}-4 & 2\\ 7 & 3\\ 1 & -5\end{pmatrix} + \begin{pmatrix}2 & 6\\ 5 & -9\\ -2 & -3\end{pmatrix} = \boxed{\begin{pmatrix}-2 & 8\\ 12 & -6\\ -1 & -8\end{pmatrix}}$$

b)
Find $\boldsymbol{A}-\boldsymbol{B}$.

$$A - B = \begin{pmatrix}-4 & 2\\ 7 & 3\\ 1 & -5\end{pmatrix} - \begin{pmatrix}2 & 6\\ 5 & -9\\ -2 & -3\end{pmatrix} = \boxed{\begin{pmatrix}-6 & -4\\ 2 & 12\\ 3 & -2\end{pmatrix}}$$

Matrix Multiplication

Matrices can also be **multiplied** either by a **scalar** or by **another matrix**.

How do I multiply a matrix by a scalar?

- Multiply **each element** in the matrix by the **scalar** value
  - $k\boldsymbol{A} = (ka_{ij})$
- The **resultant** matrix is of the **same order** as the original matrix
- Multiplication by a **negative** scalar changes the **sign** of each element in the matrix

How do I multiply a matrix by another matrix?

- To multiply a matrix by another matrix, the **number of columns** in the **first** matrix must be **equal** to the **number of rows** in the **second** matrix
- If the order of the **first** matrix is $m \times n$ and the order of the **second** matrix is $n \times p$, then the order of the **resultant** matrix will be $m \times p$
- The product of two matrices is found by multiplying the corresponding elements in the **row** of the **first** matrix with the corresponding elements in the **column** of the **second** matrix and finding the **sum** to place in the resultant matrix
  - E.g. If $\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix}$
    - then $\boldsymbol{AB} = \begin{bmatrix} (ag+bi+ck) & (ah+bj+cl) \\ (dg+ei+fk) & (dh+ej+fl) \end{bmatrix}$
    - then $\boldsymbol{BA} = \begin{bmatrix} (ga+hd) & (gb+he) & (gc+hf) \\ (ia+jd) & (ib+je) & (ic+jf) \\ (ka+ld) & (kb+le) & (kc+lf) \end{bmatrix}$

How do I square an expression involving matrices?

- If an expression involving matrices is squared then you are multiplying the expression by itself, so write it out in bracket form first, e.g. $(A+B)^2 = (A+B)(A+B)$
  - remember, the regular rules of algebra do not apply here and you cannot expand these brackets, instead, add together the matrices inside the brackets and then multiply the matrices together

What are the properties of matrix multiplication?

- $\boldsymbol{AB} \neq \boldsymbol{BA}$ (non-commutative)
- $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$ (associative)
- $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{AB}+\boldsymbol{AC}$ (distributive)
- $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC}+\boldsymbol{BC}$ (distributive)
- $\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$ (identity law)
- $\boldsymbol{AO} = \boldsymbol{OA} = \boldsymbol{O}$, where $\boldsymbol{O}$ is a zero matrix
- Powers of **square** matrices: $\boldsymbol{A}^2 = \boldsymbol{AA},\ \boldsymbol{A}^3 = \boldsymbol{AAA}$ etc.

Exam Tip

- Make sure that you are clear on the properties of matrix algebra and show each step of your calculations

Worked Example

Consider the matrices $\boldsymbol{A}=\begin{bmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix} 5 & 1 \\ -2 & 5 \\ 9 & 7 \end{bmatrix}$.

a)
Find $\boldsymbol{AB}$.

$$AB = \begin{pmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix} \times \begin{pmatrix} 5 & 1 \\ -2 & 5 \\ 9 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} (4\times5 + 2\times-2 + -5\times9) & (4\times1 + 2\times5 + -5\times7) \\ (-3\times5 + 8\times-2 + 1\times9) & (-3\times1 + 8\times5 + 1\times7) \\ (-1\times5 + -2\times-2 + 2\times9) & (-1\times1 + -2\times5 + 2\times7) \end{pmatrix}$$

$$= \begin{pmatrix} (20 - 4 - 45) & (4 + 10 - 35) \\ (-15 - 16 + 9) & (-3 + 40 + 7) \\ (-5 + 4 + 18) & (-1 - 10 + 14) \end{pmatrix}$$

$$AB = \begin{pmatrix} -29 & -21 \\ -22 & 44 \\ 17 & 3 \end{pmatrix}$$

b)
Explain why you cannot find $\boldsymbol{BA}$.

BA cannot be found because the number of columns in B is different to the number of rows in A

c)
Find $\boldsymbol{A}^2$.

$$A^2 = \begin{pmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}^2 = \begin{pmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}\begin{pmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} (4\times4 + 2\times-3 + -5\times-1) & (4\times2 + 2\times8 + -5\times-2) & (4\times-5 + 2\times1 + -5\times2) \\ (-3\times4 + 8\times-3 + 1\times-1) & (-3\times2 + 8\times8 + 1\times-2) & (-3\times-5 + 8\times1 + 1\times2) \\ (-1\times4 + -2\times-3 + 2\times-1) & (-1\times2 + -2\times8 + 2\times-2) & (-1\times-5 + -2\times1 + 2\times2) \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 15 & 34 & -28 \\ -37 & 56 & 25 \\ 0 & -22 & 7 \end{pmatrix}$$

1.7.3 Determinants & Inverses

Determinants

What is a determinant?

- The **determinant** is a **numerical value** (positive or negative) calculated from the elements in a matrix and is used to find the **inverse** of a matrix
- You can only find the determinant of a **square** matrix
- The method for finding the determinant of a 2×2 matrix is given in your **formula booklet**:

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det \boldsymbol{A} = |\boldsymbol{A}| = ad - bc$$

- You only need to be able to find the determinant of a 2×2 matrix **by hand**
  - For larger $n \times n$ matrices you are expected to **use your GDC**
- The determinant of an **identity matrix** is $\det(\boldsymbol{I}) = 1$
- The determinant of a **zero matrix** is $\det(\boldsymbol{O}) = 0$
- When finding the determinant of a **multiple** of a matrix or the **product** of two matrices:
  - $\det(k\boldsymbol{A}) = k^2 \det(\boldsymbol{A})$ (for a 2×2 matrix)
  - $\det(\boldsymbol{AB}) = \det(\boldsymbol{A}) \times \det(\boldsymbol{B})$

Worked Example

Consider the matrix $A = \begin{pmatrix} 3 & -6 \\ p & 7 \end{pmatrix}$, where $p \in \mathbb{R}$ is a constant.

a)
Given that $\det A = -3$, find the value of p.

Determinant of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = \lvert A \rvert = ad - bc$

$\det A = 3 \times 7 - -6 \times p = 21 + 6p$

So, $-3 = 21 + 6p$

$-24 = 6p$

$p = -4$

b)
Find the determinant of $4A$.

$\det(4A) = 4^2 \times -3 = -48$

Inverse Matrices

How do I find the inverse of a matrix?

- The determinant can be used to find out if a matrix is invertible or not:
  - If $\det \boldsymbol{A} \neq 0$, then $\boldsymbol{A}$ is **invertible**
  - If $\det \boldsymbol{A} = 0$, then $\boldsymbol{A}$ is singular and does **not** have an inverse
- The method for finding the inverse of a 2×2 matrix is given in your **formula booklet**:

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \ ad \neq bc$$

- You only need to be able to find the inverse of a 2×2 matrix **by hand**
  - For larger $n \times n$ matrices you are expected to **use your GDC**
- The inverse of a square matrix $\boldsymbol{A}$ is the matrix $\boldsymbol{A}^{-1}$ such that the product of these matrices is an **identity** matrix, $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$
  - As a result of this property:
    - $\boldsymbol{AB} = \boldsymbol{C} \Rightarrow \boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{C}$ (pre-multiplying by $\boldsymbol{A}^{-1}$)
    - $\boldsymbol{BA} = \boldsymbol{C} \Rightarrow \boldsymbol{B} = \boldsymbol{C}\boldsymbol{A}^{-1}$ (post-multiplying by $\boldsymbol{A}^{-1}$)

Worked Example

Consider the matrices $\boldsymbol{P}=\begin{pmatrix}4 & -2\\ 8 & 2\end{pmatrix}$, $\boldsymbol{Q}=\begin{pmatrix}k & 6\\ -5 & 3\end{pmatrix}$ and $\boldsymbol{R}=\begin{pmatrix}18 & 18\\ 6 & 54\end{pmatrix}$, where k is a constant.

a)
Find $\boldsymbol{P}^{-1}$.

Determinant of a 2 × 2 matrix	$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow \det A = \lvert A\rvert = ad-bc$

Inverse of a 2 × 2 matrix	$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow A^{-1}=\frac{1}{\det A}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix},\ ad \neq bc$

$$P^{-1} = \frac{1}{4\times2-(-2)\times8}\begin{pmatrix}2 & 2\\ -8 & 4\end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix}2 & 2\\ -8 & 4\end{pmatrix}$$

$$P^{-1} = \begin{pmatrix}\frac{1}{12} & \frac{1}{12}\\ -\frac{1}{3} & \frac{1}{6}\end{pmatrix}$$

b)
Given that $\boldsymbol{PQ}=\boldsymbol{R}$ find the value of k.

$$PQ = R \Rightarrow Q = P^{-1}R$$

$$\begin{pmatrix}k & 6\\ -5 & 3\end{pmatrix} = \begin{pmatrix}\frac{1}{12} & \frac{1}{12}\\ -\frac{1}{3} & \frac{1}{6}\end{pmatrix}\begin{pmatrix}18 & 18\\ 6 & 54\end{pmatrix}$$

$$\begin{pmatrix}k & 6\\ -5 & 3\end{pmatrix} = \begin{pmatrix}(\frac{1}{12}\times18+\frac{1}{12}\times6) & (\frac{1}{12}\times18+\frac{1}{12}\times54)\\ (-\frac{1}{3}\times18+\frac{1}{6}\times6) & (-\frac{1}{3}\times18+\frac{1}{6}\times54)\end{pmatrix}$$

$$\begin{pmatrix}k & 6\\ -5 & 3\end{pmatrix} = \begin{pmatrix}2 & 6\\ -5 & 3\end{pmatrix}$$

$$k = 2$$

1.7.4 Solving Systems of Linear Equations with Matrices

Solving Systems of Linear Equations with Matrices

Matrices are used in a huge variety of applications within engineering, computing and business. They are particularly useful for encrypting data and forecasting from given data. Using matrices allows for much larger and more complex systems of linear equations to be solved easily.

How do you set up a system of linear equations using matrices?

- A linear equation can be written in the form $\boldsymbol{A}x = b$, where $\boldsymbol{A}$ is a matrix
- Note that for a system of linear equations to have a **unique** solution, the matrix must be **invertible** and therefore must be a **square** matrix
  - In exams, only invertible matrices will be given (except when solving for eigenvectors)
- You should be able to use matrices to solve a system of up to **two** linear equations both **with and without** your GDC
- You should be able to use a mixture of matrices and technology to solve a system of up to **three** linear equations

How do you solve a system of linear equations with matrices?

- STEP 1
  Write the information in a matrix equation, e.g. for a system of three linear equations $\boldsymbol{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{B}$, where the entries into matrix $\boldsymbol{A}$ are the coefficients of x, y and z and matrix $\boldsymbol{B}$ is a column matrix
- STEP 2
  Re-write the equation using the inverse of $\boldsymbol{A}$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{A}^{-1}\boldsymbol{B}$
- STEP 3
  Evaluate the right-hand side to find the values of the unknown variables x, y and z

Exam Tip

- If you are asked to solve a system of linear equations by hand you can check your work afterwards by solving the same question on your GDC

Worked Example

a)

Write the system of equations

$$\begin{cases} x+3y-z=-3 \\ 2x+2y+z=2 \\ 3x-y+2z=1 \end{cases}$$

in matrix form.

$$\begin{pmatrix} 1 & 3 & -1 \\ 2 & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

b)

Hence solve the simultaneous linear equations.

Re-write the equation in part a) using the inverse matrix

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{10} & \frac{1}{2} & -\frac{3}{10} \\ -\frac{4}{5} & 1 & -\frac{2}{5} \end{pmatrix}\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

Use your GDC to find A^{-1} if it is larger than a 2x2 matrix

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

$x = -2$
$y = 1$
$z = 4$

EXAM PAPERS PRACTICE

1.8.1 Eigenvalues & Eigenvectors

1.8.1 Eigenvalues & Eigenvectors

Characteristic Polynomials

Eigenvalues and **eigenvectors** are properties of square matrices and are used in a lot of real-life applications including geometrical transformations and probability scenarios. In order to find these eigenvalues and eigenvectors, the **characteristic polynomial** for a matrix must be found and solved.

What is a characteristic polynomial?

- For a matrix $\boldsymbol{A}$, if $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ when $\boldsymbol{x}$ is a non-zero vector and λ a **constant**, then λ is an **eigenvalue** of the matrix $\boldsymbol{A}$ and $\boldsymbol{x}$ is its corresponding **eigenvector**
- If $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow (\lambda\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = 0$ or $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = 0$ and for $\boldsymbol{x}$ to be a non-zero vector, $\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$
- The characteristic polynomial of an $n \times n$ matrix is:

$$p(\lambda) = \det(\lambda\boldsymbol{I} - \boldsymbol{A})$$

- In this course you will only be expected to find the characteristic equation for a 2×2 matrix and this will always be a **quadratic**

How do I find the characteristic polynomial?

- STEP 1
  Write $\lambda\boldsymbol{I} - \boldsymbol{A}$, remembering that the identity matrix must be of the same order as $\boldsymbol{A}$
- STEP 2
  Find the determinant of $\lambda\boldsymbol{I} - \boldsymbol{A}$ using the formula given to you in the formula booklet

$$\det \boldsymbol{A} = |\boldsymbol{A}| = ad - bc$$

- STEP 3
  Re-write as a polynomial

Exam Tip

- You need to remember the **characteristic equation** as it is **not** given in the formula booklet

Worked Example

Find the characteristic polynomial of the following matrix

$$A = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}.$$

$$p(\lambda) = \det(\lambda I - A)$$

$$= \det\left(\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} \lambda - 5 & -4 \\ -3 & \lambda - 1 \end{pmatrix}$$

Determinant of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = \lvert A \rvert = ad - bc$

$$= (\lambda - 5)(\lambda - 1) - (-4)(-3)$$

$$= \lambda^2 - 5\lambda - \lambda + 5 - 12$$

$$p(\lambda) = \lambda^2 - 6\lambda - 7$$

Eigenvalues & Eigenvectors

How do you find the eigenvalues of a matrix?

- The eigenvalues of matrix $\boldsymbol{A}$ are found by solving the **characteristic polynomial** of the matrix
- For this course, as the characteristic polynomial will always be a **quadratic**, the polynomial will always generate one of the following:
  - **two real and distinct** eigenvalues,
  - **one real repeated** eigenvalue or
  - **complex** eigenvalues

How do you find the eigenvectors of a matrix?

- A value for $\boldsymbol{x}$ that satisfies the equation is an **eigenvector** of matrix $\boldsymbol{A}$
- Any scalar multiple of $\boldsymbol{x}$ will also satisfy the equation and therefore there an **infinite number** of eigenvectors that correspond to a particular eigenvalue
- STEP 1
  Write $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$
- STEP 2
  Substitute the eigenvalues into the equation $(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = 0$, and form two equations in terms of x and y
- STEP 3
  There will be an infinite number of solutions to the equations, so choose one by letting one of the variables be equal to 1 and using that to find the other variable

Exam Tip

- You can do a quick check to on your calculated eigenvalues as the values along the **leading diagonal** of the matrix you are analysing should **sum** to the **total of the eigenvalues** for the matrix

Worked Example

Find the eigenvalues and associated eigenvectors for the following matrices.

a)

$$\boldsymbol{A} = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}.$$

Solve the characteristic polynomial to find the eigenvalues

$$p(\lambda) = \lambda^2 - 6\lambda - 7$$ ← From worked example above in Characteristic Polynomials

$$(\lambda - 7)(\lambda + 1)$$

$$\Rightarrow \boxed{\lambda = 7} \quad \boxed{\lambda = -1}$$

Use the eigenvalues in the equation $(\lambda I - A)\,x = 0$ to find the eigenvectors

For $\lambda = 7 \Rightarrow \left(7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 2x - 4y &= 0 \\ -3x + 6y &= 0 \end{aligned}\right\} \; 2y = x$$

The eigenvector associated with $\lambda = 7$ is any multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For $\lambda = -1 \Rightarrow \left(-1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -6 & -4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -6x - 4y &= 0 \\ -3x - 2y &= 0 \end{aligned}\right\} \; 2y = -3x$$

The eigenvector associated with $\lambda = -1$ is any multiple of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

b)

$$\boldsymbol{B} = \begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix}.$$

Find the characteristic polynomial

$$p(\lambda) = \det\begin{pmatrix}\lambda-1 & 5\\ -2 & \lambda-3\end{pmatrix}$$

$$= (\lambda-1)(\lambda-3) - (5)(-2)$$

$$= \lambda^2 - 3\lambda - \lambda + 3 + 10$$

$$p(\lambda) = \lambda^2 - 4\lambda + 13$$

Solve the characteristic polynomial to find the eigenvalues by hand or using the GDC

$$p(\lambda) = \lambda^2 - 4\lambda + 13 = 0$$

$$(\lambda-2)^2 - 4 + 13 = 0$$

$$(\lambda-2)^2 = -9$$

$$\lambda = 2 \pm \sqrt{-9}$$

$$\lambda = 2 \pm 3i$$

Use the eigenvalues in the equation $(\lambda I - A)\,x = 0$ to find the eigenvectors

For $\lambda = 2+3i \Rightarrow \left((2+3i)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}1 & -5\\ 2 & 3\end{pmatrix}\right)\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$

$$\left(\begin{pmatrix}2+3i & 0\\ 0 & 2+3i\end{pmatrix} - \begin{pmatrix}1 & -5\\ 2 & 3\end{pmatrix}\right)\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}1+3i & 5\\ -2 & -1+3i\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\left.\begin{aligned}(1+3i)x + 5y &= 0\\ -2x + (-1+3i)y &= 0\end{aligned}\right\} 2x = (-1+3i)y$$

Both equations can be simplified to the same thing

The eigenvector associated with $\lambda = 2+3i$ is any multiple of $\begin{pmatrix}-1+3i\\ 2\end{pmatrix}$

For $\lambda = 2-3i \Rightarrow \left((2-3i)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}1 & -5\\ 2 & 3\end{pmatrix}\right)\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$

$$\left(\begin{pmatrix}2-3i & 0\\ 0 & 2-3i\end{pmatrix} - \begin{pmatrix}1 & -5\\ 2 & 3\end{pmatrix}\right)\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}1-3i & 5\\ -2 & -1-3i\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\left.\begin{aligned}(1-3i)x + 5y &= 0\\ -2x + (-1-3i)y &= 0\end{aligned}\right\} 2x = (-1-3i)y$$

Both equations can be simplified to the same thing

The eigenvector associated with $\lambda = 2-3i$ is any multiple of $\begin{pmatrix}-1-3i\\ 2\end{pmatrix}$

1.8.2 Applications of Matrices

Diagonalisation

What is matrix diagonalisation?

- A **non-zero, square** matrix is considered to be **diagonal** if all elements **not** along its leading diagonal are **zero**
- A matrix $\boldsymbol{P}$ can be said to diagonalise matrix $\boldsymbol{M}$, if $\boldsymbol{D}$ is a diagonal matrix where $\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P}$
- If matrix $\boldsymbol{M}$ has **eigenvalues** λ_1, λ_2 and **eigenvectors** $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and is diagonisable by $\boldsymbol{P}$, then
    - $\boldsymbol{P} = (\boldsymbol{x}_1\ \boldsymbol{x}_2)$, where the first column is the eigenvector $\boldsymbol{x}_1$ and the second column is the eigenvector $\boldsymbol{x}_2$
    - $\boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
- You will only need to be able to diagonalise 2×2 matrices
- You will only need to consider matrices with **real, distinct eigenvalues**
    - If there is only one eigenvalue, the matrix is either already diagonalised or cannot be diagonalised
    - Diagonalisation of matrices with complex or imaginary eigenvalues is outside the scope of the course

Exam Tip

- Remember to use the formula booklet for the **determinant** and **inverse** of a matrix

Worked Example

The matrix $\boldsymbol{M} = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}$ has the eigenvalues $\lambda_1 = 7$ and $\lambda_2 = -1$ with eigenvectors $\boldsymbol{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{x}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ respectively.

Show that $\boldsymbol{P}_1 = (\boldsymbol{x}_1 \boldsymbol{x}_2)$ and $\boldsymbol{P}_2 = (\boldsymbol{x}_2 \boldsymbol{x}_1)$ both diagonalise $\boldsymbol{M}$.

Show that $P^{-1}MP$ produces a diagonal matrix

Inverse of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad \neq bc$

$$P_1 = \begin{pmatrix} 2 & 2 \\ 1 & -3 \end{pmatrix} \Rightarrow P_1^{-1} = -\frac{1}{8}\begin{pmatrix} -3 & -2 \\ -1 & 2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 3 & 2 \\ 1 & -2 \end{pmatrix}$$

$$D_1 = P_1^{-1} M P_1 = \frac{1}{8}\begin{pmatrix} 3 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & -3 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} 3 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 14 & -2 \\ 7 & 3 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} 56 & 0 \\ 0 & -8 \end{pmatrix}$$

$$D = \begin{pmatrix} 7 & 0 \\ 0 & -1 \end{pmatrix}$$

D = Diagonal matrix of eigenvalues

$$P_2 = \begin{pmatrix} 2 & 2 \\ -3 & 1 \end{pmatrix} \Rightarrow P_2^{-1} = \frac{1}{8}\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$$

$$D_2 = P_2^{-1} M P_2 = \frac{1}{8}\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ -3 & 1 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -2 & 14 \\ 3 & 7 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} -8 & 0 \\ 0 & 56 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 7 \end{pmatrix}$$

D = Diagonal matrix of eigenvalues

Matrix Powers

One of the main applications of diagonalising a matrix is to make it easy to find **powers** of the matrix, which is useful when modelling transient situations such as the movement of populations between two towns.

How can the diagonalised matrix be used to find higher powers of the original matrix?

- The equation to find the diagonalised matrix can be re-arranged for $\boldsymbol{M}$:

$$\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P} \Rightarrow \boldsymbol{M} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$$

- Finding higher powers of a matrix when it is diagonalised is straight forward:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

- Therefore, we can easily find higher powers of the matrix using the **power formula** for a matrix found in the formula booklet:

$$\boldsymbol{M}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}$$

Exam Tip

- If you are asked to show this by hand, don't forget to use your GDC to **check** your answer afterwards!

Worked Example

The matrix $\boldsymbol{M} = \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}$ has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenvectors $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively.

a)
Show that $\boldsymbol{M}^n$ can be expressed as

$$\boldsymbol{M}^n = -\frac{1}{3}\begin{pmatrix} (-(-1)^n - 2(5)^n) & (-(-1)^n + (5)^n) \\ (-2(-1)^n + 2(5)^n) & (-2(-1)^n - (5)^n) \end{pmatrix}$$

Find D, P and P^{-1}

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \Rightarrow P^{-1} = -\frac{1}{3}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}$$

Use the matrix power formula from the formula booklet

Power formula for a matrix	$\boldsymbol{M}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}$	$\boldsymbol{P}$ is the matrix of eigenvectors and $\boldsymbol{D}$ is the diagonal matrix of eigenvalues

$$M^n = -\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}^n\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1^n & 0 \\ 0 & 5^n \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}$$

When multiplying these expressions, be careful! $-2 \times 5^n = -2(5^n)$ NOT -10^n

$$= -\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -(-1)^n & -(-1)^n \\ -2(5)^n & (5)^n \end{pmatrix}$$

$$M^n = -\frac{1}{3}\begin{pmatrix} (-(-1)^n - 2(5)^n) & (-(-1)^n + (5)^n) \\ (-2(-1)^n + 2(5)^n) & (-2(-1)^n - (5)^n) \end{pmatrix}$$

b)
Hence find $\boldsymbol{M}^5$.

Substitute $n = 5$

$$M^5 = -\frac{1}{3}\begin{pmatrix} (-(-1)^5 - 2(5)^5) & (-(-1)^5 + (5)^5) \\ (-2(-1)^5 + 2(5)^5) & (-2(-1)^5 - (5)^5) \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} -6249 & 3126 \\ 6252 & -3123 \end{pmatrix}$$

$$M^5 = \begin{pmatrix} 2083 & -1042 \\ -2084 & 1041 \end{pmatrix}$$