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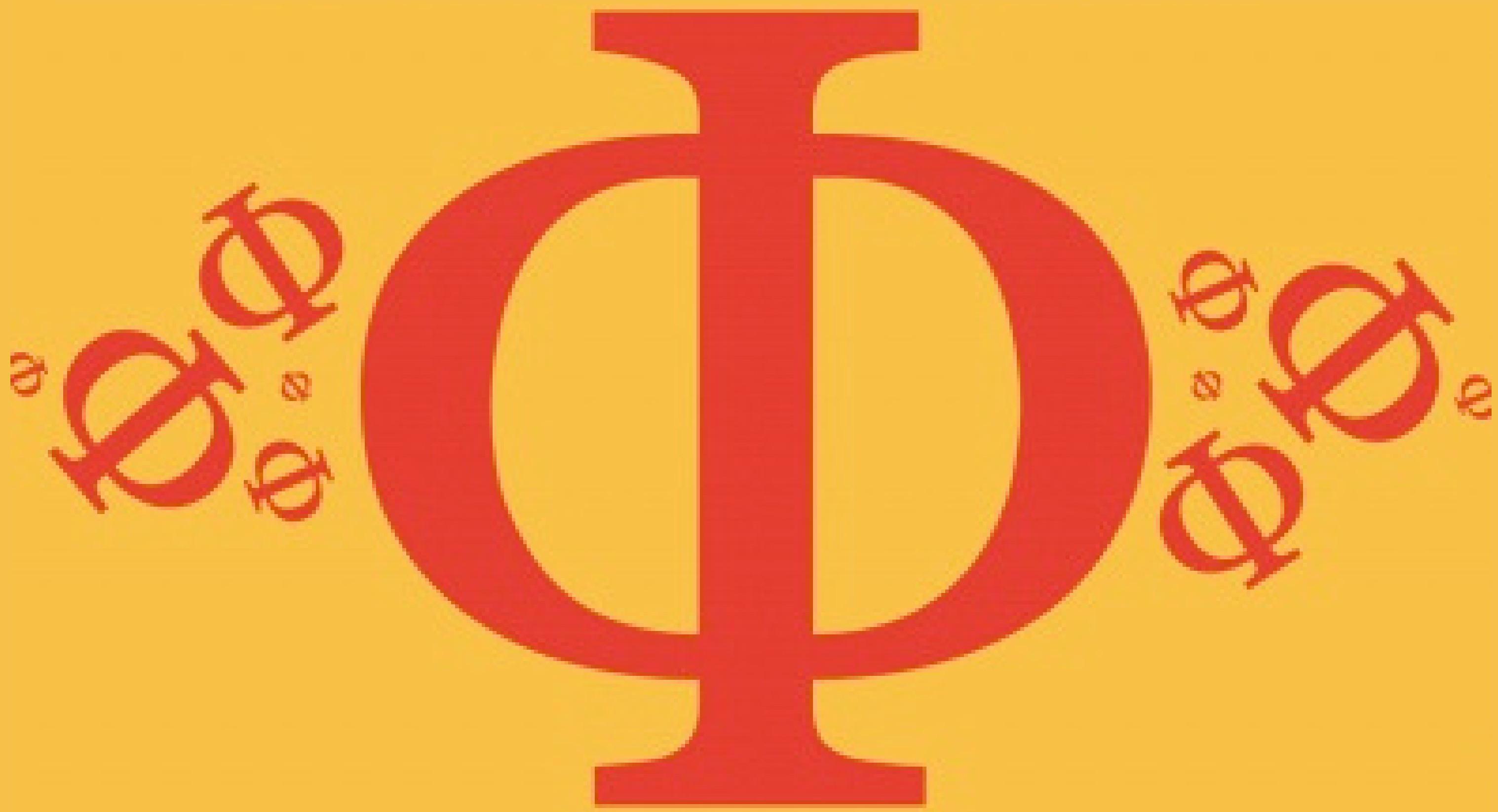
# 3.Geometry & Trigonometry

## 3.6 Matrix Transformations



# MATH

## IB AI HL

EXAM PAPERS PRACTICE

# IB Maths DP

## 3. Geometry & Trigonometry

**CONTENTS**

# 3.1 Geometry Toolkit

## 3.1.1 Coordinate Geometry

### Basic Coordinate Geometry

### What are cartesian coordinates?

- **Cartesian** coordinates are basically the *x*-*y* coordinate system
  - They allow us to label where things are in a two-dimensional plane
- In the 2D cartesian system, the horizontal axis is labelled *x* and the vertical axis is labelled *y*

### What can we do with coordinates?

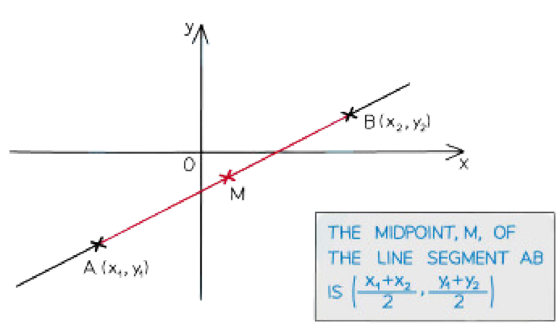
- If we have two points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ then we should be able to find
  - The **midpoint** of the two points
  - The **distance** between the two points
  - The **gradient** of the line between them

### How do I find the midpoint of two points?

- The midpoint is the **average (middle) point**
  - It can be found by finding the middle of the *x*-coordinates and the middle of the *y*-coordinates
- The coordinates of the midpoint will be

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

  - This is given in the formula booklet under the prior learning section at the beginning



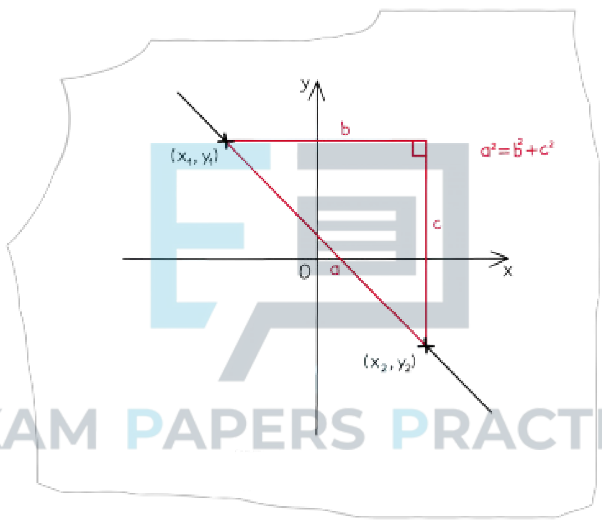
## How do I find the distance between two points?

- The distance between two points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

  - This is given in the formula booklet in the *prior learning* section at the beginning
- Pythagoras' Theorem $a^2 = b^2 + c^2$ is used to find the length of a line between two coordinates

- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]




## How do I find the gradient of the line between two points?

- The gradient of a line between two points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ can be found using the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  - This is given in the formula booklet under section 2.1 Gradient formula
  - This is usually known as $m = \frac{\text{rise}}{\text{run}}$

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Worked Example

Point $A$ has coordinates (3, -4) and point $B$ has coordinates (-5, 2).

i)

Calculate the distance of the line segment $AB$.

$A: (3, -4)$ $\quad B: (-5, 2)$

$x_1 \quad y_1 \quad x_2 \quad y_2$

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Sub coordinates for A and B into the formula:

$$d = \sqrt{(3-(-5))^2 + (-4-2)^2}$$

$$= \sqrt{8^2 + (-6)^2} = \sqrt{100}$$

$$\boxed{d = 10 \text{ units}}$$

ii)

Find the gradient of the line connecting points $A$ and $B$.

$A: (3, -4)$ $\quad B: (-5, 2)$

$x_1 \quad y_1 \quad x_2 \quad y_2$

Formula for gradient of a line segment:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Sub coordinates for A and B into the formula:

$$m = \frac{2 - -4}{-5 - 3} = \frac{6}{-8} = -\frac{3}{4}$$

$$\boxed{m = -\frac{3}{4}}$$

iii)

Find the midpoint of $[AB]$.

A: $(3, -4)$ B: $(-5, 2)$

$x_1 = 3$, $y_1 = -4$, $x_2 = -5$, $y_2 = 2$

Formula for the midpoint of two coordinates:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Sub values in:

$$\text{Midpoint} = \left(\frac{3 + (-5)}{2}, \frac{-4 + 2}{2}\right) = (-1, -1)$$

$$\boxed{\text{Midpoint} = (-1, -1)}$$

# Perpendicular Bisectors

## What is a perpendicular bisector?

- A perpendicular bisector of a line segment cuts the line segment in half at a right angle
  - Perpendicular lines meet at right angles
  - Bisect means to cut in half
- Two lines are perpendicular if the **product of their gradients is -1**

## How do I find the equation of the perpendicular bisector of a line segment?

- To find the equation of a straight line you need to find
  - The gradient of the line
  - A coordinate of a point on the line
- To find the equation of the **perpendicular bisector** of a line segment follow these steps:
  - STEP 1: Find the coordinates of the midpoint of the line segment
    - We know that the perpendicular bisector will cut the line segment in half so we can use the midpoint of the line segment as the known coordinate on the bisector
  - STEP 2: Find the gradient of the line segment
  - STEP 3: Find the gradient of the perpendicular bisector
    - This will be -1 divided by the gradient of the line segment
  - STEP 4: Substitute the gradient of the perpendicular bisector and the coordinates of the midpoint into an equation for a straight line
    - The **point-gradient** form $y - y_1 = m(x - x_1)$ is the easiest
  - STEP 5: Rearrange into the required form
    - Either $y = mx + c$ or $ax + by + d = 0$
    - These equations for a straight line are given in the formula booklet

## Worked Example

Point $A$ has coordinates (4, -6) and point $B$ has coordinates (8, 6). Find the equation of the perpendicular bisector to [$AB$]. Give your answer in the form $ax + by + d = 0$.

Step 1: find the coordinates of the midpoint:

A: (4, −6) B: (8, 6)

$x_1$ $y_1$ $x_2$ $y_2$

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Sub values in:

$$\text{Midpoint} = \left(\frac{4+8}{2}, \frac{-6+6}{2}\right) = (6, 0)$$

Step 2: Find the gradient of [AB]:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - -6}{8 - 4} = \frac{12}{4} = 3$$

Step 3: Find the gradient of the perpendicular bisector:

$$m_{\perp r} = -\frac{1}{m} = -\frac{1}{3}$$

Step 4: Substitute gradient and coordinate into an equation for a straight line.

insert coordinates of the midpoint.

$$(y - y_1) = m(x - x_1)$$

$$(y - 0) = -\frac{1}{3}(x - 6)$$

Step 5: Rearrange into the form $ax + by + d = 0$

$$(y - 0) = -\frac{1}{3}(x - 6) \quad (\times -3)$$

$$-3y = x - 6 \quad (+3y)$$

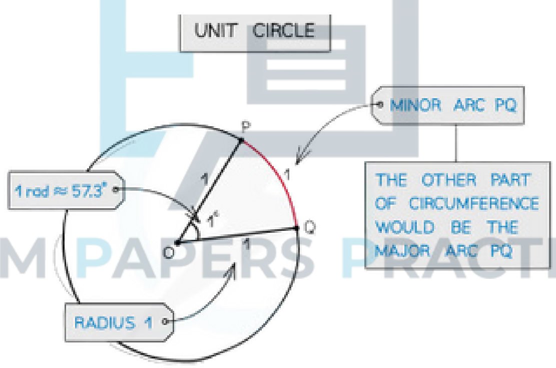
$$\boxed{x + 3y - 6 = 0}$$

# 3.1.2 Radian Measure

## Radian Measure

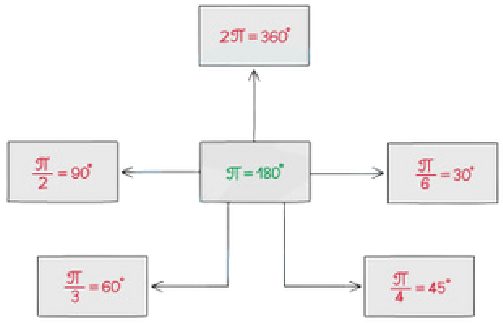
### What are radians?

- Radians are an alternative to degrees for measuring angles
- 1 radian is the angle in a **sector** of radius 1 and arc length 1
  - A circle with radius 1 is called a **unit circle**
- Radians are normally quoted in terms of π
  - 2π radians = 360°
  - π radians = 180°
- The symbol for radians is $^c$ but it is more usual to see **rad**
  - Often, when π is involved, no symbol is given as it is obvious it is in radians
  - Whilst it is okay to omit the symbol for radians, you should never omit the symbol for degrees
- In the exam you should use radians unless otherwise indicated




### How do I convert between radians and degrees?

- Use $\pi^c = 180°$ to convert between radians and degrees
  - To convert from radians to degrees multiply by $\frac{180}{\pi}$
  - To convert from degrees to radians multiply by $\frac{\pi}{180}$
- Some of the common conversions are:
  - $2\pi^c = 360°$
  - $\pi^c = 180°$
  - $\frac{\pi}{2}^c = 90°$
  - $\frac{\pi}{3}^c = 60°$
  - $\frac{\pi}{4}^c = 45°$
  - $\frac{\pi}{6}^c = 30°$
- It is a good idea to remember some of these and use them to work out other conversions
- Your GDC will be able to work with both radians and degrees




### Exam Tip

- Sometimes an exam question will specify whether you should be using degrees or radians and sometimes it will not, if it doesn't it is expected that you will work in radians
- If the question involves π then working in radians is useful as there will likely be opportunities where you can cancel out π
- Make sure that your calculator is in the correct mode for the type of angle you are working with

## Worked Example

i)

Convert 43.8° to radians.

$$43.8^\circ \xrightarrow{\div 180^\circ} \frac{73}{300} \xrightarrow{\times \pi^c} \frac{73\pi}{300}^c$$

$(\pi^c = 180^\circ)$

$$\boxed{43.8^\circ = 0.764^c \text{ (3 s.f.)}}$$

ii) Convert $\frac{5\pi}{4}$ to degrees.

$$\frac{5\pi}{4} \xrightarrow{\div \pi^c} \frac{5}{4} \xrightarrow{\times 180^\circ} 225^\circ$$

$(\pi^c = 180^\circ)$

$$\boxed{\frac{5\pi}{4} = 225^\circ}$$

# 3.1.3 Arcs & Sectors

## Length of an Arc

### What is an arc?

- An arc is a part of the **circumference** of a circle
  - It is easiest to think of it as the crust of a single slice of pizza
- The length of an arc depends of the size of the angle at the centre of the circle
- If the angle at the centre is **less than 180°** then the arc is known as a **minor arc**
  - This could be considered as the crust of a single slice of pizza
- If the angle at the centre is **more than 180°** then the arc is known as a **major arc**
  - This could be considered as the crust of the remaining pizza after a slice has been taken away

### How do I find the length of an arc?

- The length of an arc is simply a fraction of the circumference of a circle
  - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the length, $l$, of an arc is

$$l = \frac{\theta}{360} \times 2\pi r$$

  - Where $\theta$ is the angle measured in degrees
  - $r$ is the radius
  - This is **in the formula booklet**, you do not need to remember it



### Exam Tip

- Make sure that you read the question carefully to determine if you need to calculate the arc length of a sector, the perimeter or something else that incorporates the arc length!

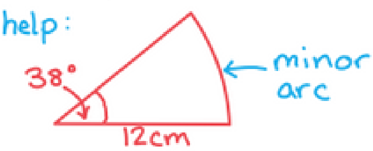
## Worked Example

A circular pizza has had a slice cut from it, the angle of the slice that was cut was 38 °. The radius of the pizza is 12 cm. Find

i)

the length of the outside crust of the slice of pizza (the minor arc),

A diagram will help:




Formula for the length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

$$l = \frac{38}{360} \times 2\pi(12)$$

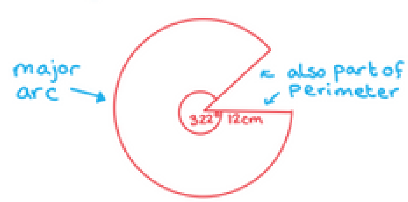
$$= \frac{38\pi}{15} = 7.9587\ldots \text{ cm}$$

$$\boxed{\text{Length of crust} = 7.96 \text{ cm (3s.f)}}$$

ii)

the perimeter of the remaining pizza.

A diagram will help:




Formula for the length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

$$l = \frac{322}{360} \times 2\pi(12)$$

$$= \frac{322\pi}{15}$$ ← Length of major arc

Find perimeter:

$$P = \text{major arc} + \text{radius} + \text{radius}$$

$$= \frac{322\pi}{15} + 12 + 12 = 91.4395\ldots \text{ cm}$$

$$\boxed{\text{Perimeter} = 91.4 \text{ cm (3s.f)}}$$

# Area of a Sector

## What is a sector?

- A sector is a part of a circle enclosed by two radii (radiuses) and an arc
  - It is easier to think of this as the shape of a single slice of pizza
- The area of a sector depends of the size of the angle at the centre of the sector
- If the angle at the centre is **less than 180°** then the sector is known as a **minor sector**
  - This could be considered as the shape of a single slice of pizza
- If the angle at the centre is **more than 180°** then the sector is known as a **major sector**
  - This could be considered as the shape of the remaining pizza after a slice has been taken away

## How do I find the area of a sector?

- The area of a sector is simply a fraction of the area of the whole circle
  - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the area, $A$, of a sector is

$$A = \frac{\theta}{360} \times \pi r^2$$

  - Where $\theta$ is the angle measured in degrees
  - $r$ is the radius
  - This is **in the formula booklet**, you do not need to remember it

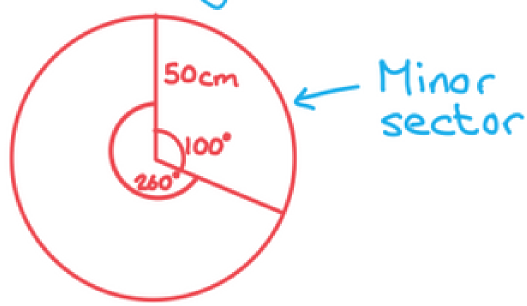
## Worked Example

Jamie has divided a circle of radius 50 cm into two sectors; a minor sector of angle 100° and a major sector of angle 260°. He is going to paint the minor sector blue and the major sector yellow. Find

i)

the area Jamie will paint blue,

Start with a diagram:




Formula for the area of a sector:

$$A = \frac{\theta}{360°} \times \pi r^2$$

Substitute: $A = \frac{100}{360} \times \pi \times 50^2$

$$= \frac{6250}{9}\pi$$

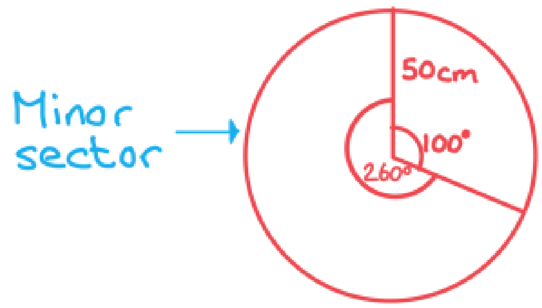
$$= 2181.66\ldots \text{ cm}^2$$

$$\boxed{\text{Blue area} = 2180 \text{ cm}^2 \text{ (3sf)}}$$

ii)

the area Jamie will paint yellow.

Start with a diagram:




Formula for the area of a sector:

$$A = \frac{\theta}{360°} \times \pi r^2$$

Substitute: $A = \frac{260}{360} \times \pi \times 50^2$

$$= \frac{16250}{9}\pi$$

$$= 5672.32\ldots \text{cm}^2$$

$\boxed{\text{Yellow area} = 5670 \text{ cm}^2 \text{ (3sf)}}$

# Arcs & Sectors Using Radians

## How do I use radians to find the length of an arc?

- As the radian measure for a **full turn** is $2\pi$, the fraction of the circle becomes $\frac{\theta}{2\pi}$
- Working in radians, the formula for the length of an arc will become

$$l = \frac{\theta}{2\pi} \times 2\pi r$$

- Simplifying, the formula for the length, $l$, of an arc is

$$l = r\theta$$

  - $\theta$ is the angle measured in **radians**
  - $r$ is the radius
  - This is **given in the formula booklet**, you do not need to remember it

## How do I use radians to find the area of a sector?

- As the radian measure for a **full turn** is $2\pi$, the fraction of the circle becomes $\frac{\theta}{2\pi}$
- Working in radians, the formula for the area of a sector will become

$$A = \frac{\theta}{2\pi} \times \pi r^2$$



- Simplifying, the formula for the area, $A$, of a sector is

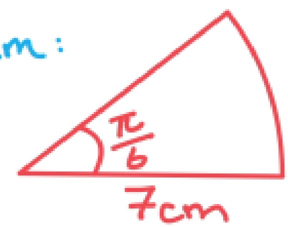
$$A = \frac{1}{2} r^2 \theta$$

  - $\theta$ is the angle measured in **radians**
  - $r$ is the radius
  - This is **given in the formula booklet**, you do not need to remember it

## Worked Example

A slice of cake forms a sector of a circle with an angle of $\frac{\pi}{6}$ radians and radius of 7 cm. Find the area of the surface of the slice of cake and its perimeter.

Draw a diagram:




Area of a sector: $A = \frac{1}{2}r^2\theta$

Substitute: $r = 7$, $\theta = \frac{\pi}{6}$

$$A = \frac{1}{2}(7)^2\left(\frac{\pi}{6}\right) = \frac{49\pi}{12}$$

$$\boxed{\text{Area} = 12.8\text{ cm}^2 \text{ (3 s.f.)}}$$



Perimeter = arc length + 2(radius)

Length of an arc: $l = r\theta$

$$P = 7\left(\frac{\pi}{6}\right) + 2(7)$$

$$\boxed{\text{Perimeter} = 17.7\text{ cm (3 s.f.)}}$$

# 3.2 Geometry of 3D Shapes

## 3.2.1 3D Coordinate Geometry

### 3D Coordinate Geometry

#### How does the 3D coordinate system work?

- In three-dimensional space we can label where any object is using the *x*-*y*-*z* coordinate system
- In the 3D cartesian system, the *x*- and *y*- axes usually represent lateral space (length and width) and the *z*-axis represents vertical height

#### What can we do with 3D coordinates?

- If we have two points with coordinates $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ then we should be able to find:
  - The **midpoint** of the two points
  - The **distance** between the two points
- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]



#### How do I find the midpoint of two points in 3D?

- The midpoint is the **average (middle) point**
  - It can be found by finding the middle of the *x*-coordinates and the middle of the *y*-coordinates
- The coordinates of the midpoint will be

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

  - This is given in the formula booklet, you do not need to remember it

#### How do I find the distance between two points in 3D?

- The distance between two points with coordinates ($(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

  - This is given in the formula booklet, you do not need to remember it

## Worked Example

The points $A$ and $B$ have coordinates (-2, 1, 5) and (4, -3, 2) respectively.

i)

Calculate the distance of the line segment $AB$.

Formula for the distance of a line segment:

in formula booklet

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

A : ($\underset{x_1}{-2}$, $\underset{y_1}{1}$, $\underset{z_1}{5}$)    B : ($\underset{x_2}{4}$, $\underset{y_2}{-3}$, $\underset{z_2}{2}$)

Substitute:

$$d = \sqrt{(-2-4)^2 + (1-(-3))^2 + (5-2)^2}$$

$$= \sqrt{(-6)^2 + 4^2 + 3^2}$$

$$= \sqrt{36 + 16 + 9}$$

$$= \sqrt{61}$$

$$\boxed{d = 7.81 \text{ units (3 sf)}}$$



ii)

Find the midpoint of [$AB$].

Formula for the midpoint of a line segment:

in formula booklet

$$MP = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

A : ($\underset{x_1}{-2}$, $\underset{y_1}{1}$, $\underset{z_1}{5}$)    B : ($\underset{x_2}{4}$, $\underset{y_2}{-3}$, $\underset{z_2}{2}$)

Substitute:

$$MP = \left(\frac{-2+4}{2}, \frac{1+(-3)}{2}, \frac{5+2}{2}\right)$$

$$= \left(\frac{2}{2}, -\frac{2}{2}, \frac{7}{2}\right)$$

$$\boxed{MP = (1, -1, 3.5)}$$

# 3.2.2 Volume & Surface Area

## Volume of 3D Shapes

### What is volume?

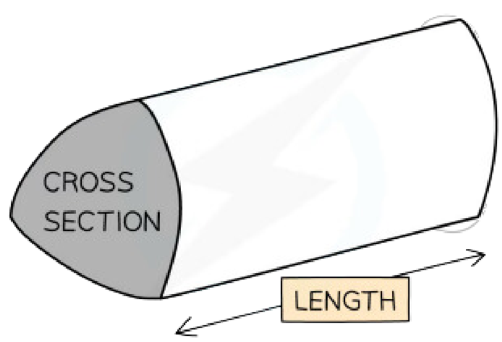
- The volume of a 3D shape is a measure of how much 3D space it takes up
  - A 3D shape is also called a **solid**
- You need to be able to calculate the volume of a number of common shapes

### How do I find the volume of cuboids, prisms and cylinders?

- A prism is a 3–D shape that has two identical **base** shapes connected by parallel **edges**
  - A prism has the same base shape all the way through
  - A **prism** takes its name from its base
- To find the **volume** of any prism use the formula:

$$\text{Volume of a prism} = Ah$$

  - Where $\boldsymbol{A}$ is the area of the cross section and $\boldsymbol{h}$ is the base height
    - $h$ could also be the length of the prism, depending on how it is oriented
  - This is in the formula booklet in the **prior learning** section at the beginning
  - The base could be any shape so as long as you know its area and length you can calculate the volume of any prism



- Note two special cases:
  - To find the volume of a cuboid use the formula:

$$\text{Volume of a cuboid} = \text{length} \times \text{width} \times \text{height}$$
$$V = lwh$$

  - The volume of a **cylinder** can be found in the same way as a prism using the formula:

$$\text{Volume of a cylinder} = \pi r^2 h$$

  - where $r$ is the radius, $h$ is the height (or length, depending on the orientation
  - Note that a cylinder is technically not a prism as its base is not a polygon, however the method for finding its volume is the same
- Both of these are **in the formula booklet** in the **prior learning** section

## How do I find the volume of pyramids and cones?

- In a **right-pyramid** the apex (the joining point of the triangular faces) is vertically above the centre of the base
  - The base can be any shape but is usually a square, rectangle or triangle
- To calculate the volume of a **right-pyramid** use the formula

$$V = \frac{1}{3} Ah$$

  - Where $A$ is the area of the base, $h$ is the height
  - Note that the height must be **vertical to the base**

- **A right cone** is a circular-based pyramid with the vertical height joining the apex to the centre of the circular base
- To calculate the volume of a **right-cone** use the formula

$$V = \frac{1}{3} \pi r^2 h$$

  - Where $r$ is the radius, $h$ is the height

- These formulae are both given in the formula booklet

## How do I find the volume of a sphere?

- To calculate the volume of a **sphere** use the formula

$$V = \frac{4}{3}\pi r^3$$

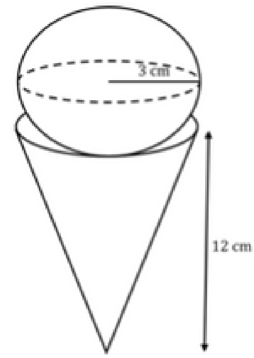
  - Where $r$ is the radius
    - the line segment from the centre of the sphere to the surface
  - This formula is given in the formula booklet

### Exam Tip

- Remember to make use of the formula booklet in the exam as all the volume formulae you need will be here
  - Formulae for basic 3D objects (cuboid, cylinder and prism) are in the **prior learning** section
  - Formulae for other 3D objects (pyramid, cone and sphere) are in the **Topic 3: Geometry** section

## Worked Example

A dessert can be modelled as a right-cone of radius 3 cm and height 12 cm and a scoop of ice-cream in the shape of a sphere of radius 3 cm. Find the total volume of the ice-cream and cone.




Volume of a sphere: $V = \frac{4}{3}\pi r^3$ (In formula booklet)

Substitute: $r = 3 \Rightarrow V = \frac{4}{3}\pi \times 3^3$

$= 36\pi$

Volume of a right cone: $V = \frac{1}{3}\pi r^2 h$ (In formula booklet)

Substitute: $r = 3,\ h = 12 \Rightarrow V = \frac{1}{3}\pi(3)^2(12)$

$= 36\pi$

Total Volume $= 72\pi \text{ cm}^3$

$\boxed{\text{Total Volume} = 226\text{cm}^3 \text{ (3sf)}}$

# Surface Area of 3D Shapes

## What is surface area?

- The surface area of a 3D shape is the sum of the areas of all the **faces** that make up a shape
  - A **face** is one of the flat or curved surfaces that make up a 3D shape
  - It often helps to consider a 3D shape in the form of its 2D net

## How do I find the surface area of cuboids, pyramids and prisms?

- Any prisms and pyramids that have polygons as their bases have only flat faces
  - The surface area is simply found by adding up the areas of these flat faces
  - Drawing a 2D net will help to see which faces the 3D shape is made up of

## How do I find the surface area of cylinders, cones and spheres?

- Cones, cylinders and spheres all have curved faces so it is not always as easy to see their shape
  - The net of a **cylinder** is made up of two identical circles and a rectangle
  - The rectangle is the curved surface area and is harder to identify
  - The length of the rectangle is the same as the circumference of the circle
  - The area of the **curved surface area** is

$$A = 2\pi rh$$

    - where $r$ is the radius, $h$ is the height
  - This is given in the formula book in the prior learning section
  - The area of the **total surface area of a cylinder** is

$$A = 2\pi rh + 2\pi r^2$$

  - This is **not** given in the formula book, however it is easy to put together as both the area of a circle and the area of the curved surface area are given

- The net of a **cone** consists of the circular base along with the curved surface area
  - The area of the **curved surface area** is

$$A = \pi rl$$

    - Where $r$ is the radius and $l$ is the **slant height**
  - This is **given in the formula book**
    - Be careful not to confuse the slant height, $l$, with the vertical height, $h$
    - Note that $r$, $h$ and $l$ will create a **right-triangle** with $l$ as the hypotenuse
  - The area of the **total surface area of a cone** is

$$A = \pi rl + \pi r^2$$

  - This is **not** given in the formula book, however it is easy to put together as both the area of a circle and the area of the curved surface area are given

- To find the surface area of a **sphere** use the formula

$$A = 4\pi r^2$$

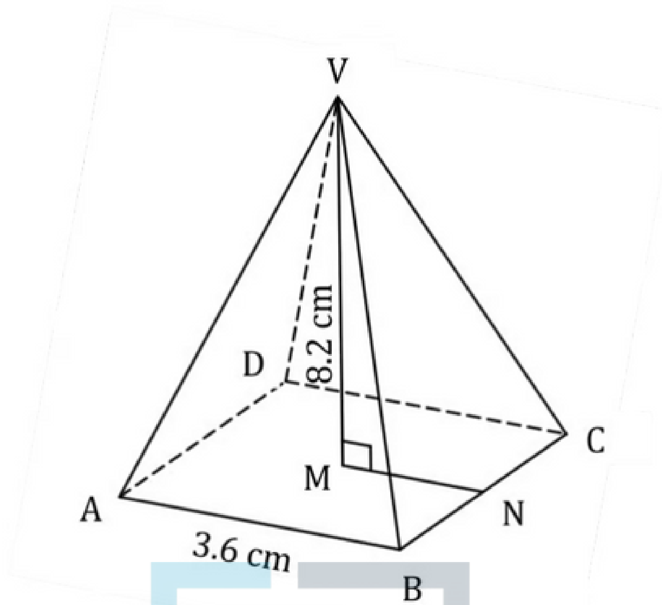
- where $r$ is the radius (line segment from the centre to the surface)
- This is given in the formula booklet, you do not have to remember it

### Exam Tip

- Remember to make use of the formula booklet in the exam as all the area formulae you need will be here
  - Formulae for basic 2D shapes (parallelogram, triangle, trapezoid, circle, curved surface of a cylinder) are in the **prior learning** section
  - Formulae for other 2D shapes (curved surface area of a cone and surface area of a sphere ) are in the **Topic 3: Geometry** section

## Worked Example

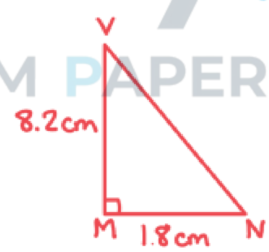
In the diagram below *ABCD* is the square base of a right pyramid with vertex *V*. The centre of the base is *M*. The sides of the square base are 3.6 cm and the vertical height is 8.2 cm.




i)

Use the Pythagorean Theorem to find the distance VN.

Sketch the triangle MNV:




M is the midpoint
so MN = 3.6 ÷ 2

By the Pythagorean Theorem:

$$VN^2 = \sqrt{VM^2 + MN^2}$$

$$= \sqrt{8.2^2 + 1.8^2}$$

$$= \sqrt{70.48}$$

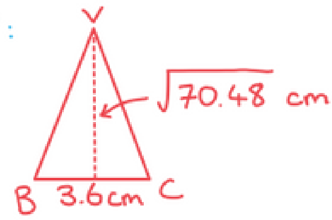
$$\boxed{VN = 8.40 \text{ cm (3sf)}}$$

ii)

Calculate the area of the triangle ABV.

Area $\Delta ABV$ = area $\Delta BCV$

Sketch $\Delta BCV$:




Area of a triangle $= \frac{1}{2}bh$

Substitute $b = 3.6$, $h = \sqrt{70.48}$

$$\text{Area} = \frac{1}{2}(3.6)(\sqrt{70.48})$$

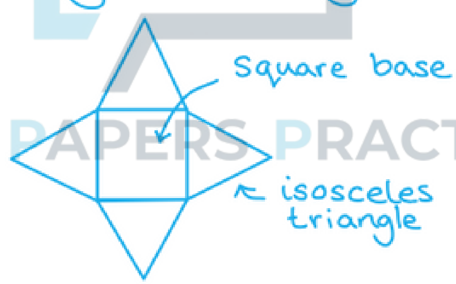
$$= 15.111\ldots \text{ cm}^2$$

$$\boxed{\text{Area } \Delta ABV = 15.1 \text{ cm}^2}$$

iii)

Find the surface area of the right pyramid.

Considering the net may help:




$$\text{Surface area} = \text{area square} + 4(\text{area triangle})$$

$$SA = 3.6^2 + 4(15.111\ldots)$$

$$= 73.405\ldots \text{ cm}^2$$

$$\boxed{SA = 73.4 \text{ cm}^2 \text{ (3sf)}}$$

# 3.3 Trigonometry

## 3.3.1 Pythagoras & Right-Angled Trigonometry

### Pythagoras

#### What is the Pythagorean theorem?

- Pythagoras' theorem is a formula that works for **right-angled triangles** only
- It states that for any right-angled triangle, the **square of the hypotenuse is equal to the sum of the squares of the two shorter sides**
  - The **hypotenuse** is the **longest side** in a right-angled triangle
    - It will always be **opposite** the right angle
  - If we label the hypotenuse c, and label the other two sides *a* and *b*, then Pythagoras' theorem tells us that

$$a^2 + b^2 = c^2$$

- The formula for Pythagoras' theorem is assumed prior knowledge and is **not in the formula booklet**
  - You will need to remember it

#### How can we use Pythagoras' theorem?

- If you know two sides of any right-angled triangle you can use Pythagoras' theorem to find the length of the third side
  - Substitute the values you have into the formula and either solve or rearrange
- To find the length of the **hypotenuse** you can use:

$$c = \sqrt{a^2 + b^2}$$

- To find the length of **one of the other sides** you can use:

$$a = \sqrt{c^2 - b^2} \text{ or } b = \sqrt{c^2 - a^2}$$

- Note that when finding the **hypotenuse** you should **add** inside the square root and when finding **one of the other sides** you should **subtract** inside the square root
- Always **check** your answer carefully to make sure that the hypotenuse is the longest side
- Note that Pythagoras' theorem questions will rarely be standalone questions and will often be 'hidden' in other geometry questions

#### What is the converse of the Pythagorean theorem?

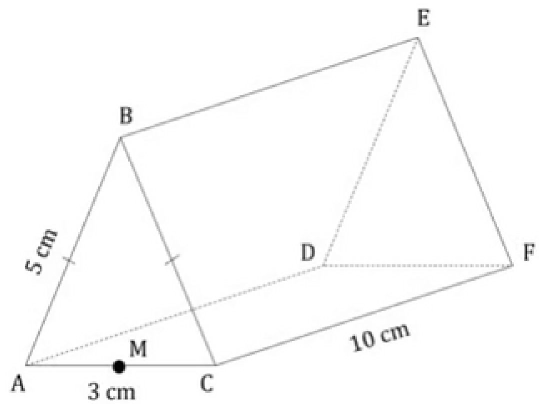
- The converse of the Pythagorean theorem states that if $a^2 + b^2 = c^2$ is true then the triangle must be a right-angled triangle
  - This is a very useful way of determining whether a triangle is right-angled
- If a diagram in a question does not clearly show that something is right-angled, you may need to use Pythagoras' theorem to check

**Exam Tip**

- Pythagoras' theorem pops up in lots of exam questions so bear it in mind whenever you see a right-angled triangle in an exam question!

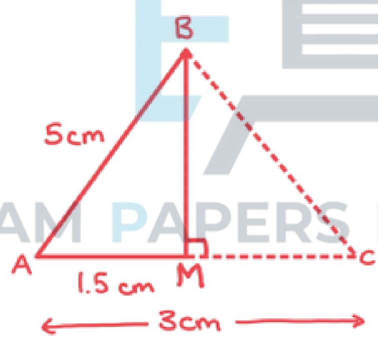
## Worked Example

ABCDEF is a chocolate bar in the shape of a triangular prism. The end of the chocolate bar is an isosceles triangle where AC = 3 cm and AB = BC = 5 cm. M is the midpoint of AC. This information is shown in the diagram below.




Calculate the length BM.

Sketch the triangle ABM:






By the Pythagorean Theorem:

$$BM^2 = \sqrt{AB^2 - AM^2}$$

shorter side — hypotenuse — shorter side

$$= \sqrt{5^2 - 1.5^2}$$

$$= \sqrt{22.75}$$

$$\boxed{BM = 4.77 \text{ cm (3sf)}}$$

# Right-Angled Trigonometry

## What is Trigonometry?

- Trigonometry is the mathematics of angles in triangles
- It looks at the relationship between side lengths and angles of triangles
- It comes from the Greek words *trigonon* meaning 'triangle' and *metron* meaning 'measure'

## What are Sin, Cos and Tan?

- The three trigonometric functions Sine, Cosine and Tangent come from ratios of side lengths in right-angled triangles
- To see how the ratios work you must first label the sides of a right-angled triangle in relation to a chosen angle
    - The **hypotenuse, *H*,** is the **longest side** in a right-angled triangle
        - It will always be **opposite** the right angle
    - If we label one of the other angles $\theta$, the side opposite $\theta$ will be labelled **opposite**, ***O*,** and the side next to $\theta$ will be labelled **adjacent**, ***A***
- The functions Sine, Cosine and Tangent are the ratios of the lengths of these sides as follows

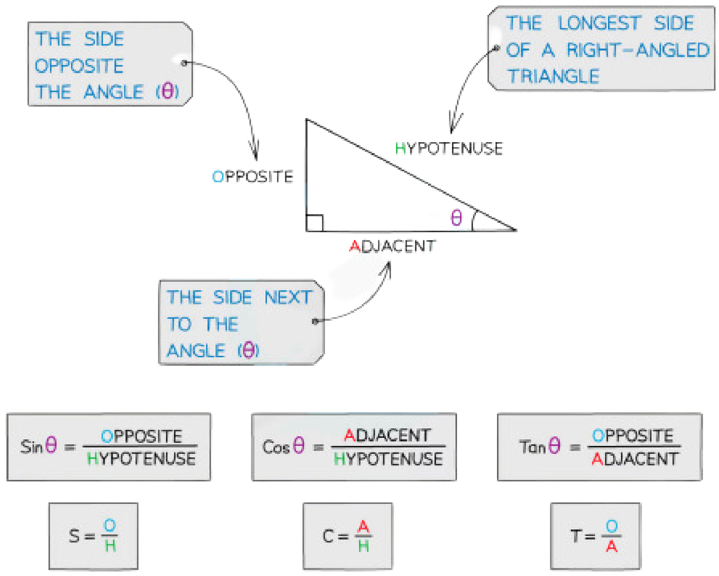
$$\text{Sin } \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H}$$

$$\text{Cos } \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{A}}{\text{H}}$$

$$\text{Tan } \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{O}}{\text{A}}$$



- These are **not in the formula book**, you must remember them
- The mnemonic **SOHCAHTOA** is often used as a way of remembering which ratio is which
    - **S**in is **O**pposite over **H**ypotenuse
    - **C**os is **A**djacent over **H**ypotenuse
    - **T**an is **O**pposite over **A**djacent




## How can we use SOHCAHTOA to find missing lengths?

- If you know the length of one of the sides of any right-angled triangle and one of the angles you can use SOHCAHTOA to find the length of the other sides
    - Always start by **labelling the sides** of the triangle with H, O and A
    - Choose the correct ratio by looking only at the values that you have and that you want
        - For example if you know the angle and the side opposite it (O) and you want to find the hypotenuse (H) you should use the sine ratio
    - Substitute the values into the ratio
    - Use your calculator to find the solution

## How can we use SOHCAHTOA to find missing angles?

- If you know two sides of any right-angled triangle you can use SOHCAHTOA to find the size of one of the angles
- Missing angles are found using the **inverse functions**:

$$\theta = \text{Sin}^{-1}\frac{O}{H}\ ,\ \theta = \text{Cos}^{-1}\frac{\text{A}}{\text{H}}\ ,\ \theta = \text{Tan}^{-1}\frac{\text{O}}{\text{A}}$$

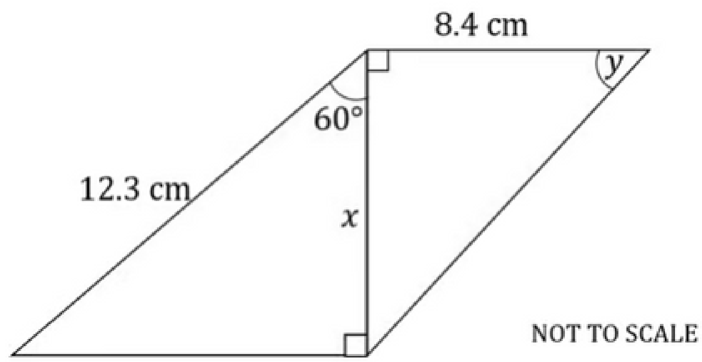
- After choosing the correct ratio and substituting the values use the inverse trigonometric functions on your calculator to find the correct answer

### Exam Tip

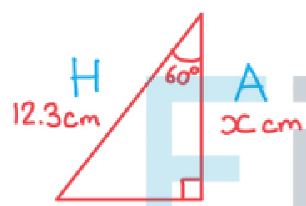
- You need to remember the sides involved in the different trig ratios as they are not given to you in the exam

## Worked Example

Find the values of $x$ and $y$ in the following diagram. Give your answers to 3 significant figures.




Start by labelling the sides of the triangle:



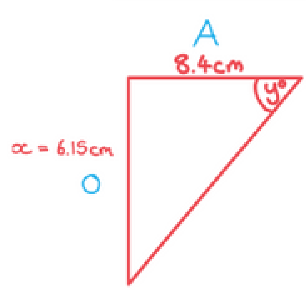

SOH$\underbrace{\text{CAH}}$TOA

We know H and we want to find A so we need to use $\cos\theta = \frac{A}{H}$

$$\cos 60° = \frac{x}{12.3}$$

$$x = 12.3\cos 60°$$

$$\boxed{x = 6.15\text{cm}}$$




SOHCAH$\underbrace{\text{TOA}}$

$$\tan y° = \frac{O}{A}$$

$$\tan y° = \frac{6.15}{8.4}$$

$$y° = \tan^{-1}\left(\frac{6.15}{8.4}\right)$$

$$\boxed{y° = 36.2° \ (3\text{ s.f.})}$$

# 3D Problems

## How does Pythagoras work in 3D?

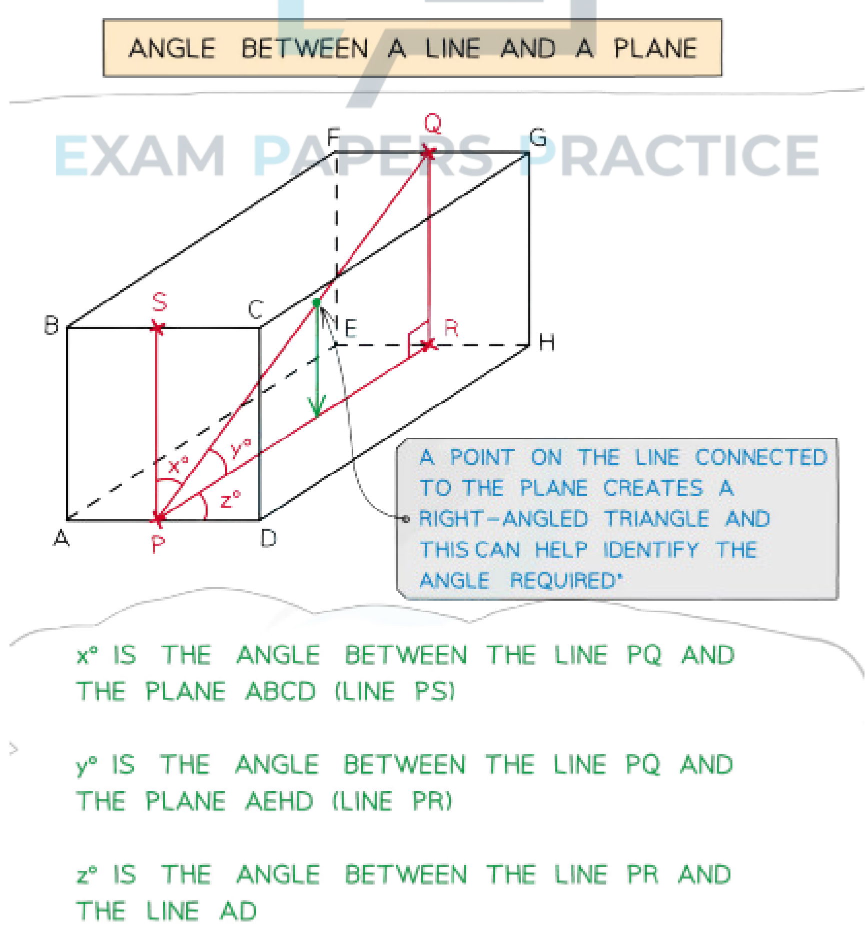
- 3D shapes can often be broken down into several 2D shapes
- With Pythagoras' Theorem you will be specifically looking for right-angled triangles
  - The right-angled triangles you need will have two known sides and one unknown side
  - Look for perpendicular lines to help you spot right-angled triangles
- There is a 3D version of the Pythagorean theorem formula:

$$d^2 = x^2 + y^2 + z^2$$

  - However it is usually easier to see a problem by breaking it down into two or more 2D problems

## How does SOHCAHTOA work in 3D?

- Again look for a combination of right-angled triangles that would lead to the missing angle or side
- The angle you are working with can be awkward in 3D
  - The angle between a line and a plane is not always obvious
  - If unsure put a point on the line and draw a new line to the plane
    - This should create a right-angled triangle



### Exam Tip

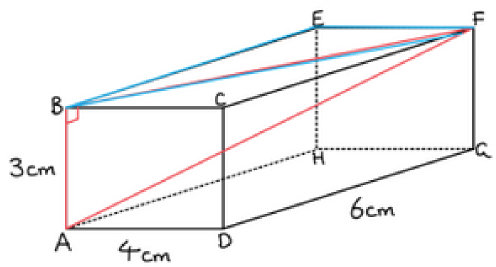
- Annotate diagrams that are given to you with values that you have calculated
- It can be useful to make additional sketches of parts of any diagrams that are given to you, especially if there are multiple lengths/angles that you are asked to find
- If you are not given a diagram, sketch a nice, big, clear one!

## Worked Example

A pencil is being put into a cuboid shaped box which has dimensions 3 cm by 4 cm by 6 cm. Find:

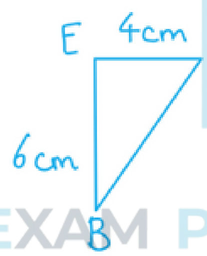
a)
the length of the longest pencil that could fit inside the box,

Draw a diagram:




The longest pencil could fit on any of the diagonals, e.g. AF.

To find AF we must first find BF:

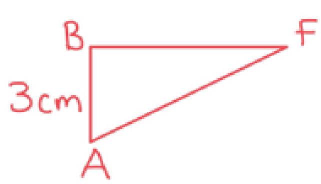



$BF^2 = 4^2 + 6^2$

$BF^2 = 16 + 36$

$BF^2 = 52$ ← can leave as $BF^2$ for now.






$AF^2 = 3^2 + BF^2$

$= 9 + 52$

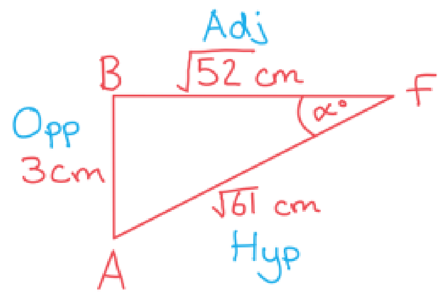
$AF^2 = 61$

$AF = \sqrt{61} = 7.8102...$

$\boxed{7.81 \text{ cm } (3 \text{ s.f.})}$

b)
the angle that the pencil would make with the top of the box.

Find $A\hat{F}B$ :




All three sides are known so can use any of the trig ratios.

S$^{O}$H C$^{A}$H T$^{O}$A

Choose $\tan\alpha = \dfrac{opp}{adj}$

$$\tan\alpha = \frac{3}{\sqrt{52}}$$

$$\alpha = \tan^{-1}\left(\frac{3}{\sqrt{52}}\right)$$

$$= 22.588...$$

$$\boxed{A\hat{F}B = 22.6° \text{ (3 s.f.)}}$$

# 3.3.2 Non Right-Angled Trigonometry

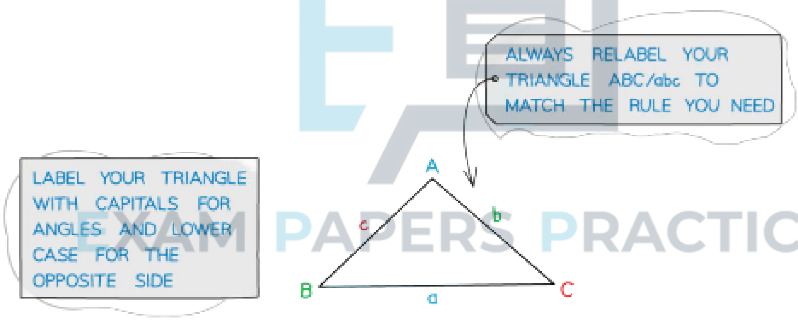
## Sine Rule

### What is the sine rule?

- The sine rule allows us to find missing side lengths or angles in **non-right-angled triangles**
- It states that for any triangle with angles *A*, *B* and *C*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

  - Where
    - *a* is the side **opposite** angle *A*
    - *b* is the side **opposite** angle *B*
    - c is the side **opposite** angle C
- This formula **is in the formula booklet**, you do not need to remember it
- Sin 90° = 1 so if one of the angles is 90° this becomes SOH from **SOHCAHTOA**




### How can we use the sine rule to find missing side lengths or angles?

- The sine rule can be used when you have any opposite pairs of sides and angles
- Always **start by labelling your triangle** with the angles and sides
  - Remember the sides with the lower-case letters are **opposite** the angles with the equivalent upper-case letters
- Use the formula in the formula booklet to find the **length of a side**
- To find a missing angle you can rearrange the formula and use the form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

  - This is **not in the formula booklet** but can easily be found by rearranging the one given

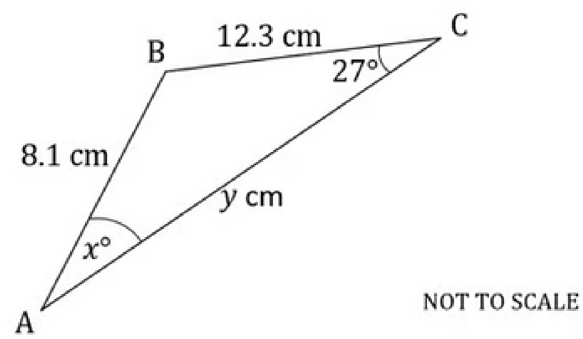
- Substitute the values you have into the formula and solve

**Exam Tip**

- If you're using a calculator make sure that it is in the correct mode (degrees/radians)
- Remember to give your answers as exact values if you are asked too

## Worked Example

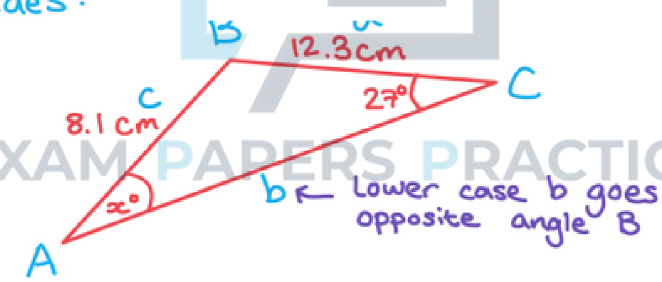
The following diagram shows triangle ABC. AB = 8.1 cm, AC = 12.3 cm, $B\hat{C}A = 27°$.




Use the sine rule to calculate the value of:

i)
$x$,

Sketch the diagram and label the sides:




Using the sine rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ ← We are looking for an angle so this version is easier.

$$\frac{\sin x}{12.3} = \frac{\sin 27}{8.1}$$

$$\sin x = \frac{12.3 \sin 27}{8.1}$$

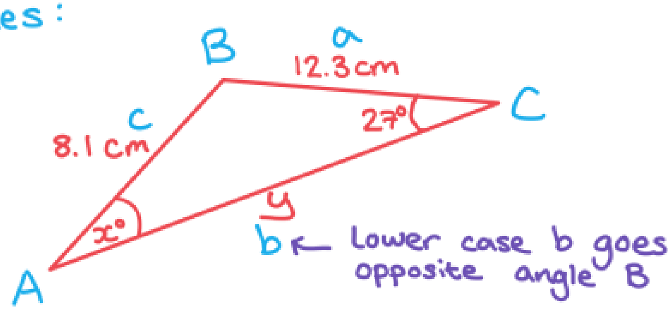
$$x = \sin^{-1}\left(\frac{12.3 \sin 27}{8.1}\right)$$

$$\boxed{x = 43.6° \text{ (3s.f.)}}$$

ii)

$y$.

Sketch the diagram and label the sides:




Find $A\hat{B}C$: $180 - (27 + 43.582...)$

$A\hat{B}C = 109.417...$

Using the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

We are looking for a side so this version is easier.

$$\frac{y}{\sin(109.417...)} = \frac{8.1}{\sin 27}$$

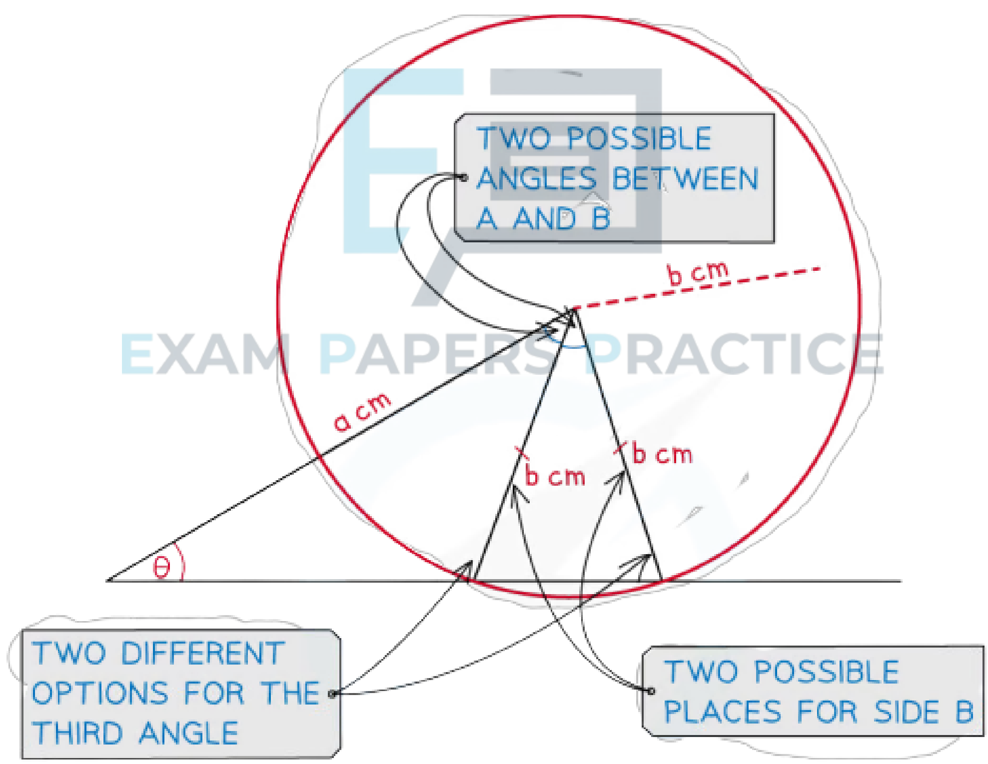
$$y = \frac{8.1 \sin(109.417...)}{\sin 27}$$

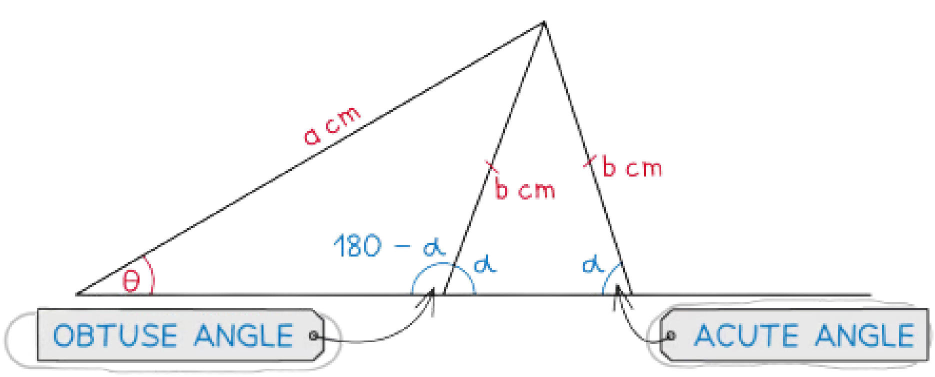
$$\boxed{y = 16.8\text{cm } (3\text{s.f.})}$$

## Ambiguous Sine Rule

### What is the ambiguous case of the sine rule?

- If the sine rule is used in a triangle **given two sides and an angle which is not the angle between them** there **may** be more than one possible triangle which could be drawn
- The side **opposite** the given angle could be in two possible positions
- This will create two possible values for each of the missing angles and two possible lengths for the missing side
- The two angles found **opposite** the given side (not the ambiguous side) will **add up to 180°**
    - In IB the question will usually tell you whether the angle you are looking for is **acute** or **obtuse**
    - The sine rule will always give you the acute option but you can **subtract from 180°** to find the obtuse angle
    - Sometimes the obtuse angle will not be valid
        - It could cause the sum of the three interior angles of the triangle to exceed 180°



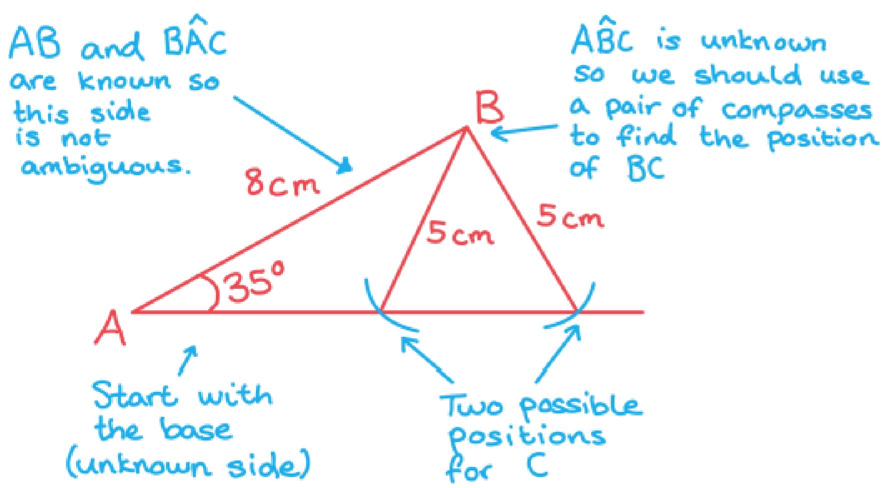
## Exam Tip

- Make sure that you are clear which of the two answers is the one that is required and make sure that you communicate this clearly to the examiner by writing it on the answer line!

## Worked Example

Given triangle $\mathrm{ABC}$, $\mathrm{AB} = 8\ \mathrm{cm}$, $\mathrm{BC} = 5\ \mathrm{cm}$, $\mathrm{B\widehat{A}C} = 35°$. Find the two possible options for $\mathrm{A\widehat{C}B}$, giving both answers to 1 decimal place.

There are two ways triangle ABC can be drawn:




Find $\mathrm{A\hat{C}B}$ : $\dfrac{\sin 35°}{5} = \dfrac{\sin C}{8}$

$$C = \sin^{-1}\left(\frac{8\sin 35°}{5}\right)$$

$$= 66.59...$$

$$\boxed{\mathrm{A\hat{C}B} = 66.6° \text{ or } 113.4° \text{ (1dp)}}$$

# Cosine Rule

## What is the cosine rule?

- The cosine rule allows us to find missing side lengths or angles in **non-right-angled triangles**
- It states that for any triangle

$$c^2 = a^2 + b^2 - 2ab\cos C \; ; \; \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

  - Where
    - *c* is the side **opposite** angle *C*
    - *a* and *b* are the other two sides
- Both of these formulae **are in the formula booklet**, you do not need to remember them
  - The first version is used to find a missing side
  - The second version is a rearrangement of this and can be used to find a missing angle
- Cos 90° = 0 so if C = 90° this becomes **Pythagoras' Theorem**

## How can we use the cosine rule to find missing side lengths or angles?

- The cosine rule can be used when you have two sides and the angle between them or all three sides
- Always **start by labelling your triangle** with the angles and sides
  - Remember the sides with the lower-case letters are **opposite** the angles with the equivalent upper-case letters
- Use the formula $c^2 = a^2 + b^2 - 2ab\cos C$ to find an unknown side
- Use the formula $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ to find an unknown angle
  - *C* is the angle **between** sides *a* and *b*
- Substitute the values you have into the formula and solve

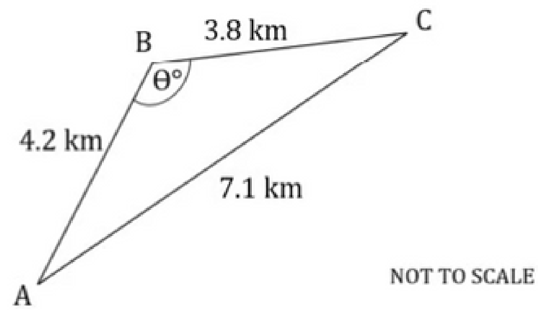


### Exam Tip

- If you're using a calculator make sure that it is in the correct mode (degrees/radians)
- Remember to give your answers as exact values if you are asked too

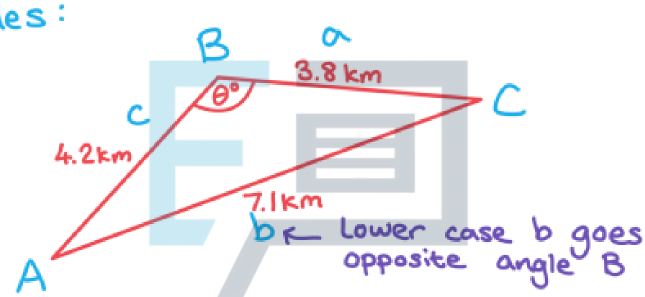
## Worked Example

The following diagram shows triangle $ABC$. $AB = 4.2$ km, $BC = 3.8$ km, $AC = 7.1$ km.




Calculate the value of $A\hat{B}C$.

Sketch the diagram and label the sides:




Using the cosine rule:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$ ← We are looking for an angle so this version is easier.

$$\cos \theta = \frac{4.2^2 + 3.8^2 - 7.1^2}{2(4.2)(3.8)}$$

$$\theta = \cos^{-1}\left(\frac{4.2^2 + 3.8^2 - 7.1^2}{2(4.2)(3.8)}\right)$$

$$\boxed{\theta = 128^\circ \ (3 \text{ s.f.})}$$

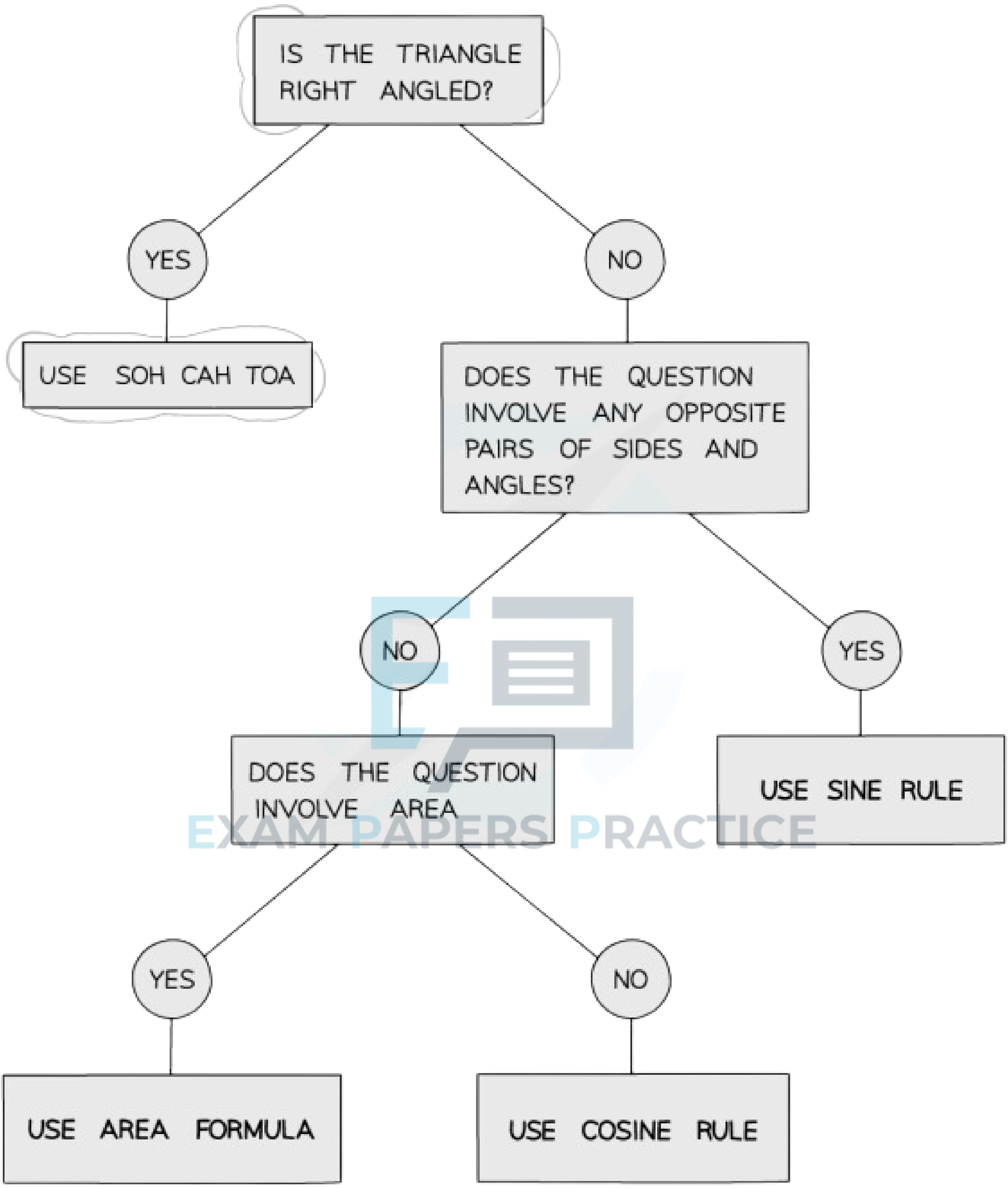
# Area of a Triangle

## How do I find the area of a non-right triangle?

- The area of **any triangle** can be found using the formula

$$A = \frac{1}{2}ab\sin C$$

  - Where C is the angle between sides *a* and *b*
  - This formula **is in the formula booklet**, you do not need to remember it
- Be careful to label your triangle correctly so that C is always the angle **between** the two sides
- Sin 90° = 1 so if C = 90° this becomes Area = ½ × base × height

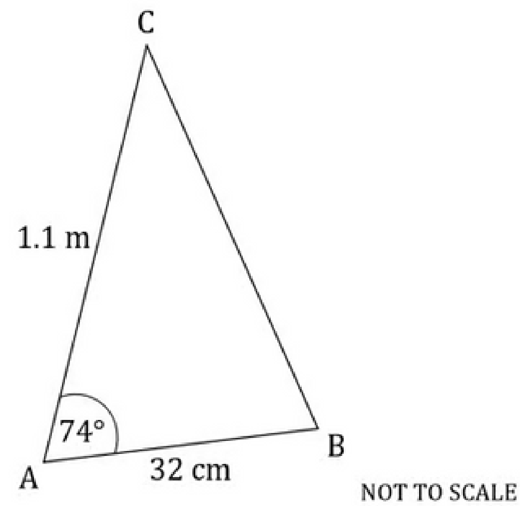



### Exam Tip

- If you're using a calculator make sure that it is in the correct mode (degrees/radians)
- Remember to give your answers as exact values if you are asked too

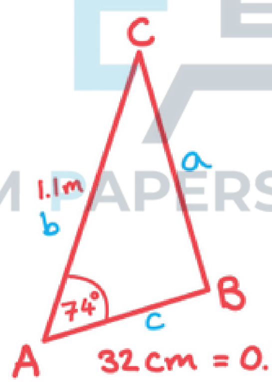
## Worked Example

The following diagram shows triangle ABC. $AB = 32$ cm, $AC = 1.1$ m, $B\widehat{A}C = 74°$.




Calculate the area of triangle .

Label the sides of the triangle:




32 cm = 0.32m

change all units to be the same

Area of a triangle: $A = \frac{1}{2}ab\sin C$

$A = \frac{1}{2}(1.1)(0.32)\sin 74°$

$$\boxed{A = 0.169 \text{ m}^2}$$

# 3.3.3 Applications of Trigonometry & Pythagoras

## Bearings

### What are bearings?

- **Bearings** are a way of describing and using **directions** as **angles**
- They are specifically defined for use in navigation because they give a precise **location** and/or **direction**

### How are bearings defined?

- There are **three rules** which must be followed every time a bearing is defined
  - They are **measured** from the **North** direction
    - An arrow showing the North line should be included on the diagram
  - They are **measured clockwise**
  - The angle is always written in **3 figures**
    - If the angle is less than 100° the first digit will be a zero

### What are bearings used for?

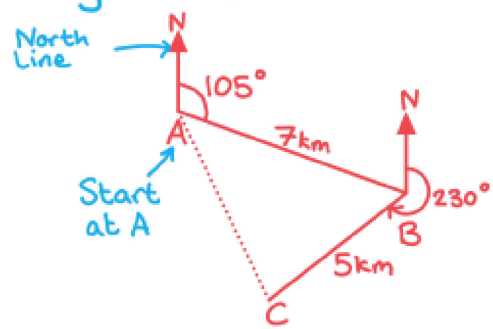
- Bearings questions will normally involve the use of Pythagoras or trigonometry to find missing distances (lengths) and directions (angles) within navigation questions
  - You should always **draw a diagram**
- There may be a scale given or you may need to consider using a scale
  - However normally in IB you will be using triangle calculations to find the distances
- Some questions may also involve the use of angle facts to find the missing directions
- To answer a question involving **drawing bearings** the following steps may help:
  - STEP 1: Draw a diagram adding in any points and distances you have been given
  - STEP 2: Draw a North line (arrow pointing vertically up) at the point you wish to measure the bearing **from**
    - If you are given the bearing **from A to B** draw the North line at **A**
  - STEP 3: Measure the angle of the bearing given **from the North line** in the **clockwise direction**
  - STEP 4: Draw a line and add the point B at the given distance
- You will likely then need to use trigonometry to calculate the shortest distance or another given distance

> Exam Tip
>
> - **Always** draw a big, clear diagram and annotate it, be especially careful to label the angles in the correct places!

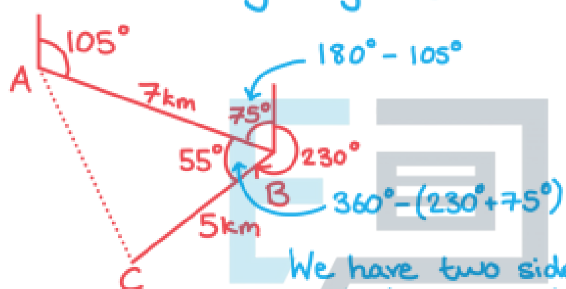
## Worked Example

The point B is 7 km from A on a bearing of 105°. The distance from B to C is 5 km and the bearing from B to C is 230°. Find the distance from A to C.

Always start with a diagram:




Fill in the angles you can on the diagram




We have two sides and the angle between them so we can use the cosine rule for the third side



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$AC^2 = 7^2 + 5^2 - 2(7)(5)\cos(55°)$$

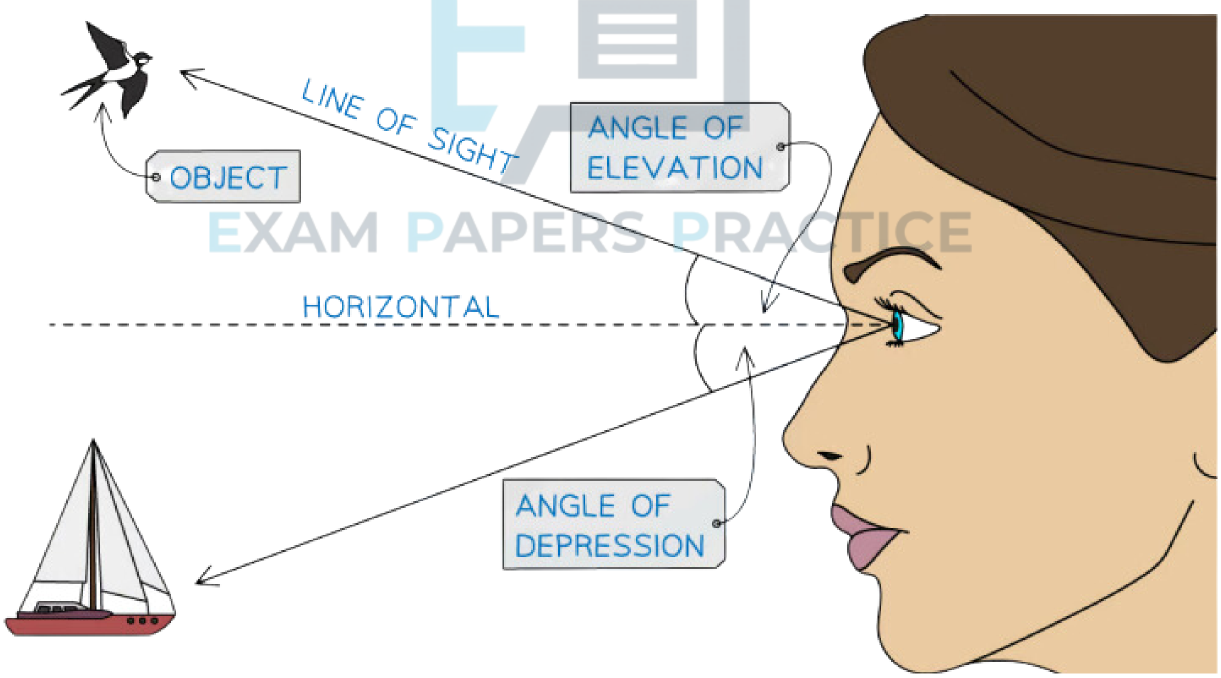
$$= 33.849...$$

$$\boxed{AC = 5.82 \text{ km } (3 \text{ s.f.})}$$

# Elevation & Depression

## What are the angles of elevation and depression?

- If a person looks at an **object** that is not on the same horizontal line as their eye-level they will be looking at either an angle of **elevation** or **depression**
    - If a person looks **up** at an object their line of sight will be at an **angle of elevation** with the horizontal
    - If a person looks **down** at an object their line of sight will be at an **angle of depression** with the horizontal
- Angles of elevation and depression are measured **from the horizontal**
- **Right-angled trigonometry** can be used to find an angle of elevation or depression or a missing distance
- Tan is often used in real-life scenarios with angles of elevation and depression
    - For example if we know the distance we are standing from a tree and the angle of elevation of the top of the tree we can use Tan to find its height
    - Or if we are looking at a boat at to sea and we know our height above sea level and the angle of depression we can find how far away the boat is




### Exam Tip

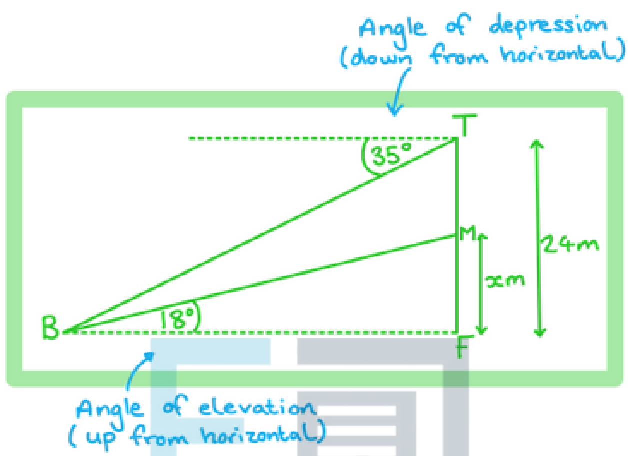
- It may be useful to draw more than one diagram if the triangles that you are interested in overlap one another

## Worked Example

A cliff is perpendicular to the sea and the top of the cliff stands 24 m above the level of the sea. The angle of depression from the cliff to a boat at sea is 35°. At a point $x$ m up the cliff is a flag marker and the angle of elevation from the boat to the flag marker is 18°.

a)

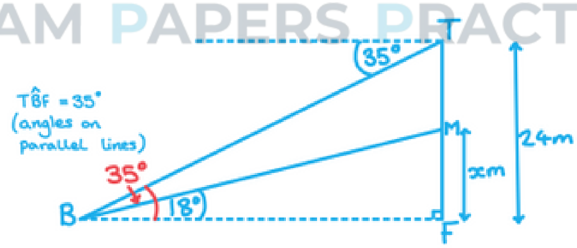
Draw and label a diagram to show the top of the cliff, T, the foot of the cliff, F, the flag marker, M, and the boat, B, labelling all the angles and distances given above.



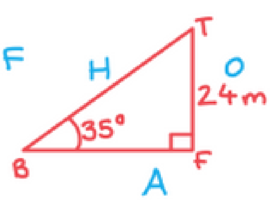

b)

Find the distance from the boat to the foot of the cliff.






Consider triangle TBF




SOHCAHTOA

we have opposite and adjacent so use Tan

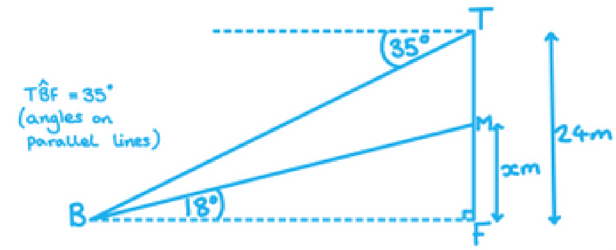
$$\tan 35° = \frac{24}{BF}$$

$$BF = \frac{24}{\tan 35°}$$

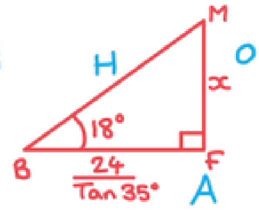
$$\boxed{BF = 34.3\text{m (3s.f.)}}$$

c)

Find the value of $x$.




Consider triangle FBM




SOHCAHTOA
we have opposite
and adjacent so
use Tan

$$\text{Tan } 18° = \frac{x}{\left(\frac{24}{\text{Tan } 35°}\right)}$$

$$x = \text{Tan } 18° \times \left(\frac{24}{\text{Tan } 35°}\right)$$

$$= 11.136...$$

$$\boxed{x = 11.1\text{m (3s.f.)}}$$

# Constructing Diagrams

## What diagrams will I need to construct?

- In IB you will be expected to construct diagrams based on information given
- The information will include **compass directions**, **bearings**, **angles**
    - Look out for the **plane** the diagram should be drawn in
    - It will either be **horizontal** (something occurring at sea or on the ground)
    - Or it will be **vertical** (Including height)
- Work through the statements given in the instructions systematically

## What do I need to know?

- Your diagrams will be sketches, they do not need to be accurate or to scale
    - However the more accurate your diagram is the easier it is to work with
- Read the full set of instructions once before beginning to draw the diagram so you have a rough idea of where each object is
- Make sure you know your **compass directions**
    - **Due east** means on a **bearing of 090°**
        - Draw the line directly to the right
    - **Due south** means on a **bearing of 180°**
        - Draw the line vertically downwards
    - **Due west** means on a **bearing of 270°**
        - Draw the line directly to the left
    - **Due north** means on a **bearing of 360° (or 000°)**
        - Draw the line vertically upwards
- Using the above bearings for compass directions will help you to estimate angles for other bearings on your diagram



### Exam Tip

- Draw your diagrams in pencil so that you can easily erase any errors

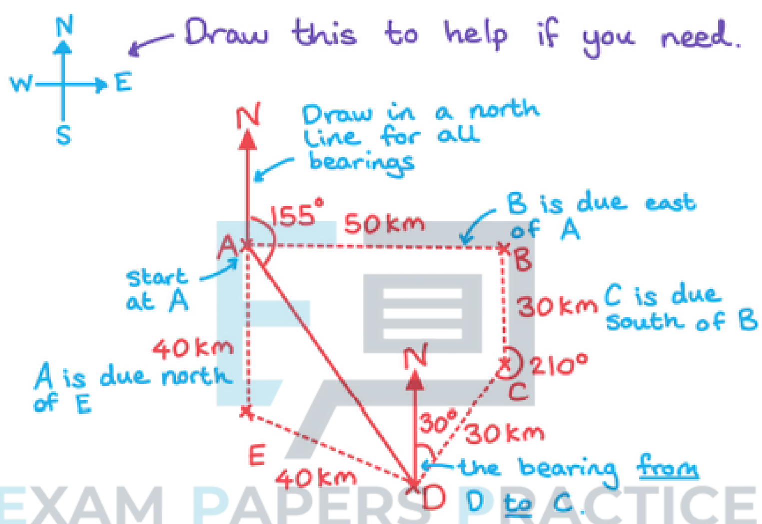
## Worked Example

A city at B is due east of a city at A and A is due north of a city at E. A city at C is due south of B.

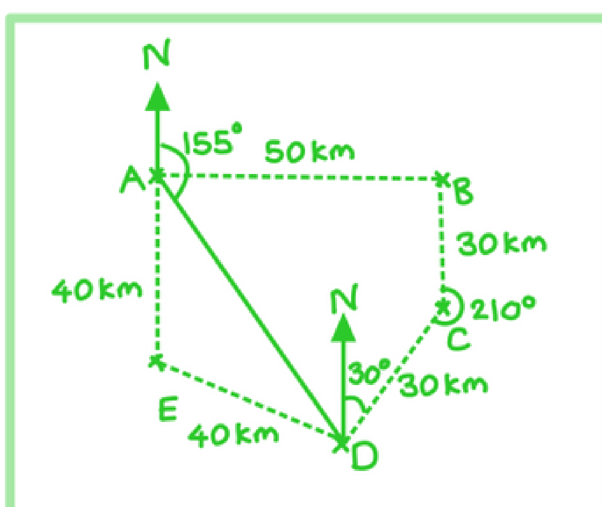
The bearing from A to D is 155° and the bearing from D to C is 30°.

The distance AB = 50 km, the distances BC = CD = 30 km and the distances DE = AE = 40 km.

Draw and label a diagram to show the cities A, B, C, D and E and clearly mark the bearings and distances given.





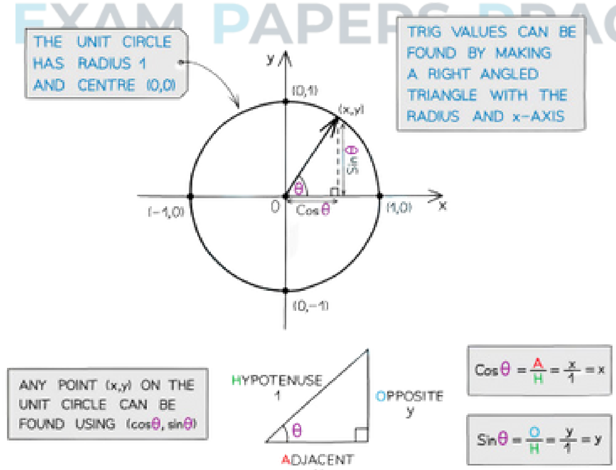
# 3.4 Further Trigonometry

## 3.4.1 The Unit Circle

### Defining Sin, Cos and Tan

**What is the unit circle?**

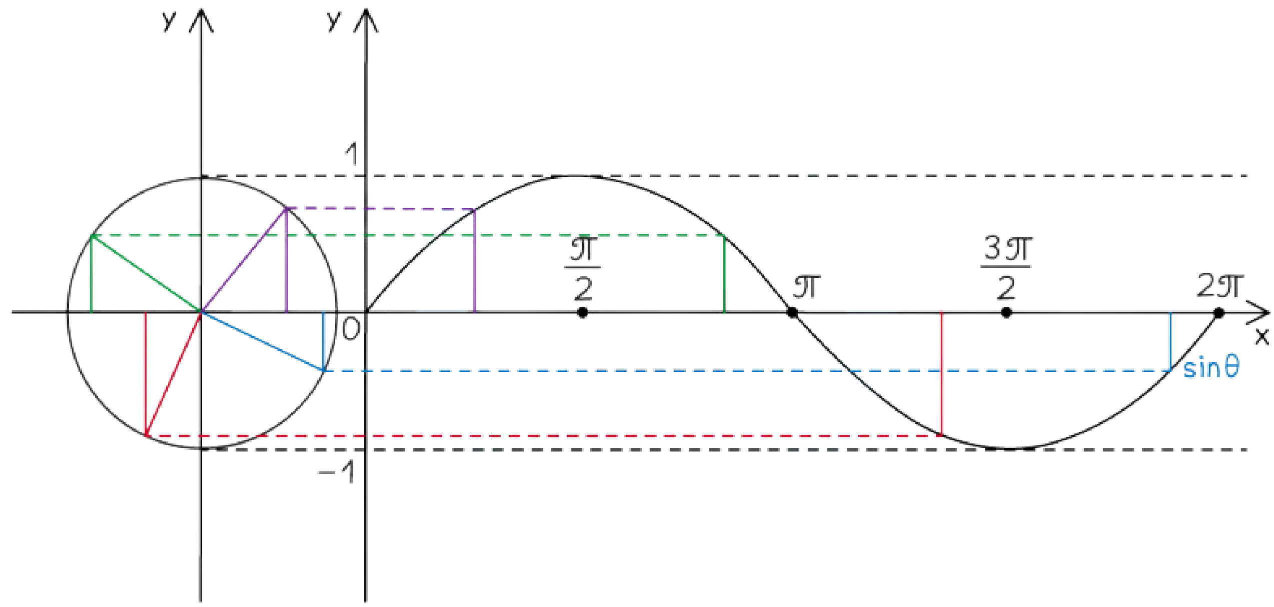
- The unit circle is a circle with radius 1 and centre (0, 0)
- Angles are always measured from the positive x-axis and turn:
  - **anticlockwise** for **positive** angles
  - **clockwise** for **negative** angles
- It can be used to calculate trig values as a coordinate point (x, y) on the circle
  - Trig values can be found by making a right triangle with the radius as the hypotenuse
  - $\theta$ is the angle measured anticlockwise from the positive $x$-axis
  - The $x$-axis will always be adjacent to the angle, $\theta$
- SOHCAHTOA can be used to find the values of sin$\theta$, cos$\theta$ and tan$\theta$ easily
- As the radius is 1 unit
  - the ***x* coordinate** gives the value of **cos$\theta$**
  - the ***y* coordinate** gives the value of **sin$\theta$**
- As the origin is one of the end points - dividing the y coordinate by the x coordinate gives the gradient
  - the **gradient** of the line gives the value of **tan$\theta$**
- It allows us to calculate sin, cos and tan for angles greater than 90° ($\frac{\pi}{2}$ rad)



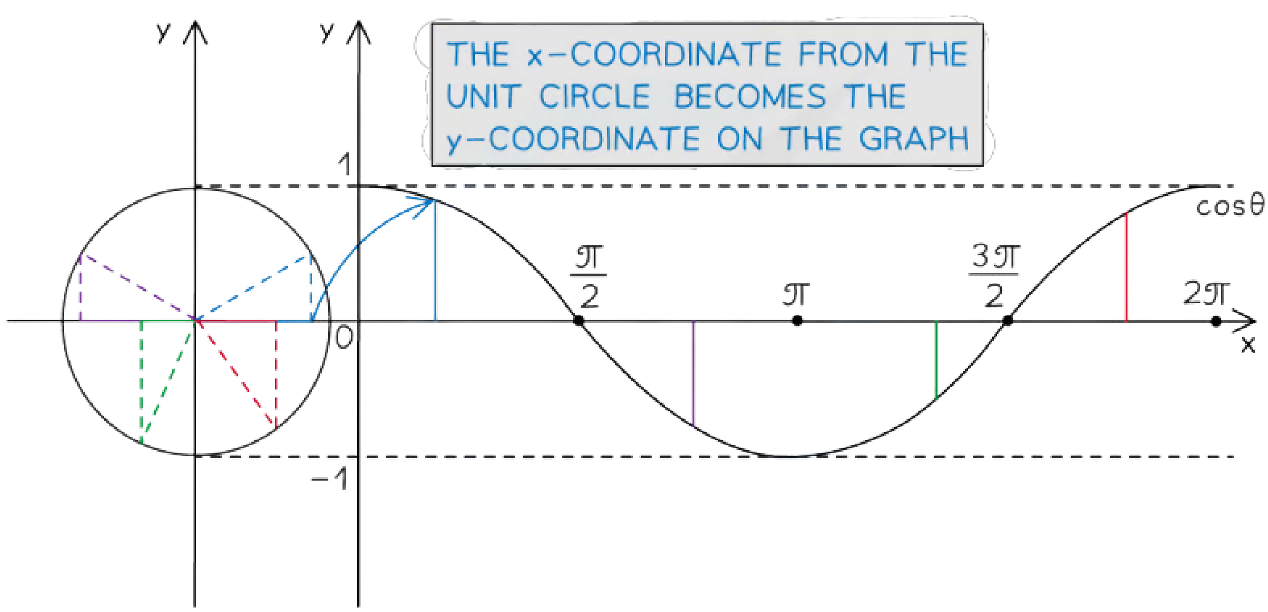

**How is the unit circle used to construct the graphs of sine and cosine?**

- On the unit circle the ***y*-coordinates** give the value of **sine**
  - Plot the $y$-coordinate from the unit circle as the $y$-coordinate on a trig graph for $x$-coordinates of $\theta = 0, \pi/2, \pi, 3\pi/2$ and $2\pi$
  - Join these points up using a smooth curve

- To get a clearer idea of the shape of the curve the points for $x$-coordinates of $\theta$ = π/4, 3π/4, 5π/4 and 7π/4 could also be plotted



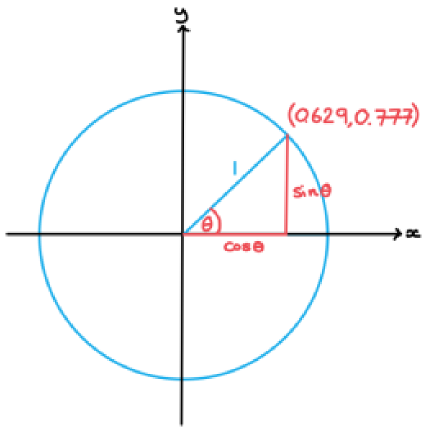

- On the unit circle the **x-coordinates** give the value of **cosine**
  - Plot the $x$-coordinate from the unit circle as the $y$-coordinate on a trig graph for $x$-coordinates of $\theta$ = 0, π/4, π/2, 3π/4 and 2π
  - Join these points up using a smooth curve
    - To get a clearer idea of the shape of the curve the points for $x$-coordinates of $\theta$ = π/4, 3π/4, 5π/4 and 7π/4 could also be plotted




- Looking at the unit circle alongside of the sine or cosine graph will help to visualise this clearer

## Worked Example

The coordinates of a point on a unit circle, to 3 significant figures, are (0.629, 0.777).
Find $\theta$° to the nearest degree.




We know $(x, y) = (\cos\theta, \sin\theta)$

so,

$\cos\theta = 0.629$

$\sin\theta = 0.777$

Using either ratio:

$\theta = \cos^{-1}(0.629)$

$= 51.023...$

$$\boxed{\theta = 51° \text{ (nearest degree)}}$$

# Using The Unit Circle

## What are the properties of the unit circle?

- The unit circle can be split into four **quadrants** at every 90° ($\frac{\pi}{2}$ rad)
  - The first quadrant is for angles between 0 and 90°
    - **A**ll three of Sinθ, Cosθ and Tanθ are positive in this quadrant
  - The second quadrant is for angles between 90° and 180° ($\frac{\pi}{2}$ rad and π rad)
    - **S**inθ is positive in this quadrant
  - The third quadrant is for angles between 180° and 270° (π rad and $\frac{3\pi}{2}$)
    - **T**anθ is positive in this quadrant
  - The fourth quadrant is for angles between 270° and 360° ($\frac{3\pi}{2}$ rad and 2π)
    - **C**osθ is positive in this quadrant
  - Starting from the **fourth** quadrant (on the bottom right) and working anti-clockwise the positive trig functions spell out **CAST**
    - This is why it is often thought of as the **CAST** diagram
    - You may have your own way of remembering this
    - A popular one starting from the first quadrant is **All Students Take Calculus**
  - To help picture this better try sketching all three trig graphs on one set of axes and look at which graphs are positive in each 90° section

## How is the unit circle used to find secondary solutions?

- Trigonometric functions have more than one input to each output
  - For example sin 30° = sin 150° = 0.5
  - This means that trigonometric equations have more than one solution
  - For example both 30° and 150° satisfy the equation sin $x$ = 0.5
- The unit circle can be used to find all solutions to trigonometric equations in a given interval
  - Your calculator will only give you the first solution to a problem such as $x = \sin^{-1}(0.5)$
    - This solution is called the **primary value**
  - However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
    - Further solutions are called the **secondary values**
  - This is why you will be given a **domain** in which your solutions should be found
    - This could either be in degrees or in radians
    - If you see π or some multiple of π then you must work in radians
- The following steps may help you use the unit circle to find **secondary values**

STEP 1: Draw the angle into the first quadrant using the $x$ or $y$ coordinate to help you

- If you are working with sin $x = k$, draw the line from the origin to the circumference of the circle at the point where the **y coordinate** is $k$
- If you are working with cos $x = k$, draw the line from the origin to the circumference of the circle at the point where the **x coordinate** is $k$
- If you are working with tan $x = k$, draw the line from the origin to the circumference of the circle such that the gradient of the line is $k$

- Note that whilst this method works for tan, it is complicated and generally unnecessary, tan $x$ repeats every 180° ($\pi$ radians) so the quickest method is just to add or subtract multiples of 180° to the primary value
- This will give you the angle which should be measured from the **positive x-axis ...**
  - ... anticlockwise for a positive angle
  - ... clockwise for a negative angle

STEP 2: Draw the radius in the other quadrant which has the same...

- ... $x$-coordinate if solving $\cos x = k$
  - This will be the quadrant which is vertical to the original quadrant
- ... $y$-coordinate if solving $\sin x = k$
  - This will be the quadrant which is horizontal to the original quadrant
- ... gradient if solving $\tan x = k$
  - This will be the quadrant diagonally across from the original quadrant

STEP 3: Work out the size of the second angle, measuring from the positive x-axis

- ... anticlockwise for a positive angle
- ... clockwise for a negative angle
  - You should look at the given range of values to decide whether you need the negative or positive angle

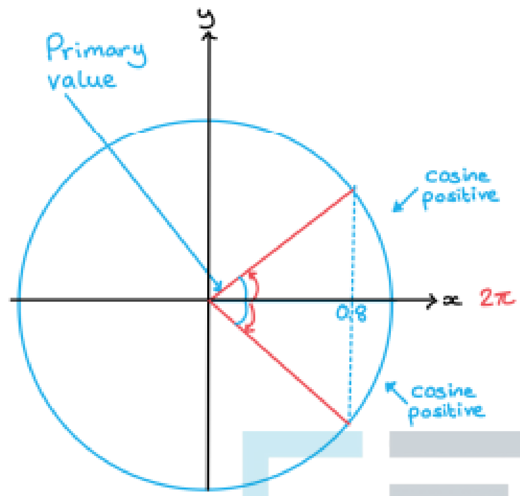
STEP 4: Add or subtract either 360° or $2\pi$ radians to both values until you have all solutions in the required range

### Exam Tip

- Being able to sketch out the unit circle and remembering CAST can help you to find all solutions to a problem in an exam question

## Worked Example

Given that one solution of $\cos\theta = 0.8$ is $\theta = 0.6435$ radians correct to 4 decimal places, find all other solutions in the range $-2\pi \le \theta \le 2\pi$. Give your answers correct to 3 significant figures.




Cosine is positive in the first and fourth quadrants so draw the angle from the horizontal axis in both quadrants.

Primary value $= 0.6435$

Using diagram, secondary value $= -0.6435$

Therefore all values are: $0.6435 \pm 2\pi n$

and $-0.6435 \pm 2\pi n$

Within given domain: $-2\pi \le \theta \le 2\pi$

$$\boxed{\theta = -5.64^c,\ -0.644^c,\ 0.644^c,\ 5.64^c}$$

# 3.4.2 Simple Identities

## Simple Identities

### What is a trigonometric identity?

- Trigonometric identities are statements that are true for all values of $x$ or $\theta$
- They are used to help simplify trigonometric equations before solving them
- Sometimes you may see identities written with the symbol $\equiv$
  - This means 'identical to'

### What trigonometric identities do I need to know?

- The two trigonometric identities you must know are
  - $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
    - This is the identity for $\tan\theta$
  - $\sin^2\theta + \cos^2\theta = 1$
    - This is the Pythagorean identity
    - Note that the notation $\sin^2\theta$ is the same as $(\sin\theta)^2$
- Both identities can be found **in the formula booklet**
- Rearranging the second identity often makes it easier to work with
  - $\sin^2\theta = 1 - \cos^2\theta$
  - $\cos^2\theta = 1 - \sin^2\theta$

### Where do the trigonometric identities come from?

- You do not need to know the proof for these identities but it is a good idea to know where they come from
- From SOHCAHTOA we know that
  - $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{O}{H}$
  - $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{A}{H}$
  - $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{O}{A}$
- The identity for $\tan\theta$ can be seen by diving $\sin\theta$ by $\cos\theta$?
  - $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{O}{H}}{\frac{A}{H}} = \dfrac{O}{A} = \tan\theta$
  - This can also be seen from the unit circle by considering a right-triangle with a hypotenuse of 1
    - $\tan\theta = \dfrac{O}{A} = \dfrac{\sin\theta}{\cos\theta}$
- The Pythagorean identity can be seen by considering a right-triangle on the unit circle with a hypotenuse of 1
  - Then $(\text{opposite})^2 + (\text{adjacent})^2 = 1$
  - Therefore $\sin^2\theta + \cos^2\theta = 1$
- Considering the equation of the unit circle also shows the Pythagorean identity

- The equation of the unit circle is $x^2 + y^2 = 1$
- The coordinates on the unit circle are $(\cos\theta, \sin\theta)$
- Therefore the equation of the unit circle could be written $\cos^2\theta + \sin^2\theta = 1$

- A third very useful identity is $\sin\theta = \cos(90° - \theta)$ or $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$
  - This is not included in the formula booklet but is useful to remember

## How are the trigonometric identities used?

- Most commonly trigonometric identities are used to change an equation into a form that allows it to be solved
- They can also be used to prove further identities such as the **double angle formulae**

### Exam Tip

- If you are asked to show that one thing is identical (≡) to another, look at what parts are missing – for example, if tan x has gone it must have been substituted

## Worked Example

Show that the equation $2\sin^2 x - \cos x = 0$ can be written in the form $a\cos^2 x + b\cos x + c = 0$, where $a$, $b$ and $c$ are integers to be found.

$2\sin^2 x - \cos x = 0$

Equation has both $\sin x$ and $\cos x$ so will need changing before it can be solved.

Use the identity $\sin^2 x = 1 - \cos^2 x$

Substitute : $2(1 - \cos^2 x) - \cos x = 0$

Expand : $2 - 2\cos^2 x - \cos x = 0$

Rearrange : $2\cos^2 x + \cos x - 2 = 0$

$\boxed{a = 2, \; b = 1, \; c = -2}$

# 3.4.3 Solving Trigonometric Equations

## Graphs of Trigonometric Functions

### What are the graphs of trigonometric functions?

- The trigonometric functions sin, cos and tan all have special **periodic graphs**
- You'll need to know their properties and how to sketch them for a given domain in either **degrees** or **radians**
- Sketching the trigonometric graphs can help to
  - Solve trigonometric equations and find all solutions
  - Understand transformations of trigonometric functions

### What are the properties of the graphs of sin x and cos x?

- The graphs of sin x and cos x are both **periodic**
  - They **repeat every 360°** (2π radians)
  - The angle will always be on the x-axis
    - Either in degrees or radians
- The graphs of sin x and cos x are always in the **range −1 ≤ y ≤ 1**
  - **Domain:** $\{x \mid x \in \mathbb{R}\}$
  - **Range:** $\{y \mid -1 \le y \le 1\}$
  - The graphs of sin x and cos x are identical however one is a **translation** of the other
    - sin x passes through the origin
    - cos x passes through (0, 1)
- The **amplitude** of the graphs of sin x and cos x is 1

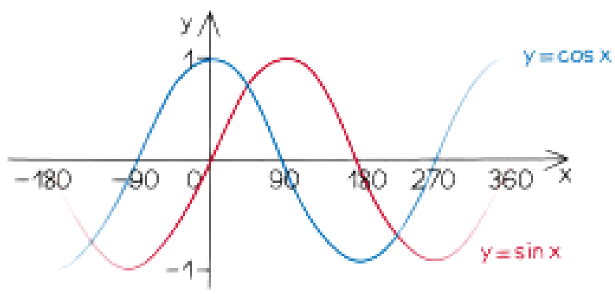
### What are the properties of the graph of tan x?

- The graph of tan x is **periodic**
  - It **repeats every 180°** (π radians)
  - The angle will always be on the x-axis
    - Either in degrees or radians
- The graph of tan x is **undefined** at the points ±90°, ±270° etc
  - There are **asymptotes** at these points on the graph
  - In radians this is at the points $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ etc
- The range of the graph of tan x is
  - **Domain:** $\left\{x \mid x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$
  - **Range:** $\{y \mid y \in \mathbb{R}\}$

## $y = \sin x$ AND $y = \cos x$

SIN X AND COS X ARE ALWAYS IN THE RANGE −1 TO 1

SIN X PASSES THROUGH THE ORIGIN
COS X PASSES THROUGH 1

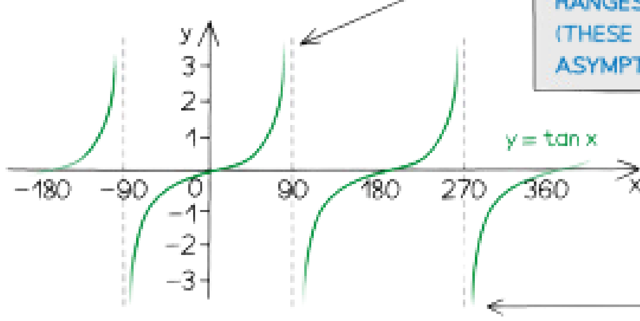



SIN X AND COS X ARE PERIODIC REPEATING EVERY 360°

SIN X HAS ROTATIONAL SYMMETRY ABOUT THE ORIGIN SO $\sin(-x) = -\sin(x)$
COS X IS SYMMETRICAL THROUGH THE y-AXIS SO $\cos(-x) = \cos(x)$

## $y = \tan x$

TAN X IS UNDEFINED AT ±90°, ±270°, ±450°... MEANING IT RANGES FROM −∞ TO +∞ (THESE POINTS ARE CALLED ASYMPTOTES)




TAN X IS PERIODIC REPEATING EVERY 180°

## How do I sketch trigonometric graphs?

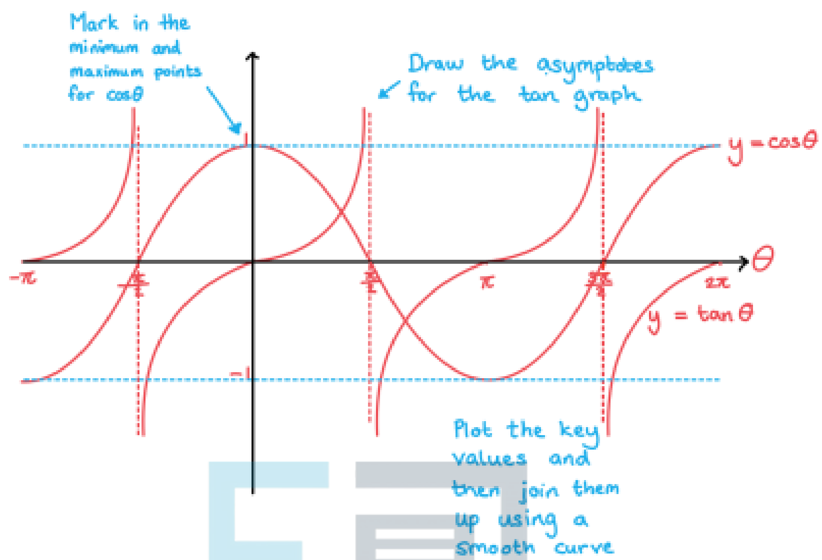
- You may need to sketch a trigonometric graph so you will need to remember the key features of each one
- The following steps may help you sketch a trigonometric graph
  - STEP 1: Check whether you should be working in degrees or radians
    - You should check the domain given for this
    - If you see π in the given domain then you should work in radians
  - STEP 2: Label the x-axis in multiples of 90°
    - This will be multiples of $\frac{\pi}{2}$ if you are working in radians
    - Make sure you cover the whole domain on the x-axis
  - STEP 3: Label the y-axis
    - The range for the y-axis will be $-1 \le y \le 1$ for sin or cos
    - For tan you will not need any specific points on the y-axis
  - STEP 4: Draw the graph
    - Knowing exact values will help with this, such as remembering that sin(0) = 0 and cos(0) = 1
    - Mark the important points on the axis first
    - If you are drawing the graph of tan x put the asymptotes in first
    - If you are drawing sin x or cos x mark in where the maximum and minimum points will be
    - Try to keep the symmetry and rotational symmetry as you sketch, as this will help when using the graph to find solutions

### Exam Tip

- Sketch all three trig graphs on your exam paper so you can refer to them as many times as you need to!

## Worked Example

Sketch the graphs of y = cosθ and y = tanθ on the same set of axes in the interval -π ≤ θ ≤ 2π. Clearly mark the key features of both graphs.



# Using Trigonometric Graphs

## How can I use a trigonometric graph to find extra solutions?

- Your calculator will only give you the first solution to a problem such as $\sin^{-1}(0.5)$
  - This solution is called the **primary value**
- However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
  - Further solutions are called the **secondary values**
- This is why you will be given a **domain** (interval) in which your solutions should be found
  - This could either be in degrees or in radians
    - If you see π or some multiple of π then you must work in radians
- The following steps will help you use the **trigonometric graphs** to find **secondary values**
  - STEP 1: Sketch the graph for the given function and interval
    - Check whether you should be working in degrees or radians and label the axes with the key values
  - STEP 2: Draw a horizontal line going through the y-axis at the point you are trying to find the values for
    - For example if you are looking for the solutions to $\sin^{-1}(-0.5)$ then draw the horizontal line going through the y-axis at -0.5
    - The number of times this line cuts the graph is the number of solutions within the given interval
  - STEP 3: Find the primary value and mark it on the graph
    - This will either be an exact value and you should know it
    - Or you will be able to use your calculator to find it
  - STEP 4: Use the symmetry of the graph to find all the solutions in the interval by adding or subtracting from the key values on the graph

## What patterns can be seen from the graphs of trigonometric functions?

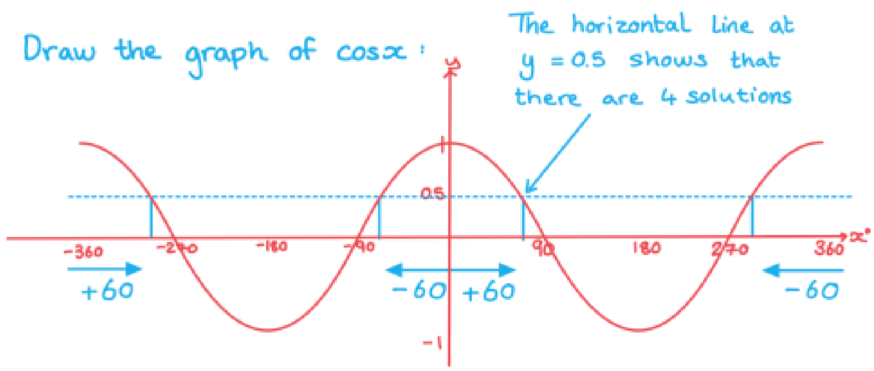
- The graph of sin x has rotational symmetry about the origin
  - So $\sin(-x) = -\sin(x)$
  - $\sin(x) = \sin(180° - x)$ or $\sin(\pi - x)$
- The graph of cos x has reflectional symmetry about the y-axis
  - So $\cos(-x) = \cos(x)$
  - $\cos(x) = \cos(360° - x)$ or $\cos(2\pi - x)$
- The graph of tan x repeats every 180° (π radians)
  - So $\tan(x) = \tan(x \pm 180°)$ or $\tan(x \pm \pi)$
- The graphs of sin x and cos x repeat every 360° (2π radians)
  - So $\sin(x) = \sin(x \pm 360°)$ or $\sin(x \pm 2\pi)$
  - $\cos(x) = \cos(x \pm 360°)$ or $\cos(x \pm 2\pi)$

### Exam Tip

- Take care to always check what the **interval** for the angle is that the question is focused on

## Worked Example

One solution to cos x = 0.5 is 60°. Find all the other solutions in the range -360° ≤ x ≤ 360°.




Solutions are : 60°, 360° − 60°, − 60°, −360° + 60°

$\boxed{-60°, \ -300°, \ 60°, \ 300°}$

# 3.5 Voronoi Diagrams

## 3.5.1 Voronoi Diagrams

### Drawing Voronoi Diagrams

#### What are Voronoi Diagrams?

- A **Voronoi diagram** shows the region containing the set of all points which are **closer** to one given **site** than to any other **site** on the diagram
    - A **site** is located at the coordinates of a specific place of interest on a Voronoi diagram
- It will be partitioned into a number of **regions**
    - These regions are often called **Voronoi cells** and will be **polygons**
        - There will be the same number of **regions** as **sites** on the diagram
    - For example, if a city contains five parks a Voronoi diagram could be drawn for that city dividing it into five regions based on their closest park
- The **edges** of each region will be the **perpendicular bisector** of two of the sites
    - The **edges** may also be called **boundaries**
- The **vertices** of each region are the **intersections of three** of these perpendicular bisectors
    - The perpendicular bisectors of three individual points will always intersect at the point that is **equidistant** from the three points

#### How are Voronoi diagrams drawn for three sites?

- You will **not** be expected to draw a Voronoi diagram from scratch, however you should understand how one is constructed
    - First, the perpendicular bisector of the line segment joining each pair of sites will be constructed
        - These should be constructed using dashed lines as only a part of each line will be needed for the final diagram
    - The **points of intersection** of these perpendicular bisectors will create the **vertices**
    - Each perpendicular bisector should stop when it meets another perpendicular bisector
        - Remove the part of the perpendicular bisector that is not in the region of the two sites
        - No perpendicular bisector should cross over another
        - This will form the **regions**, or **cells**

## How are Voronoi diagrams drawn for more than three sites?

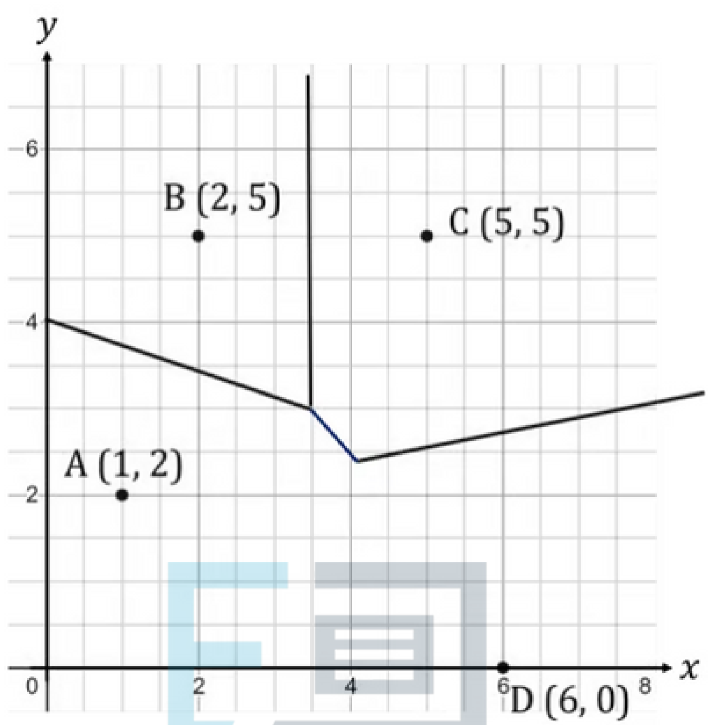
- It is challenging to draw a Voronoi diagram from scratch if it has **more than three sites**
- In this case it is easiest to draw the Voronoi diagram for three sites first and then add the next sites one by one following these steps
  - STEP 1: The fourth site will be in one of the cells containing an existing site
    - Draw the perpendicular bisector of the line segment between these two sites
  - STEP 2: Stop this new line at the point where it meets an existing **boundary** in the Voronoi diagram
  - STEP 3: There will now be an existing edge in the region of the new site
    - This should be **shortened** to meet the new boundary
  - STEP 4: The fourth site will now be in the same cell as a different existing site
    - Draw the perpendicular bisector of the line segment between these two sites
    - This is the step you will most likely carry out in an exam
- You may be asked to find the **equation of a missing edge**
  - This will mean finding the equation of the **perpendicular bisector** between the **two sites** that are both within **one region**
- You may be asked to add the **location of a missing site** to the Voronoi diagram
  - This will mean using the given **edge** of one or two of the regions and finding the second site that would make this edge a perpendicular bisector
    - Draw a perpendicular line from the site to the edge
    - Check the distance of this line and then continue it on the other site of the edge for the same distance
    - This will be the location of your new site
  - You may need to find the gradients of the edges you have and then use the negative reciprocal to find the gradient of the perpendicular bisector of the current and new site

### Exam Tip

- Make sure that you have a straight edge and an eraser with you in the exam so that any perpendicular bisectors that you draw are clear and any mistakes that are made can be erased
- If you are asked to adjust a given Voronoi diagram and a perpendicular bisector that needs to be removed or shortened, you can put a series of little lines along it to indicate that it is crossed out

## Worked Example

The Voronoi diagram below shows sites A, B, C and D.




a)
Explain how you know that the Voronoi diagram is incomplete.

The Voronoi diagram has four sites but only three Voronoi cells.

b)
Find the equation of the line which would complete the Voronoi cell containing site A.
Give your answer in the form $ax + by + d = 0$ where $a, b, d \in \mathbb{Z}$.

Sites A and D are both in the same region so find the perpendicular bisector of the line segment connecting A and D.

$A:(1,2) \quad D:(6,0)$

Find the midpoint:

$$MP = \left(\frac{1+6}{2}, \frac{2+0}{2}\right) = (3.5, 1)$$

gradient AD

Perpendicular gradient.

$$m_{AD} = \frac{0-2}{6-1} = -\frac{2}{5} \quad \therefore \; m_{\perp AD} = \frac{5}{2}$$

Sub MP and $m_{\perp AD}$ into equation for a straight line:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{5}{2}\left(x - \frac{7}{2}\right)$$

multiply by 2 to remove the fraction and rearrange

$$2y - 2 = 5x - \frac{35}{2}$$

$$4y - 4 = 10x - 35$$

$$\boxed{10x - 4y - 31 = 0}$$

# Interpreting Voronoi Diagrams

## What is a Voronoi diagram used for?

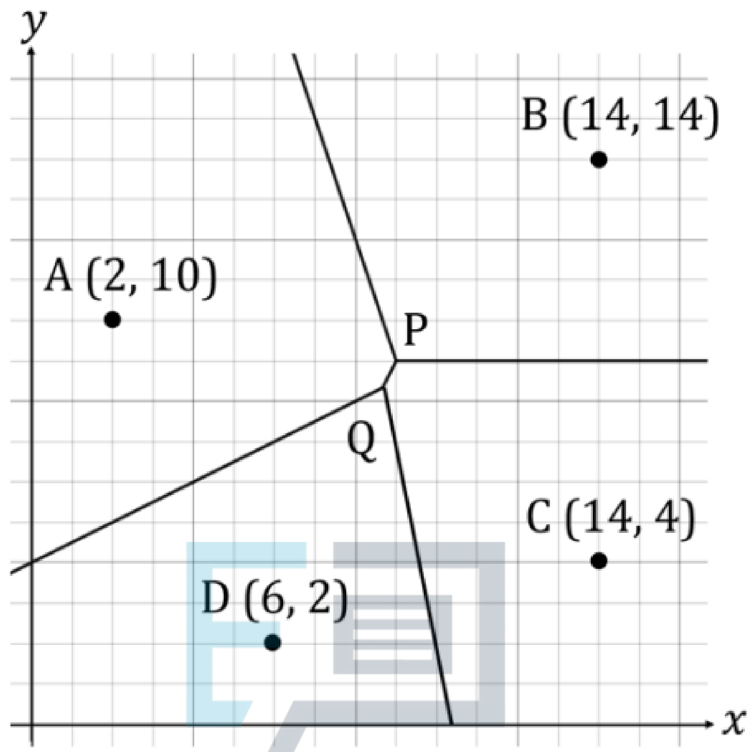
- Voronoi diagrams are often used in land management to work out where the best location would be according to where sites are already situated
- They can show where to put something to make sure that it is
  - Closest to a particular site
  - Closer to one site than another
  - Equidistant from two or three specific sites
  - As far as possible from any other site

## What do I need to know about Voronoi diagrams?

- You may be asked to find the shortest distance from a point to its closest site
  - Use Pythagoras' Theorem to find the distance between the given coordinate and the site in the same region as it
  - If the coordinate is on an edge then there will be two sites **equidistant** from it
- You may be asked to find the point which is furthest from any of the sites
  - This will be one of the vertices
  - To choose which vertex look at which is the centre of the **largest empty circle**
- You may be asked to estimate the success of a new site
  - This is done by looking at the data for the **nearest site**
  - The prediction for the new site would be assumed to be the same
    - This is called **nearest neighbour interpolation**

## Worked Example

The Voronoi diagram below shows the four sites A, B, C and D with coordinates (2, 10), (14, 14), (14, 4), and (6, 2) respectively. 1 unit represents 10 km.

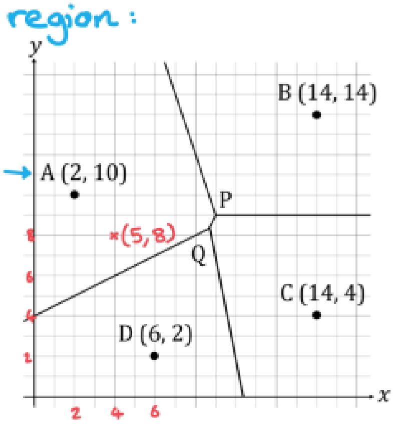



i)

State which site a new business opening at the coordinate (5, 8) should look at to predict future sales.

Plot the point and look for the site in the same region:

The new business is in the same region as site A.




**Site A**

ii)

Find the shortest distance from the point (5, 8) to its nearest site.

The point $(5,8)$ is closest to site A.

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$P: (\underset{x_1}{5}, \underset{y_1}{8})$ $\quad A: (\underset{x_2}{2}, \underset{y_2}{10})$

Sub coordinates:

$$d = \sqrt{(2-5)^2 + (10-8)^2}$$

$$= \sqrt{(-3)^2 + (2)^2} = \sqrt{13}$$

$$\text{distance} = \sqrt{13} \times 10\text{km} = 36.055\ldots\text{ km}$$

$$\boxed{\text{distance} = 36.1\text{km } (3\text{s.f.})}$$

# 3.5.2 Toxic Waste Dump Problem

## Toxic Waste Dump Problem

### What is the toxic waste dump problem?

- The **toxic waste dump problem** is the name given to the general idea of finding the point on a **Voronoi diagram** which is furthest from any of the **sites**
  - A **site** is the coordinates of a specific place of interest on a Voronoi diagram
- It is given this name because of the common problem of finding a place to put a toxic waste dump that is **equally far** away from any inhabited area
  - For example, if a province contains five towns a Voronoi diagram could be used to find the point within the province which is furthest from each town
- The toxic waste dump problem is more of an idea than a specific problem
  - The same concept could be applied to other contexts such as
    - Finding a position for a new supermarket that is equally far from all competitors
    - Finding a place to plant a new tree that is equally far from other trees competing for water resources
    - Finding the quietest place to enjoy a picnic that is equally far from other noisy groups of people
  - Note that the term **equally far** is used in all of the above examples

### How is a Voronoi diagram used to find the furthest point from any site?

- Within any Voronoi diagram the furthest point from any site will always be either
  - one of the cell vertices, or
  - somewhere on a boundary of the diagram
- In an IB exam, the solution will always be one of the **cell vertices**
- To find the furthest point you will need to consider each of the cell vertices separately and find which one is furthest from all of the sites
- This is done by constructing the **largest empty circle**

### What is the largest empty circle?

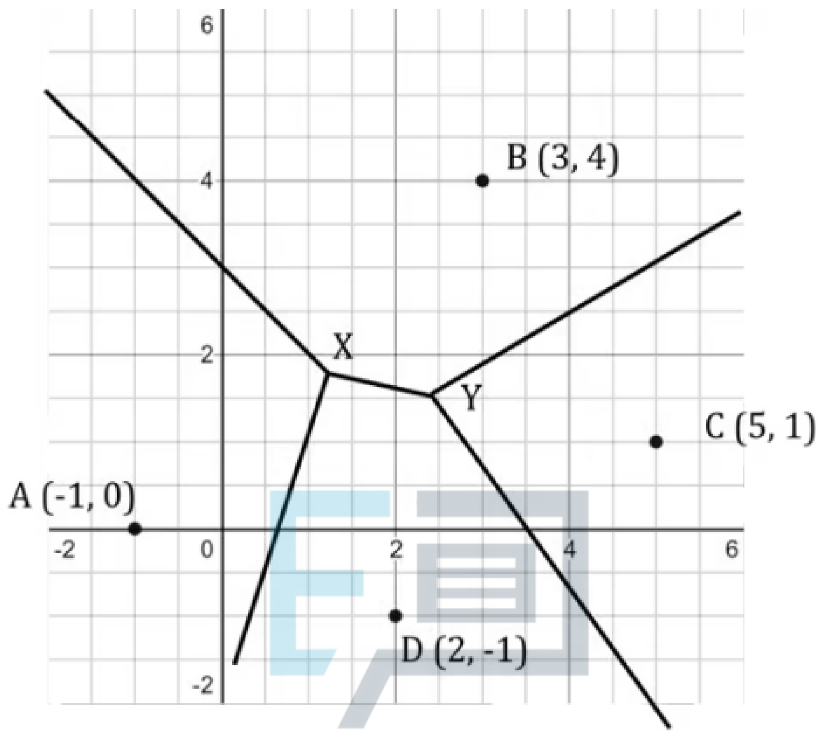
- The **largest empty circle** is the largest possible circle constructed on a Voronoi diagram that contains **no sites**
- The **centre** of the circle will be one of the vertices of a **cell** or **region**
  - The **vertices** of each region are the **intersections** of **the boundaries**
- The **radius** of the circle will be the **distance** from the vertex to the closest site
  - The closest site will be on the circumference
  - Use Pythagoras' Theorem to find the distance
- There may be a **scale** to convert the distance found on the Voronoi diagram into a distance in real life
  - For example if the scale is 1 unit represents 5 km then 5 units represents 25 km

### Exam Tip

- The solution to the toxic waste dump will always be one of the points of intersection between the perpendicular bisectors, so you need to know the coordinates of these points
  - Remember that you can use your GDC to solve a pair of the simultaneous equations quickly if you know the equations of two of the perpendicular bisectors that intersect at that point

## Worked Example

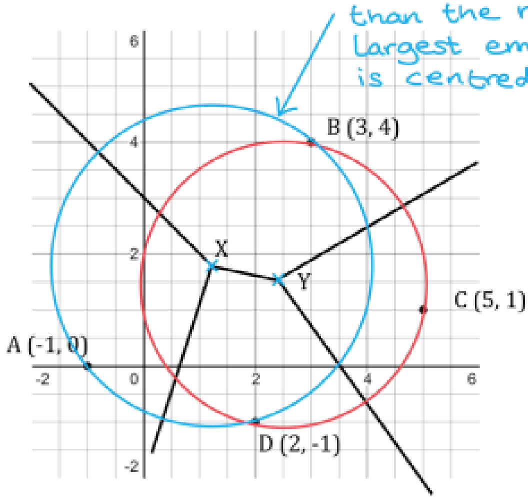
The Voronoi diagram below shows four cities at the sites A, B, C and D. The coordinates of the points X and Y are $\left(\frac{5}{4}, \frac{7}{4}\right)$ and $\left(\frac{5}{2}, \frac{3}{2}\right)$ respectively.




Determine the optimal position where a toxic waste site could be located and, given that 1 unit represents 50 km, find the distance from this point to its nearest city.

The optimal position would be at the point X or Y

Draw the largest possible circle centred at X and Y.




Choose any of the sites on the circle to find the distance

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$X: \left(\frac{5}{4}, \frac{7}{4}\right)$ $B: (3, 4)$

$x_1$ $y_1$ $x_2$ $y_2$

sub coordinates

$$d = \sqrt{\left(\frac{5}{4} - 3\right)^2 + \left(\frac{7}{4} - 4\right)^2} = \sqrt{\left(\frac{7}{4}\right)^2 + \left(\frac{9}{4}\right)^2} = \sqrt{\frac{65}{8}}$$

$= 2.8504...$ units

distance $= 2.8504... \times 50\text{km} = 142.52...$ km

$\boxed{\text{distance} = 143 \text{ km (3s.f.)}}$

# 3.6 Matrix Transformations

## 3.6.1 Matrix Transformations

### Transformation by a Matrix

#### What is a transformation matrix?

- A transformation matrix is used to determine the coordinates of an **image** from the **transformation** of an **object**
  - Commonly used transformation matrices include
    - reflections, rotations, enlargements and stretches
- (In 2D) a multiplication by any 2×2 matrix could be considered a transformation (in the 2D plane)
- An individual point in the plane can be represented as a position vector, $\begin{pmatrix} x \\ y \end{pmatrix}$
  - Several points, that create a shape say, can be written as a position matrix $\begin{pmatrix} x_1 & x_2 & x_3 & \cdots \\ y_1 & y_2 & y_3 & \cdots \end{pmatrix}$
- A matrix transformation will be of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} e \\ f \end{pmatrix}$
  - where $\begin{pmatrix} x \\ y \end{pmatrix}$ represents any point in the 2D plane
  - $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} e \\ f \end{pmatrix}$ are given matrices

## How do I find the coordinates of an image under a transformation?

- The coordinates $(x', y')$ - the image of the point $(x, y)$ under the transformation with matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} e \\ f \end{pmatrix}$ - are given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

- Similarly, for a position matrix

$$\begin{pmatrix} x'_1 & x'_2 & x'_3 & \cdots \\ y'_1 & y'_2 & y'_3 & \cdots \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 & x_2 & x_3 & \cdots \\ x_1 & x_2 & x_3 & \cdots \end{pmatrix} + \begin{pmatrix} e & e & e & \ldots \\ f & f & f & \ldots \end{pmatrix}$$

  - If you use this method then remember to add e and *f* to each column
- A GDC can be used for matrix multiplication
  - If matrices involved are small, it may be as quick to do this manually

- **STEP 1**
  Determine the transformation matrix (**T**) and the position matrix (**P**)
  The transformation matrix, if uncommon, will be given in the question
  The position matrix is determined from the coordinates involved, it is best to have the coordinates in order, to avoid confusion

- **STEP 2**
  Set up and perform the matrix multiplication and addition required to determine the image position matrix, **P'**
  **P' = TP**



- **STEP 3**
  Determine the coordinates of the image from the image position matrix, **P'**

## How do I find the coordinates of the original point given the image under a transformation?

- To 'reverse' a transformation we would need the **inverse transformation matrix**
  - i.e. $\mathbf{T}^{-1}$
  - For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the inverse is given by $\frac{1}{\det \boldsymbol{T}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
    - where $\det \boldsymbol{T} = ad - bc$
  - A GDC can be used to work out inverse matrices
- You would rearrange $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$
  - $\frac{1}{\det \boldsymbol{T}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\left[\begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} e \\ f \end{pmatrix}\right] = \begin{pmatrix} x \\ y \end{pmatrix}$

### Exam Tip

- The formula for the determinant and inverse of a 2×2 matrix can be found in the **Topic 1: Number and Algebra** section of the formula booklet

## Worked Example

A quadrilateral, Q, has the four vertices A(2, 5), B(5, 9), C(11, 9) and D(8, 5).

Find the coordinates of the image of Q under the transformation $\boldsymbol{T} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$.

STEP 1: Determine the transformation and position matrices

$$\underset{\sim}{T} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \qquad \underset{\sim}{P} = \begin{array}{c} \begin{pmatrix} 2 & 5 & 11 & 8 \\ 5 & 9 & 9 & 5 \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ A & B & C & D \end{matrix} \end{array}$$

STEP 2: $\underset{\sim}{P}' = \underset{\sim}{T}\,\underset{\sim}{P}$

$$\underset{\sim}{P}' = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 & 11 & 8 \\ 5 & 9 & 9 & 5 \end{pmatrix}$$

$$\underset{\sim}{P}' = \begin{pmatrix} 6-5 & 15-9 & 33-9 & 24-5 \\ -2+10 & -5+18 & -11+18 & -8+10 \end{pmatrix}$$

$$\underset{\sim}{P}' = \begin{pmatrix} 1 & 6 & 24 & 19 \\ 8 & 13 & 7 & 2 \end{pmatrix}$$

Alternatively use a GDC for matrix multiplication

STEP 3: Determine the image coordinates from P'

$$\boxed{A'(1,8) \quad B'(6,13) \quad C'(24,7) \quad D'(19,2)}$$

# Matrices of Geometric Transformations

## What is meant by a geometric transformation?

- The following transformations can be represented (in 2D) using **multiplication** of a **2×2** matrix
    - rotations (about the origin)
    - reflections
    - enlargements
    - (horizontal) stretches parallel to the x-axis
    - (vertical) stretches parallel to the y-axis
- The following transformations can be represented (in 2D) using **addition** of a **2×1** matrix
    - translations

## What are the matrices for geometric transformations?

- All of the following transformation matrices are given in the **formula booklet**
- **Rotation**
    - Anticlockwise (or counter-clockwise) through angle $\theta$ about the origin
        - $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
    - Clockwise through angle $\theta$ about the origin
        - $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
    - In both cases
        - $\theta > 0$
        - $\theta$ may be measured in degrees or radians
- **Reflection**
    - In the line $y = (\tan\theta)x$
        - $\begin{pmatrix} \cos2\theta & \sin2\theta \\ \sin2\theta & -\cos2\theta \end{pmatrix}$
    - $\theta$ may be measured in degrees or radians
    - for a reflection in the x-axis, $\theta = 0°$ (0 radians)
    - for a reflection in the y-axis, $\theta = 90°$ ($\pi/2$ radians)
- **Enlargement**
    - Scale factor $k$, centre of enlargement at the origin (0, 0)
        - $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

- **Horizontal stretch** (or stretch parallel to the x-axis)
  - Scale factor $k$
    - $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
- **Vertical stretch** (or stretch parallel to the y-axis)
  - Scale factor $k$
    - $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
- **Translation (vector)**
  - $p$ units in the (positive) x-direction
  - $q$ units in the (positive) y direction
    - $\begin{pmatrix} p \\ q \end{pmatrix}$
    - This is not given in the formula booklet

## How do I solve problems involving geometric transformations?

- The matrix equations involved in problems will be of the form
  - **P'=AP** or
  - **P'=AP+b** where **b** is a translation vector
    - (sometimes called an **affine** transformation)
  - where
    - **P** is the position vector of the object coordinates
    - **P'** is the position vector of the image coordinates
    - **A** is the transformation matrix
    - **b** is a translation vector
- Problems may ask you to
  - find the coordinates of point(s) on the image
  - find the coordinates of point(s) on the object using an inverse matrix ($\mathbf{A}^{-1}$)
  - deduce/identify a matrix corresponding to one of the common geometric transformations
    - E.g. Find the matrix of a rotation of 45° clockwise about the origin

### Exam Tip

- The formulae for the all of the transformation matrices can be found in the **Topic 3: Geometry and Trigonometry** section of the formula booklet

## Worked Example

Triangle PQR has coordinates P(-1, 4), Q(5, 4) and R(2, -1).

The transformation ***T*** is a reflection in the line $y = x\sqrt{3}$.

a)

Find the matrix ***T*** that represents a reflection in the line $y = x\sqrt{3}$.

From formula booklet: Reflection in line $y = (\tan\theta)x$ is $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

$y = x\sqrt{3}$, $\therefore \tan\theta = \sqrt{3}$, $\theta = 60°$

$\therefore \underset{\sim}{T} = \begin{pmatrix} \cos 120° & \sin 120° \\ \sin 120° & -\cos 120° \end{pmatrix}$

$$\boxed{\underset{\sim}{T} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}}$$

b)

Find the position matrix, ***P'***, representing the coordinates of the images of points P, Q and R under the transformation ***T***.

$\underset{\sim}{P'} = \underset{\sim}{T}\underset{\sim}{P}$ ("$\underset{\sim}{P'} = \underset{\sim}{A}\underset{\sim}{P}$")

$$\therefore \underset{\sim}{P'} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{array}{c} \begin{pmatrix} -1 & 5 & 2 \\ 4 & 4 & -1 \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ P & Q & R \end{matrix} \end{array} \leftarrow \text{position matrix P}$$

(Use a GDC for matrix multiplication)

$$\boxed{\underset{\sim}{P'} = \begin{pmatrix} \frac{1}{2}(1+4\sqrt{3}) & -\frac{1}{2}(5-4\sqrt{3}) & -\frac{1}{2}(2+\sqrt{3}) \\ \frac{1}{2}(4-\sqrt{3}) & \frac{1}{2}(4+5\sqrt{3}) & -\frac{1}{2}(1-2\sqrt{3}) \end{pmatrix}}$$

Be careful copying a calculator display

$$-\frac{2+\sqrt{3}}{2} \neq \frac{-2+\sqrt{3}}{2}$$

# Matrices of Composite Transformations

The order in which transformations occur can lead to different results – for example a reflection in the x-axis followed by clockwise rotation of 90° is different to rotation first, followed by the reflection.

Therefore, when one transformation is followed by another order is critical.

## What is a composite transformation?

- A composite function is the result of applying more than one function to a point or set of points
  - e.g. a **rotation**, followed by an **enlargement**
- It is possible to find a **single** composite function **matrix** that does the same job as applying the individual transformation matrices

## How do I find a single matrix representing a composite transformation?

- Multiplication of the transformation matrices
- However, the order in which the matrices is important
  - If the transformation represented by matrix **M** is applied first, and is then followed by another transformation represented by matrix **N**
    - the composite matrix is **NM**
      e. **P' = NMP**
      (**NM** is not necessarily equal to **MN**)
    - The matrices are **applied** right to left
    - The composite function matrix is **calculated** left to right
  - Another way to remember this is, starting from **P**, always **pre-multiply** by a transformation matrix
    - This is the same as applying **composite functions** to a value
    - The function (or matrix) furthest to the right is applied first

## How do I apply the same transformation matrix more than once?

- If a transformation, represented by the matrix **T**, is applied twice we would write the composite transformation matrix as $\mathbf{T}^2$
  - $\mathbf{T}^2 = \mathbf{TT}$
- This would be the case for any number of repeated applications
  - $\mathbf{T}^5$ would be the matrix for five applications of a transformation
- A GDC can quickly calculate $\mathbf{T}^2$, $\mathbf{T}^5$, etc
- Problems may involve considering patterns and sequences formed by repeated applications of a transformation
  - The coordinates of point(s) follow a particular pattern
    - (20, 16) – (10, 8) – (5, 4) – (2.5, 2) ...
  - The area of a shape increases/decreases by a constant factor with each application

e.g. if one transformation doubles the area then three applications will increase the (original) area eight times ($2^3$)

### Exam Tip

- When performing multiple transformations on a set of points, make sure you put your transformation matrices in the correct order, you can check this in an exam but sketching a diagram and checking that the transformed point ends up where it should
- You may be asked to show your workings but you can still check that you have performed you matrix multiplication correctly by putting it through your GDC

## Worked Example

The matrix **E** represents an enlargement with scale factor 0.25, centred on the origin.
The matrix **R** represents a rotation, 90° anticlockwise about the origin.

a)
Find the matrix, **C**, that represents a rotation, 90° anticlockwise about the origin followed by an enlargement of scale factor 0.25, centred on the origin.

$\underset{\sim}{C} = \underset{\sim}{E}\underset{\sim}{R}$

$$\underset{\sim}{C} = \underbrace{\begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix}}_{\substack{\text{enlargement} \\ k=0.25}} \underbrace{\begin{pmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{pmatrix}}_{\substack{\text{rotation, anti-clockwise} \\ \theta=90°}}$$

Use the formula booklet

$$\underset{\sim}{C} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Use a GDC for matrix multiplication

$$\boxed{\therefore \underset{\sim}{C} = \begin{pmatrix} 0 & -0.25 \\ 0.25 & 0 \end{pmatrix}}$$

b)

A square has position matrix $T_0 = \begin{pmatrix} 0 & 0 & 256 & 256 \\ 0 & 256 & 256 & 0 \end{pmatrix}$. $T_n$ represents the position matrix of the image square after it has been transformed $n$ times by matrix **C**. Find $T_4$

$$\underset{\sim}{T_4} = \underset{\sim}{C}^4 \underset{\sim}{T_0} = \begin{pmatrix} 0 & -0.25 \\ 0.25 & 0 \end{pmatrix}^4 \begin{pmatrix} 0 & 0 & 256 & 256 \\ 0 & 256 & 256 & 0 \end{pmatrix}$$

Use a GDC, typing this in carefully as one calculation

$$\boxed{\therefore \underset{\sim}{T_4} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}}$$

c)
Find the single transformation matrix that would map $T_4$ to $T_0$.

$\underset{\sim}{T_4}$ to $\underset{\sim}{T_0}$ would be the inverse of $\underset{\sim}{C}^4$.

(Note that $[C^4]^{-1}$ does not mean $\underset{\sim}{C}^{-4}$)

Use a GDC to find $[C^4]^{-1}$ in one calculation

$$\boxed{[C^4]^{-1} = \begin{pmatrix} 256 & 0 \\ 0 & 256 \end{pmatrix}}$$

# 3.6.2 Determinant of a Transformation Matrix

## Determinant of a Transformation Matrix

### What is a determinant?

- For the 2×2 matrix $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
  - the determinant is det $\boldsymbol{A}$ = $ad - bc$

### What does the determinant of a transformation matrix ($A$) represent?

- The **absolute value** of the **determinant** of a transformation matrix is the **area scale factor**
  - Area scale factor = |det $\boldsymbol{A}$|
- The area of the **image** will be **product** of the **area** of the **object** and |det $\boldsymbol{A}$|
  - Area of image = |det $\boldsymbol{A}$| × Area of object
- Note the area will reduce if |det $\boldsymbol{A}$| < 1
- If the determinant is **negative** then the **orientation** of the shape will be **reversed**
  - For example: the shape has been reflected

### How do I solve problems involving the determinant of a transformation matrix?

- Problems may involve comparing areas of **objects** and **images**
  - This could be as a percentage, proportion, etc
- Missing value(s) from the transformation matrix (and elsewhere) can be deduced if the determinant of the transformation matrix is known
- Remember to use the **absolute value** of the determinant
  - This can lead to multiple answers to equations
  - Use your GDC to solve these

Exam Tip

- Remember that the formula for finding the determinant of a matrix is given in the formula booklet!

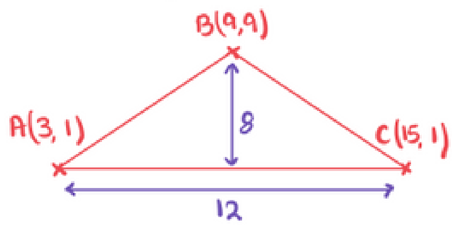
## Worked Example

An isosceles triangle has vertices A(3, 1), B(15, 1) and C(9, 9).

a)

Find the area of the isosceles triangle.

A sketch or plot on GDC will help find the area




Area $= \frac{1}{2} \times 12 \times 8$ ("$A = \frac{1}{2}bh$")

$\boxed{\therefore \text{Area of } \Delta ABC = 48 \text{ square units}}$

b)

Triangle $\Delta ABC$ is transformed using the matrix $\boldsymbol{T} = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$. Find the area of the transformed triangle.

Area scale factor is $|\det \underset{\sim}{T}|$

$|\det \underset{\sim}{T}| = 3 \times 2 - 2 \times -1 = 8$

$\therefore$ Area of image $= 48 \times 8 = 384$

$\boxed{\text{Area of transformed triangle} = 384 \text{ square units}}$

c)

Triangle $\Delta ABC$ is now transformed using the matrix $\boldsymbol{U} = \begin{pmatrix} a & -2 \\ 3 & a^2 \end{pmatrix}$ where $a \in \mathbb{Z}$.

Given that the area of the image is twice as large as the area of the object, find the value of $a$.

$\det \underset{\sim}{U} = a \times a^2 - -2 \times 3 = a^3 + 6$

$\therefore |a^3 + 6| = 2$

For $a^3 + 6 = 2$, $a^3 = -4$, $a \notin \mathbb{Z}^-$, reject

For $a^3 + 6 = -2$, $a^3 = -8$, $a = -2$, $a \in \mathbb{Z}^-$

$\boxed{\therefore a = -2}$

EXAM PAPERS PRACTICE

# 3.7 Vector Properties

## 3.7.1 Introduction to Vectors

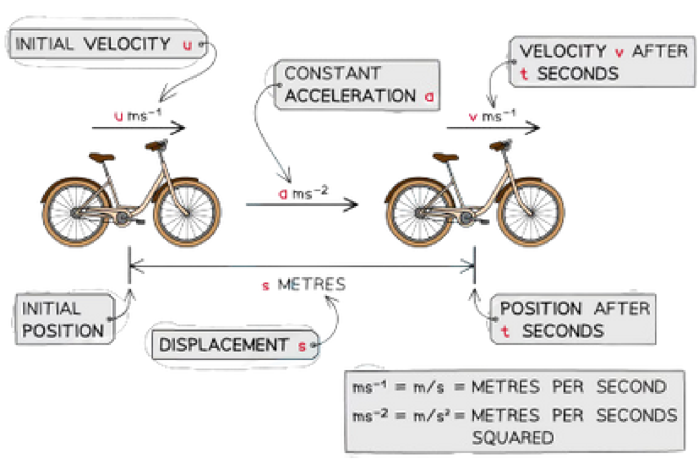
### Scalars & Vectors

#### What are scalars?

- **Scalars** are quantities without **direction**
  - They have only a **size (magnitude)**
  - For example: **speed, distance, time, mass**
- **Most scalar quantities** can never be **negative**
  - You cannot have a negative speed or distance

#### What are vectors?

- **Vectors** are quantities which also have a **direction**, this is what makes them more than just a scalar
  - For example: two objects with **velocities** of 7 m/s and -7 m/s are travelling at the **same speed** but in **opposite directions**
- A **vector quantity** is described by both its **magnitude** and **direction**
- A vector has **components** in the direction of the *x*- , *y*-, and *z*- axes
  - Vector quantities can have **positive** or **negative** components
- Some examples of vector quantities you may come across are **displacement**, **velocity**, **acceleration**, **force/weight**, **momentum**
  - **Displacement** is the position of an object from a starting point
  - **Velocity** is a speed in a given direction (displacement over time)
  - **Acceleration** is the change in velocity over time
- Vectors may be given in either 2– or 3– dimensions




**Exam Tip**

- Make sure you fully understand the definitions of all the words in this section so that you can be clear about what your exam question is asking of you

## Worked Example

State whether each of the following is a scalar or a vector quantity.

a)

A speed boat travels at 3 m/s on a bearing of 052°

Speed with a given direction → velocity

Vector

b)

A garden is 1.7 m wide

Length with no direction

Scalar

c)

A car accelerates forwards at 5.4 $ms^{-2}$

Acceleration has direction

Vector



d)

A film lasts 2 hours 17 minutes

Time has no direction

Scalar

e)

An athlete runs at an average speed of 10.44 $ms^{-1}$

Speed with no direction is a scalar

Scalar

f)

A ball rolls forwards 60 cm before stopping

Displacement has direction

$\boxed{\text{Vector}}$

EXAM PAPERS PRACTICE

# Vector Notation

## How are vectors represented?

- **Vectors** are usually represented using an arrow in the direction of movement
  - The length of the arrow represents its magnitude
- They are written as lowercase letters either in **bold** or <u>underlined</u>
  - For example a vector from the point O to A will be written **a** or <u>a</u>
    - The vector from the point A to O will be written **-a** or -<u>a</u>
- If the start and end point of the vector is known, it is written using these points as capital letters with an arrow showing the direction of movement
  - For example: $\overrightarrow{AB}$ or $\overrightarrow{BA}$
- Two vectors are equal only if their corresponding components are equal
- Numerically, vectors are either represented using **column vectors** or **base vectors**
  - Unless otherwise indicated, you may carry out all working and write your answers in either of these two types of vector notation

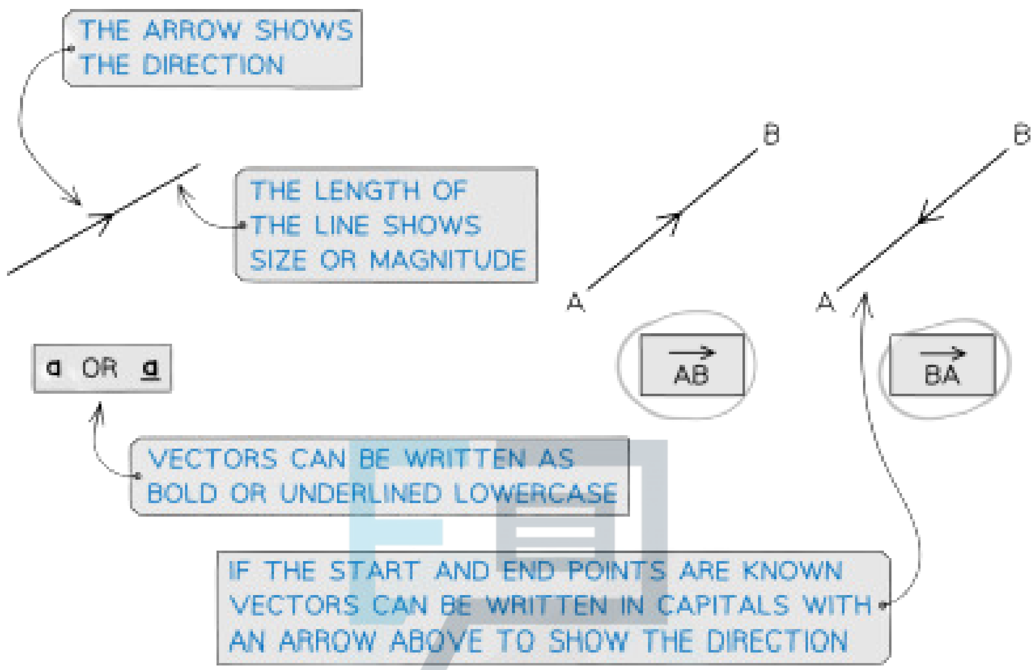
## What are column vectors?

- **Column vectors** are where one number is written above the other enclosed in brackets
- **In 2-dimensions the top number represents movement in the horizontal direction (right/left) and the bottom number represents movement in the vertical direction (up/down)**
- **A positive value represents movement in the positive direction (right/up) and a negative value represents movement in the negative direction (left/down)**
  - For example: The column vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ represents **3 units** in the **positive horizontal** ($x$) direction (i.e., **right**) and **2 units** in the **negative vertical** ($y$) direction (i.e., **down**)
- **In 3-dimensions the top number represents the movement in the *x* direction (length), the middle number represents movement in the *y* direction (width) and the bottom number represents the movement in the z direction (depth)**
  - For example: The column vector $\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$ represents **3 units** in the **positive *x* direction**, **4 units** in the **negative *y* direction** and **2 units** in the **positive *z* direction**

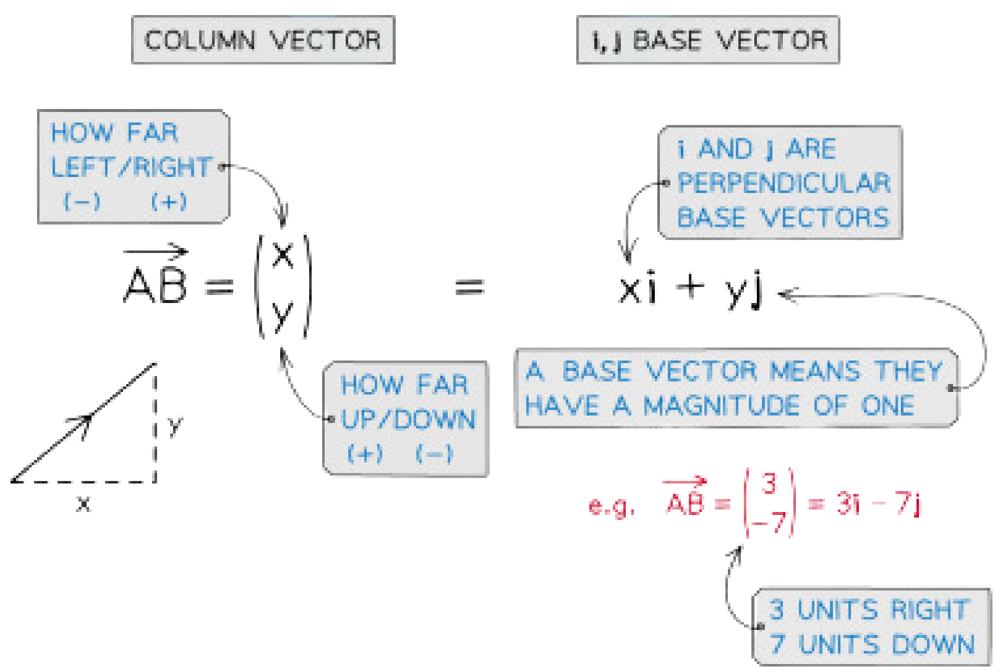
## What are base vectors?

- **Base vectors use i**, **j** and **k** notation **where i**, **j** and **k** are **unit vectors** in the positive $x$, $y$, and $z$ directions respectively
  - **This is sometimes also called unit vector notation**
  - **A unit vector has a magnitude of 1**
- **In 2-dimensions i represents movement in the horizontal direction (right/left) and j represents the movement in the vertical direction (up/down)**
  - **For example:** The vector (-4**i** + 3**j**) would mean **4 units** in the **negative horizontal** ($x$) direction (i.e., **left**) and **3 units** in the **positive vertical** ($y$) direction (i.e., **up**)

- **In 3-dimensions i represents movement in the *x* direction (length), j represents movement in the *y* direction (width) and k represents the movement in the *z* direction (depth)**
  - **For example:** The vector (-4**i** + 3**j** - **k**) would mean **4 units** in the **negative *x* direction**, **3 units** in the **positive *y* direction** and **1 unit** in the **negative *z* direction**
- As they are vectors, **i, j and k** are displayed in **bold** in textbooks and online but in handwriting they would be <u>underlined</u> (<u>i</u>, j and <u>k</u>)





### Exam Tip

- Practice working with all types of vector notation so that you are prepared for whatever comes up in the exam
    - Your working and answer in the exam can be in any form unless told otherwise
    - It is generally best to write your final answer in the same form as given in the question, however you will not lose marks for not doing this unless it is specified in the question
- Vectors appear in **bold** (non-italic) font in textbooks and on exam papers, etc (i.e. $\boldsymbol{F}$, $\boldsymbol{\alpha}$) but in handwriting should be underlined (i.e. $\underline{F}$, $\underline{\alpha}$)

## Worked Example

a)

Write the vector $\begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$ using **base vector** notation.

$$\begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix} = -4\underline{i} + 0\underline{j} + 5\underline{k}$$

$0\underline{j}$ is not needed when giving answer in base vector form.

$$\boxed{5\underline{k} - 4\underline{i}}$$

b)

Write the vector $\mathbf{k} - 2\mathbf{j}$ using **column vector** notation.

$$\underline{k} - 2\underline{j} = 0\underline{i} - 2\underline{j} + 1\underline{k}$$

Be careful with negative components and missing terms when working with base vectors

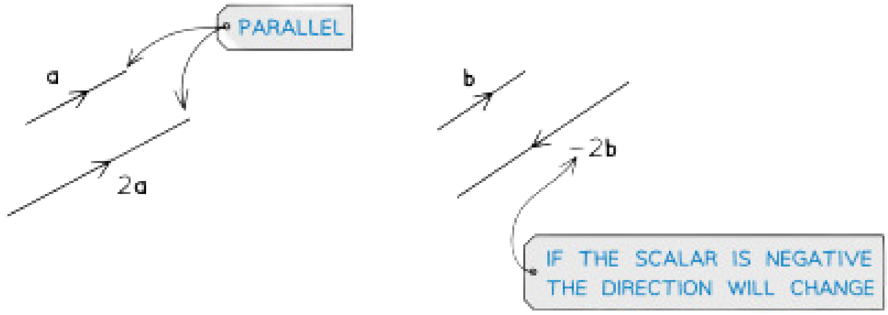
$$\boxed{\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}}$$

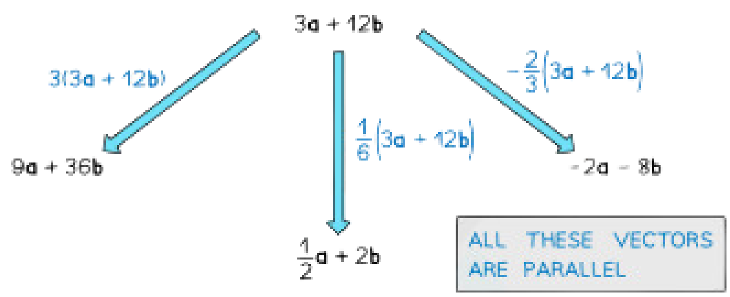
The zero term is needed when using column vector notation

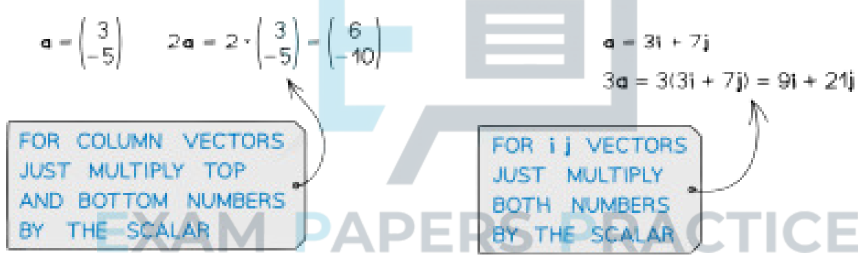
# Parallel Vectors

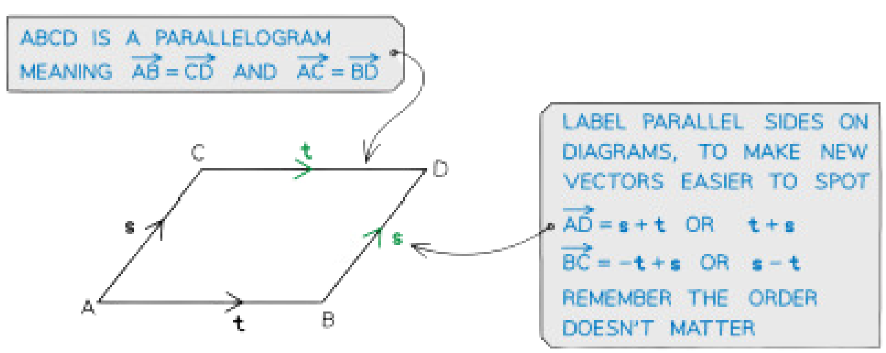
## How do you know if two vectors are parallel?

- Two vectors are parallel if one is a **scalar multiple** of the other
    - This means that all components of the vector have been multiplied by a **common constant (scalar)**
- Multiplying every component in a vector by a **scalar** will change the **magnitude** of the vector but not the **direction**
    - For example: the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{b} = 2\mathbf{a} = 2\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$ will have the **same direction** but the vector **b** will have twice the magnitude of **a**
        - They are **parallel**
- If a vector can be factorised by a **scalar** then it is parallel to any **scalar multiple** of the factorised vector
    - For example: The vector $9\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ can be factorised by the scalar 3 to $3(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ so the vector $9\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ is parallel to any **scalar multiple** of $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
- If a vector is multiplied by a **negative scalar** its direction will be **reversed**
    - It will still be **parallel** to the original vector
- Two vectors are **parallel** if they have the same or reverse **direction** and **equal** if they have the same **size and direction**




## Exam Tip

- It is easiest to spot that two vectors are parallel when they are in column vector notation
  - in your exam by writing vectors in column vector form and looking for a scalar multiple you will be able to quickly determine whether they are parallel or not

### Worked Example

Show that the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$ and $\mathbf{b} = 6\mathbf{k} - 3\mathbf{i}$ are parallel and find the scalar multiple that maps **a** onto **b**.

Convert both vectors into the same form and then look for a value of k such that $\underline{a} = k\underline{b}$, where k is a scalar.

$\underline{a} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

$\underline{b} = 6\underline{k} - 3\underline{i} = -3\underline{i} + 0\underline{j} + 6\underline{k}$

$= \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

$= -\frac{3}{2}\underline{a}$

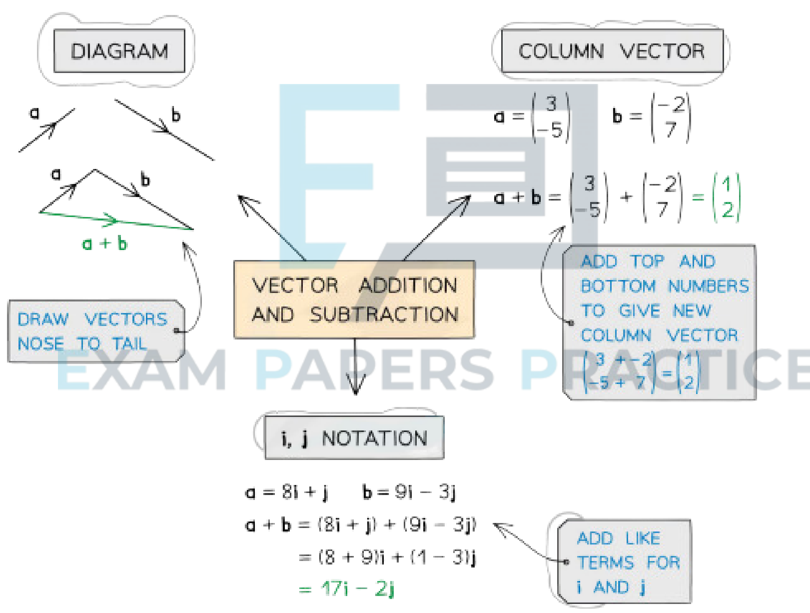
$\underline{b} = -\frac{3}{2}\underline{a}$, $k = -\frac{3}{2}$

# 3.7.2 Position & Displacement Vectors

## Adding & Subtracting Vectors

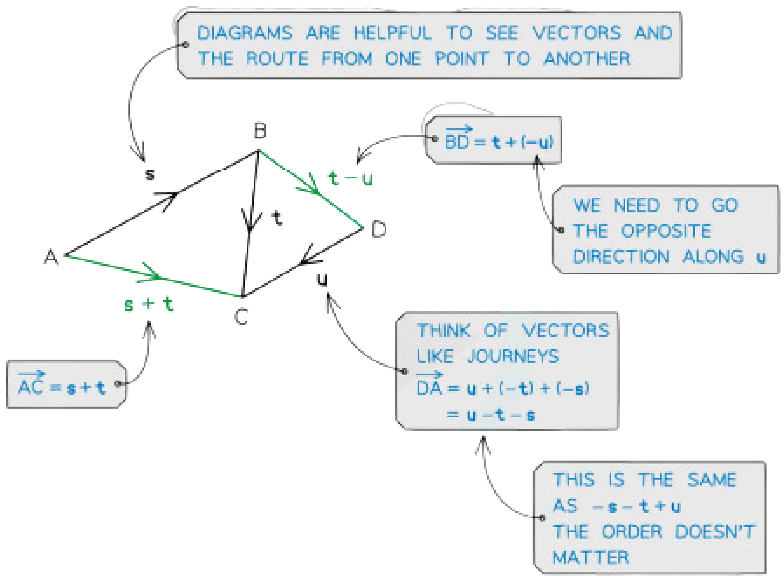
### How are vectors added and subtracted numerically?

- To **add** or **subtract** vectors numerically simply add or subtract each of the corresponding components
- In **column vector** notation just add the top, middle and bottom parts together
  - For example: $\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -8 \end{pmatrix}$
- In **base vector** notation add each of the **i**, **j**, and **k** components together separately
  - For example: (2**i** + **j** – 5**k**) – (**i** + 4**j** + 3**k**) = (**i** – 3**j** – 8**k**)




### How are vectors added and subtracted geometrically?

- Vectors can be **added** geometrically by joining the end of one vector to the start of the next one
- The **resultant** vector will be the shortest route from the start of the first vector to the end of the second
  - A **resultant** vector is a vector that results from **adding** or **subtracting** two or more vectors
- If the two vectors have the same **starting position**, the second vector can be **translated** to the end of the first vector to find the resultant vector
  - This results in a **parallelogram** with the resultant vector as the diagonal
- To **subtract** vectors, consider this as **adding on the negative vector**
  - For example: **a** – **b** = **a** + (–**b**)
  - The end of the **resultant vector a – b** will not be anywhere near the end of the vector **b**
    - Instead, it will be at the point where the end of the vector –**b** would be




## Exam Tip

- Working in column vectors tends to be easiest when adding and subtracting
  - in your exam, it can help to convert any vectors into column vectors before carrying out calculations with them
- If there is no diagram, drawing one can be helpful to help you visualise the problem

## Worked Example

Find the resultant of the vectors **a** = 5**i** – 2**j** and **b** = $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$.

$\underline{a} = 5\underline{i} - 2\underline{j} + 0\underline{k} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

Writing as a column vector makes adding and subtracting easier.

$\underline{a} + \underline{b} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$\boxed{\text{Resultant vector} = 2\underline{i} - \underline{j} + 2\underline{k}}$

# Position Vectors

## What is a position vector?

- A position vector describes the **position** of a point in relation to the **origin**
  - It describes the **direction** and the **distance** from the point O: 0**i** + 0**j** + 0**k** or $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
  - It is different to a **displacement vector** which describes the direction and distance between any two points
- The position vector of point A is written with the notation **a** = $\overrightarrow{OA}$
  - The origin is always denoted O
- The individual components of a position vector are the coordinates of its end point
  - For example the point with coordinates (3, -2, -1) has position vector 3**i** – 2**j** – **k**

### Worked Example

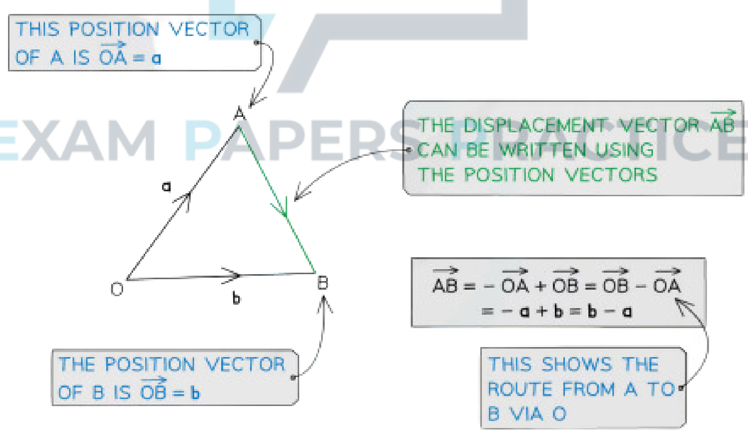
Determine the position vector of the point with coordinates (4, -1, 8).

$\boxed{4\underline{i} - \underline{j} + 8\underline{k}}$

# Displacement Vectors

## What is a displacement vector?

- A **displacement vector** describes the shortest route between any two points
  - It describes the **direction** and the **distance** between any two points
  - It is different to a **position vector** which describes the direction and distance from the point O: 0**i** + 0**j** or $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- The displacement vector of point B from the point A is written with the notation $\overrightarrow{AB}$
- A displacement vector between two points can be written in terms of the displacement vectors of a third point
  - $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$
- A displacement vector can be written in terms of its position vectors
  - For example the displacement vector $\overrightarrow{AB}$ can be written in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$
  - $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$
  - For position vector **a** = $\overrightarrow{OA}$ and **b** = $\overrightarrow{OB}$ the displacement vector $\overrightarrow{AB}$ can be written **b** − **a**



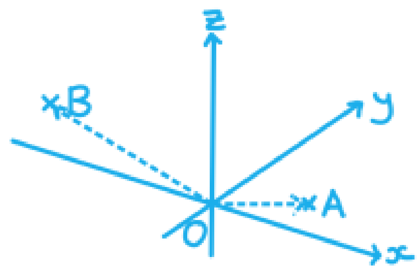

### Exam Tip

- In an exam, sketching a quick diagram can help to make working out a displacement vector easier

## Worked Example

The point A has coordinates (3, 0, -1) and the point B has coordinates (-2, -5, 7).
Find the displacement vector $\overrightarrow{AB}$.

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \qquad \overrightarrow{OB} = \begin{pmatrix} -2 \\ -5 \\ 7 \end{pmatrix}$$




$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$= -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} -2 \\ -5 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$



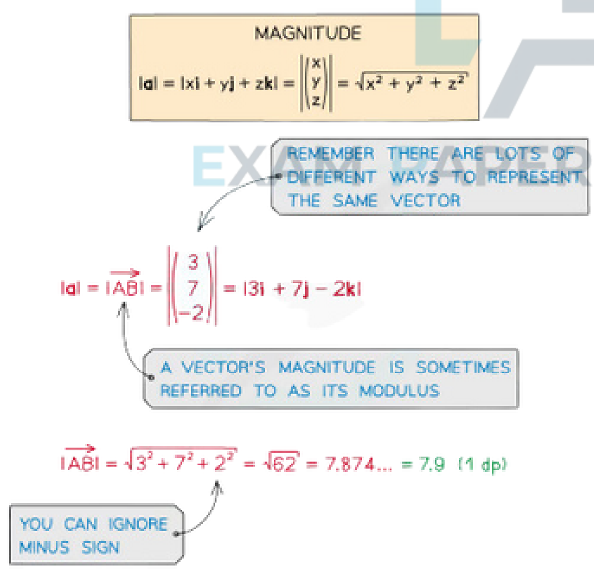
$$\boxed{\overrightarrow{AB} = \begin{pmatrix} -5 \\ -5 \\ 8 \end{pmatrix}}$$

# 3.7.3 Magnitude of a Vector

## Magnitude of a Vector

### How do you find the magnitude of a vector?

- The **magnitude** of a vector tells us its **size** or **length**
  - For a **displacement** vector it tells us the **distance** between the two points
  - For a **position** vector it tells us the **distance** of the point from the **origin**
- The magnitude of the vector $\overrightarrow{AB}$ is denoted $|\overrightarrow{AB}|$
  - The magnitude of the vector **a** is denoted |**a**|
- The magnitude of a vector can be found using **Pythagoras' Theroem**
- The magnitude of a vector $\boldsymbol{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is found using
  - $|\boldsymbol{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
  - where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
  - This is **given in the formula booklet**




### How do I find the distance between two points?

- **Vectors** can be used to find the distance (or displacement) between two points
  - It is the **magnitude** of the vector between them
- Given the **position vectors** of two points:
  - Find the displacement vector between them
  - Find the magnitude of the displacement vector between them

### Exam Tip

- Finding the magnitude of a vector is the same as finding the distance between two coordinates, it is a useful formula to commit to memory in order to save time in the exam, however it is in your formula booklet if you need it

### Worked Example

Find the magnitude of the vector AB = 4**i** – **j** + 2**k**.

| | |
|---|---|
| Magnitude of a vector | $\lvert \mathbf{v} \rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$, where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ |

$$|\overrightarrow{AB}| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$$

$$\boxed{|\overrightarrow{AB}| = \sqrt{21}}$$

# Unit Vectors

### What is a unit vector?

- A **unit vector** has a **magnitude** of 1
- It can be found by dividing a vector by its **magnitude**
  - This will result in a vector with a size of 1 unit in the direction of the original vector
- A unit vector in the direction of **a** is denoted $\frac{\mathbf{a}}{|\mathbf{a}|}$
  - For example a unit vector in the direction 3**i** – 4**j** is $\frac{(3\mathbf{i}-4\mathbf{j})}{\sqrt{3^2+4^2}}=\frac{3}{5}\mathbf{i}-\frac{4}{5}\mathbf{j}$

Exam Tip

- Finding the unit vector will not be a question on its own but will be a useful skill for further vectors problems so it is important to be confident with it

## Worked Example

Find the unit vector in the direction $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Let $\underline{a} = 2\underline{i} - 2\underline{j} + \underline{k}$

Find the magnitude of $\underline{a}$

| | |
|---|---|
| Magnitude of a vector | $\lvert \mathbf{v} \rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$, where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ |

$$|\underline{a}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

Divide $\underline{a}$ by its magnitude:

$$\text{Unit vector} = \frac{\underline{a}}{|\underline{a}|} = \frac{2\underline{i} - 2\underline{j} + \underline{k}}{3}$$

$$\boxed{\frac{2}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{1}{3}\underline{k}}$$

# 3.7.4 The Scalar Product

## The Scalar ('Dot') Product

### What is the scalar product?

- The scalar product (also known as the dot product) is one form in which two vectors can be combined together
- The scalar product between two vectors **a** and **b** is denoted $\mathbf{a} \cdot \mathbf{b}$
- The result of taking the scalar product of two vectors is a **real number**
  - i.e. a scalar
- The scalar product of two vectors gives information about the angle between the two vectors
  - If the scalar product is **positive** then the angle between the two vectors is **acute** (less than 90°)
  - If the scalar product is **negative** then the angle between the two vectors is **obtuse** (between 90° and 180°)
  - If the scalar product is **zero** then the angle between the two vectors is **90°** (the two vectors are **perpendicular**)

### How is the scalar product calculated?

- There are **two methods** for calculating the scalar product
- The most common method used to find the scalar product between the two vectors ***v*** and ***w*** is to find the **sum of the product of each component** in the two vectors
  - $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$
  - Where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
  - This is **given in the formula booklet**
- The scalar product is also equal to the **product of the magnitudes** of the two vectors and the **cosine of the angle between them**
  - $\boldsymbol{v} \cdot \boldsymbol{w} = |v||w|\cos\theta$
  - Where $\theta$ is the angle between ***v*** and ***w***
    - The two vectors ***v*** and ***w*** are joined at the start and pointing away from each other
- The scalar product can be used in the second formula to find the angle between the two vectors

### What properties of the scalar product do I need to know?

- If two vectors, ***v*** and ***w***, are **parallel** then the magnitude of the scalar product is equal to the **product** of the magnitudes of the vectors
  - $|\boldsymbol{v} \cdot \boldsymbol{w}| = |\boldsymbol{w}||\boldsymbol{v}|$
  - This is because cos 0° = 1 and cos 180° = -1
- If two vectors are **perpendicular** the scalar product is **zero**
  - This is because cos 90° = 0

## Exam Tip

- Whilst the formulae for the scalar product are given in the formula booklet, the properties of the scalar product are not, however they are important and it is likely that you will need to recall them in your exam so be sure to commit them to memory

## Worked Example

Calculate the scalar product between the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ and

$\boldsymbol{w} = 3\mathbf{j} - 2\mathbf{k} - \mathbf{i}$ using:

i)
the formula $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$,

$$\underline{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} = 2\underline{i} + 0\underline{j} - 5\underline{k}$$

$$\underline{w} = 3\underline{j} - 2\underline{k} - \underline{i} = -1\underline{i} + 3\underline{j} - 2\underline{k}$$

Be aware of the order of the terms.

| | |
|---|---|
| Scalar product | $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ |

$$\underline{v} \cdot \underline{w} = (2 \times -1) + (0 \times 3) + (-5 \times -2) = -2 + 10$$

$$\boxed{\underline{v} \cdot \underline{w} = 8}$$

ii)
the formula $\boldsymbol{v} \cdot \boldsymbol{w} = |v||w|\cos\theta$, given that the angle between the two vectors is 66.6°.

$$\underline{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} = 2\underline{i} + 0\underline{j} - 5\underline{k} \qquad \underline{w} = -1\underline{i} + 3\underline{j} - 2\underline{k}$$

| | |
|---|---|
| Scalar product | $\boldsymbol{v} \cdot \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \cos\theta$ |

Find the magnitude of both vectors:

$$|\underline{v}| = \sqrt{2^2 + (-5)^2} = \sqrt{29} \qquad |\underline{w}| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$$

$$v \cdot w = \sqrt{29} \times \sqrt{14} \cos 66.6°$$

$$\boxed{\underline{v} \cdot \underline{w} = 8}$$

# Angle Between Two Vectors

## How do I find the angle between two vectors?

- If two vectors with different directions are placed at the same starting position, they will form an angle between them
- The two formulae for the scalar product can be used together to find this angle
  - $\cos\theta = \dfrac{v_1w_1 + v_2w_2 + v_3w_3}{|\boldsymbol{v}||\boldsymbol{w}|}$
  - This is given in the formula booklet
- To find the angle between two vectors:
  - Calculate the scalar product between them
  - Calculate the magnitude of each vector
  - Use the formula to find $\cos\theta$
  - Use inverse trig to find $\theta$

### Exam Tip

- The formula for this is given in the formula booklet so you do not need to remember it but make sure that you can find it quickly and easily in your exam

## Worked Example

Calculate the angle formed by the two vectors $\boldsymbol{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

$\underline{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ , $\underline{w} = 3\underline{i} + 4\underline{j} - \underline{k} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

Start by finding the scalar product:

$$\underline{v} \cdot \underline{w} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$$

$$= (-1 \times 3) + (3 \times 4) + (2 \times -1) = 7$$

Find the magnitude of both vectors:

$$|\underline{v}| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}$$

$$|\underline{w}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}$$

| | |
|---|---|
| Angle between two vectors | $\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert v \rvert \lvert w \rvert}$ |

$$\cos\theta = \frac{7}{\sqrt{14} \times \sqrt{26}} = 0.3668...$$

$$\theta = \cos^{-1}(0.3668...)$$

$$\boxed{\theta = 68.5^\circ \ (3sf)}$$

# Perpendicular Vectors

## How do I know if two vectors are perpendicular?

- If the **scalar product** of two (non-zero) vectors is **zero** then they are **perpendicular**
  - If $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ then ***v*** and ***w*** must be perpendicular to each other
- Two vectors are **perpendicular** if their **scalar product** is **zero**
  - The value of $\cos\theta = 0$ therefore $|\mathbf{v}||\mathbf{w}|\cos\theta = 0$

### Worked Example

Find the value of $t$ such that the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ t \\ 5 \end{pmatrix}$ and $\boldsymbol{w} = (t-1)\mathbf{i} - \mathbf{j} + \mathbf{k}$ are perpendicular to each other.

The two vectors $\underline{v}$ and $\underline{w}$ are perpendicular if $\underline{v} \cdot \underline{w} = 0$.

$\underline{v} = \begin{pmatrix} 2 \\ t \\ 5 \end{pmatrix}$, $\underline{w} = \begin{pmatrix} t-1 \\ -1 \\ 1 \end{pmatrix}$

$\underline{v} \cdot \underline{w} = 2(t-1) + t(-1) + 5(1)$

$= 2t - 2 - t + 5$

Therefore $\underline{v}$ and $\underline{w}$ are perpendicular if

$t + 3 = 0$

$\boxed{t = -3}$

# 3.7.5 The Vector Product

## The Vector ('Cross') Product

### What is the vector (cross) product?

- The **vector product** (also known as the **cross** product) is a form in which two vectors can be combined together
- The vector product between two vectors ***v*** and ***w*** is denoted ***v*** × ***w***
- The result of taking the vector product of two vectors is a **vector**
- The **vector product** is a vector **in a plane** that is **perpendicular** to the two vectors from which it was calculated
  - This could be in either direction, depending on the angle between the two vectors
  - The **right-hand** rule helps you see which direction the vector product goes in
    - By pointing your index finger and your middle finger in the direction of the two vectors your thumb will automatically go in the direction of the vector product

### How do I find the vector (cross) product?

- There are **two methods** for calculating the vector product
- The **vector product** of the two vectors ***v*** and ***w*** can be written in **component form** as follows:
  - $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$
  - Where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
  - This is **given in the formula booklet**
- The vector product can also be found in terms of its **magnitude** and **direction**
- The **magnitude of the vector product** is equal to the **product of the magnitudes** of the two vectors and the **sine of the angle between them**
  - $|\boldsymbol{v} \times \boldsymbol{w}| = |v||w|\sin\theta$
  - Where $\theta$ is the angle between ***v*** and ***w***
    - The two vectors ***v*** and ***w*** are joined at the start and pointing away from each other
  - This is **given in the formula booklet**
- The **direction of the vector product** is **perpendicular** to both ***v*** and ***w***

### What properties of the vector product do I need to know?

- If two vectors are **parallel** then the vector product is **zero**
  - This is because sin 0° = sin 180° = 0
- If $\boldsymbol{v} \times \boldsymbol{w} = 0$ then ***v*** and ***w*** are parallel if they are non-zero
- If two vectors, ***v*** and ***w***, are **perpendicular** then the magnitude of the vector product is equal to the **product** of the magnitudes of the vectors

- $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{w}||\boldsymbol{v}|$
- This is because $\sin 90° = 1$

### Exam Tip

- The formulae for the vector product are given in the formula booklet, make sure you use them as this is an easy formula to get wrong
- The properties of the vector product are not given in the formula booklet, however they are important and it is likely that you will need to recall them in your exam so be sure to commit them to memory

## Worked Example

Calculate the magnitude of the vector product between the two vectors

$\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ using

i)

the formula $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$,

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} \qquad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

Use the formula to find the cross-product:

$$\underline{v} \times \underline{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} = \begin{pmatrix} (0)(-1) - (-5)(-2) \\ (-5)(3) - (2)(-1) \\ (2)(-2) - (0)(3) \end{pmatrix} = \begin{pmatrix} -10 \\ -13 \\ -4 \end{pmatrix}$$

Find the magnitude of $\underline{v} \times \underline{w}$:

$$|\underline{v} \times \underline{w}| = \sqrt{(-10)^2 + (-13)^2 + (-4)^2} = \sqrt{285}$$

$$\boxed{|\underline{v} \times \underline{w}| = 16.9 \text{ (3sf)}}$$



ii)

the formula , given that the angle between them is 1 radian.

Find the magnitude of $\underline{v}$ and $\underline{w}$:

$$|\underline{v}| = \sqrt{2^2 + 0^2 + (-5)^2} = \sqrt{29}$$

$$|\underline{w}| = \sqrt{3^2 + (-2)^2 + (-1)^2} = \sqrt{14}$$

$$|\underline{v} \times \underline{w}| = |\underline{v}||\underline{w}| \sin\theta$$

$$= \sqrt{29} \times \sqrt{14} \sin(1^c)$$

$$\boxed{|\underline{v} \times \underline{w}| = 17.0 \text{ (3sf)}}$$

# Areas using Vector Product

## How do I use the vector product to find the area of a parallelogram?

- The **area of the parallelogram** with two adjacent sides formed by the vectors ***v*** and ***w*** is equal to **the magnitude of the vector product** of two vectors ***v*** and ***w***
    - $A = |\boldsymbol{v} \times \boldsymbol{w}|$ where ***v*** and ***w*** form two **adjacent sides** of the parallelogram
        - This is **given in the formula booklet**

## How do I use the vector product to find the area of a triangle?

- The **area of the triangle** with two sides formed by the vectors ***v*** and ***w*** is equal to **half of the magnitude of the vector product** of two vectors ***v*** and ***w***
    - $A = \frac{1}{2}|\boldsymbol{v} \times \boldsymbol{w}|$ where ***v*** and ***w*** form two **sides** of the triangle
        - This is **not** given in the formula booklet

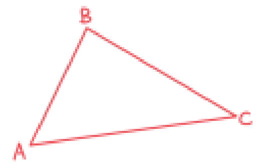
### Exam Tip

- The formula for the area of the parallelogram is given in the formula booklet but the formula for a triangle is not
    - Remember that the area of a triangle is half the area of a parallelogram

## Worked Example

Find the area of the triangle enclosed by the coordinates (1, 0, 5), (3, -1, 2) and (2, 0, -1).

Let A be (1, 0, 5), B be (3, -1, 2) and C be (2, 0, -1)




You can use any two direction vectors moving away from any vertex.

Find the two direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix}$$

Find the cross product of the two direction vectors:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} (-1)(-6) - (-3)(0) \\ (-3)(1) - (2)(-6) \\ (2)(0) - (-1)(1) \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 1 \end{pmatrix}$$

Find the magnitude of the cross product

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{6^2 + 9^2 + 1^2} = \sqrt{118}$$

Area of the triangle is half the magnitude

$$\text{Area} = \tfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \tfrac{1}{2}\sqrt{118}$$

$$\boxed{\text{Area} = 5.43 \text{ u}^2 \text{ (3sf)}}$$

# 3.7.6 Components of Vectors

## Components of Vectors

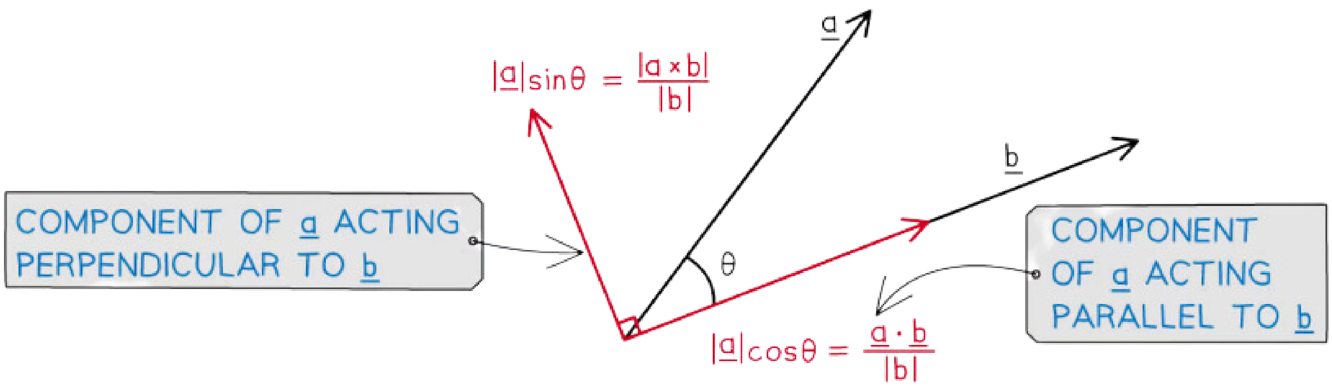
### Why do we write vectors in component form?

- When working with vectors in context it is often useful to break them down into components acting in a direction that is not one of the base vectors
- The **base vectors** are vectors acting in the directions **i, j** and **k**
- The vector will need to be resolved into components that are acting **perpendicular** to each other
- Usually, one component will be acting **parallel** to the direction of another vector and the other will act **perpendicular** to the direction of the vector
- For example: the components of a **force** parallel and perpendicular to the **line of motion** allows different types of problems to be solved
  - The **parallel** component of a force acting directly on a particle will be the component that causes an effect on the particle
  - The **perpendicular** component of a force acting directly on a particle will be the component that has no effect on the particle
- The two components of the force will have the same combined effect as the original vector



### How do we write vectors in component form?

- Use **trigonometry** to resolve a vector acting at an angle
- Given a vector ***a*** acting at an angle $\theta$ to another vector ***b***
  - Draw a vector triangle by decomposing the vector ***a*** into its components parallel and perpendicular to the direction of the vector ***b***
- The vector ***a*** will be the **hypotenuse** of the triangle and the two components will make up the **opposite** and **adjacent** sides
- The component of ***a*** acting **parallel** to ***b*** will be equal to the product of the magnitude of ***a*** and the cosine of the angle $\theta$
  - The component of ***a*** acting in the direction of ***b*** equals $|\boldsymbol{a}|\cos\theta$
  - This is equivalent to $\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}$
- The component of ***a*** acting **perpendicular** to ***b*** will be equal to the product of the magnitude of ***a*** and the sine of the angle $\theta$
  - The component of ***a*** acting perpendicular to the direction of ***b*** equals $|\boldsymbol{a}|\sin\theta$
  - This is equivalent to $\dfrac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{b}|}$
- The formulae for the components using the **scalar product** and the **vector product** are particularly useful as the angle is not needed
- The question may give you the angle the vector is acting in as a bearing
  - Bearings are always the angle taken from the north




### Exam Tip

- If a question asks you to find a component of a vector it is a good idea to sketch a quick diagram so that you can visualise which vectors are going in which direction
  - This is especially important if the question involves forces

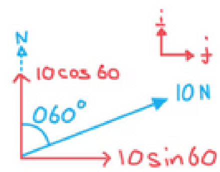
## Worked Example

A force with magnitude 10 N is acting on a bearing of 060° on an object which is moving with velocity vector **v** = 2**i** – 3**j**.

a)

By finding the components of the force in the **i** and **j** direction, write down the force as a vector.

$$F = \begin{pmatrix} 10\sin 60^\circ \\ 10\cos 60^\circ \end{pmatrix} = \begin{pmatrix} 5\sqrt{3} \\ 5 \end{pmatrix}$$




$$\boxed{F = 5\sqrt{3}\,\underline{i} + 5\underline{j}}$$

b)

Find the component of the force acting parallel to the direction of the object.

Method 1: Component of $F$ acting parallel to $\underline{v} = \dfrac{\underline{F}\cdot\underline{v}}{|\underline{v}|}$

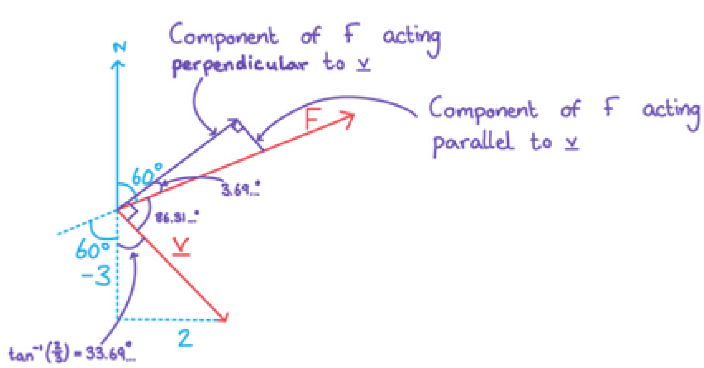
$$\underline{F}\cdot\underline{v} = \begin{pmatrix} 5\sqrt{3} \\ 5 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ -3 \end{pmatrix} = (5\sqrt{3})(2) + (5)(-3)$$

$$= -15 + 10\sqrt{3}$$

$$|\underline{v}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$$

$$\frac{\underline{F}\cdot\underline{v}}{|\underline{v}|} = \frac{-15 + 10\sqrt{3}}{\sqrt{13}} = 0.644 \text{ N (3 s.f.)}$$

Method 2: Use a diagram:




Component of $F$ acting parallel to $\underline{v} = 10\sin 3.69°$

$$\boxed{0.644\text{N (3 s.f.)}}$$

# 3.7.7 Geometric Proof with Vectors

## Geometric Proof with Vectors

### How can vectors be used to prove geometrical properties?

- If two vectors can be shown to be **parallel** then this can be used to prove parallel lines
  - If two vectors are **scalar multiples** of each other then they are **parallel**
  - To prove that two vectors are parallel simply show that one is a scalar multiple of the other
- If two vectors can be shown to be **perpendicular** then this can be used to prove perpendicular lines
  - If the **scalar product** is zero then the two vectors are **perpendicular**
- If two vectors can be shown to have equal **magnitude** then this can be used to prove two lines are the **same length**
- To prove a 2D shape is a **parallelogram** vectors can be used to
  - Show that there are two pairs of **parallel sides**
  - Show that the **opposite sides** are of **equal length**
    - The vectors opposite each other with be **equal**
  - If the angle between two of the vectors is shown to be 90° then the parallelogram is a **rectangle**
- To prove a 2D shape is a **rhombus** vectors can be used to
  - Show that there are two pairs of **parallel sides**
    - The vectors opposite each other with be **equal**
  - Show that **all four sides** are of **equal length**
  - If the angle between two of the vectors is shown to be 90° then the rhombus is a **square**



### How are vectors used to follow paths through a diagram?

- In a geometric diagram the vector $\overrightarrow{AB}$ forms a path from the point A to the point B
  - This is specific to the path AB
  - If the vector $\overrightarrow{AB}$ is labelled **a** then any other vector with the same **magnitude** and **direction** as **a** could also be labelled **a**
- The vector $\overrightarrow{BA}$ would be labelled **-a**
  - It is **parallel** to **a** but pointing in the **opposite direction**
- If the point M is exactly halfway between A and B it is called the midpoint of A and the vector $\overrightarrow{AM}$ could be labelled $\frac{1}{2}\mathbf{a}$
- If there is a point X on the line AB such that $\overrightarrow{AX} = 2\overrightarrow{XB}$ then X is two-thirds of the way along the line $\overrightarrow{AB}$
  - Other ratios can be found in similar ways
  - A diagram often helps to visualise this
- If a point X divides a line segment AB into the ratio p : q then
  - $\overrightarrow{AX} = \frac{p}{p+q}\overrightarrow{AB}$

- $\overrightarrow{XB} = \frac{q}{p+q}\overrightarrow{AB}$

## How can vectors be used to find the midpoint of two vectors?

- If the point A has position vector **a** and the point B has position vector **b** then the **position vector** of the **midpoint** of $\overrightarrow{AB}$ is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$
  - The **displacement vector** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
  - Let **M** be the midpoint of $\overrightarrow{AB}$ then $\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB}) = \frac{1}{2}(\mathbf{b}-\mathbf{a})$
  - The **position vector** $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \mathbf{a} + \frac{1}{2}(\mathbf{b}-\mathbf{a}) = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$

## How can vectors be used to prove that three points are collinear?

- Three points are collinear if they all **lie on the same line**
  - The vectors between the three points will be **scalar multiples** of each other
- The points A, B and C are collinear if $\overrightarrow{AB} = k\overrightarrow{BC}$
- If the points A, B and M are collinear and $\overrightarrow{AM} = \overrightarrow{MB}$ then M is the **midpoint** of $\overrightarrow{AB}$

### Exam Tip

- Think of vectors like a journey from one place to another
  - You may have to take a detour e.g. A to B might be A to O then O to B
- Diagrams can help, if there isn't one, draw one
  - If a diagram has been given begin by labelling all known quantities and vectors

## Worked Example

Use vectors to prove that the points $A$, $B$, $C$ and $D$ with position vectors $\mathbf{a} = (3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k})$, $\mathbf{b} = (8\mathbf{i} - 7\mathbf{j} - 5\mathbf{k})$, $\mathbf{c} = (3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$ and $\mathbf{d} = (5\mathbf{k} - 2\mathbf{i})$ are the vertices of a parallelogram.

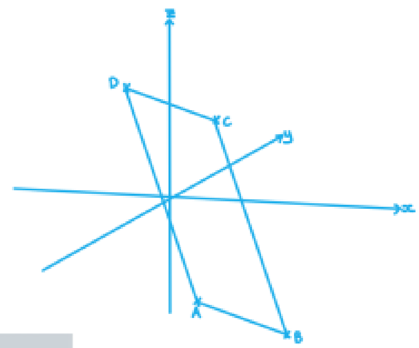
Find the displacement vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$$

$$\overrightarrow{BC} = \underline{c} - \underline{b} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 9 \end{pmatrix}$$

$$\overrightarrow{CD} = \underline{d} - \underline{c} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{DA} = \underline{a} - \underline{d} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ -9 \end{pmatrix}$$




$\overrightarrow{AB} = -\overrightarrow{CD}$ and $\overrightarrow{BC} = -\overrightarrow{DA}$ $\therefore$ ABCD must be a parallelogram

# 3.8 Vector Equations of Lines

## 3.8.1 Vector Equations of Lines

### Equation of a Line in Vector Form

### How do I find the vector equation of a line?

- The formula for finding the **vector equation** of a line is
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
    - Where ***r*** is the **position vector** of any point on the line
    - ***a*** is the **position vector** of a known point on the line
    - ***b*** is a **direction** (displacement) **vector**
    - $\lambda$ is a scalar
  - This is **given in the formula booklet**
  - This equation can be used for vectors in both 2– and 3– dimensions
- This formula is similar to a regular equation of a straight line in the form $y = mx + c$ but with a vector to show both a point on the line and the direction (or gradient) of the line
  - In 2D the gradient can be found from the direction vector
  - In 3D a numerical value for the direction cannot be found, it is given as a vector
- As ***a*** could be the position vector of **any** point on the line and ***b*** could be **any scalar multiple** of the direction vector there are infinite vector equations for a single line
- Given any two points on a line with position vectors ***a*** and ***b*** the **displacement** vector can be written as **b - a**
  - So the formula **r = a +** $\lambda$**(b - a)** can be used to find the vector equation of the line
  - This is **not given in the formula booklet**



### How do I determine whether a point lies on a line?

- Given the equation of a line $\boldsymbol{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ the point **c** with position vector $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ is on the line if there exists a value of $\lambda$ such that
  - $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$
  - This means that there exists a single value of $\lambda$ that satisfies the three equations:
    - $c_1 = a_1 + \lambda b_1$
    - $c_2 = a_2 + \lambda b_2$
    - $c_3 = a_3 + \lambda b_3$
- A GDC can be used to solve this system of linear equations for
  - The point only lies on the line if a single value of $\lambda$ exists for all three equations
- Solve one of the equations first to find a value of $\lambda$ that satisfies the first equation and then check that this value also satisfies the other two equations

- If the value of $\lambda$ does not satisfy all three equations, then the point **c** does not lie on the line

### Exam Tip

- Remember that the vector equation of a line can take many different forms
  - This means that the answer you derive might look different from the answer in a mark scheme
- You can choose whether to write your vector equations of lines using unit vectors or as column vectors
  - Use the form that you prefer, however column vectors is generally easier to work with

## Worked Example

a)

Find a vector equation of a straight line through the points with position vectors **a** = 4**i** – 5**k** and **b** = 3**i** - 3**k**

Use the position vectors to find the displacement vector between them.

$$\vec{OA} = \begin{pmatrix}4\\0\\-5\end{pmatrix}, \quad \vec{OB} = \begin{pmatrix}3\\0\\-3\end{pmatrix} \Rightarrow \vec{AB} = \begin{pmatrix}3\\0\\-3\end{pmatrix} - \begin{pmatrix}4\\0\\-5\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

| Vector equation of a line | $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ |
|---|---|

position vector of point a

$$r = \begin{pmatrix}4\\0\\-5\end{pmatrix} + \lambda \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

direction vector

or

position vector of point b

$$r = \begin{pmatrix}3\\0\\-3\end{pmatrix} + \lambda \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

direction vector

$$\boxed{r = \begin{pmatrix}4\\0\\-5\end{pmatrix} + \lambda \begin{pmatrix}-1\\0\\2\end{pmatrix}}$$

b)

Determine whether the point C with coordinate (2, 0, -1) lies on this line.

Let $c = \begin{pmatrix}2\\0\\-1\end{pmatrix}$, then check to see if there exists a value of $\lambda$ such that

$$\begin{pmatrix}2\\0\\-1\end{pmatrix} = \begin{pmatrix}4\\0\\-5\end{pmatrix} + \lambda \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

From the 'i' component : $4 - \lambda = 2$ ①

From the 'j' component : $0 + 0\lambda = 0$ ② (✓) Works for all $\lambda$

From the 'k' component : $-5 + 2\lambda = -1$ ③

① $\Longrightarrow \lambda = 2$ sub into ③ $\Longrightarrow -5 + (2 \times 2) = -5 + 4 = -1$ ✓

$\boxed{\text{Point C lies on the line}}$

# Equation of a Line in Parametric Form

## How do I find the vector equation of a line in parametric form?

- By considering the three separate components of a vector in the *x*, *y* and *z* directions it is possible to write the **vector equation** of a line as **three separate equations**
  - Letting $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ becomes
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
    - Where $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ is a position vector and $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is a direction vector
  - This vector equation can then be split into its three separate component forms:
    - $x = x_0 + \lambda l$
    - $y = y_0 + \lambda m$
    - $z = z_0 + \lambda n$
  - These are **given in the formula booklet**

## Worked Example

Write the parametric form of the equation of the line which passes through the point (-2, 1, 0) with direction vector $\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$.

| | |
|---|---|
| Parametric form of the equation of a line | $x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$ |

Use $r = a + \lambda b$ to write the equation in vector form first:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$$

position vector of a point

direction vector

Separate the components into their 3 separate equations.

$$\boxed{\begin{aligned} x &= -2 + 3\lambda \\ y &= 1 + \lambda \\ z &= -4\lambda \end{aligned}}$$

# Angle Between Two Lines

## How do we find the angle between two lines?

- The angle between two lines is equal to the angle between their **direction vectors**
  - It can be found using the **scalar product** of their direction vectors
- Given two lines in the form $\boldsymbol{r} = \boldsymbol{a}_1 + \lambda \boldsymbol{b}_1$ and $\boldsymbol{r} = \boldsymbol{a}_2 + \lambda \boldsymbol{b}_2$ use the formula
  - $\theta = \cos^{-1}\left(\dfrac{\boldsymbol{b}_1 \bullet \boldsymbol{b}_2}{|\boldsymbol{b}_1||\boldsymbol{b}_2|}\right)$
- If you are given the equations of the lines in a different form or two points on a line you will need to find their direction vectors first
- To find the angle ABC the vectors BA and BC would be used, both starting from the point B
- The intersection of two lines will always create **two angles**, an acute one and an obtuse one
  - A **positive scalar product** will result in the **acute angle** and a **negative scalar product** will result in the **obtuse angle**
    - Using the **absolute value** of the scalar product will **always result in the acute angle**

### Exam Tip

- In your exam read the question carefully to see if you need to find the acute or obtuse angle
  - When revising, get into the practice of double checking at the end of a question whether your angle is acute or obtuse and whether this fits the question

## Worked Example

Find the acute angle, in radians between the two lines defined by the equations:

$$l_1:\ \mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \text{ and } l_2:\ \mathbf{b} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$$

STEP 1: Find the scalar product of the direction vectors:

$$\begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} = (1 \times -3) + (-4 \times 2) + (-3 \times 5) = -3 + (-8) + (-15) = -26$$

negative, so the angle will be the obtuse angle.

STEP 2: Find the magnitudes of the direction vectors:

$$\sqrt{(1)^2 + (-4)^2 + (-3)^2} = \sqrt{26} \qquad \sqrt{(-3)^2 + (2)^2 + (5)^2} = \sqrt{38}$$

STEP 3: Find the angle:

$$\cos\theta = \frac{|-26|}{\sqrt{26}\sqrt{38}}$$

Using the absolute value will result in the acute angle

$$\theta = \cos^{-1}\left(\frac{26}{\sqrt{26}\sqrt{38}}\right)$$

$$\boxed{\theta = 0.597 \text{ radians (3sf)}}$$

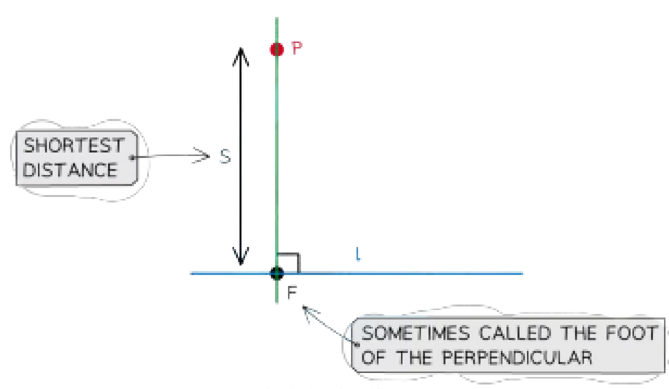
# 3.8.2 Shortest Distances with Lines

## Shortest Distance Between a Point and a Line

### How do I find the shortest distance from a point to a line?

- The shortest distance from any point to a line will always be the **perpendicular** distance
  - Given a line *l* with equation $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ and a point *P* not on *l*
  - The **scalar product** of the direction vector, **b**, and the vector in the direction of the **shortest distance** will be zero
- The shortest distance can be found using the following steps:
  - STEP 1: Let the vector equation of the line be **r** and the point not on the line be *P*, then the point on the line closest to *P* will be the point *F*
    - The point *F* is sometimes called the foot of the perpendicular
  - STEP 2: Sketch a diagram showing the line *l* and the points *P* and *F*
    - The vector $\overrightarrow{FP}$ will be **perpendicular** to the line *l*
  - STEP 3: Use the equation of the line to find the position vector of the point *F* in terms of $\lambda$
  - STEP 4: Use this to find the displacement vector $\overrightarrow{FP}$ in terms of $\lambda$
  - STEP 5: The scalar product of the direction vector of the line *l* and the displacement vector $\overrightarrow{FP}$ will be zero
    - Form an equation $\overrightarrow{FP} \cdot \mathbf{b} = 0$ and solve to find $\lambda$
  - STEP 6: Substitute $\lambda$ into $\overrightarrow{FP}$ and find the magnitude $\left|\overrightarrow{FP}\right|$
    - The shortest distance from the point to the line will be the magnitude of $\overrightarrow{FP}$

- Note that the shortest distance between the point and the line is sometimes referred to as the **length of the perpendicular**




### How do we use the vector product to find the shortest distance from a point to a line?

- The vector product can be used to find the shortest distance from any point to a line on a 2-dimensional plane
- Given a point, P, and a line $r = a + \lambda b$
  - The shortest distance from P to the line will be $\dfrac{\left|\overrightarrow{AP} \times b\right|}{|b|}$
  - Where *A* is a point on the line
  - This is **not** given in the formula booklet

## Exam Tip

- Column vectors can be easier and clearer to work with when dealing with scalar products.

## Worked Example

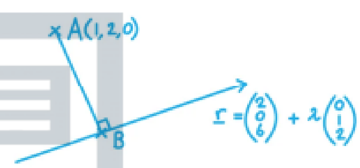
Point $A$ has coordinates (1, 2, 0) and the line $l$ has equation $\mathbf{r}=\begin{pmatrix}2\\0\\6\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\2\end{pmatrix}$.

Point $B$ lies on the $l$ such that $[AB]$ is perpendicular to $l$.

Find the shortest distance from $A$ to the line $l$.

B is on $l$ so can be written in terms of $\lambda$:

$$\vec{OB}=\begin{pmatrix}2\\0\\6\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\2\end{pmatrix}=\begin{pmatrix}2\\\lambda\\6+2\lambda\end{pmatrix}$$




Find $\vec{AB}$ using $\vec{AB}=\vec{OB}-\vec{OA}$

$$\vec{AB}=\begin{pmatrix}2\\\lambda\\6+2\lambda\end{pmatrix}-\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}1\\\lambda-2\\6+2\lambda\end{pmatrix}$$

$\vec{AB}$ is perpendicular to $l$: $\vec{AB}\cdot\begin{pmatrix}0\\1\\2\end{pmatrix}=0$

$$\begin{pmatrix}1\\\lambda-2\\6+2\lambda\end{pmatrix}\cdot\begin{pmatrix}0\\1\\2\end{pmatrix}=0$$

$$\lambda-2+2(6+2\lambda)=0$$

$$5\lambda+10=0$$

$$\lambda=-2$$

Substitute back into $\vec{AB}$ and find the magnitude:

$$\vec{AB}=\begin{pmatrix}1\\-2-2\\6+2(-2)\end{pmatrix}=\begin{pmatrix}1\\-4\\2\end{pmatrix}$$

$$|\vec{AB}|=\sqrt{1^2+(-4)^2+2^2}=\sqrt{21}$$

$$\boxed{\text{Shortest distance }=\sqrt{21}\text{ units}}$$

# Shortest Distance Between Two Lines

## How do we find the shortest distance between two parallel lines?

- Two **parallel** lines will never intersect
- The shortest distance between two **parallel lines** will be the **perpendicular distance** between them
- Given a line $l_1$ with equation $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and a line $l_2$ with equation $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$ then the shortest distance between them can be found using the following steps:
  - STEP 1: Find the vector between $\mathbf{a}_1$ and a general coordinate from $l_2$ in terms of $\mu$
  - STEP 2: Set the scalar product of the vector found in STEP 1 and the direction vector $\mathbf{d}_1$ equal to zero
    - Remember the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are scalar multiples of each other and so either can be used here
  - STEP 3: Form and solve an equation to find the value of $\mu$
  - STEP 4: Substitute the value of $\mu$ back into the equation for $l_2$ to find the coordinate on $l_2$ closest to $l_1$
  - STEP 5: Find the distance between $\mathbf{a}_1$ and the coordinate found in STEP 4
- Alternatively, the formula $\dfrac{|\overrightarrow{AB} \times \mathbf{d}|}{|\mathbf{d}|}$ can be used
  - Where $\overrightarrow{AB}$ is the vector connecting the two given coordinates $\mathbf{a}_1$ and $\mathbf{a}_2$
  - **d** is the simplified vector in the direction of $\mathbf{d}_1$ and $\mathbf{d}_2$
  - This is **not** given in the formula booklet



## How do we find the shortest distance from a given point on a line to another line?

- The shortest distance from any point on a line to another line will be the **perpendicular** distance from the point to the line
- If the angle between the two lines is known or can be found then right-angled trigonometry can be used to find the perpendicular distance
  - The formula $\dfrac{|\overrightarrow{AB} \times \mathbf{d}|}{|\mathbf{d}|}$ given above is derived using this method and can be used
- Alternatively, the equation of the line can be used to find a general coordinate and the steps above can be followed to find the shortest distance

## How do we find the shortest distance between two skew lines?

- Two **skew** lines are not parallel but will never intersect
- The shortest distance between two **skew lines** will be perpendicular to **both** of the lines
  - This will be at the point where the two lines pass each other with the perpendicular distance where the point of intersection would be
  - The **vector product** of the two direction vectors can be used to find a vector in the direction of the shortest distance
  - The shortest distance will be a vector **parallel** to the vector product

- To find the shortest distance between two skew lines with equations $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$,
  - STEP 1: Find the vector product of the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$
    - $\mathbf{d} = \mathbf{d}_1 \times \mathbf{d}_2$
  - STEP 2: Find the vector in the direction of the line between the two general points on $l_1$ and $l_2$ in terms of $\lambda$ and $\mu$
    - $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
  - STEP 3: Set the two vectors parallel to each other
    - $\mathbf{d} = k\overrightarrow{AB}$
  - STEP 4: Set up and solve a system of linear equations in the three unknowns, $k$, $\lambda$ and $\mu$

### Exam Tip

- Exam questions will often ask for the shortest, or minimum, distance within vector questions
- If you're unsure start by sketching a quick diagram
- Sometimes calculus can be used, however vector methods are usually required

## Worked Example

A drone travels in a straight line and at a constant speed. It moves from an initial point (-5, 4, -8) in the direction of the vector $\begin{pmatrix}-1\\2\\1\end{pmatrix}$. At the same time as the drone begins moving a bird takes off from initial point (6, -4, 3) and moves in a straight line at a constant speed in the direction of the vector $\begin{pmatrix}2\\-3\\4\end{pmatrix}$.

Find the minimum distance between the bird and the drone during this movement.

Find the vector product of the direction vectors.

$$\begin{pmatrix}2\\-3\\4\end{pmatrix} \times \begin{pmatrix}-1\\2\\1\end{pmatrix} = \begin{pmatrix}(-3)(1)-(4)(2)\\(4)(-1)-(2)(1)\\(2)(2)-(-3)(-1)\end{pmatrix} = \begin{pmatrix}-11\\-6\\1\end{pmatrix}$$

Find the vector in the direction of the line between the general coordinates.

$$\overrightarrow{AB} = \underbrace{\begin{pmatrix}-5-\mu\\4+2\mu\\-8+\mu\end{pmatrix}}_{\text{A point on } l_2} - \underbrace{\begin{pmatrix}6+2\lambda\\-4-3\lambda\\3+4\lambda\end{pmatrix}}_{\text{A point on } l_1} = \begin{pmatrix}-11-\mu-2\lambda\\8+2\mu+3\lambda\\-11+\mu-4\lambda\end{pmatrix}$$

$$\begin{pmatrix}-11-\mu-2\lambda\\8+2\mu+3\lambda\\-11+\mu-4\lambda\end{pmatrix} = k\begin{pmatrix}-11\\-6\\1\end{pmatrix}$$

$\overrightarrow{AB}$ is parallel to $\begin{pmatrix}-11\\-6\\1\end{pmatrix}$ so $\overrightarrow{AB} = k\begin{pmatrix}-11\\-6\\1\end{pmatrix}$

Set up and solve a system of equations.

$$\left.\begin{aligned}11k-2\lambda-\mu &= 11\\6k+3\lambda+2\mu &= -8\\\mu-4\lambda-k &= 11\end{aligned}\right\}$$

Solve using GDC: $k=\frac{31}{79}$ $\lambda=-\frac{238}{79}$ $\mu=-\frac{52}{79}$

Substitute back into the expression for $\overrightarrow{AB}$ and find the magnitude:

$$|\overrightarrow{AB}| = \left|\begin{pmatrix}-11-\left(-\frac{52}{79}\right)-2\left(-\frac{238}{79}\right)\\8+2\left(-\frac{52}{79}\right)+3\left(-\frac{238}{79}\right)\\-11+\left(-\frac{52}{79}\right)-4\left(-\frac{238}{79}\right)\end{pmatrix}\right| = \left|\begin{pmatrix}-\frac{341}{79}\\-\frac{186}{79}\\\frac{31}{79}\end{pmatrix}\right| = \sqrt{\left(-\frac{341}{79}\right)^2+\left(-\frac{186}{79}\right)^2+\left(\frac{31}{79}\right)^2}$$

$$\boxed{\text{Shortest distance} = 4.93 \text{ units (3 s.f.)}}$$

# 3.9 Modelling with Vectors

## 3.9.1 Kinematics with Vectors

### Kinematics using Vectors

#### How are vectors related to kinematics?

- Kinematics is the use of mathematics to model motion in objects
- If an object is moving in **one dimension** then its velocity, displacement and time are related using the formula $s = vt$
  - where $s$ is **displacement**, $v$ is **velocity** and $t$ is the **time taken**
- If an object is moving in **more than one dimension** then **vectors** are needed to represent its **velocity** and **displacement**
  - Whilst **time** is a **scalar quantity**, **displacement** and **velocity** are both **vector quantities**
- Vectors are often used in questions in the context of **forces**, **acceleration** or **velocity**
- The position of an object at a particular time can be modelled using a vector equation

#### How do I find the direction of a vector?

- Vectors have opposite directions if they are the same size but opposite signs
- The direction of a vector is what makes it more than just a scalar
  - E.g. two objects with velocities of 7 m/s and -7 m/s are travelling at the **same speed** but in **opposite directions**
- Two vectors are **parallel** if and only if one is a **scalar multiple** of the other
- For real-life contexts such as mechanics, direction can be calculated from a given vector using **trigonometry**
  - **Given the i and j components a right-triangle can be created and the angle found using SOHCAHTOA**
- It is usually given as a **bearing** or as an angle calculated **anticlockwise** from the positive $x$-axis

#### How do I find the distance between two moving objects?

- If two objects are moving with constant velocity in non-parallel directions the distance between them will change
- The distance between them can be found by finding the magnitude of their position vectors at any point in time
- The **shortest distance** between the two objects at a particular time can be found by finding the value of the time at which the magnitude is at its minimum value
  - Let the time when the objects are at the shortest distance be $t$
  - Find the distance, $d$, in terms of $t$ by substituting into the equation for the magnitude of their position vectors
  - $d^2$ will be an expression in terms of $t$ which can be differentiated and set to 0
  - Solving this will give the time at which the distance is at a minimum
  - Substitute this back into the expression for $d$ to find the shortest distance

## Exam Tip

- Kinematics questions can have a lot of information in, read them carefully and pick out the parts that are essential to the question
- Look out for where variables used are the same and/or different within vector equations, you will need to use different techniques to find these

## Worked Example

Two objects, A and B, are moving so that their position relative to a fixed point, O at time $t$, in minutes can be defined by the position vectors $\boldsymbol{r}_A = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\boldsymbol{r}_B = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

The unit vectors **i** and **j** are a displacement of 1 metre due East and North of O respectively.

a)
Find the coordinates of the initial position of the two objects.

The initial position is when $t = 0$

$$r_A = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + 0\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$r_B = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 0\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$\boxed{A\,(3,-1) \text{ and } B\,(2,5)}$



b)
Find the shortest distance between the two objects and the time at which this will occur.

Let the shortest distance occur at time, $t$, then:

A: $(3-2t, -1+4t)$ B: $(2+3t, 5-t)$

Find the distance between A and B in terms of $t$:

$$d = \sqrt{[(2+3t)-(3-2t)]^2 + [(5-t)-(-1+4t)]^2}$$

$$= \sqrt{(-1+5t)^2 + (6-5t)^2}$$

$$= \sqrt{(1-10t+25t^2)+(36-60t+25t^2)}$$

$$d^2 = 37 - 70t + 50t^2$$

Find the minimum point of $d^2$:

$$\frac{dd^2}{dt} = -70 + 100t \quad \therefore \quad -70 + 100t = 0$$

$$t = \frac{70}{100} = 0.7$$

When $t = 0.7$, $d = \sqrt{37 - 70(0.7) + 50(0.7)^2} = \sqrt{12.5}$

$$\boxed{d = 3.54 \text{ m } (3 \text{ s.f.})}$$

# 3.9.2 Constant & Variable Velocity

## Vectors & Constant Velocity

### How are vectors used to model linear motion?

- If an object is moving with **constant velocity** it will travel in a **straight line**
- For an object moving in a **straight line** in two or three dimensions its velocity, displacement and time can be related using the vector equation of a line
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
  - Letting
    - $r$ be the position of the object at the time, $t$
    - $\boldsymbol{a}$ be the position vector, $\boldsymbol{r_0}$ at the start ($t = 0$)
    - $\lambda$ represent the time, $t$
    - $\boldsymbol{b}$ be the **velocity** vector, $\boldsymbol{v}$
  - Then the position of the object at the time, $t$ can be given by
    - $\boldsymbol{r} = \boldsymbol{r_0} + t\boldsymbol{v}$
- The **velocity vector** is the direction vector in the equation of the line
- The speed of the object will be the magnitude of the velocity $|\boldsymbol{v}|$



### Worked Example

A car, moving at constant speed, takes 2 minutes to drive in a straight line from point A (-4, 3) to point B (6, -5).

At time $t$, in minutes, the position vector ($\boldsymbol{p}$) of the car relative to the origin can be given in the form $\boldsymbol{p} = \boldsymbol{a} + t\boldsymbol{b}$.

Find the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Vector $\underline{a}$ represents the initial position and vector $\underline{b}$ represents the direction vector per minute.

Position vector $\overrightarrow{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

At $t = 0$ minutes, $\underline{p} = \underline{a}$ so $\underline{a} = \overrightarrow{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

Position vector $\overrightarrow{OB} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$

At $t = 2$ minutes, the car is at the point B and so $\overrightarrow{OB} = \underline{a} + 2\underline{b}$

$\begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + 2\underline{b}$

Direction vector $2\underline{b} = \begin{pmatrix} 6 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \end{pmatrix}$

$\boxed{\underline{a} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}}$

## Vectors & Variable Velocity

### How are vectors used to model motion with variable velocity?

- The **velocity** of a particle is the rate of change of its **displacement** over time
- In one dimension velocity, ***v***, is found be taking the derivative of the displacement, ***s***, with respect to time, ***t***
  - $v = \frac{\mathrm{d}s}{\mathrm{d}t}$
- In more than one dimension **vectors** are used to represent motion
- For displacement given as a function of time in the form
  - $\mathbf{r}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$
- The velocity vector can be found by differentiating each component of the vector individually
  - $\mathbf{v} = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$
  - $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \begin{pmatrix} f_1'(t) \\ f_2'(t) \end{pmatrix}$
  - The velocity should be left as a **vector**
  - The speed is the magnitude of the velocity
- If the velocity vector is known, displacement can be found by **integrating** each component of the vector individually
  - The constant of integration for each component will need to be found
- The **acceleration** of a particle is the rate of change of its **velocity** over time
- In one dimension acceleration, ***a***, is found be taking the derivative of the velocity, ***v***, with respect to time, ***t***
  - $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$
- In two dimensions acceleration can be found by differentiating each component of the velocity vector individually
  - $\mathbf{a} = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}$
  - $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \begin{pmatrix} v_1'(t) \\ v_2'(t) \end{pmatrix}$
  - $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} f_1''(t) \\ f_2''(t) \end{pmatrix}$
- If the acceleration vector is known, the velocity vector can be found by **integrating** each component of the acceleration vector individually
  - The constant of integration for each component will need to be found

## Exam Tip

- Look out for clues in the question as to whether you should treat the question as a constant or variable velocity problem
  - 'moving at a constant speed' will imply using a linear model
  - an object falling or rolling would imply variable velocity

## Worked Example

A ball is rolling down a hill with velocity $v = \begin{pmatrix} 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ -0.8 \end{pmatrix}$. At the time $t = 0$ the coordinate of the ball are (3, -2).

a)
Find the acceleration vector of the ball's motion.

$$\underline{v} = \begin{pmatrix} 5 \\ 3 - 0.8t \end{pmatrix} \Rightarrow \underline{a} = \frac{d\underline{v}}{dt} = \begin{pmatrix} 0 \\ -0.8 \end{pmatrix}$$

$$\boxed{\underline{a} = -0.8\underline{j}}$$

b)
Find the position vector of the ball at the time, *t*.

$$\underline{r} = \int \underline{v}\, dt = \int \begin{pmatrix} 5 \\ 3 - 0.8t \end{pmatrix} dt = \begin{pmatrix} 5t + c \\ 3t - \frac{0.8t^2}{2} + d \end{pmatrix}$$

A constant of integration is needed for both components.

at $t = 0$, $\underline{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$$\begin{pmatrix} 5(0) + c \\ 3(0) - \frac{0.8(0)^2}{2} + d \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad \therefore \quad c = 3, \; d = -2$$

$$\boxed{\underline{r} = (5t + 3)\underline{i} + (3t - 0.4t^2 - 2)\underline{j}}$$

# 3.10 Graph Theory

## 3.10.1 Introduction to Graph Theory

### Parts of a Graph

A **graph** is a mathematical structure that is used to represent objects and the connections between them. They can be used in modelling many real-life applications, e.g. electrical circuits, flight paths, maps etc.

### What are the different parts of a graph?

- A **vertex** (point) represents an object or a place
  - **Adjacent vertices** are connected by an edge
  - The **degree** of a vertex can be defined by how many edges are connected to it
- An **edge** (line) forms a connection between two vertices
  - **Adjacent edges** share a common vertex
  - An edge that starts and ends at the same vertex is called a **loop**
  - There may be **multiple edges** connecting two vertices

# Types of Graphs

## What are the types of graphs?

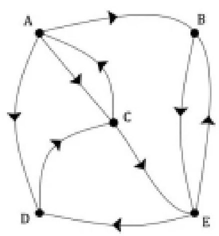
- A **complete graph** is a graph in which each vertex is connected by an edge to each of the other vertices
- The edges in a **weighted graph** are assigned numerical values such as distance or money
- The edges in a **directed graph** can only be travelled along in the direction indicated
  - the **in-degree** of a vertex is the number of edges that lead to that vertex
  - the **out-degree** is the number of edges that leave from that vertex
- A **simple graph** is undirected and unweighted and contains **no loops** or **multiple edges**
- Given a graph G, a **subgraph** will only contain edges and vertices that appear in G
- In a **connected graph** it is possible to move along the edges and vertices to find a route between any two vertices
  - If the graph is **strongly connected**, this route can be in either direction between the two vertices
- A **tree** is a graph in which any two vertices are connected by exactly one path
- A **spanning tree** is a subgraph, which is also a tree, of a graph G that contains all the vertices from G

### Exam Tip

- There are a lot of specific terms involved in graph theory and you are often asked to describe them in an exam - make sure you learn the definitions
- Make sure that any graphs you draw are big and clear so they are easy for the examiner to read

## Worked Example

The graph G shown below is a strongly connected, unweighted, directed graph with 5 vertices.




a)

State the in-degree of vertex A.

Only the edge connecting A and C is going into A

In - degree of vertex A = 1

b)

Explain why the graph is considered to be strongly connected.

The graph is strongly connected because it is possible to construct a walk in either direction between any two vertices

# 3.10.2 Walks & Adjacency Matrices

## Walks & Adjacency Matrices

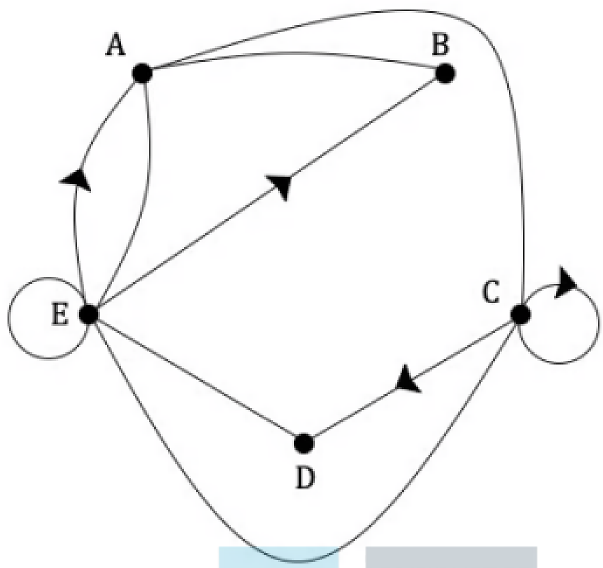
**Adjacency matrices** are another way to represent graphs and connections between the different vertices.

### What is an adjacency matrix?

- An adjacency matrix is a **square** matrix where all of the vertices in the graph are listed as the headings for both the rows ($i$) and columns ($j$)
- An adjacency matrix can be used to show the **number of direct connections** between two vertices
- An entry of **0** in the matrix means that there is **no direct connection** between that pair of vertices
- In a **simple** graph the only entries are either 0 or 1
- A **loop** is indicated in an adjacency matrix with a value in the **leading diagonal** (the line from top left to bottom right)
  - In an **undirected matrix** the value in the leading diagonal will be **2** because you can use the loop to travel out of and into the vertex in two different directions
  - In a **directed matrix**, if the loop has been given a direction, the value in the leading diagonal will be **1** as you can only travel along the loop out of and back into the vertex in one direction
  - For a graph with **no loops** every entry in the leading diagonal will be **0**
- An **undirected graph** will be **symmetrical** in the leading diagonal
- The sum of the entries in a **row** is the **in degree** of that vertex
- The sum of the entries in a **column** is the **out degree** of that vertex

## Worked Example

Let G be the graph below.




Write down the adjacency matrix for G.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{pmatrix} \end{array}$$

# Number of Walks

## What is a walk?

- A **walk** is a **sequence of vertices** that are visited when moving through a graph along its **edges**
- Both **edges** and **vertices** can be revisited in a walk
- The **length of a walk** is the **total number of edges** that are traversed in the walk

## How do you find the number of walks in a graph?

- Let $\boldsymbol{M}$ denote the adjacency matrix of a graph. The $(i, j)$ entry in the matrix $\boldsymbol{M}^k$ will give the number of walks of length $k$ from vertex $i$ to vertex $j$
- If there is an entry of **2** in the leading diagonal of the matrix, this should be changed to a **1** **before** the matrix is raised to a power
- The number of walks, between vertex $i$ and vertex $j$, of length $n$ or less can be given by the matrix $\boldsymbol{S}^n$, where $\boldsymbol{S}^n = \boldsymbol{M}^1 + \boldsymbol{M}^2 + ... + ... \boldsymbol{M}^n$
    - If all of the entries in a single row of $\boldsymbol{S}^n$ are non-zero values then the graph is **connected**

### Exam Tip

- Read the question carefully to determine if you need to choose a specific power for the adjacency matrix or if you need to play around with different powers!

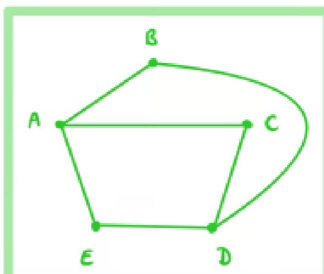
## Worked Example

The adjacency matrix $\boldsymbol{M}$ of a graph G is given by

$$\boldsymbol{M} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

a)

Draw the graph described by the adjacency matrix $\boldsymbol{M}$.




b)

Find the number of walks of length 4 from vertex B to Vertex E.

Enter the matrix into your GDC and raise it to the power 4

$$M^4 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}^4$$

$$M^4 = \begin{pmatrix} 18 & 0 & 0 & 18 & 0 \\ 0 & 12 & 12 & 0 & 12 \\ 0 & 12 & 12 & 0 & 12 \\ 18 & 0 & 0 & 18 & 0 \\ 0 & 12 & 12 & 0 & 12 \end{pmatrix}$$

Value in row B and column E

The number of walks of length 4 from vertex B to vertex E is 12

c)

Find the number of walks of 3 or less from vertex A to vertex C.

Enter the matrix into your GDC and add successive powers of it

$S^3 = M + M^2 + M^3$

Value in row A and column C

$$S^3 = \begin{pmatrix} 3 & 7 & 7 & 3 & 7 \\ 7 & 2 & 2 & 7 & 2 \\ 7 & 2 & 2 & 7 & 2 \\ 3 & 7 & 7 & 3 & 7 \\ 7 & 2 & 2 & 7 & 2 \end{pmatrix}$$

The number of walks of length 3 or less from vertex A to vertex C is 7

# Weighted Adjacency Tables

A **weighted adjacency table** gives more detailed information about the connection between different vertices in a **weighted graph**.

## What is a weighted adjacency table?

- A weighted adjacency table is different to an adjacency matrix as the **value** in each cell is the **weight** of the edge connecting that pair of vertices
  - Weight could be cost, distance, time etc.
- An **empty cell** can be used to indicate that there is **no connection** between a pair of vertices
- A directed graph is **not** symmetrical along the leading diagonal (the line from top left to bottom right)
- When drawing a graph from its adjacency table be careful when labelling the edges
  - For an **un-directed graph** the two cells between a specific pair of vertices will be the same so connect the vertices with **one edge** labelled with the relevant weight
  - For a **directed graph** if the two cells between a specific pair of vertices have different values draw **two lines** between the vertices and label each with the correct weight and direction
- A weighted adjacency table can be used to work out the weight of different **walks** in the graph

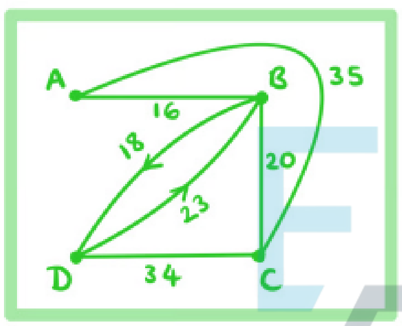
## Worked Example

The table below shows the time taken in minutes to travel by car between 4 different towns.

| | A | B | C | D |
|---|---|---|---|---|
| A | | 16 | 35 | |
| B | 16 | | 20 | 18 |
| C | 35 | 20 | | 34 |
| D | | 23 | 34 | |

a)

Draw the graph described by the adjacency table.






b)

State the time taken to drive from Town B to Town D.

$\boxed{\text{18 minutes}}$

# 3.10.3 Minimum Spanning Trees

## Kruskal's Algorithm

In a situation that can be modelled by a graph, **Kruskal's algorithm** is a mathematical tool that can be used to **reduce** costs, materials or time.

### Why do we use Kruskal's Algorithm?

- Kruskal's algorithm is a series of steps that when followed will produce the **minimum spanning tree** for a **connected graph**
- Finding the minimum spanning tree is useful in a lot of practical applications to connect all of the vertices in the most efficient way possible
- The **number of edges** in a minimum spanning tree will always be **one less** than the **number of vertices** in the graph
- A **cycle** is a **walk** that starts at a given vertex and ends at the **same** vertex.
    - A minimum spanning tree **cannot** contain any cycles.

### What is Kruskal's Algorithm?

- STEP 1
  Sort the edges in terms of increasing weight
- STEP 2
  Select the edge of least weight (if there is more than one edge of the same weight, either may be used)
- STEP 3
  Select the next edge of least weight that has not already been chosen and add it to your tree provided that it does not make a cycle with any of the previously selected edges
- STEP 4
  Repeat STEP 3 until all of the vertices in the graph are connected

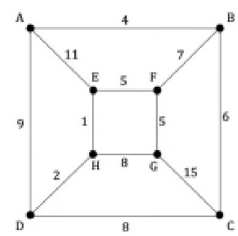


**Exam Tip**

- When using any of the algorithms for finding the minimum spanning tree, make sure that you state the **order** in which the edges are selected to get full marks for working!

## Worked Example

Consider the weighted graph G below.




a)

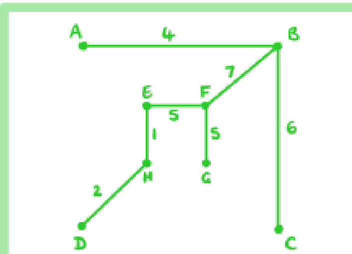
Use Kruskal's algorithm to find the minimum spanning tree. Show each step of the algorithm clearly.

List the edges in order of weight

EH (1)
DH (2)
AB (4)
EF (5)
FG (5)
BC (6)
BF (7)
CD (8)
GH (8)
AD (9)
AE (11)
CG (15)



Add the edge of least weight, then the next unused edge of least weight (without making a cycle) until all vertices are connected




Edges added in this order:

EH, DH, AB, $\underbrace{\text{EF, FG}}$, BC, BF

Note: EF and FG could be added in either order

b)

State the total weight of the minimum spanning tree.

Add up the weights of the edges in the minimum spanning tree

$1 + 2 + 4 + 5 + 5 + 6 + 7 = \boxed{30}$

# Prim's Algorithm

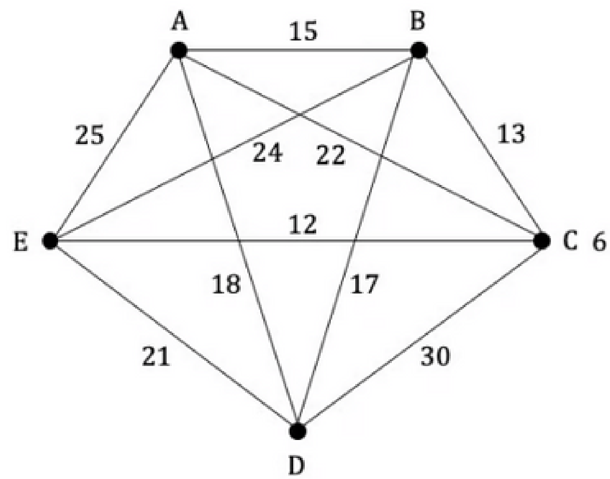
**Prim's algorithm** is a second method of finding the minimum spanning tree of a graph.

## What is Prim's Algorithm?

- **Prim's algorithm** involves adding edges from vertices that are **already connected** to the tree.
- Cycles are **avoided** by only adding edges that are **not** already connected at one end.
- STEP 1
  Start at any vertex and choose the edge of least weight that is connected to it
- STEP 2
  Choose the edge of least weight that is incident (connected) to any of the vertices already connected and does not connect to another vertex that is already in the tree
- STEP 3
  Repeat STEP 2 until all of the vertices are added to the tree

## Worked Example

Consider the weighted graph below.




a)

Using Prim's algorithm, find the minimum spanning tree.

Select a starting vertex (A) and choose the edge of least weight that is connected to it

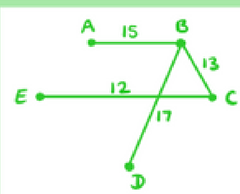

AB (15)

Continue to select the edge of least weight that is connected to any of the other vertices that are already connected in the tree


AB (15)
BC (13)



Repeat this process until all vertices are connected, remembering to record the order in which the edges were added




AB (15)
BC (13)
CE (12)
BD (17)

Add up the weights of the edges in the minimum spanning tree

$15 + 13 + 12 + 17 = \boxed{57}$

b)

State the total weight of the minimum spanning tree.

Add up the weights of the edges in the minimum spanning tree

$15 + 13 + 12 + 17 = \boxed{57}$

# Prim's Algorithm Using a Matrix

Information may be given to you either in the form of a graph or as a **weighted adjacency table**. Prim's algorithm can be adapted to be used from the adjacency matrix.

## How do you apply Prim's algorithm to a matrix?

- A minimum spanning tree is built up from the least weight edges that are incident to vertices already in the tree by looking at the **relevant rows** in the **adjacency table**
- STEP 1
  Select any vertex to start from, cross out the values in the column associated with that vertex and label the row associated with the vertex 1
- STEP 2
  Circle the lowest value in any cell along that row and add the edge to your tree, cross out the remaining values in the column of the cell that you have circled
- STEP 3
  Label the row associated with the same vertex as the column in the previous STEP with the next
- STEP 4
  Circle the lowest value in any cell along any of the rows that have been labelled and add the edge to your tree, cross out the remaining values in the column of the cell that you have circled
- STEP 5
  Repeat STEPS 3 and 4 until all rows have been labelled and all vertices have been added to the tree

## Which should I use Prim's or Kruskal's Algorithm?

- **Kruskal's algorithm** can be used when the information is in graph form whereas **Prim's algorithm** can be used in either graph or matrix form.
- **Prim's algorithm** is sometimes considered to be more **efficient** that **Kruskal's algorithm** as
  - the edges **do not need to be ordered** at the start and
  - it does not rely on **checking for cycles** at each step

- An **exam question** will usually specify which method should be used, otherwise you have the choice
- If you are asked to find the **minimum spanning tree** and the information given in the question is in the form of a **table**, you should use **Prim's algorithm**

### Exam Tip

- Look out for questions that ask you to minimise the cost or length etc. from a weighted graph – they are implying that they want you to find the minimum spanning tree!

## Worked Example

Celeste is building a model city incorporating 6 main buildings that need to be connected to an electrical supply.

Each vertex listed in the table below represents a building and the weighting of each edge is the cost in USD of creating a link to the electrical supply between the given vertices.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| A | - | 4 | 9 | 8 | 11 | 3 |
| B | 4 | - | 13 | 2 | 5 | 12 |
| C | 9 | 13 | - | 7 | 1 | 4 |
| D | 8 | 2 | 7 | - | 10 | 3 |
| E | 11 | 5 | 1 | 10 | - | 15 |
| F | 3 | 12 | 4 | 3 | 15 | - |

Celeste wants to find the lowest cost solution that links all 6 buildings up to the electrical supply.

a)
Starting from vertex A, use Prim's algorithm on the table to find and draw the minimum spanning tree. Show each step of the process clearly.

Start at any vertex and label it 1, cross out the values in colum A and select the edge of least weight in row A

| | | A | B | C | D | E | F |
|---|---|---|---|---|---|---|---|
| 1 | A | - | 4 | 9 | 8 | 11 | 3 |
| | B | 4 | - | 13 | 2 | 5 | 12 |
| | C | 9 | 13 | - | 7 | 1 | 4 |
| | D | 8 | 2 | 7 | - | 10 | 3 |
| | E | 11 | 5 | 1 | 10 | - | 15 |
| | F | 3 | 12 | 4 | 3 | 15 | - |

AF (3)

Label row F 2, delete the remaining values in column F and select the edge of least weight from rows A and F

| | | A | B | C | D | E | F |
|---|---|---|---|---|---|---|---|
| 1 | A | - | 4 | 9 | 8 | 11 | 3 |
| | B | 4 | - | 13 | 2 | 5 | 12 |
| | C | 9 | 13 | - | 7 | 1 | 4 |
| | D | 8 | 2 | 7 | - | 10 | 3 |
| | E | 11 | 5 | 1 | 10 | - | 15 |
| 2 | F | 3 | 12 | 4 | 3 | 15 | - |

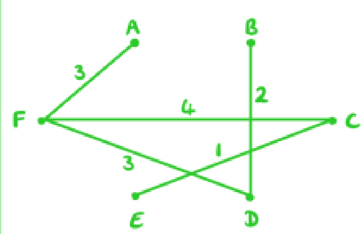
AF (3)
DF (3)

Continue in this way until all vertices are selected, remembering to record the order in which the edges are added

| | | A | B | C | D | E | F |
|---|---|---|---|---|---|---|---|
| 1 | A | - | 4 | 9 | 8 | 11 | 3 |
| 4 | B | 4 | - | 13 | 2 | 5 | 12 |
| 5 | C | 9 | 13 | - | 7 | 1 | 4 |
| 3 | D | 8 | 2 | 7 | - | 10 | 3 |
| 6 | E | 11 | 5 | 1 | 10 | - | 15 |
| 2 | F | 3 | 12 | 4 | 3 | 15 | - |

AF (3)
DF (3)
BD (2)
CF (4)
CE (1)






Edges added in this order:
AF, DF, BD, CF, CE

b)
State the lowest cost of connecting all of the buildings to the electricity supply.

Add up the weights of the edges in the minimum spanning tree

$3 + 3 + 2 + 4 + 1 = \boxed{13}$

# 3.10.4 Chinese Postman Problem

## Eulerian Trails & Circuits

### What are Eulerian trails and circuits?

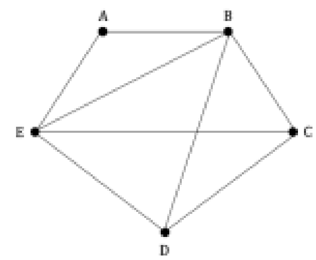
- A **trail** is a walk in which **no edge is repeated**
- An **Eulerian trail** is a trail that visits each **edge** in a graph **exactly once**
- A **circuit** is a trail that begins and ends at the **same vertex**
- An **Eulerian circuit** is a trail that visits each **edge** in a graph **exactly once** and begins and ends at the **same vertex**
- A graph which contains an **Eulerian circuit** is called an **Eulerian graph**
  - In an **Eulerian graph** the degree of each vertex is **even**
- A **semi-Eulerian graph** contains an **Eulerian trail** but **not** an **Eulerian circuit**
  - In a **semi-Eulerian** graph exactly **one pair** of vertices have an **odd** degree
  - These are the **start** and **finish** points of any **Eulerian trail**
- An **adjacency matrix** can be used to determine if a graph is Eulerian or semi-Eulerian as the **degree** of each vertex can be found by inspecting the **sum of the entries** in the rows (out-degree) or columns (in-degree)

Exam Tip

- If you can draw a graph without taking your pen off the paper and without going over any edge more than once then you have an Eulerian or semi-Eulerian graph!

## Worked Example

Let G be the graph shown below.




a)

Show that G is a semi-Eulerian graph.

Look at the degree of each vertex

A : 2
B : 4
C : 3
D : 3
E : 4

G is a semi-Eulerian graph because it has exactly one pair of odd vertices C and D

b)

Write down an Eulerian trail for G.

An Eulerian trail must start and finish at C/D

There are several possible Eulerian trails one solution is:

D E A B E C D B C

# Chinese Postman Problem

The **Chinese postman problem** requires you to find the **route of least weight** that **starts** and **finishes** at the **same vertex** and traverses **every edge** in the graph. Some edges may need to be traversed twice and the challenge is to **minimise** the total weight of these **repeated edges**.

## How do I solve the Chinese postman problem?

- If **all** of the vertices in a graph are **even** then the shortest route will be the sum of the weights of the edges in an **Eulerian circuit**
- If there is **one pair** of **odd vertices** in the graph then the **shortest** route between them will need to be found and repeated before finding an Eulerian circuit
  - There will always be an **even number of odd vertices** as the **total sum of the degrees** of the vertices is **double** the **number of edges**
- If there are **more than two odd vertices**, then **each possible pairing** of the odd vertices must be considered in order to find the **minimum weight** of the edges that need to be repeated
- The **maximum** number of odd vertices that could appear in an exam question is **4**

## What are the steps of the Chinese postman algorithm?

- STEP 1
  Inspect the degree of all of the vertices and identify any odd vertices
- STEP 2
  Find the possible pairings between the odd vertices
- STEP 3
  For each possible combination of vertices, find the shortest walk between the vertices and add the edges to be repeated to the graph
- STEP 4
  Write down an Eulerian circuit of the adjusted graph to find a possible route and find the sum of the edges traversed to find the total weight



## What variations may there be on the Chinese postman algorithm?

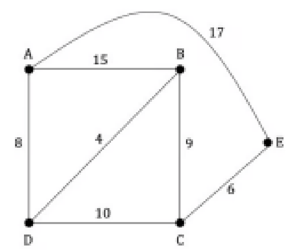
- The **weighting** of the edge between a pair of vertices may be different depending on if it is the **first time** it is being traversed or a **repeat**.
  - For example, if an inspector was checking a pipeline for defects then the first time going along a section of pipeline could take longer during inspection than if it is being repeated in order to get from one vertex to another
- If there are **4 odd vertices** you may be asked to **start** and **finish** at **different vertices**. Find the length of the routes for **all** possible pairings of the odd vertices and choose the **shortest route** between any 2 of them to be **repeated**. The other two odd vertices will be your start and finish points.

### Exam Tip

- Look carefully for the shortest route between two vertices exam questions often have graphs where a combination of edges will turn out to be a shorter distance than a more direct route

## Worked Example

The graph G shown below displays the distances, in kilometres, of the main roads between towns A, B, C, D and E. Each road is to be inspected for potholes.




a)

Explain why G does not contain an Eulerian circuit.

Inspect the degree of each vertex

A : 3 (odd)
B : 3 (odd)
C : 3 (odd)
D : 3 (odd)
E : 2 (even)

The graph G does not contain an Eulerian circuit as some of the vertices are odd



b)

Find the shortest route that starts and finishes at town A and allows for each road to be inspected.

There are 4 odd vertices so all of the possible pairings must be considered to find the shortest distance that needs to be repeated

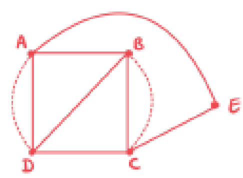
Possible pairings :

$AB + CD = \overset{ADB}{12} + \overset{CD}{10} = 22$

$AC + BD = \overset{ADC}{18} + \overset{BD}{4} = 22$

$AD + BC = \overset{AD}{8} + \overset{BC}{9} = 17$ * shortest

Add the edges to be repeated onto the graph and find an Eulerian circuit starting from A




One possible Eulerian circuit is :

A D A B D C B C E A

c)

State the total length of the shortest route.

Add the length of each edge in the graph then add the weight of the repeated edges

$\underbrace{15 + 17 + 8 + 4 + 9 + 10 + 6}_{\text{Edges in the original graph}} + \overset{\text{Repeated edges}}{17} = \boxed{86 \text{ km}}$

# 3.10.5 Travelling Salesman Problem

## Hamiltonian Paths & Cycles

### What are Hamiltonian paths and cycles?

- A **path** is a walk in which **no vertices are repeated**
  - A **Hamiltonian path** is a path in which each **vertex** in a graph is visited **exactly once**
- A **cycle** is a walk that starts and ends at the **same vertex** and **repeats no other vertices**
  - A **Hamiltonian cycle** is a cycle which visits each **vertex** in a graph **exactly once**
- If a graph contains a **Hamiltonian cycle** then it is known as a **Hamiltonian graph**
- A graph is **semi-Hamiltonian** if it contains a **Hamiltonian path** but not a **Hamiltonian cycle**
- The only way to show that a graph is **Hamiltonian** or **semi-Hamiltonian** is to find a **Hamiltonian cycle** or **Hamiltonian trail**

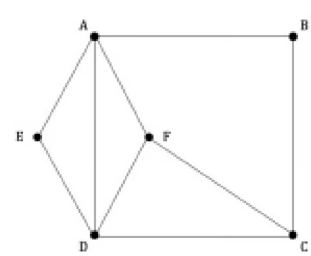
#### Exam Tip

- If you are given an adjacency matrix and are asked to find a Hamiltonian cycle, make sure that you sketch out the graph first



#### Worked Example

Let G be the graph shown below.




Show that G is a Hamiltonian graph.

To show that the graph is Hamiltonian, identify a Hamiltonian cycle

One possible Hamiltonian cycle is:

A B C F D E A

# Travelling Salesman Problem

The **travelling salesman problem** requires you to find the **route of least weight** that **starts** and **finishes** at the **same vertex** and visits every other **vertex** in the graph **exactly once**.

## How do I solve the travelling salesman problem?

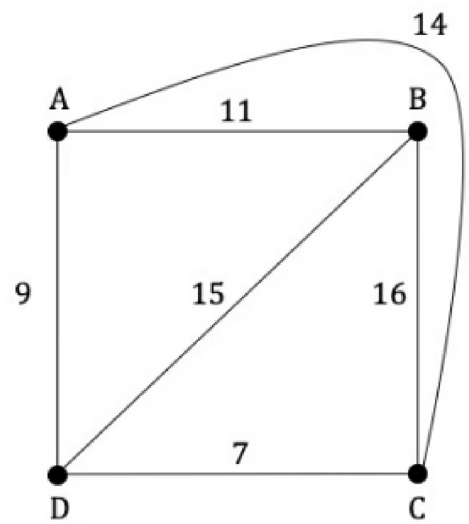
- In the **classical travelling salesman problem** the following conditions are observed:
  - the graph is **complete**
  - the **direct route** between two vertices is the **shortest route** (it satisfies the triangle inequality)
- List **all** of the possible **Hamiltonian cycles** and find the cycle of **least weight**
  - A complete graph with 3 vertices will have 2 possible Hamiltonian cycles, 4 vertices will have 6 possible cycles and a graph with 5 vertices will have 24 possible cycles
- There is **no known algorithm** that guarantees finding the shortest Hamiltonian cycle in a graph so this method is only suitable for **small graphs**

### Exam Tip

- To remember the difference between the travelling salesman problem and the Chinese postman problem, remember that the salesman is interested in selling at each destination (vertex) whereas the postman wants to walk along every road (edge) in order to deliver the letters

## Worked Example

The graph below shows four towns and the distances between them in km.




A salesman lives in city A and wishes to travel to each of the other three cities before returning home.

Find the shortest route that the salesman could take and state the total length of the route.

List all six possible Hamiltonian cycles and their weight

A B C D A = 11 + 16 + 7 + 9 = 43

A D C B A = 43

The route is the same but in reverse of the one above and therefore has the same weight

A B D C A = 11 + 15 + 7 + 14 = 47

A C D B A = 47

A D B C A = 9 + 15 + 16 + 14 = 54

A C B D A = 54

Route of least weight = 43 km

A B C D A

OR

A D C B A

# 3.10.6 Bounds for Travelling Salesman Problem

This revision note discusses more complex situations for the **travelling salesman problem** and you may wish to refer to the revision note **3.10.5 Travelling Salesman Problem**.

## Table of Least Distances

In some real-life contexts a graph may not be **complete** nor satisfy the **triangle inequality**, for example, when looking at a rail network, not every stop will be connected to every other stop and it may be quicker to travel from stop A to stop B via stop C rather than to travel from A to B directly. Thus, the problem is considered to be a **practical travelling salesman problem**.

Finding the **table of least distances** (or weights) can convert a **practical travelling salesman problem** into a **classical travelling salesman problem** that can then be analysed.

### What is a table of least distances?

- A **table of least distances** shows the **shortest** distance between any **two vertices** in a graph
    - In some cases, the **direct route** between two vertices may **not** be the **shortest**
- By finding the **table of least distances**, a graph can be converted into a **complete** graph that satisfies the **triangle inequality**
- STEP 1
  Fill in the information for vertices that are adjacent in the graph (at this stage check if the direct connections are actually the shortest route)
- STEP 2
  Complete the rest of the table by finding the shortest route that can be travelled between each pair of vertices that are not adjacent

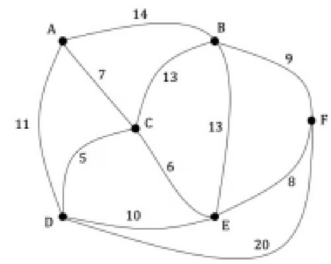


Exam Tip

- Remember that the table of least values has a line of symmetry along the leading diagonal for an undirected graph, so complete one half carefully first, then map over to the second half

## Worked Example

The graph G below contains six vertices representing villages and the roads that connect them. The weighting of the edges represents the time, in minutes, that it takes to walk along a particular road between two villages.




a)

Explain why G is not complete graph.

G is not a complete graph as each vertex is not connected to every other vertex by a single edge, e.g. there is no single edge connecting B and D

b)

Complete the table of least weights below.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| A | | | | | | |
| B | | | | | | |
| C | | | | | | |
| D | | | | | | |
| E | | | | | | |
| F | | | | | | |

Fill in the table with the direct connections from the graph but check that they are the shortest routes as you go along

Shortest route between D and F : D → F = 20
D → E → F = 18

Remember that in an undirected graph there will be a line of symmetry along the leading diagonal so start by filling in one half

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| A | – | | | | | |
| B | 14 | – | | | | |
| C | 7 | 13 | – | | | |
| D | 11 | | 5 | – | | |
| E | | 13 | 6 | 10 | – | |
| F | | 9 | | 18 | 8 | – |

Find the shortest routes between unconnected vertices



A and E : A → C → E = 13

A and F : A → C → E → F = 21

B and D : B → C → D = 18

C and F : C → E → F = 14

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| A | – | 14 | 7 | 11 | 13 | 21 |
| B | 14 | – | 13 | 18 | 13 | 9 |
| C | 7 | 13 | – | 5 | 6 | 14 |
| D | 11 | 18 | 5 | – | 10 | 18 |
| E | 13 | 13 | 6 | 10 | – | 8 |
| F | 21 | 9 | 14 | 18 | 8 | – |

# Nearest Neighbour Algorithm

As the number of vertices in a graph increases, so does the number of possible Hamiltonian cycles and it can become impractical to solve. The **nearest neighbour algorithm** can be used to find the **upper bound** for the **minimum** weight **Hamiltonian cycle**.

## What is the nearest neighbour algorithm?

- For a complete graph with at least 3 vertices, performing the **nearest neighbour algorithm** will generate a **low** (but not necessarily least) **weight** Hamiltonian cycle
- This low weight cycle can be considered the **upper bound**
- The **best upper bound** is the upper bound with the **smallest value**
- The nearest neighbour algorithm can only be used on a graph that is **complete** and satisfies the **triangle inequality** so the **table of least distances** should be found first

## What are the steps of the nearest neighbour algorithm?

- STEP 1
  Choose a starting vertex
- STEP 2
  Follow the edge of least weight from the current vertex to an adjacent unvisited vertex (if there is more than one edge of least weight pick one at random)
- STEP 3
  Repeat STEP 2 until all vertices have been visited
- STEP 4
  Add the final edge to return to the starting vertex



### Exam Tip

- If asked to write down the route for the lower bound, don't forget that some of the entries in the table of lowest distances may not be direct routes between vertices!

## Worked Example

The table below contains six vertices representing villages and the roads that connect them. The weighting of the edges represents the time in minutes that it takes to walk along a particular road between two villages.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| A | - | 14 | 7 | 11 | 13 | 21 |
| B | 14 | - | 13 | 18 | 13 | 9 |
| C | 7 | 13 | - | 5 | 6 | 14 |
| D | 11 | 18 | 5 | - | 10 | 18 |
| E | 13 | 13 | 6 | 10 | - | 8 |
| F | 21 | 9 | 14 | 18 | 8 | - |

Starting at village A, use the nearest neighbour algorithm to find the upper bound of the time it would take to visit each village and return to village A.

Start at vertex A (column A) and select the edge of least weight (AC), write it down

Move to column C, cross out the value that would join C to A as vertex A has already been visited. Select the edge of least weight (CD) and write down

Move to column D, cross out the values for the vertices already visited (A and C) and select the edge of least weight from the remaining values and write it down

Continue until all vertices have been visited then choose the final edge to get back from the last vertex to the starting position

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| A | — | 14 | ~~7~~ | ~~11~~ | ~~13~~ | ~~21~~ |
| B | 14 | — | 13 | 18 | 13 | 9 |
| C | 7 | 13 | — | ~~5~~ | ~~6~~ | ~~14~~ |
| D | 11 | 18 | 5 | — | ~~10~~ | ~~18~~ |
| E | 13 | 13 | 6 | 10 | — | ~~8~~ |
| F | 21 | 9 | 14 | 18 | 8 | — |

$$\begin{tikzcd} A \arrow[r, "7"] & C \arrow[r, "5"] & D \arrow[r, "10"] & E \arrow[r, "8"] & F \arrow[r, "9"] & B \arrow[lllll, bend left, "14"] \end{tikzcd}$$

$\boxed{\text{53 mins is an upper bound}}$

# Deleted Vertex Algorithm

The **deleted vertex algorithm** can be used to find the **lower bound** for the **minimum** weight **Hamiltonian cycle**.

## What is the deleted vertex algorithm?

- The deleted vertex algorithm can only be used on a graph that is **complete** and satisfies the **triangle inequality** so the **table of least distances** should be found first
- Deleting **different** vertices may give different results, the **best lower bound** is the lower bound with the **highest value**
- If you have found a **cycle** the **same length** as the **lower bound** then you have found the **shortest route** for the travelling salesman problem
- If the **lower bound** and the **upper bound** are the **same weight** then you have found the **shortest route** for the travelling salesman problem

## What are the steps of the deleted vertex algorithm?

- STEP 1
  Choose a vertex and delete it along with all edges that are connected to it
- STEP 2
  Find the minimum spanning tree for the remaining graph (see revision note **3.10.3 Minimum Spanning Trees**)
- STEP 3
  Add the two shortest edges that were deleted from the original graph to the weight of the minimum spanning tree



### Exam Tip

- Be careful when using a **weighted adjacency table** not to get confused between using Prim's algorithm and the nearest neighbour algorithm.
  - Remember that **Prim's** is used to find a minimum spanning tree, so vertices can be connected to **several** other vertices and hence can have **more than one value** in a column circled
  - When using the table for the **nearest neighbour** algorithm, vertices **cannot be revisited** so **only one value** will be circled in each column

## Worked Example

The table below contains six vertices representing villages and the roads that connect them. The weighting of the edges represents the time in minutes that it takes to walk along a particular road between two villages.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| A | - | 14 | 7 | 11 | 13 | 21 |
| B | 14 | - | 13 | 18 | 13 | 9 |
| C | 7 | 13 | - | 5 | 6 | 14 |
| D | 11 | 18 | 5 | - | 10 | 8 |
| E | 13 | 13 | 6 | 10 | - | 8 |
| F | 21 | 9 | 14 | 18 | 8 | - |

a)

By deleting vertex A and using Prim's algorithm, find a lower bound for the time taken to start at village A, visit each of the other villages and return to village A

| | | A | B | C | D | E | F |
|---|---|---|---|---|---|---|---|
| | ~~A~~ | ~~-~~ | ~~14~~ | ~~7~~ | ~~11~~ | ~~13~~ | ~~21~~ |
| 1 | B | 14 | — | ~~13~~ | 18 | ~~13~~ | 9 |
| 4 | C | 7 | ~~13~~ | — | 5 | ~~6~~ | ~~14~~ |
| 5 | D | 11 | ~~18~~ | ~~5~~ | — | ~~10~~ | ~~18~~ |
| 3 | E | 13 | ~~13~~ | 6 | 10 | — | ~~8~~ |
| 2 | F | 21 | ~~9~~ | ~~14~~ | 18 | 8 | — |

Add edges in the order: BF (9)
FE (8)
CE (6)
CD (5)

Total weight of minimum spanning tree = 28

Add weights of two edges of least weight connected to A: AC (7)
AD (11)

Lower bound = 28 + 7 + 11 = $\boxed{\text{46 minutes}}$

b)

Show that by deleting vertex B instead, a higher lower bound can be found.

| | | A | B | C | D | E | F |
|---|---|---|---|---|---|---|---|
| 1 | A | – | 14 | 7 | ~~11~~ | ~~13~~ | 21 |
| | B | 14 | – | 13 | 18 | 13 | 9 |
| 2 | C | ~~7~~ | 13 | – | 5 | 6 | 14 |
| 3 | D | ~~11~~ | 18 | ~~5~~ | – | ~~10~~ | 18 |
| 4 | E | ~~13~~ | 13 | ~~6~~ | ~~10~~ | – | 8 |
| 5 | F | ~~21~~ | 9 | ~~14~~ | ~~18~~ | ~~8~~ | – |

Total weight of minimum spanning tree = 7 + 8 + 6 + 5
= 26

Weight of two edges of least weight connected to B : BF (9)
BC/D (13)

Lower bound = 48 mins

This is a higher lower bound than found when deleting vertex A