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5. Calculus

5.1 Differentiation

MATHS

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5. Calculus

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5.1 Differentiation

5.1.1 Introduction to Differentiation

Introduction to Derivatives

- Before introducing a **derivative**, an understanding of a **limit** is helpful

What is a limit?

- The **limit** of a **function** is the value a function (of x) approaches as x approaches a particular value from either side
  - Limits are of interest when the function is undefined at a particular value
  - For example, the function $f(x) = \frac{x^4 - 1}{x - 1}$ will approach a limit as x approaches 1 from both below and above but is undefined at $x = 1$ as this would involve dividing by zero

What might I be asked about limits?

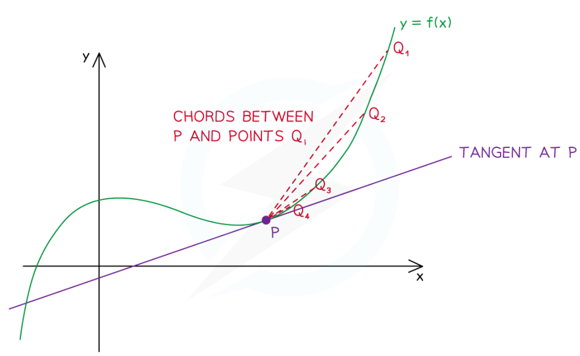
- You may be asked to predict or estimate limits from a table of function values or from the graph of $y = f(x)$
- You may be asked to use your GDC to plot the graph and use values from it to estimate a limit

What is a derivative?

- **Calculus** is about **rates** of **change**
  - the way a car's position on a road changes is its speed (velocity)
  - the way the car's speed changes is its acceleration
- The **gradient** (rate of change) of a (non-linear) **function** varies with x
- The **derivative** of a function is a function that relates the **gradient** to the value of x
- The derivative is also called the **gradient function**

How are limits and derivatives linked?

- Consider the point P on the graph of $y = f(x)$ as shown below
  - $[PQ_i]$ is a series of chords



- The **gradient** of the **function** $f(x)$ at the point P is **equal** to the **gradient** of the **tangent** at point P
- The **gradient** of the **tangent** at point P is the **limit** of the **gradient** of the chords $[PQ_i]$ as point Q 'slides' down the curve and gets ever closer to point P
- The **gradient** of the function changes as x changes

- The **derivative** is the function that calculates the gradient from the value x

What is the notation for derivatives?

- For the function $y = f(x)$, the **derivative**, with respect to x, would be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$$

- Different variables may be used
  - e.g. If $V = f(s)$ then $\frac{\mathrm{d}V}{\mathrm{d}s} = f'(s)$

Worked Example

The graph of $y = f(x)$ where $f(x) = x^3 - 2$ passes through the points $P(2, 6)$, $A(2.3, 10.167)$, $B(2.1, 7.261)$ and $C(2.05, 6.615125)$.

a)
Find the gradient of the chords $[PA]$, $[PB]$ and $[PC]$.

Gradient of a line (chord) is " $\frac{y_2 - y_1}{x_2 - x_1}$ "

$[PA]$: $\frac{10.167 - 6}{2.3 - 2} = 13.89$

$[PB]$: $\frac{7.261 - 6}{2.1 - 2} = 12.61$

$[PC]$: $\frac{6.615125 - 6}{2.05 - 2} = 12.3$

Gradient of chords are: $[PA]$ 13.89
$[PB]$ 12.61
$[PC]$ 12.3025

b)
Estimate the gradient of the tangent to the curve at the point P.

There will be a limit the gradient of the chord reaches as the difference in the x-coordinates approaches zero.

Estimate of gradient of tangent at $x = 2$ is 12

Differentiating Powers of x

What is differentiation?

- **Differentiation** is the process of finding an expression of the **derivative** (**gradient function**) from the expression of a function

How do I differentiate powers of x?

- **Powers** of x are **differentiated** according to the following formula:
  - If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ where $n \in \mathbb{Q}$
  - This is given in the **formula booklet**
- If the power of x is **multiplied** by a **constant** then the derivative is also multiplied by that constant
  - If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$ where $n \in \mathbb{Q}$ and a is a constant
- The **alternative notation** (to $f'(x)$) is to use $\frac{dy}{dx}$
  - If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$
    - e.g. If $y = -4x^{\frac{1}{2}}$ then $\frac{dy}{dx} = -4 \times \frac{1}{2} \times x^{\frac{1}{2}-1} = -2x^{-\frac{1}{2}}$
- Don't forget these **two** special cases:
  - If $f(x) = ax$ then $f'(x) = a$
    - e.g. If $y = 6x$ then $\frac{dy}{dx} = 6$
  - If $f(x) = a$ then $f'(x) = 0$
    - e.g. If $y = 5$ then $\frac{dy}{dx} = 0$
  - These allow you to differentiate **linear terms** in x and **constants**
- Functions involving **roots** will need to be rewritten as **fractional powers** of x first
  - e.g. If $f(x) = 2\sqrt{x}$ then rewrite as $f(x) = 2x^{\frac{1}{2}}$ and differentiate
- Functions involving **fractions** with **denominators** in terms of x will need to be rewritten as **negative powers** of x first
  - e.g. If $f(x) = \frac{4}{x}$ then rewrite as $f(x) = 4x^{-1}$ and differentiate

How do I differentiate sums and differences of powers of x?

- The formulae for differentiating powers of x apply to **all rational** powers so it is possible to differentiate any expression that is a **sum** or **difference** of **powers** of x
  - e.g. If $f(x) = 5x^4 - 3x^{\frac{2}{3}} + 4$ then

$$f'(x) = 5 \times 4x^{4-1} - 3 \times \frac{2}{3}x^{\frac{2}{3}-1} + 0$$

$$f'(x) = 20x^3 - 2x^{-\frac{1}{3}}$$

- **Products** and **quotients cannot** be differentiated in this way so would need **expanding/simplifying** first
  - e.g. If $f(x) = (2x - 3)(x^2 - 4)$ then expand to $f(x) = 2x^3 - 3x^2 - 8x + 12$ which **is** a **sum/difference** of powers of x and can be differentiated

Exam Tip

- A common mistake is not simplifying expressions before differentiating
  - The derivative of $(x^2 + 3)(x^3 - 2x + 1)$ can **not** be found by multiplying the derivatives of $(x^2 + 3)$ and $(x^3 - 2x + 1)$

Worked Example

The function $f(x)$ is given by

$$f(x) = 2x^3 + \frac{4}{\sqrt{x}}, \text{ where } x > 0$$

Find the derivative of $f(x)$

Rewrite $f(x)$ so every term is a power of x

$f(x) = 2x^3 + 4x^{-\frac{1}{2}}$

Differentiate by applying the formula

$f'(x) = 6x^2 - 2x^{-\frac{3}{2}}$

anx^{n-1}

take care with negatives

$-\frac{1}{2} - 1 = -\frac{3}{2}$

$\therefore f'(x) = 6x^2 - 2x^{-\frac{3}{2}}$

5.1.2 Applications of Differentiation

Finding Gradients

How do I find the gradient of a curve at a point?

- The **gradient of a curve** at a point is the **gradient of the tangent** to the curve at that point
- **Find the gradient** of a curve at a point by **substituting the value of x** at that point into the curve's **derivative function**
- For example, if $f(x) = x^2 + 3x - 4$
  - then $f'(x) = 2x + 3$
  - and the gradient of $y = f(x)$ when $x = 1$ is $f'(1) = 2(1) + 3 = 5$
  - and the gradient of $y = f(x)$ when $x = -2$ is $f'(-2) = 2(-2) + 3 = -1$
- Although your GDC won't find a derivative function for you, it is possible to use your **GDC** to **evaluate the derivative** of a function at a point, using $\frac{d}{dx}(\square)_{x=\square}$

Worked Example

A function is defined by $f(x) = x^3 + 6x^2 + 5x - 12$.

(a) Find $f'(x)$.

Find $f'(x)$ by differentiating

$f'(x) = 3x^2 + 2 \times 6x^1 + 5x^0$

$f'(x) = 3x^2 + 12x + 5$

(b) Hence show that the gradient of $y = f(x)$ when $x = 1$ is 20.

Substitute $x = 1$ into $f'(x)$

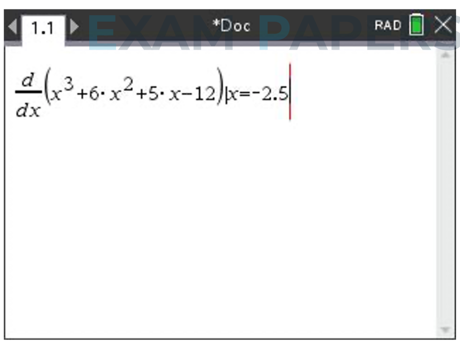
$f'(1) = 3(1)^2 + 12(1) + 5$

$= 3 + 12 + 5$

$f'(1) = 20$

(c) Find the gradient of $y = f(x)$ when $x = -2.5$.

Use the GDC to evaluate the derivative of $f(x)$ at $x = -2.5$

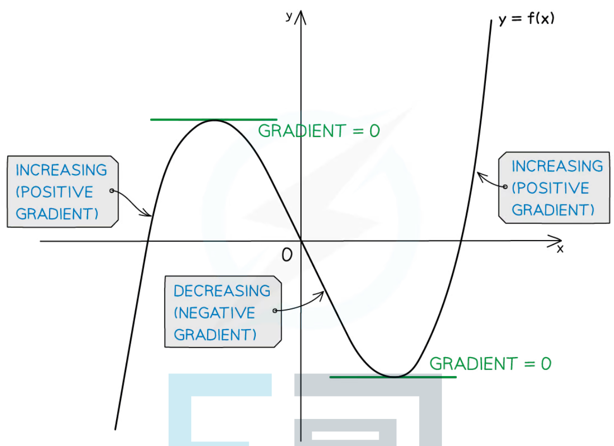


$f'(-2.5) = -6.25$

Increasing & Decreasing Functions

What are increasing and decreasing functions?

- A function, $f(x)$, is **increasing** if $\boldsymbol{f'(x) > 0}$
    - This means the **value** of the **function** ('output') **increases** as x **increases**
- A function, $f(x)$, is **decreasing** if $\boldsymbol{f'(x) < 0}$
    - This means the **value** of the **function** ('output') **decreases** as x **increases**
- A function, $f(x)$, is **stationary** if $\boldsymbol{f'(x) = 0}$



How do I find where functions are increasing, decreasing or stationary?

- To identify the **intervals** on which a function is increasing or decreasing

STEP 1
Find the derivative **f'(*x*)**

STEP 2
Solve the inequalities
$\boldsymbol{f'(x) > 0}$ (for increasing intervals) and/or
$\boldsymbol{f'(x) < 0}$ (for decreasing intervals)

- Most functions are a combination of **increasing, decreasing** and **stationary**
    - a range of values of x (**interval**) is given where a function satisfies each condition
    - e.g. The function $f(x) = x^2$ has **derivative** $f'(x) = 2x$ so
        - $f(x)$ is **decreasing** for $x < 0$
        - $f(x)$ is **stationary** at $x = 0$
        - $f(x)$ is **increasing** for $x > 0$

Worked Example

$f(x) = x^2 - x - 2$

a)
Determine whether $f(x)$ is increasing or decreasing at the points where $x = 0$ and $x = 3$.

Differentiate

$f'(x) = 2x - 1$

At $x = 0$, $f'(0) = 2 \times 0 - 1 = -1 < 0$ $\therefore$ decreasing

At $x = 3$, $f'(3) = 2 \times 3 - 1 = 6 > 0$ $\therefore$ increasing

$\therefore$ At $x = 0$, $f(x)$ is decreasing
At $x = 3$, $f(x)$ is increasing

b)
Find the values of x for which $f(x)$ is an increasing function.

$f(x)$ is increasing when $f'(x) > 0$

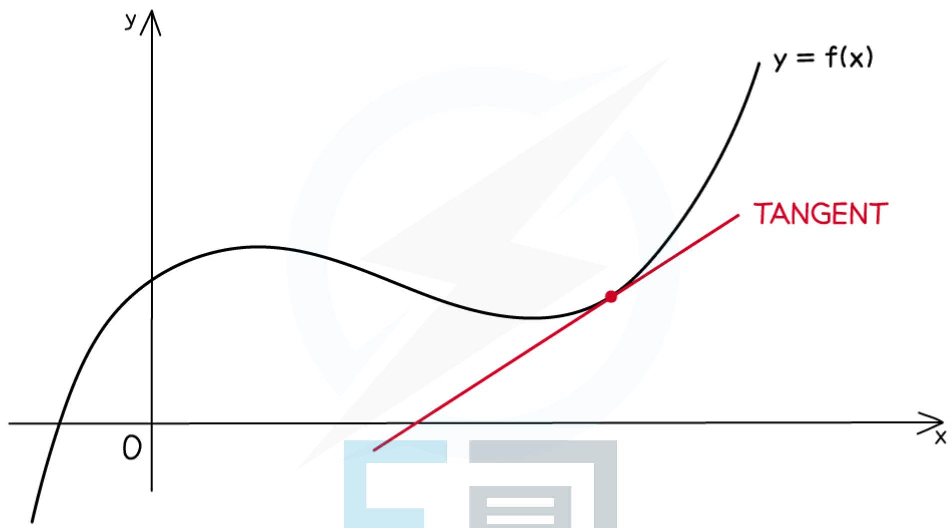
$f'(x) > 0$
$2x - 1 > 0$
$x > \frac{1}{2}$

$\therefore$ $f(x)$ is increasing for $x > \frac{1}{2}$

Tangents & Normals

What is a tangent?

- At any point on the graph of a (non-linear) **function**, the **tangent** is the straight line that **touches** the graph at a point **without crossing** through it
- Its **gradient** is given by the **derivative function**

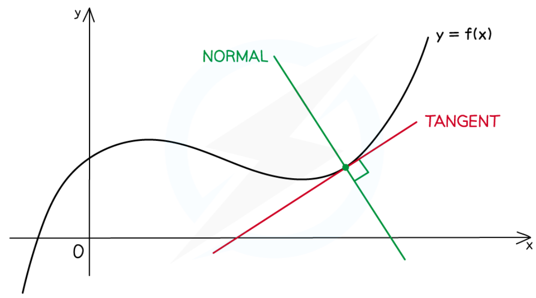


How do I find the equation of a tangent?

- To find the **equation of a straight line**, a **point** and the **gradient** are needed
- The **gradient**, m, of the **tangent** to the function $y = f(x)$ at (x_1, y_1) is $\boldsymbol{f'(x_1)}$
- Therefore find the **equation** of the **tangent** to the function $y = f(x)$ at the point (x_1, y_1) by substituting the gradient, $f'(x_1)$, and point (x_1, y_1) into $y - y_1 = m(x - x_1)$, giving:
  - $y - y_1 = \boldsymbol{f'(x_1)}(x - x_1)$
- (You could also substitute into $y = mx + c$ but it is usually quicker to substitute into $y - y_1 = m(x - x_1)$)

What is a normal?

- At any point on the graph of a (non-linear) function, the **normal** is the straight line that passes through that point and is **perpendicular** to the **tangent**



How do I find the equation of a normal?

- The **gradient** of the **normal** to the function $y = f(x)$ at (x_1, y_1) is $\frac{-1}{f'(x_1)}$
- Therefore find the **equation** of the **normal** to the function $y = f(x)$ at the point (x_1, y_1) by using $y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$

Exam Tip

- You are not given the formula for the equation of a tangent or the equation of a normal
- But both can be derived from the equations of a straight line which are given in the formula booklet

Worked Example

The function $f(x)$ is defined by

$$f(x) = 2x^4 + \frac{3}{x^2} \qquad x \neq 0$$

a)

Find an equation for the tangent to the curve $y = f(x)$ at the point where $x = 1$, giving your answer in the form $y = mx + c$.

First find $f'(x)$ by differentiating

$f(x) = 2x^4 + 3x^{-2}$ Rewrite as powers of x

$f'(x) = 8x^3 - 6x^{-3}$

For a tangent, "$y - y_1 = f'(a)(x - x_1)$"

At $x = 1$, $y = 2(1)^4 + \frac{3}{(1)^2} = 5$

$f'(1) = 8(1)^3 - \frac{6}{(1)^3} = 2$

$\therefore$ $y - 5 = 2(x - 1)$

Tangent at $x = 1$, is $y = 2x + 3$

b)

Find an equation for the normal at the point where $x = 1$, giving your answer in the form $ax + by + d = 0$, where a, b and d are integers.

For a normal, "$y - y_1 = \frac{-1}{f'(a)}(x - x_1)$"

Using results from part (a):

$y - 5 = \frac{-1}{2}(x - 1)$

$y = -\frac{1}{2}x + \frac{11}{2}$

$2y = -x + 11$

$\therefore$ Equation of normal is $x + 2y - 11 = 0$

5.2 Further Differentiation

5.2.1 Differentiating Special Functions

Differentiating Trig Functions

How do I differentiate sin, cos and tan?

- The derivative of $y = \sin x$ is $\frac{dy}{dx} = \cos x$
- The derivative of $y = \cos x$ is $\frac{dy}{dx} = -\sin x$
- The derivative of $y = \tan x$ is $\frac{dy}{dx} = \sec^2 x$
    - This result can be derived using **quotient rule**
- For the **linear** function $ax + b$, where a and b are constants,
    - the derivative of $y = \sin(ax + b)$ is $\frac{dy}{dx} = a\cos(ax + b)$
    - the derivative of $y = \cos(ax + b)$ is $\frac{dy}{dx} = -a\sin(ax + b)$
    - **the derivative of** $y = \tan(ax + b)$ **is** $\frac{dy}{dx} = a\sec^2(ax + b)$
- For the **general** function $f(x)$,
    - the derivative of $y = \sin(f(x))$ is $\frac{dy}{dx} = f'(x)\cos(f(x))$
    - the derivative of $y = \cos(f(x))$ is $\frac{dy}{dx} = -f'(x)\sin(f(x))$
    - the derivative of $y = \tan(f(x))$ is $\frac{dy}{dx} = f'(x)\sec^2(f(x))$
- These last three results can be derived using the **chain rule**
- For calculus with trigonometric functions angles must be measured in **radians**
    - Ensure you know how to change the angle mode on your GDC

Exam Tip

- As soon as you see a question involving differentiation and trigonometry put your GDC into radians mode

Worked Example

a)

Find $f'(x)$ for the functions

i. $f(x) = \sin x$

ii. $f(x) = \cos(5x + 1)$

i. $f'(x) = \cos x$

ii. $f'(x) = -5\sin(5x+1)$ (Linear function $ax+b$)

b) A curve has equation $y = \tan\left(6x^2 - \frac{\pi}{4}\right)$.

Find the gradient of the tangent to the curve at the point where $x = \frac{\sqrt{\pi}}{2}$.

Give your answer as an exact value.

This is of the form $y = \tan(f(x))$

so $\frac{dy}{dx} = f'(x)\sec^2(f(x))$

$f(x) = 6x^2 - \frac{\pi}{4}$

$\therefore f'(x) = 12x$

$\therefore \frac{dy}{dx} = 12x\sec^2\left(6x^2 - \frac{\pi}{4}\right)$

At $x = \frac{\sqrt{\pi}}{2}$, $\frac{dy}{dx} = 12\left(\frac{\sqrt{\pi}}{2}\right)\sec^2\left[6\left(\frac{\sqrt{\pi}}{2}\right)^2 - \frac{\pi}{4}\right]$

$= \frac{6\sqrt{\pi}}{\cos^2\left(\frac{5\pi}{4}\right)}$ $\left(\sec^2 x = \frac{1}{\cos^2 x}\right)$

$\therefore \frac{dy}{dx} = 12\sqrt{\pi}$ at $x = \frac{\sqrt{\pi}}{2}$

Differentiating e^x & lnx

How do I differentiate exponentials and logarithms?

- The derivative of $y = e^x$ is $\frac{dy}{dx} = e^x$ where $x \in \mathbb{R}$
- The derivative of $y = \ln x$ is $\frac{dy}{dx} = \frac{1}{x}$ where $x > 0$
- For the **linear** function $ax + b$, where a and b are constants,
  - the derivative of $y = e^{(ax+b)}$ is $\frac{dy}{dx} = ae^{(ax+b)}$
  - the derivative of $y = \ln(ax + b)$ is $\frac{dy}{dx} = \frac{a}{(ax+b)}$
    - in the special case $b = 0$, $\frac{dy}{dx} = \frac{1}{x}$ (a's cancel)
- For the **general** function $f(x)$,
  - the derivative of $y = e^{f(x)}$ is $\frac{dy}{dx} = f'(x)e^{f(x)}$
  - the derivative of $y = \ln(f(x))$ is $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
- The last two sets of results can be derived using the **chain rule**

Exam Tip

- Remember to avoid the common mistakes:
  - the derivative of $\ln kx$ with respect to x is $\frac{1}{x}$, NOT $\frac{k}{x}$
  - the derivative of e^{kx} with respect to x is ke^{kx}, NOT kxe^{kx-1}

Worked Example

A curve has the equation $y = e^{-3x+1} + 2\ln 5x$.

Find the gradient of the curve at the point where $x = 2$ gving your answer in the form $y = a + be^c$, where a, b and c are integers to be found.

$$y = e^{-3x+1} + 2(\ln 5x)$$

$$\therefore \frac{dy}{dx} = -3e^{-3x+1} + 2\left(\frac{1}{x}\right)$$

"$y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$"

"$y = \ln(ax+b)$, special case $b = 0$, $\frac{dy}{dx} = \frac{1}{x}$"

At $x = 2$, $\frac{dy}{dx} = -3e^{-3(2)+1} + \frac{2}{2} = -3e^{-5} + 1$

$\therefore$ Gradient at $x = 2$ is $1 - 3e^{-5}$

i.e. $a = 1$, $b = -3$, $c = -5$

Your GDC may be able to find gradients but probably not in the exact form required. It is still helpful to check approximate answers though.

5.2.2 Techniques of Differentiation

Chain Rule

What is the chain rule?

- The **chain rule** states if y is a function of u and u is a function of x then

$$y = f(u(x))$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

  - This is given in the **formula booklet**

- In **function notation** this could be written

$$y = f(g(x))$$

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

How do I know when to use the chain rule?

- The chain rule is used when we are trying to differentiate **composite functions**
  - "function of a function"
  - these can be identified as the variable (usually x) does not 'appear alone'
    - $\sin x$ – **not** a composite function, x 'appears alone'
    - $\sin(3x+2)$ **is a composite function;** x is tripled and has 2 added to it before the sine function is applied

How do I use the chain rule?

STEP 1
Identify the two functions
Rewrite y as a function of u; $y = f(u)$
Write u as a function of x; $u = g(x)$

STEP 2
Differentiate y with respect to u to get $\frac{dy}{du}$
Differentiate u with respect to x to get $\frac{du}{dx}$

STEP 3
Obtain $\frac{dy}{dx}$ by applying the formula $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ and substitute u back in for $g(x)$

- In trickier problems **chain rule** may have to be applied **more than once**

Are there any standard results for using chain rule?

- There are **five** general results that can be useful
  - If $y = (f(x))^n$ then $\frac{dy}{dx} = nf'(x)f(x)^{n-1}$

- If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$
- If $y = \ln(f(x))$ then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
- If $y = \sin(f(x))$ then $\frac{dy}{dx} = f'(x)\cos(f(x))$
- If $y = \cos(f(x))$ then $\frac{dy}{dx} = -f'(x)\sin(f(x))$

Exam Tip

- You should aim to be able to spot and carry out the chain rule mentally (rather than use substitution)
  - every time you use it, say it to yourself in your head *"differentiate the first function ignoring the second, then multiply by the derivative of the second function"*

Worked Example

a)

Find the derivative of $y = (x^2 - 5x + 7)^7$.

STEP 1 Identify the two functions and rewrite

$y = u^7$ i.e. $f(u) = u^7$

$u = x^2 - 5x + 7$ i.e. $g(x) = x^2 - 5x + 7$

STEP 2 Find $\frac{dy}{du}$ and $\frac{du}{dx}$.

$\frac{dy}{du} = 7u^6$ $\frac{du}{dx} = 2x - 5$

STEP 3 Apply chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Chain rule is in the formula booklet

$\frac{dy}{dx} = 7u^6(2x-5)$

and substitute u back for g(x)

$\frac{dy}{dx} = 7(2x-5)(x^2-5x+7)^6$

b)

Find the derivative of $y = \sin(e^{2x})$.

$y = \sin(e^{2x})$ "... differentiate sin □, ignore e^{2x}..."

$\frac{dy}{dx} = \cos(e^{2x}) \times 2e^{2x}$ "... multiply by derivative of e^{2x}..."

"$y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$"
or by applying chain rule again

$\therefore \frac{dy}{dx} = 2e^{2x}\cos(e^{2x})$

Product Rule

What is the product rule?

- The **product rule** states if y is the product of two functions $u(x)$ and $v(x)$ then

$$y = uv$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written as

$$y = f(x)g(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g'(x) + g(x)f'(x)$$

- **'Dash notation'** may be used as a **shorter** way of writing the rule

$$y = uv$$

$$y' = uv' + vu'$$

- Final answers should match the notation used throughout the question

How do I know when to use the product rule?

- The **product rule** is used when we are trying to **differentiate** the **product** of **two functions**
  - these can easily be confused with composite functions (see **chain rule)**
    - $\sin(\cos x)$ is a composite function, "sin of cos of x"
    - $\sin x \cos x$ is a product, "sin x times cos x"

How do I use the product rule?

- Make it clear what u, v, u' and v' are
  - arranging them in a square can help
    - opposite diagonals match up

STEP 1
Identify the two functions, u and v
Differentiate both u and v with respect to x to find u' and v'

STEP 2
Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the product rule formula $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$
Simplify the answer if straightforward to do so or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding u' and v'

Exam Tip

- Use u, v, u' and v' for the elements of product rule
  - lay them out in a 'square' (imagine a 2×2 grid)
  - those that are paired together are then on opposite diagonals (u and v', v and u')
- For trickier functions chain rule may be reuqired inside product rule
  - i.e. chain rule may be needed to differentiate u and v

Worked Example

a) Find the derivative of $y = e^x \sin x$.

$y = e^x \sin x$

STEP 1 Identify functions and differentiate

$u = e^x \qquad v = \sin x$

$u' = e^x \qquad v' = \cos x$

arranging u, v, u', v' in a square makes product rule 'diagonal pairs'

STEP 2 Apply product rule: '$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$'

(As it is given in the formula booklet)

$y' = e^x \cos x + e^x \sin x$

$\therefore \frac{dy}{dx} = e^x(\cos x + \sin x)$

It is straightforward to take a factor of e^x out

b) Find the derivative of $y = 5x^2 \cos 3x^2$.

$y = 5x^2 \cos 3x^2$

STEP 1 $u = 5x^2 \qquad v = \cos 3x^2$

$u' = 10x \qquad v' = -\sin 3x^2 \times 6x$ ← chain rule

$v' = -6x \sin 3x^2$

STEP 2 $y' = -30x^3 \sin 3x^2 + 10x \cos 3x^2$

$\therefore \frac{dy}{dx} = 10x(\cos 3x^2 - 3x^2 \sin 3x^2)$

Quotient Rule

What is the quotient rule?

- The **quotient rule** states if y is the quotient $\frac{u(x)}{v(x)}$ then

$$y = \frac{u}{v}$$

$$\frac{\mathbf{d}\boldsymbol{y}}{\mathbf{d}\boldsymbol{x}} = \frac{\boldsymbol{v}\frac{\mathbf{d}\boldsymbol{u}}{\mathbf{d}\boldsymbol{x}} - \boldsymbol{u}\frac{\mathbf{d}\boldsymbol{v}}{\mathbf{d}\boldsymbol{x}}}{2}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written

$$y = \frac{f(x)}{g(x)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

- As with product rule, **'dash notation'** may be used

$$y = \frac{u}{v}$$

$$y' = \frac{vu' - uv'}{v^2}$$

- Final answers should match the notation used throughout the question

How do I know when to use the quotient rule?

- The **quotient rule** is used when trying to differentiate a fraction where **both** the **numerator** and **denominator** are **functions** of x
  - if the **numerator** is a **constant**, **negative powers** can be used
  - if the **denominator** is a **constant**, treat it as a **factor** of the expression

How do I use the quotient rule?

- Make it clear what u, v, u' and v' are
  - arranging them in a square can help
    - opposite diagonals match up (like they do for product rule)

STEP 1
Identify the two functions, u and v
Differentiate both u and v with respect to x to find u' and v'

STEP 2
Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the quotient rule formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Be careful using the formula – because of the **minus sign** in the **numerator**, the **order** of the functions is important
Simplify the answer if straightforward or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding u' and v',

Exam Tip

- Use u, v, u' and v' for the elements of quotient rule
    - lay them out in a 'square' (imagine a 2×2 grid)
    - those that are paired together are then on opposite diagonals (v and u', u and v')
- Look out for functions of the form $y = f(x)(g(x))^{-1}$
    - These can be differentiated using a combination of **chain rule** and **product rule**
      (it would be good practice to try!)
    - ... but it can also be seen as a quotient rule question in disguise
    - ... and vice versa!
        - A quotient could be seen as a product by rewriting the denominator as $(g(x))^{-1}$

Worked Example

Differentiate $f(x) = \dfrac{\cos 2x}{3x+2}$ with respect to x.

STEP 1 Identify u and v, differentiate

$u = \cos 2x \qquad v = 3x+2$

$u' = -2\sin 2x \qquad v' = 3$

chain rule

opposite diagonals match up

STEP 2 Apply quotient rule: '$\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$'

(As it is given in the formula booklet)

$$f'(x) = \frac{(3x+2)(-2\sin 2x) - (\cos 2x)(3)}{(3x+2)^2}$$

$$\therefore f'(x) = \frac{-2(3x+2)\sin 2x - 3\cos 2x}{(3x+2)^2}$$

(Nothing obvious/easy to simplify and question does not specify a particular form)

5.2.3 Higher Order Derivatives

Second Order Derivatives

What is the second order derivative of a function?

- If you **differentiate** the **derivative** of a **function** (i.e. differentiate the function a second time) you get the **second order derivative** of the function
- There are two forms of **notation** for the **second order derivative**
  - $y = f(x)$
  - $\frac{dy}{dx} = f'(x)$ (First order derivative)
  - $\frac{d^2y}{dx^2} = f''(x)$ (Second order derivative)
- Note the position of the superscript 2's
  - **d**ifferentiating twice (so $\mathbf{d^2}$) with respect to x twice (so $\boldsymbol{x^2}$)
- The **second order derivative** can be referred to simply as the **second derivative**
  - Similarly, the **first order derivative** can be just the **first derivative**
- A **first order derivative** is the **rate** of **change** of a function
  - a **second order derivative** is the **rate** of **change** of the **rate** of **change** of a function
    - i.e. the **rate** of **change** of the function's **gradient**
- **Second order derivatives** can be used to
  - test for local minimum and maximum points
  - help determine the nature of stationary points
  - help determine the concavity of a function
  - graph derivatives

How do I find a second order derivative of a function?

- By **differentiating twice!**
- This may involve
  - rewriting **fractions, roots,** etc as **negative** and/or **fractional powers**
  - differentiating **trigonometric** functions, **exponentials** and **logarithms**
  - using **chain rule**
  - using **product** or **quotient** rule

Exam Tip

- Negative and/or fractional powers can cause problems when finding second derivatives so work carefully through each term

Worked Example

Given that $\mathrm{f}(x) = 4 - \sqrt{x} + \dfrac{3}{\sqrt{x}}$

a)
Find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$.

a) $f(x) = 4 - x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$ ← REWRITE AS POWERS OF x

$f'(x) = -(\frac{1}{2})x^{\frac{1}{2}-1} + 3(-\frac{1}{2})x^{-\frac{1}{2}-1}$

$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$ ← DIFFERENTIATE ONCE TO FIND $f'(x)$

$f''(x) = -\frac{1}{2}(-\frac{1}{2})x^{-\frac{1}{2}-1} - \frac{3}{2}(-\frac{3}{2})x^{-\frac{3}{2}-1}$

$f''(x) = \frac{1}{4}x^{-\frac{3}{2}} + \frac{9}{4}x^{-\frac{5}{2}}$ ← DIFFERENTIATE A SECOND TIME TO FIND $f''(x)$

b)
Evaluate $\mathrm{f}''(3)$.
Give your answer in the form $a\sqrt{b}$, where b is an integer and a is a rational number.

b) $f''(x) = \dfrac{1}{4x\sqrt{x}} + \dfrac{9}{4x^2\sqrt{x}}$ ← $x^{\frac{3}{2}} = x\sqrt{x}$ $\quad x^{\frac{5}{2}} = x^2\sqrt{x}$

$f''(3) = \dfrac{1}{12\sqrt{3}} + \dfrac{9}{36\sqrt{3}}$

$= \dfrac{12}{36\sqrt{3}} = \dfrac{1}{3\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{9}$ ← RATIONALISE DENOMINATOR

$f''(3) = \frac{1}{9}\sqrt{3}$

Higher Order Derivatives

What is meant by higher order derivatives of a function?

- Many functions can be **differentiated** numerous times
  - The third, fourth, fifth, etc derivatives of a function are generally called **higher order derivatives**
- It may not be possible, or practical to (algebraically) differentiate complicated functions more than once or twice
- **Polynomials** will, eventually, have higher order derivatives of zero
  - Since powers of x reduce by 1 each time

What is the notation for higher order derivatives?

- The notation for higher order derivatives follows the logic from the first and second derivatives

$$f^{(n)}(x) \quad \text{or} \quad \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$$

except the 'dash' (prime) notation is replaced with numbers as this would become cumbersome after the first few

  - e.g. the fifth derivative would be

$$f^{(5)}(x) \quad \text{or} \quad \frac{\mathrm{d}^5 y}{\mathrm{d}x^5}$$

How do I find a higher order derivative of a function?

- By differentiating as many times as required!
- This may involve
  - rewriting fractions, roots, etc as negative and/or fractional powers
  - differentiating trigonometric functions, exponentials and logarithms
  - using chain rule
  - using product or quotient rule

Exam Tip

- If you are required to evaluate a higher order derivative at a specific point your GDC can help
  - Typically a GDC will only work out the first and second derivative directly from the original function
  - But, if you wanted the fourth derivative, say, you only need differentiate twice algebraically, then call this the 'original' function on your GDC

Worked Example

It is given that $f(x) = \sin 2x$.

a)

Show that $f^4(x) = 16f(x)$.

$f^4(x)$ is the FOURTH derivative

$f(x) = \sin 2x$

$f'(x) = 2\cos 2x$ ('sin → cos', chain rule)

$f''(x) = -4\sin 2x$ ('cos → -sin', chain rule)

$f^3(x) = -8\cos 2x$ You should notice a pattern by now...

$f^4(x) = 16\sin 2x$

$\therefore f^4(x) = 16\sin 2x = 16f(x)$ as required

b)

Without further working, write down an expression for $f^8(x)$.

We can see from part (a)

- the coefficient of each derivative is a power of 2
- $\sin 2x$ ($f(x)$) is involved in every even derivative
- $\sin 2x$ is positive in every other even derivative

$\therefore f^8(x) = 256\sin 2x$

5.2.4 Further Applications of Differentiation

Stationary Points & Turning Points

What is the difference between a stationary point and a turning point?

- A **stationary point** is a point at which the **gradient function** is equal to zero
  - The **tangent** to the **curve** of the **function** is **horizontal**
- A **turning point** is a type of stationary point, but in addition the **function changes** from **increasing** to **decreasing,** or **vice versa**
  - The curve **'turns'** from **'going upwards'** to **'going downwards'** or **vice versa**
  - **Turning points** will either be **(local) minimum** or **maximum** points
- A **point** of **inflection *could*** also be a **stationary point** but is **not** a turning point

How do I find stationary points and turning points?

- For the function $y = f(x)$, **stationary points** can be found using the following process

STEP 1

Find the **gradient function,** $\frac{dy}{dx} = f'(x)$

STEP 2

Solve the equation $f'(x) = 0$ to find the x-coordiante(s) of any stationary points

STEP 3

If the y-coordaintes of the stationary points are also required then substitute the x-coordinate(s) into $f(x)$

- A GDC will solve $f'(x) = 0$ and most will find the coordinates of turning points (minimum and maximum points) in graphing mode

Testing for Local Minimum & Maximum Points

What are local minimum and maximum points?

- Local **minimum** and **maximum** points are two types of **stationary** point
  - The **gradient function** (derivative) at such points equals zero
  - i.e. $f'(x) = 0$
- A **local minimum** point, $(x, f(x))$ will be the lowest value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **lower** value further afield
- Similarly, a **local maximum** point, $(x, f(x))$ will be the lowest value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **greater** value further afield
- The graphs of many functions **tend** to **infinity** for **large** values of x (and/or **minus infinity** for **large negative** values of x)
- The **nature** of a stationary point refers to whether it is a **local minimum** point, a **local maximum** point or a **point of inflection**
- A **global** minimum point would represent the **lowest** value of $f(x)$ for **all values** of x
  - similar for a **global** maximum point

How do I find local minimum & maximum points?

- The **nature** of a **stationary point** can be determined using the **first derivative** but it is *usually* quicker and easier to use the **second derivative**
  - only in cases when the second derivative is **zero** is the first derivative method needed
- For the function $f(x)$ ...

STEP 1

Find $f'(x)$ and solve $f'(x) = 0$ to find the x-coordinates of any stationary points

STEP 2 (Second derivative)

Find $f''(x)$ and evaluate it at each of the stationary points found in **STEP 1**

STEP 3 (Second derivative)

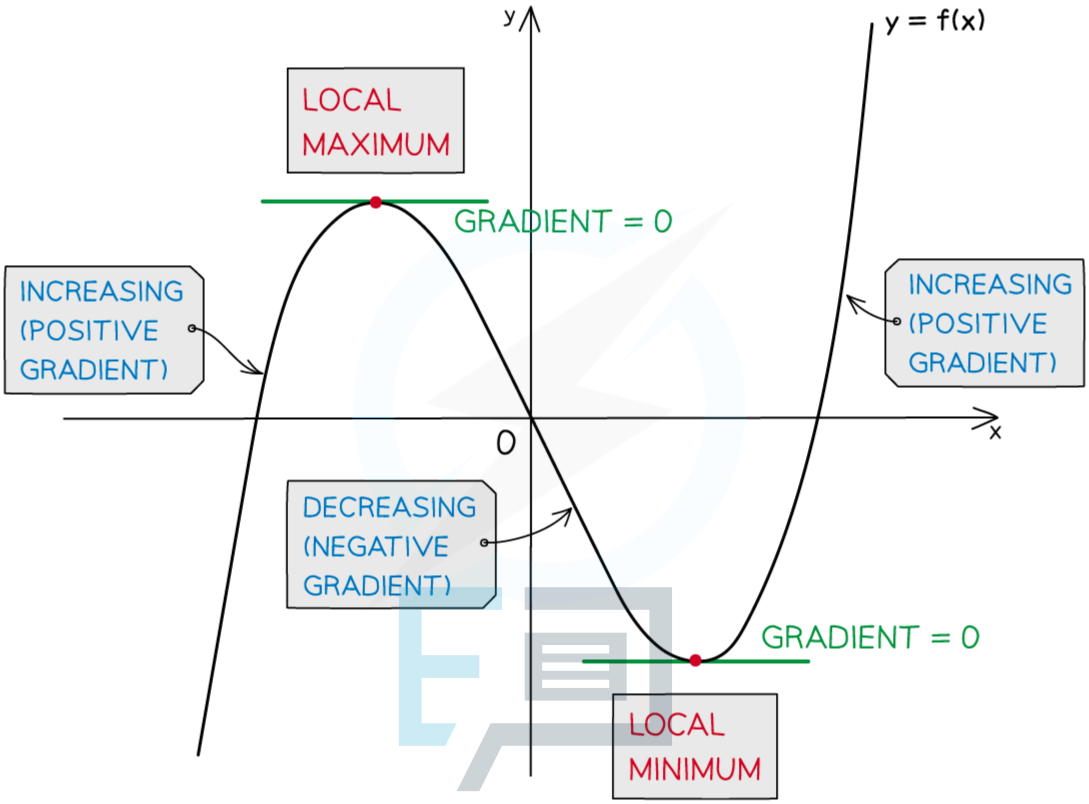
- If $f''(x) = 0$ then the nature of the stationary point **cannot** be determined; use the **first derivative** method (**STEP 4**)
- If $f''(x) > 0$ then the curve of the graph of $y = f(x)$ is **concave up** and the stationary point is a **local minimum** point
- If $f''(x) < 0$ then the curve of the graph of $y = f(x)$ is **concave down** and the stationary point is a **local maximum** point

STEP 4 (First derivative)

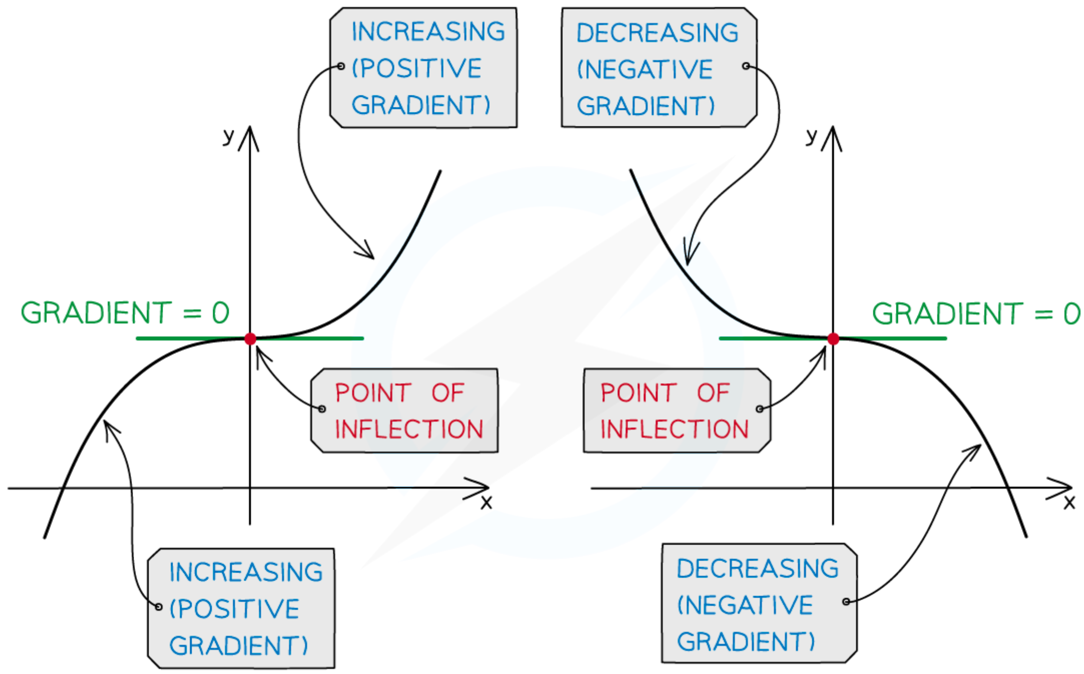
Find the sign of the first derivative just either side of the stationary point; i.e. evaluate $f'(x-h)$ and $f'(x+h)$ for small h

- A **local minimum point** changes the function from **decreasing** to **increasing**
  - the **gradient** changes from **negative** to **positive**
  - $f'(x-h) < 0, \ f'(x) = 0, \ f'(x+h) > 0$
- A **local maximum point** changes the **function** from **increasing** to **decreasing**

- the **gradient** changes from **positive** to **negative**
-



- A **stationary point** of **inflection** results from the function **either increasing or decreasing** on **both sides** of the stationary point
  - the **gradient** does **not change** sign
  - $f'(x-h)>0,\ f'(x+h)>0$ or $f'(x-h)<0,\ f'(x+h)<0$
  - a **point** of **inflection** does **not** necessarily have $f'(x)=0$
    - this method will only find those that do - and are often called **horizontal points** of **inflection**



Exam Tip

- Exam questions may use the phrase "classify turning points" instead of "find the nature of turning points"
- Using your GDC to sketch the curve is a valid test for the nature of a stationary point in an exam unless the question says "show that..." or asks for an algebraic method
- Even if required to show a full algebraic solution you can still use your GDC to tell you what you're aiming for and to check your work

Worked Example

Find the coordinates and the nature of any stationary points on the graph of $y = f(x)$ where $f(x) = 2x^3 - 3x^2 - 36x + 25$.

At stationary points, $f'(x) = 0$

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$

$6(x^2 - x - 6) = 0$

$(x-3)(x+2) = 0$

$x = 3, \quad y = f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 25 = -56$

$x = -2, \quad y = f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 25 = 69$

Using the second derivative to determine their nature

$f''(x) = 12x - 6 = 6(2x - 1)$

$f''(3) = 6(2 \times 3 - 1) = 30 > 0$

$\therefore x = 3$ is a local minimum point

$f''(-2) = 6(2 \times -2 - 1) = -30 < 0$

$\therefore x = -2$ is a local maximum point

(Note: In this case, both stationary points are turning points)

Turning points are:

(3, -56) local minimum point

(-2, 69) local maximum point

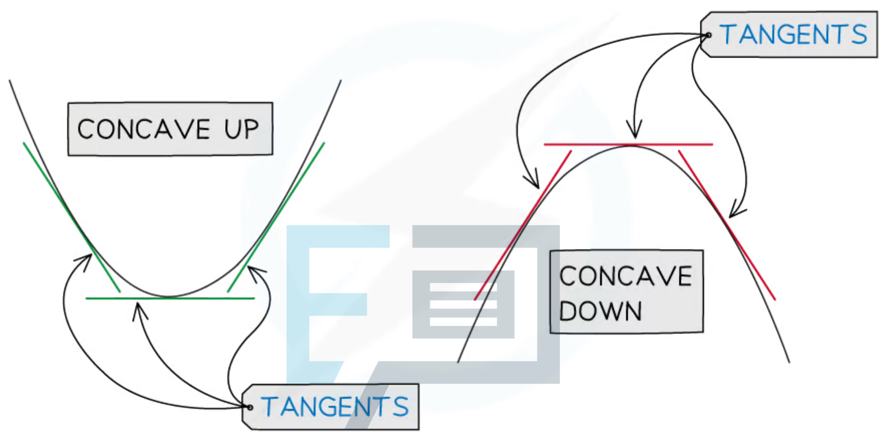
Use a GDC to graph $y = f(x)$ and the max/min solving feature to check the answers.

5.2.5 Concavity & Points of Inflection

Concavity of a Function

What is concavity?

- **Concavity** is the way in which a **curve** (or surface) **bends**
- Mathematically,
  - a curve is **CONCAVE DOWN** if $f''(x) < 0$ for all values of x in an interval
  - a curve is **CONCAVE UP** if $f''(x) > 0$ for all values of x in an interval



Exam Tip

- In an exam an easy way to remember the difference is:
  - Concave **down** is the shape of (the mouth of) a sad smiley ☹
  - **Concave up** is the shape of (the mouth of) a happy smiley ☺

Worked Example

The function $f(x)$ is given by $f(x) = x^3 - 3x + 2$.

a)
Determine whether the curve of the graph of $y = f(x)$ is concave down or concave up at the points where $x = -2$ and $x = 2$.

$f(x) = x^3 - 3x + 2$
$f'(x) = 3x^2 - 3$
$f''(x) = 6x$

$f''(-2) = 6 \times -2 = -12 < 0$ (concave down)

$f''(2) = 6 \times 2 = 12 > 0$ (concave up)

At $x = -2$, $y = f(x)$ is concave down
At $x = 2$, $y = f(x)$ is concave up

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answers are sensible

b)
Find the values of x for which the curve of the graph $y = f(x)$ of is concave up.

$f''(x) = 6x$ from part (a)
Concave up is $f''(x) > 0$
$6x > 0$ when $x > 0$

$\therefore$ $y = f(x)$ is concave up for $x > 0$

Use your GDC to check your answer

Points of Inflection

What is a point of inflection?

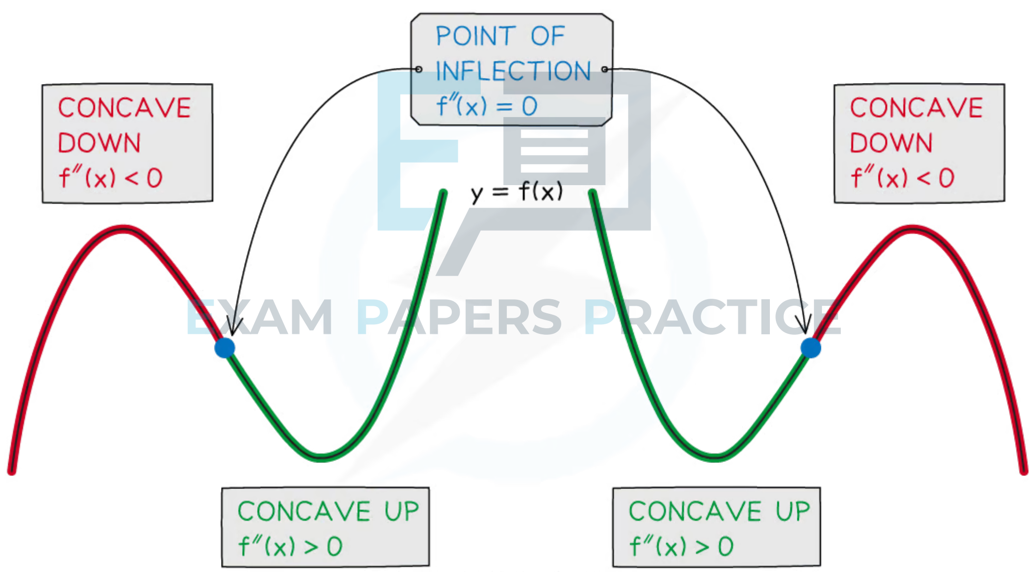
- A point at which the curve of the graph of $y = f(x)$ changes **concavity** is a **point** of **inflection**
- The alternative spelling, **inflexion**, may sometimes be used

What are the conditions for a point of inflection?

- A point of inflection requires **BOTH** of the following two conditions to hold
  - the **second derivative** is zero
    - $f''(x) = 0$

AND

  - the graph of $y = f(x)$ changes **concavity**
    - $f''(x)$ changes **sign** through a **point** of **inflection**



- It is important to understand that the first condition is **not** sufficient on its own to locate a point of inflection
  - points where $f''(x) = 0$ could be **local minimum** or **maximum** points
    - the **first derivative** test would be needed
  - However, if it is already known $f(x)$ has a point of inflection at $x = a$, say, then $f''(a) = 0$

What about the first derivative, like with turning points?

- A **point** of **inflection**, unlike a turning point, does not necessarily have to have a first derivative value of 0 ($f'(x) = 0$)
  - If it does, it is also a **stationary point** and is often called a **horizontal point** of **inflection**
    - the tangent to the curve at this point would be horizontal

- The **normal distribution** is an example of a commonly used function that has a graph with two non-stationary points of inflection

How do I find the coordinates of a point of inflection?

- For the function $f(x)$

STEP 1

Differentiate $f(x)$ **twice** to find $f''(x)$ and solve $f''(x) = 0$ to find the x-coordinates of possible points of inflection

STEP 2

Use the **second derivative** to **test** the **concavity** of $f(x)$ either side of $x = a$

- If $f''(x) < 0$ then $f(x)$ is concave down
- If $f''(x) > 0$ then $f(x)$ is concave up

If concavity changes, $x = a$ is a **point of inflection**

STEP 3

If required, the y-coordinate of a point of inflection can be found by substituting the x-coordinate into $f(x)$

Exam Tip

- You can find the *x*-coordinates of the point of inflections of $y = f(x)$ by drawing the graph $y = f'(x)$ and finding the *x*-coordinates of any local maximum or local minimum points
- Another way is to draw the graph $y = f''(x)$ and find the *x*-coordinates of the points where the graph crosses (not just touches) the *x*-axis

Worked Example

Find the coordinates of the point of inflection on the graph of $y = 2x^3 - 18x^2 + 24x + 5$.

Fully justify that your answer is a point of inflection.

STEP 1: Differentiate twice, solve $f''(x) = 0$

$f(x) = 2x^3 - 18x^2 + 24x + 5$

$f'(x) = 6x^2 - 36x + 24$

$f''(x) = 12x - 36$

$12x - 36 = 0$ when $x = 3$

STEP 2: Use the second derivative to test concavity

$f''(3) = 0$

$f''(2.9) < 0$ (concave down)

$f''(3.1) > 0$ (concave up)

$\therefore$ concavity changes through $x = 3$

STEP 3: The y-coordinate is required

$f(3) = 2(3)^3 - 18(3)^2 + 24(3) + 5 = -31$

Since $f''(3) = 0$ AND the graph of $y = f(x)$ changes concavity through $x = 3$, the point $(3, -31)$ is a point of inflection.

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answer is sensible

5.2.6 Derivatives & Graphs

Derivatives & Graphs

How are derivatives and graphs connected?

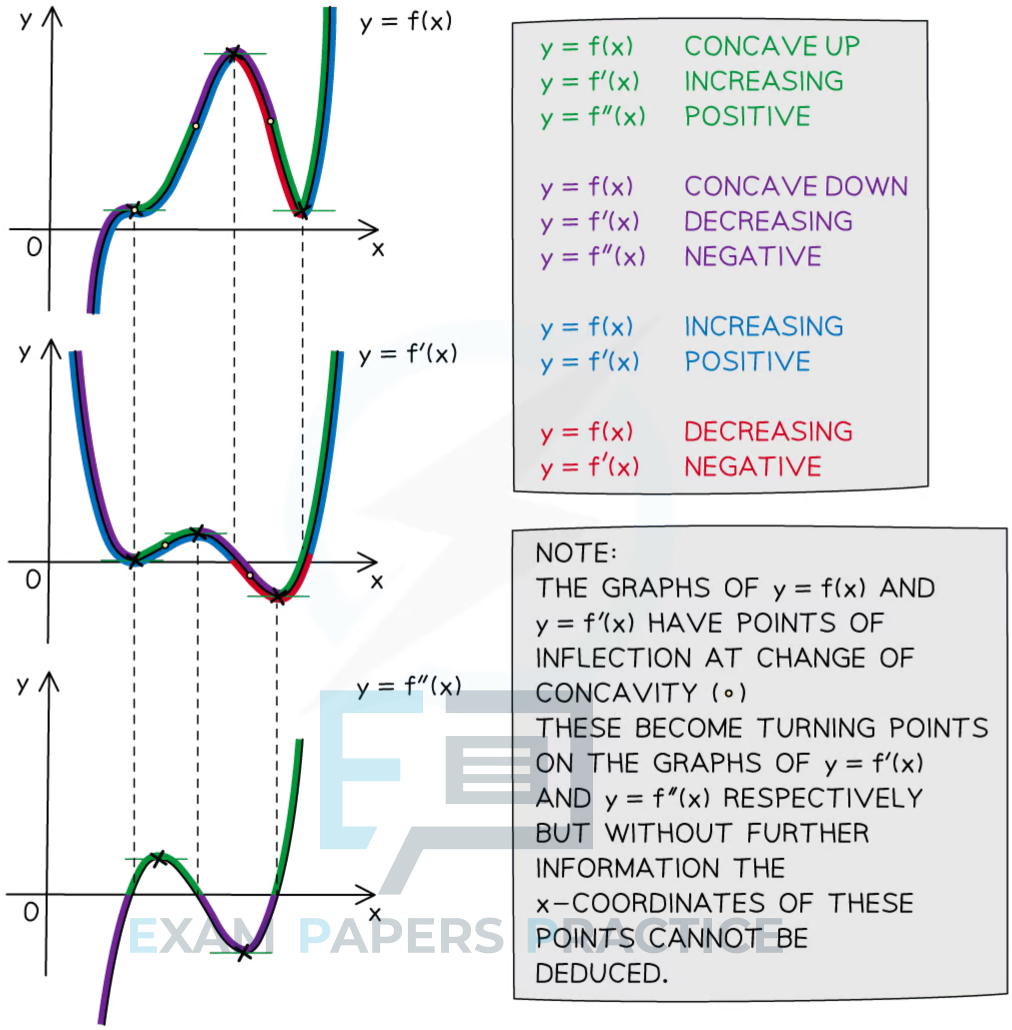
- If the graph of a function $y = f(x)$ is known, or can be sketched, then it is also possible to sketch the graphs of the derivatives $y = f'(x)$ and $y = f''(x)$
- The key properties of a graph include
  - the $\boldsymbol{y}$**-axis intercept**
  - the $\boldsymbol{x}$**-axis intercepts** – the **roots** of the function; where $f(x) = 0$
  - **stationary points**; where $f'(x) = 0$
    - turning points – (local) **minimum** and **maximum** points
    - (horizontal) **points of inflection**
  - (non-stationary, $f'(x) \neq 0$) points of inflection
  - **asymptotes** – **vertical** and **horizontal**
  - intervals where the graph is **increasing** and **decreasing**
  - intervals where the graph is **concave down** and **concave up**
- **Not** all graphs have all of these properties and **not** all can be determined without knowing the expression of the function
- However questions will provide enough information to sketch
  - the **shape** of the graph
  - **some** of the **key properties** such as **roots** or **turning points**

How do I sketch the graph of *y* = f'(*x*) from the graph of *y* = f(*x*)?

- The graph of $y = f'(x)$ will have its
  - x-axis **intercepts** at the x-coordinates of the **stationary points** of $y = f(x)$
  - **turning points** at the x-coordinates of the **points of inflection** of $y = f(x)$
- For intervals where $y = f(x)$ is concave up, $y = f'(x)$ will be increasing
- For intervals where $y = f(x)$ is concave down, $y = f'(x)$ will be decreasing
- For intervals where $y = f(x)$ is increasing, $y = f'(x)$ will be positive
- For intervals where $y = f(x)$ is decreasing, $y = f'(x)$ will be negative

How do I sketch the graph of *y* = f''(*x*) from the graph of *y* = f(*x*)?

- First sketch the graph of $y = f'(x)$ from $y = f(x)$, as per the above process
- Then, using the same process, sketch the graph of $y = f''(x)$ from the graph of $y = f'(x)$
- There are a couple of things you can deduce about the graph of $y = f''(x)$ directly from the graph of $y = f(x)$
  - The graph of $y = f''(x)$ will have its x-axis **intercepts** at the x-coordinates of the **points of inflection** of $y = f(x)$
  - For intervals where $y = f(x)$ is concave up, $y = f''(x)$ will be positive
  - For intervals where $y = f(x)$ is concave down, $y = f''(x)$ will be negative



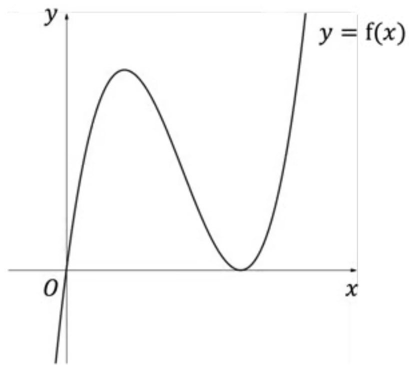
Is it possible to sketch the graph **of $y = f(x)$ from the graph of a derivative?**

- It is possible to **sketch** a graph of $y = f(x)$ by considering the reverse of the above
  - For intervals where $y = f'(x)$ is **positive**, $y = f(x)$ will be **increasing** but is **not** necessarily positive
  - For intervals where $y = f'(x)$ is **negative**, $y = f(x)$ will be **decreasing** but is **not** necessarily negative
  - **Roots** of $y = f'(x)$ give the x-coordinates of the **stationary points** of $y = f(x)$
- There are some properties of the graph of $y = f(x)$ that **cannot** be determined from the graph of $y = f'(x)$
  - the **y-axis intercept**
  - the **intervals** for which $y = f(x)$ is **positive** and **negative**
  - the roots of $y = f(x)$
- Unless a **specific** point the **curve passes through** is **known**, the **constant** of **integration cannot** be determined
  - the *exact* location of the curve will remain unknown
  - but it will still be possible to **sketch** its **shape**

- If starting from the graph of the **second derivative,** $y = f''(x)$, it is easier to sketch the graph of $y = f'(x)$ first, then sketch $y = f(x)$

Worked Example

The graph of $y = f(x)$ is shown in the diagram below.

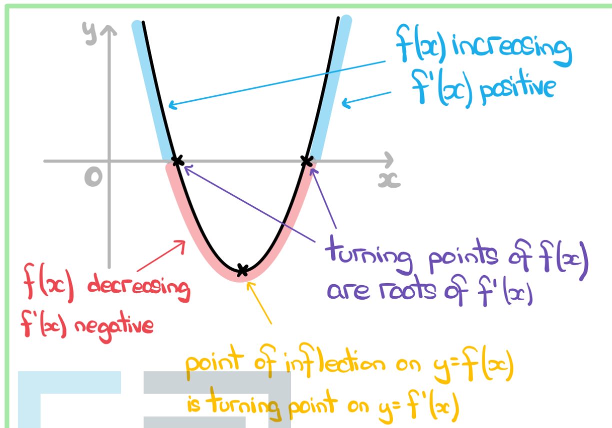


On separate diagrams sketch the graphs of $y = f'(x)$ and $y = f''(x)$, labelling any roots and turning points.

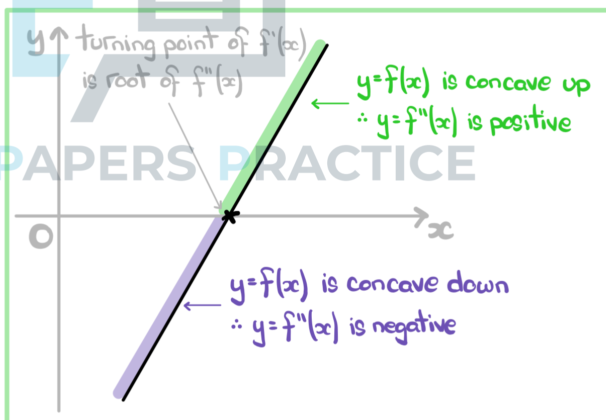
Key features from graph of $y=f(x)$ are:

- local maximum and local minimum
- $f(x)$ is increasing 'at either end'
- $f(x)$ is decreasing between max. and min.
- point of inflexion (non-stationary), graph changes from concave down to concave up

Graph of $y=f'(x)$:



Graph of $y=f''(x)$:



5.3 Integration

5.3.1 Introduction to Integration

Introduction to Integration

What is integration?

- **Integration** is the opposite to **differentiation**
  - Integration is referred to as **antidifferentiation**
  - The result of integration is referred to as the **antiderivative**
- **Integration** is the process of finding the expression of a function (**antiderivative**) from an expression of the **derivative** (**gradient function**)

What is the notation for integration?

- An **integral** is normally written in the form

$$\int f(x)\,\mathrm{d}x$$

  - the large operator $\int$ means "integrate"
  - "$\mathrm{d}x$" indicates which variable to integrate with respect to
  - $f(x)$ is the function to be integrated (sometimes called the integrand)
- The **antiderivative** is sometimes denoted by $\mathrm{F}(x)$
  - there's then no need to keep writing the whole integral; refer to it as $\mathrm{F}(x)$
- $\mathrm{F}(x)$ may also be called the **indefinite integral** of $f(x)$

What is the constant of integration?

- Recall one of the special cases from **Differentiating Powers of x**
  - If $f(x) = a$ then $f'(x) = 0$
- This means that integrating 0 will produce a **constant** term in the antiderivative
  - a zero term wouldn't be written as part of a function
  - **every** function, when integrated, potentially has a **constant** term
- This is called the **constant** of **integration** and is usually denoted by the letter c
  - it is often referred to as "plus c"
- Without more information it is impossible to deduce the value of this constant
  - there are endless antiderivatives, $\mathrm{F}(x)$, for a function $f(x)$

Integrating Powers of x

How do I integrate powers of x?

- Powers of x are integrated according to the following formulae:
    - If $f(x) = x^n$ then $\int f(x)\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$ where $n \in \mathbb{Q}$, $n \neq -1$ and c is the **constant** of **integration**
    - This is given in the **formula booklet**
- If the power of x is multiplied by a constant then the integral is also multiplied by that constant
    - If $f(x) = ax^n$ then $\int f(x)\,\mathrm{d}x = \frac{ax^{n+1}}{n+1} + c$ where $n \in \mathbb{Q}$, $n \neq -1$ and a is a constant and c is the **constant** of **integration**
- $\frac{\mathrm{d}y}{\mathrm{d}x}$ notation can still be used with integration
- Note that the formulae above do not apply when $x = -1$ as this would lead to division by zero
- Remember the special case:
    - $\int a\,\mathrm{d}x = ax + c$

      e.g. $\int 4\,\mathrm{d}x = 4x + c$
    - This allows **constant** terms to be integrated
- Functions involving **roots** will need to be rewritten as **fractional powers** of x first
    - eg. If $f(x) = 5\sqrt[3]{x}$ then rewrite as $f(x) = 5x^{\frac{1}{3}}$ and integrate
- Functions involving **fractions** with **denominators** in **terms** of $\boldsymbol{x}$ will need to be rewritten as **negative powers** of x first
    - e.g. If $f(x) = \frac{4}{x^2} + x^2$ then rewrite as $f(x) = 4x^{-2} + x^2$ and integrate
- The formulae for integrating powers of x apply to **all rational numbers** so it is possible to integrate any expression that is a sum or difference of powers of x
    - e.g. If $f(x) = 8x^3 - 2x + 4$ then $\int f(x)\,\mathrm{d}x = \frac{8x^{3+1}}{3+1} - \frac{2x^{1+1}}{1+1} + 4x + c = 2x^4 - x^2 + 4x + c$
- **Products** and **quotients** cannot be integrated this way so would need **expanding/simplifying** first
    - e.g. If $f(x) = 8x^2(2x - 3)$ then

      $\int f(x)\,\mathrm{d}x = \int (16x^3 - 24x^2)\,\mathrm{d}x = \frac{16x^4}{4} - \frac{24x^3}{3} + c = 4x^4 - 8x^3 + c$

What might I be asked to do once I've found the anti-derivative (integrated)?

- With more information the **constant of integration**, c, can be found

- The **area under** a **curve** can be found using integration

Exam Tip

- You can speed up the process of integration in the exam by committing the pattern of basic integration to memory
  - In general you can think of it as 'raising the power by one and dividing by the new power'
  - Practice this lots before your exam so that it comes quickly and naturally when doing more complicated integration questions

Worked Example

Given that

$$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - \frac{1}{\sqrt{x}}$$

find an expression for y in terms of x.

Firstly rewrite all terms as powers of x

$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - x^{-\frac{1}{2}}$ ← fractional AND negative!

$y = \int \left(3x^4 - 2x^2 + 3 - x^{-\frac{1}{2}}\right) dx$

$\therefore y = \frac{3x^5}{5} - \frac{2x^3}{3} + 3x - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

Special case (for $3x$)

take care with negatives, $-\frac{1}{2} + 1 = \frac{1}{2}$

constant of integration (for c)

$$\therefore y = \frac{3}{5}x^5 - \frac{2}{3}x^3 + 3x - 2\sqrt{x} + c$$

5.6 Kinematics

5.6.1 Kinematics Toolkit

Displacement, Velocity & Acceleration

What is kinematics?

- **Kinematics** is the branch of mathematics that models and analyses the **motion** of objects
- Common words such as **distance, speed** and **acceleration** are used in kinematics but are used according to their technical definition

What definitions do I need to be aware of?

- Firstly, only motion of an object in a **straight line** is considered
  - this could be a **horizontal** straight line
    - the **positive** direction would be to the **right**
  - or this could be a **vertical** straight line
    - the **positive** direction would be **upwards**

Particle

- A **particle** is the general term for an **object**
  - some questions may use a **specific** object such as a **car** or a **ball**

Time t seconds

- **Displacement**, **velocity** and **acceleration** are all **functions** of **time** t
- **Initially** time is zero $t = 0$

Displacement s m

- The **displacement** of a particle is its **distance relative** to a **fixed point**
  - the fixed point is often (but not always) the particle's **initial position**
- **Displacement** will be **zero** $s = 0$ if the object is at or has returned to its initial position
- **Displacement** will be negative if its **position relative** to the **fixed point** is in the **negative direction** (left or down)

Distance d m

- Use of the word **distance** needs to be considered carefully and could refer to
  - the distance **travelled** by a particle
  - the **(straight line)** distance the particle is from a **particular point**
- Be careful not to confuse **displacement** with **distance**
  - if a bus route starts and ends at a bus depot, when the bus has returned to the depot, its **displacement** will be **zero** but the distance the bus has travelled will be the length of the route
- **Distance** is always **positive**

Velocity v m s^{-1}

- The **velocity** of a particle is the **rate of change** of its **displacement** at time t
- **Velocity** will be **negative** if the **particle** is moving in the **negative direction**
- A **velocity** of **zero** means the particle is **stationary** $v = 0$

Speed $|v|$ m s⁻¹

- **Speed** is the **magnitude** (a.k.a. absolute value or modulus) of **velocity**
  - as the particle is **moving** in a **straight line, speed** is the **velocity ignoring** the **direction**

    if $v = 4,\ |v| = 4$

    if $v = -6,\ |v| = 6$

Acceleration a m s⁻²

- The **acceleration** of a particle is the **rate of change** of its **velocity** at time t
- Acceleration can be **negative** but this alone cannot fully describe the particle's motion
  - if **velocity** and **acceleration** have the **same** sign the particle is **accelerating** (speeding up)
  - if **velocity** and **acceleration** have **different** signs then the particle is **decelerating** (slowing down)
  - if **acceleration** is **zero** $a = 0$ the particle is moving with **constant** velocity
  - in all cases the **direction** of **motion** is determined by the **sign** of **velocity**

Are there any other words or phrases in kinematics I should know?

- Certain words and phrases can imply values or directions in kinematics
  - a particle described as "at **rest**" means that its velocity is zero, $= 0$
  - a particle described as moving "**due east**" or "**right**" or would be moving in the **positive horizontal** direction

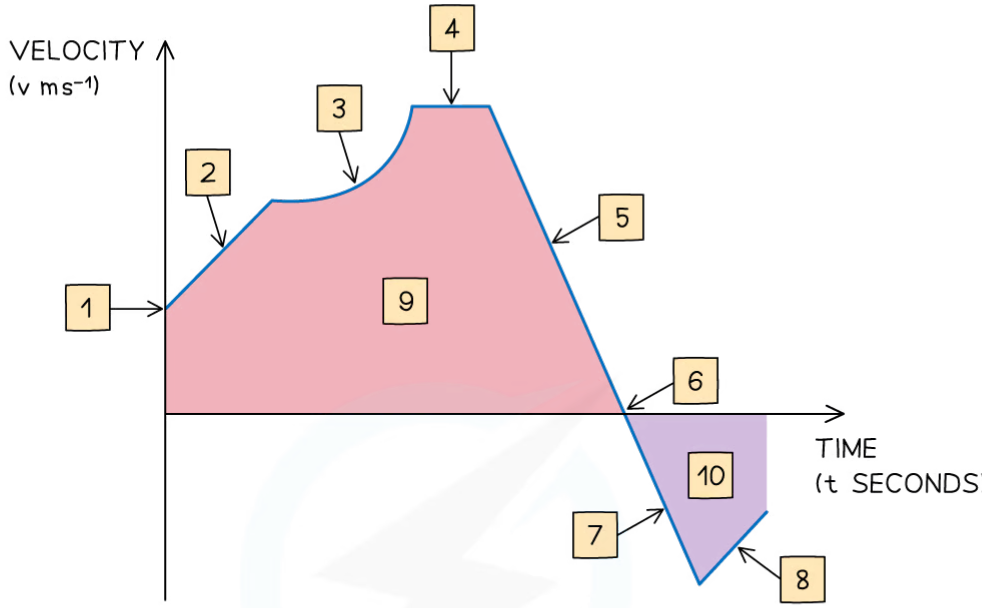
    this also means that $v > 0$
  - a particle "**dropped from the top of a cliff**" or "**down**" would be moving in the **negative vertical** direction

    this also means that $v < 0$

What are the key features of a velocity-time graph?

- The **gradient** of the graph equals the **acceleration** of an object
- A **straight line** shows that the object is **accelerating** at a **constant rate**
- A **horizontal** line shows that the object is moving at a **constant velocity**
- The **area** between graph and the x-axis tells us the **change in displacement** of the object
  - Graph **above** the x-axis means the object is moving **forwards**
  - Graph **below** the x-axis means the object is moving **backwards**

- The **total displacement** of the object from its starting point is the sum of the **areas above** the x-axis **minus** the sum of the **areas below** the x-axis
- The **total distance travelled** by the object is the sum of **all** the **areas**
- If the graph **touches** the **x-axis** then the object is **stationary** at that time
- If the graph is **above** the **x-axis** then the object has positive velocity and is **travelling forwards**
- If the graph is **below** the **x-axis** then the object has negative velocity and is **travelling backwards**



1	INITIAL VELOCITY	7	SPEEDING UP BUT MOVING BACKWARDS
2	CONSTANT ACCELERATION	8	SLOWING DOWN BUT STILL MOVING BACKWARDS
3	VARIABLE ACCELERATION	9	DISTANCE TRAVELLED FORWARDS
4	CONSTANT VELOCITY	10	DISTANCE TRAVELLED BACKWARDS
5	DECELERATING (SLOWING DOWN BUT STILL MOVING FORWARDS)		
6	INSTANTANEOUSLY AT REST (STATIONARY FOR AN INSTANT)		

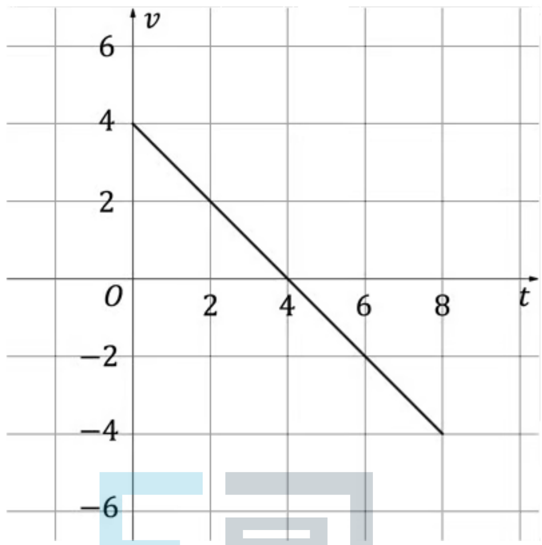
Exam Tip

- In an exam if you are given an expression for the velocity then sketching a velocity-time graph can help visualise the problem

Worked Example

A particle is projected vertically upwards from ground level, taking 8 seconds to return to the ground.

The velocity-time graph below illustrates the motion of the particle for these 8 seconds.



i)
How many seconds does the particle take to reach its maximum height?
Give a reason for your answer.

ii)
State, with a reason, whether the particle is accelerating or decelerating at time $t = 3$.

i. At maximum height, velocity is zero

$v = 0$ at $t = 4$

∴ The particle takes 4 seconds to reach its maximum height. This is because its velocity is 0 m s⁻¹ at 4 seconds.

ii. At $t = 3$, velocity is POSITIVE

Acceleration is the gradient of velocity

At $t = 3$, acceleration is NEGATIVE

∴ At 3 seconds the particle is decelerating as its velocity and acceleration have different signs.

5.7 Basic Limits & Continuity

5.7.1 Basic Limits & Continuity

Limits

What are limits in mathematics?

- When we consider a **limit** in mathematics we look at the tendency of a mathematical process as it approaches, but never quite reaches, an 'end point' of some sort
- We use a special limit notation to indicate this
  - For example $\lim_{x \to 3} f(x)$ denotes 'the limit of the function *f*(*x*) as *x* goes to (or approaches) 3'
    - I.e., what value (if any) *f*(*x*) gets closer and closer to as *x* takes on values closer and closer to 3
    - We are not concerned here with what value (if any) *f*(*x*) takes when *x* is equal to 3 – only with the behaviour of *f*(*x*) as *x* gets close to 3
- The sum of an infinite geometric sequence is a type of limit
  - When you calculate S_∞ for an infinite geometric sequence, what you are actually finding is $\lim_{n \to \infty} S_n$
    - I.e., what value (if any) the sum of the first *n* terms of the sequence gets closer and closer to as the number of terms (*n*) goes to infinity
    - The sum never actually reaches S_∞, but as more and more terms are included in the sum it gets closer and closer to that value
- In this section of the IB course we will be considering the limits of functions
  - This may include finding the limit at a point where the function is undefined
    - For example, $f(x) = \frac{\sin x}{x}$ is undefined when *x* = 0, but we might want to know how the function behaves as *x* gets closer and closer to zero
  - Or it may include finding the limit of a function *f*(*x*) as *x* gets infinitely big in the positive or negative direction
    - For this type of limit we write $\lim_{x \to \infty} f(x)$ or $\lim_{x \to -\infty} f(x)$ (the first one can also be written as $\lim_{x \to +\infty} f(x)$ to distinguish it from the second one)
  - These sorts of limits are often used to find the **asymptotes** of the graph of a function

How do I find a simple limit?

- STEP 1: To find $\lim_{x \to a} f(x)$ begin by substituting *a* into the function *f*(*x*)
  - If *f*(*a*) exists with a well-defined value, then that is also the value of the limit
  - For example, for $f(x) = \frac{x-1}{x}$ we may find the limit as *x* approaches 3 like this:

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x-1}{x} = \frac{3-1}{3} = \frac{2}{3}$$

In this case, is simply equal to *f*(3)

- STEP 2: If *f*(*a*) does not exist, it may be possible to use algebra to simplify *f*(*x*) so that substituting *a* into the simplified function gives a well-defined value
  - In that case, the well-defined value at *x* = *a* of the simplified version of the function is also the value of the limit of the function as *x* goes to *a*
  - For example, $f(x) = \frac{x^2}{x}$ is not defined at *x* = 0, but we may use algebra to find the limit as *x* approaches zero:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} \frac{x}{1} \text{ (cancelling the } x\text{'s)} = \frac{0}{1} = 0$$

Note that $f(x) = \frac{x^2}{x}$ and $g(x) = x$ are **not** the same function!

They are equal for all values of *x* except zero

But for *x* = 0, *g*(0) = 0 while *f*(0) is undefined

However *f*(*x*) gets closer and closer to zero as *x* gets closer and closer to zero

- If neither of these steps gives a well-defined value for the limit you may need to consider more advanced techniques to evaluate the limit
  - For example **l'Hôpital's Rule** or using **Maclaurin series**

How do I find a limit to infinity?

- As *x* goes to $+\infty$ or $-\infty$, a typical function *f*(*x*) may **converge** to a well-defined value, or it may **diverge** to $+\infty$ or $-\infty$
  - Other behaviours are possible – for example $\lim_{x \to \infty} \sin x$ is simply undefined, because sin *x* continues to oscillate between 1 and -1 as *x* gets larger and larger
- There are two key results to be used here:
  - $\lim_{x \to \pm\infty} \frac{k}{x^n}$ **converges** to 0 for all *n* >0 and all $k \in \mathbb{R}$
  - $\lim_{x \to +\infty} x^n$ **diverges** to $+\infty$ for all *n* > 0

    $\lim_{x \to -\infty} x^n$ for *n* > 0 will need to be considered on a case-by-case basis, because of the differing behaviour of x^n for different values of *n* when *x* is negative

- STEP 1: If necessary, use algebra to rearrange the function into a form where one or the other of the key results above may be applied

- STEP 2: Use the key results above to evaluate your limit
- For example:

$$\lim_{x\to\infty}\frac{3x^2-2x+1}{4x^2-x+2}=\lim_{x\to\infty}\frac{3-\frac{2}{x}+\frac{1}{x^2}}{4-\frac{1}{x}+\frac{2}{x^2}}=\frac{3-0+0}{4-0+0}=\frac{3}{4}$$

- Or:

$$\lim_{x\to+\infty}\frac{x^2+5x-2}{32x+3}=\lim_{x\to+\infty}\frac{x+5-\frac{2}{x}}{32+\frac{3}{x}}=\frac{(+\infty)+5-0}{32+0}=+\infty$$

  - I.e., the limit diverges to $+\infty$ (because $\frac{x^2+5x-2}{32x+3}$ it gets bigger and bigger without limit as *x* gets bigger and bigger)
- Remember that neither $\frac{0}{0}$ nor $\frac{\pm\infty}{\pm\infty}$ has a well-defined value!
  - If you attempt to evaluate a limit and get one of these two forms, you will need to try another strategy
  - This may just mean different or additional algebraic rearrangement
  - But it may also mean that you need to consider using **l'Hôpital's Rule** or **Maclaurin series** to evaluate the limit
- It is also worth remembering that if $\lim_{x\to\infty} f(x)=\infty$, then $\lim_{x\to\infty}\frac{k}{f(x)}=0$ for any non-zero $k\in\mathbb{R}$
  - This can be useful for example when evaluating the limits of functions containing exponentials

    $\lim_{x\to\infty} e^{px}=\infty$ for any *p* > 0, so we immediately have $\lim_{x\to\infty} e^{-px}=\lim_{x\to\infty}\frac{1}{e^{px}}=0$ for *p* > 0

    See the worked example below for a more involved version of this

Do limits ever have 'directions'?

- Yes they do!
- The notation $\lim_{x\to a^+} f(x)$ means 'the limit of *f*(*x*) as *x* **approaches *a* from above**'
  - I.e., this is the limit as *x* comes 'down' towards *a*, only considering the function's behaviour for values of *x* that are greater than *a*
- The notation $\lim_{x\to a^-} f\left(x\right)$ means 'the limit of *f*(*x*) as *x* **approaches *a* from below**'
  - I.e., this is the limit as *x* comes 'up' towards *a*, only considering the function's behaviour for values of *x* that are less than *a*
- One place these sorts of limits appear is for functions defined piecewise
  - In this case the limits 'from above' and 'from below' may well be different for values of *x* at which the different 'pieces' of the function are joined

- But also be aware of a situation like the following, where the limits from above and below may also be different:
  - $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ (because $\frac{1}{x} > 0$ for $x > 0$, with $\frac{1}{x}$ becoming bigger and bigger in the positive direction as x gets closer and closer to zero 'from above')
  - $\lim_{x \to 0^-} \frac{1}{x} = +\infty$ (because $\frac{1}{x} < 0$ for $x < 0$, with $\frac{1}{x}$ becoming bigger and bigger in the negative direction as x gets closer and closer to zero 'from below')
  - The graph of $y = \frac{1}{x}$ shows this limiting behaviour as x approaches zero from the two different directions

Worked Example

a)

Consider the function

$$f(x) = \frac{3 - 4x - 5x^4}{2x^4 + x^3 + 7},$$

find $\lim_{x \to \infty} f(x)$.

$$\frac{3 - 4x - 5x^4}{2x^4 + x^3 + 7} \cdot \frac{1/x^4}{1/x^4} = \frac{\frac{3}{x^4} - \frac{4}{x^3} - 5}{2 + \frac{1}{x} + \frac{7}{x^4}}$$

$$\lim_{x \to \infty} \frac{3 - 4x - 5x^4}{2x^4 + x^3 + 7} = \lim_{x \to \infty} \frac{\frac{3}{x^4} - \frac{4}{x^3} - 5}{2 + \frac{1}{x} + \frac{7}{x^4}}$$

$$= \frac{0 - 0 - 5}{2 + 0 + 0} = \boxed{-\frac{5}{2}}$$

b)

Consider the function

$$g(x) = \begin{cases} \frac{1 - 5x}{x^2}, & x < 5 \\ x^2 - 4x - 6, & x \geq 5 \end{cases}$$

find (i) $\lim_{x \to 5^-} g(x)$, and (ii) $\lim_{x \to 5^+} g(x)$.

(i) For $\lim_{x \to 5^-}$, we only consider $x < 5$

$$\lim_{x \to 5^-} g(x) = \lim_{x \to 5} \frac{1 - 5x}{x^2}$$

$$= \frac{1 - 5(5)}{(5)^2} = \boxed{-\frac{24}{25}}$$

(ii) For $\lim_{x \to 5^+}$, we only consider $x > 5$

$$\lim_{x \to 5^+} g(x) = \lim_{x \to 5} (x^2 - 4x - 6)$$

$$= (5)^2 - 4(5) - 6 = \boxed{-1}$$

c)

Consider the function

$$h(x) = \frac{2e^{3x} - 3}{4 - 5e^{3x}}$$

find $\lim_{x \to \infty} h(x)$.

$$\frac{2e^{3x}-3}{4-5e^{3x}} \cdot \frac{1/e^{3x}}{1/e^{3x}} = \frac{2-\frac{3}{e^{3x}}}{\frac{4}{e^{3x}}-5}$$

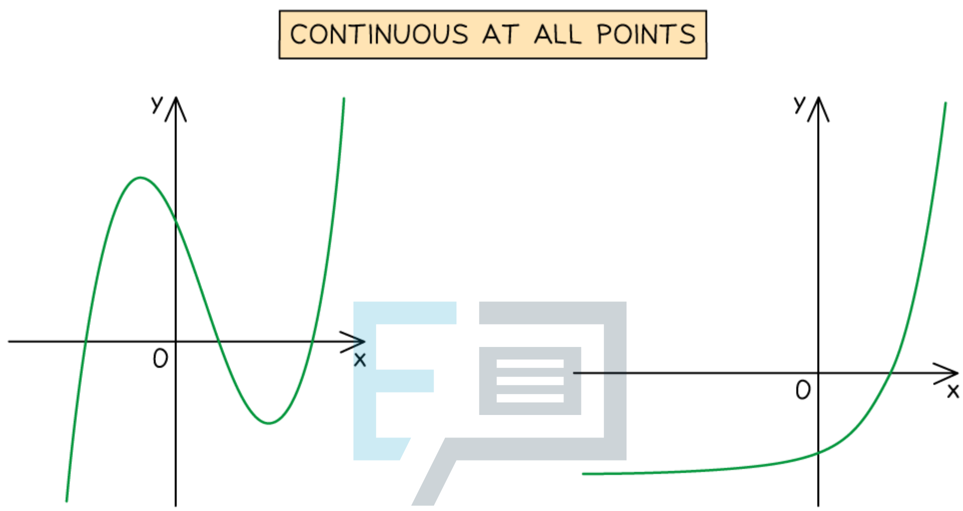
$$\lim_{x\to\infty} \frac{2e^{3x}-3}{4-5e^{3x}} = \lim_{x\to\infty} \frac{2-\frac{3}{e^{3x}}}{\frac{4}{e^{3x}}-5}$$

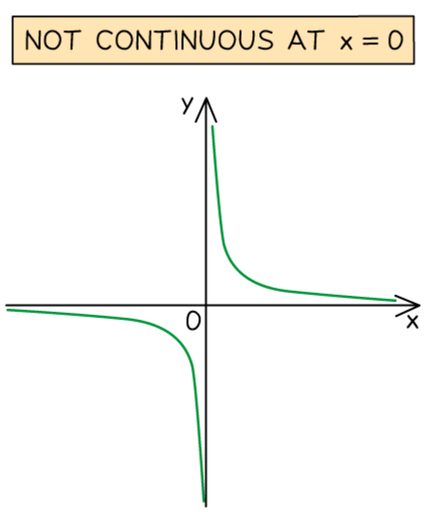
$$= \frac{2-0}{0-5} = \boxed{-\frac{2}{5}}$$

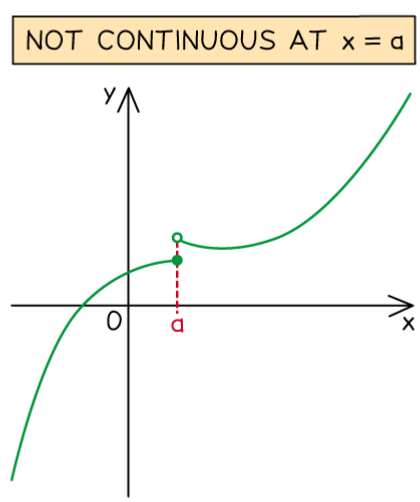
Continuity & Differentiability

What does it mean for a function to be continuous at a point?

- If a function is **continuous** at a point then the graph of the function does not have any 'holes' or any sudden 'leaps' or 'jumps' at that point
    - One way to think about this is to imagine sketching the graph
        So long as you can sketch the graph without lifting your pencil from the paper, then the function is continuous at all the points that your sketch goes through
        But if you would have to lift your pencil off the paper at some point and continue drawing the graph from another point, then the function is not continuous at any such points where the function 'jumps'







- There are two main ways a function can fail to be continuous at a point:

- If the function is not defined for a particular value of *x* then it is not continuous at that value of *x*

  For example, $f(x) = \frac{1}{x}$ is not continuous at $x = 0$

- If the function is defined for a particular value of *x*, but then the value of the function 'jumps' as *x* moves away from that x value in the positive or negative direction, then the function is not continuous at that value of *x*

  This type of discontinuity can occur in a piecewise function, for example, where the different pieces of the function's graph don't 'join up'

- You can use limits to show that a function is continuous at a point
  - Let *f*(*x*) be a function defined at *x* = *a*, such that *f*(*a*) = *b*

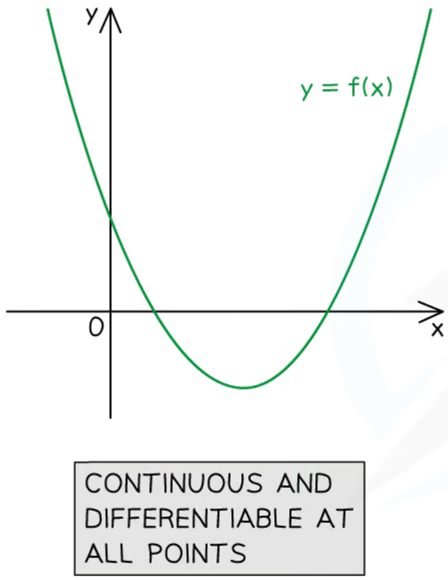
    If $\lim_{x \to a^-} f(x) = b$ and $\lim_{x \to a+} f(x) = b$, then *f*(*x*) is continuous at *x* = *a*

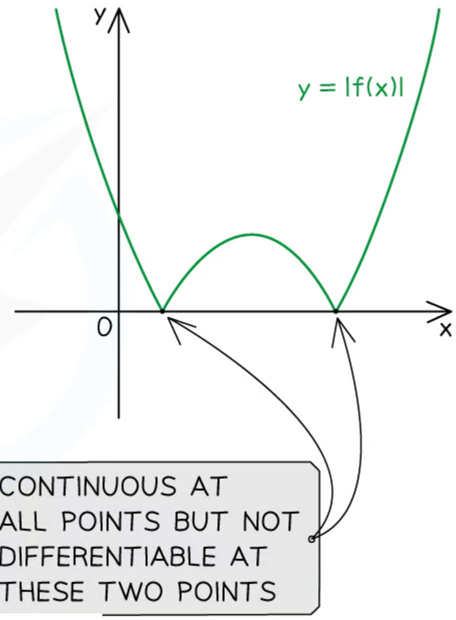
    If either of those limits is not equal to *b*, then *f*(*x*) is not continuous at *x* = *a*
  - This is a slightly more formal way of expressing the 'you don't have to lift your pencil from the paper' idea!

What does it mean for a function to be differentiable at a point?

- We say that a function *f*(*x*) is **differentiable** at a point with *x*-coordinate x_0, if the derivative *f'*(*x*) exists and has a well-defined value $f'(x_0)$ at that point
- To be differentiable at a point a function has to be continuous at that point
  - So if a function is not continuous at a point, then it is also not differentiable at that point
- But continuity by itself does not guarantee differentiability
  - This means that differentiability is a stronger condition than continuity
  - If a function is differentiable at a point, then the function is also continuous at that point
  - But a function may be continuous at a point without being differentiable at that point
  - This means there are functions that are continuous everywhere but are not differentiable everywhere
- In addition to being continuous a point, differentiability also requires that the function be **smooth** at that point
  - 'Smooth' means that the graph of the function does not have any 'corners' or sudden changes of direction at the point
  - An obvious example of a function that is not smooth at certain points is a modulus function |*f*(*x*)| at any values of *x* where *f*(*x*) changes sign from positive to negative

    At any such point a modulus function will not be differentiable





Exam Tip

- On the exam you will not usually be asked to test a function for **continuity** at a point
  - You should however be familiar with the basic ideas about continuity outlined above
- On the exam you will not be asked to test a function for **differentiability** at a point
  - You should however be familiar with the basic ideas about differentiability and its relationship with continuity as outlined above

Worked Example

Consider the function f defined by

$$f(x) = \begin{cases} x^2 - 2x - 1, & x < 3 \\ 2 & x = 3 \\ \frac{x+2}{2}, & x > 3 \end{cases}$$

a)
use limits to show that f is not continuous at $x = 3$.

$f(3) = 2$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3} (x^2 - 2x - 1) = (3)^2 - 2(3) - 1 = 2$

↑ The limit 'from below'

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3} \left(\frac{x+2}{2}\right) = \frac{3+2}{2} = \frac{5}{2}$

↑ The limit 'from above'

$\lim_{x \to 3^+} f(x) \neq f(3)$, therefore f is not continuous at $x = 3$.

b)
Hence explain why f cannot be differentiable at $x = 3$.

In order to be differentiable at a point, a function must be continuous at that point.

f is not continuous at $x = 3$, therefore it cannot be differentiable at $x = 3$.

5.8 Advanced Fifferentiation

5.8.1 First Principles Differentiation

First Principles Differentiation

What is differentiation from first principles?

- Differentiation from **first principles** uses the **definition** of the **derivative** of a function ***f(x)***
- The definition is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- $\lim_{h \to 0}$ means the '*limit as **h** tends to zero*'
- When $h = 0$, $\frac{f(x+h) - f(x)}{h} = \frac{f(x) - f(x)}{0} = \frac{0}{0}$ which is **undefined**
  - Instead we consider what happens as ***h*** gets closer and closer to zero
- Differentiation **from first principles** means using that definition to show what the **derivative** of a function is
- The first principles definition (formula) is in the **formula booklet**

How do I differentiate from first principles?

STEP 1: Identify the function ***f(x)*** and substitute this into the first principles formula

e.g. Show, from first principles, that the derivative of $3x^2$ is $6x$

$f(x) = 3x^2$ so $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$

STEP 2: Expand ***f(x+h)*** in the numerator

$$f'(x) = \lim_{h \to 0} \frac{3(x^2 + 2hx + h^2) - 3x^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3x^2 + 6hx + 3h^2 - 3x^2}{h}$$

STEP 3: Simplify the numerator, factorise and cancel h with the denominator

$$f'(x) = \lim_{h \to 0} \frac{h(6x + 3h)}{h}$$

STEP 4: Evaluate the remaining expression as ***h*** *tends to zero*

$f'(x) = \lim_{h \to 0} (6x + 3h) = 6x$ As $h \to 0$, $(6x + 3h) \to (6x + 0) \to 6x$

∴ The derivative of $3x^2$ is $6x$

Exam Tip

- Most of the time you will not use first principles to find the derivative of a function (there are much quicker ways!)
  However, you can be asked to demonstrate differentiation from first principles
- To get full marks make sure you are are writing lim h -> 0 right up until the concluding sentence!

Worked Example

Prove, from first principles, that the derivative of $5x^3$ is $15x^2$.

STEP 1: For $f(x) = 5x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is given in the formula booklet

$$= \lim_{h \to 0} \frac{5(x+h)^3 - 5x^3}{h}$$

STEP 2:

$$= \lim_{h \to 0} \frac{5(x^3 + 3x^2h + 3xh^2 + h^3) - 5x^3}{h}$$

Expand $(x+h)^3$ using binomial theorem

$$= \lim_{h \to 0} \frac{5x^3 + 15x^2h + 15xh^2 + 5h^3 - 5x^3}{h}$$

STEP 3:

$$= \lim_{h \to 0} \frac{15x^2h + 15xh^2 + 5h^3}{h}$$

$$= \lim_{h \to 0} (15x^2 + 15xh + 5h^3)$$

STEP 4: As $h \to 0$

$$(15x^2 + 15xh + 5h^2) \to (15x^2 + 15x(0) + 5(0)^2) = 15x^2$$

∴ The derivative of $5x^3$ is $15x^2$

As with other problems in integration the results in this revision note may have further uses such as

- evaluating a definite integral
- finding the constant of integration
- finding areas under a curve, between a line and a curve or between two curves

5.9 Advanced Integration

5.9.1 Integrating Further Functions

Integrating with Reciprocal Trigonometric Functions

cosec *(cosecant, csc)*, **sec** *(secant)* and **cot** *(cotangent)* are the reciprocal functions of sine, cosine and tangent respectively.

What are the antiderivatives involving reciprocal trigonometric functions?

- $\int \sec^2 x \, dx = \tan x + c$
- $\int \sec x \tan x \, dx = \sec x + c$
- $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$
- $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$
- These are **not** given in the **formula booklet** directly
  - they are listed the other way round as 'standard derivatives'
  - be careful with the negatives in the last two results
  - and remember "+c" !

How do I integrate these if a *linear* function of x is involved?

- All integration rules could apply alongside the results above
- The use of reverse chain rule is particularly common
  - For linear functions the following results can be useful

$$\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + \mathrm{c}$$

$$\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + \mathrm{c}$$

$$\int \cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}^2(ax+b) + \mathrm{c}$$

  - These are **not** in the formula booklet
    - they can be deduced by spotting reverse chain rule
    - they are not essential to remember but can make problems easier

> **Exam Tip**
>
> - Even if you think you have remembered these antiderivatives, always use the formula booklet to double check
>   - those squares, negatives and "1 over"'s are easy to get muddled up!
> - Remember to use 'adjust' and 'compensate' for reverse chain rule when coefficients are involved

Worked Example

The graph of $y = f(x)$ where $f(x) = \int 2\sec^2 5x \, dx$ passes through the point $\left(\frac{\pi}{3}, 0\right)$.

Show that $5y = 2\left(\sqrt{3} + \tan 5x\right)$.

Reverse chain rule is needed

$$\int 2\sec^2 5x \, dx = 2 \times \frac{1}{5} \int 5 \sec^2 5x \, dx$$

'compensate' 'adjust'

$$\therefore y = \frac{2}{5}\tan 5x + c \qquad \text{"}\int \sec^2 x \, dx = \tan x + c\text{"}$$

At $x = \frac{\pi}{3}$, $y = 0$, $\quad 0 = \frac{2}{5}\tan\frac{5\pi}{3} + c$ (with $\tan\frac{5\pi}{3} = -\sqrt{3}$)

$$c = \frac{2\sqrt{3}}{5}$$

$$\therefore y = \frac{2}{5}\tan 5x + \frac{2}{5}\sqrt{3}$$

$$y = \frac{2}{5}\left(\tan 5x + \sqrt{3}\right)$$

$$\therefore 5y = 2\left(\sqrt{3} + \tan 5x\right)$$

Integrating with Inverse Trigonometric Functions

arcsin, **arccos** and **arctan** are (one-to-one) functions defined as the inverse functions of sine, cosine and tangent respectively.

What are the antiderivatives involving the inverse trigonometric functions?

- $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + c$
- $\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + c$
- Note that the antiderivative involving $\arccos x$ would arise from

$$\int -\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arccos x + c$$

  - However, the negative can be treated as a coefficient of -1 and so

$$\int -\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\arcsin x + c$$

  - Similarly,

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\int -\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\arccos x + c$$

- Unless a question requires otherwise, stick to the first two results
- These are listed in the **formula booklet** the other way round as 'standard derivatives'
- For the antiderivative involving $\arctan x$, note that $(1+x^2)$ is the same as (x^2+1)

How do I integrate these expressions if the denominator is not in the correct form?

- Some problems involve integrands that look very similar to the above but the denominators start with a number other than one
  - denominators of the form $a^2 \pm (bx)^2$ (with or without the square root!)
  - the integrand can be rewritten by taking a factor of a^2

    this means the denominator will start with 1

    e.g. $9+4x^2 = 9\left(1+\frac{4}{9}x^2\right) = 9\left(1+\left(\frac{2}{3}x\right)^2\right)$

    $\frac{2}{3}x$ is a linear function of x, so use 'adjust and compensate'
  - Another type of problem has a quadratic denominator

    denominators of the form ax^2+bx+c

    a rearrangement of this is more likely but it is still quadratic

    the integrand can be rewritten by completing the square

    e.g. $5-x^2+4x = 5-(x^2+4x) = 5-[(x+2)^2-4] = 9-(x+2)^2$

    This can then be dealt with like the first type of problem above

Exam Tip

- Always start integrals involving the inverse trig functions by rewrtiting the denominator into the correct form
  - With or without the square root, the denominator should be of the form $1 \pm [g(x)]^2$
  - The numerator can be dealt with afterwards using 'adjust' and 'compensate' if necessary

Worked Example

a) Find $\int \frac{1}{9+4x^2}\,dx$.

Rewrite the integral so the denominator is in the form $1+(bx)^2$

$$I = \int \frac{1}{9+4x^2}\,dx = \int \frac{1}{9\left(1+\frac{4}{9}x^2\right)}\,dx$$

$$= \frac{1}{9}\int \frac{1}{1+\left(\frac{2}{3}x\right)^2}\,dx$$

'Adjust and compensate' for $\frac{2}{3}x$

$$I = \frac{1}{9} \times \frac{3}{2}\int \frac{\frac{2}{3}}{1+\left(\frac{2}{3}x\right)^2}\,dx$$

('adjust' points to the $\frac{2}{3}$ in the numerator; 'compensate' points to the $\frac{3}{2}$; the 9 and 3 cancel to give $\frac{1}{3} \times \frac{1}{2}$)

Now reverse chain rule applies

$$\therefore I = \frac{1}{6}\arctan\left(\frac{2}{3}x\right) + c$$

b) Find $\int \frac{1}{\sqrt{5-x^2+4x}}\,dx$.

The denominator is quadratic so complete the square

$$I = \int \frac{1}{\sqrt{5 - x^2 + 4x}}\,dx = \int \frac{1}{\sqrt{5 - (x^2 - 4x)}}\,dx$$

$$= \int \frac{1}{\sqrt{5 - [(x-2)^2 - 4]}}\,dx$$

$$= \int \frac{1}{\sqrt{9 - (x-2)^2}}\,dx$$

Now rewrite into an integratable form

$$I = \int \frac{1}{\sqrt{9\left[1 - \frac{(x-2)^2}{9}\right]}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1 - \left(\frac{x-2}{3}\right)^2}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1 - \left(\frac{1}{3}x - \frac{2}{3}\right)^2}}\,dx$$

$\left(\frac{1}{3}x - \frac{2}{3}\right)$ is a linear function of x so 'adjust and compensate'

$$I = \frac{1}{3} \times 3 \int \frac{\frac{1}{3}}{\sqrt{1 - \left(\frac{1}{3}x - \frac{2}{3}\right)^2}}\,dx$$

'compensate' ($\times 3$) 'adjust' ($\frac{1}{3}$)

$$\therefore\ I = \arcsin\left(\frac{1}{3}x - \frac{2}{3}\right) + c$$

Integrating Exponential & Logarithmic Functions

Exponential functions have the general form $y = a^x$. Special case: $y = \mathrm{e}^x$.

Logarithmic functions have the general form $y = \log_a x$. Special case: $y = \log_\mathrm{e} x = \ln x$.

What are the antiderivatives of exponential and logarithmic functions?

- Those involving the special cases have been met before
  - $\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + c$
  - $\int \frac{1}{x} \, \mathrm{d}x = \ln |x| + c$
  - These are given in the **formula booklet**
- Also
  - $\int a^x \, \mathrm{d}x = \frac{1}{\ln a} a^x + c$
  - This is also given in the **formula booklet**
- By reverse chain rule
  - $\int \frac{1}{x \ln a} \, \mathrm{d}x = \log_a |x| + c$
  - This is **not** in the formula booklet
- There is also the reverse chain rule to look out for
  - this occurs when the numerator is (almost) the derivative of the denominator
  - $\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \ln | f(x) | + c$

How do I integrate exponentials and logarithms with a *linear* function of x involved?

- In the case of linear functions of the form $ax + b$
  - $\int \mathrm{e}^{ax+b} \, \mathrm{d}x = \frac{1}{a} \mathrm{e}^{ax+b} + \mathrm{c}$
  - $\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{1}{a} \ln | ax + b | + c$
  - These are **not** in the formula booklet but can be derived using 'adjust and compensate' from **reverse chain rule**

Exam Tip

- Remember to always use the modulus signs for logarithmic terms in the antiderivative
  - Once it is deduced that $g(x)$ in $\ln |g(x)|$ (say) is guaranteed to be positive, the modulus signs can be replaced with brackets

Worked Example

a) Find $\int 3^{2x+4}\,dx$.

Using reverse chain rule with 'adjust and compensate'

$$I = \int 3^{2x+4}\,dx = \frac{1}{2}\int 2 \times 3^{2x+4}\,dx$$

'adjust' (the 2), 'compensate' (the $\frac{1}{2}$)

$$= \frac{1}{2}\left(\frac{1}{\ln 3} \times 3^{2x+4}\right) + c$$

$$\therefore I = \frac{3^{2x+4}}{2\ln 3} + c$$

b) Given $\frac{dy}{dx} = \frac{5}{1-7x}$ show that $y = -\frac{5}{7}\ln|1-7x| + c$, where c is a constant.

$1-7x \equiv -7x+1$ so is of the form '$ax+b$'

Using reverse chain rule with 'adjust and compensate'

$$y = \int \frac{5}{1-7x}\,dx = 5 \times -\frac{1}{7}\int \frac{-7}{1-7x}\,dx$$

'adjust' (the -7), 'compensate' (the $-\frac{1}{7}$)

$$\therefore y = -\frac{5}{7}\ln|1-7x| + c$$

5.9.2 Further Techniques of Integration

Integration by Substitution

What is integration by substitution?

- Integration by substitution is used when an integrand where reverse chain rule is either not obvious or is not spotted
  - in the latter case it is like a "back-up" method for reverse chain rule

How do I use integration by substitution?

- For instances where the substitution is not obvious it will be given in a question
  - e.g. Find $\int \cot x \, \mathrm{d}x$ using the substitution $u = \sin x$
- Substitutions are usually of the form $u = g(x)$
  - in some cases $u^2 = g(x)$ and other variations are more convenient
    as these would not be obvious, they would be given in a question
  - if need be, this can be rearranged to find x in terms of u
- Integration by substitution then involves rewriting the integral, including "$\mathrm{d}x$" in terms of u

STEP 1
Name the integral to save rewriting it later
Identify the given substitution $u = g(x)$

STEP 2
Find $\frac{\mathrm{d}u}{\mathrm{d}x}$ and rearrange into the form $f(u)\,\mathrm{d}u = g(x)\,\mathrm{d}x$ such that (some of) the integral can be rewritten in terms of u

STEP 3
If limits are involved, use $u = g(x)$ to change them from x values to u values

STEP 4
Rewrite the integral so everything is in terms of u rather than x
This is the step when it may become apparent that x is needed in terms of u

STEP 5
Integrate with respect to u and either rewrite in terms of x or apply the limits using their u values

- For quotients the substitution usually involves the denominator
- It may be necessary to use 'adjust and compensate' to deal with any coefficients in the integrand
- Although $\frac{\mathrm{d}u}{\mathrm{d}x}$ can be treated like a fraction it should be appreciated that this is a 'shortcut' and the maths behind it is beyond the scope of the IB course

Exam Tip

- If a substitution is not given in a question, it is usually because it is obvious
  - If you can't see anything obvious, or you find that your choice of substitution doesn't reduce the integrand to something easy to integrate, consider that it may not be a substitution question

Worked Example

Use the substitution $u = (1 + 2x)$ to evaluate $\int_0^1 x(1+2x)^7 \, dx$.

STEP 1: Name the integral, identify the substitution

$I = \int_0^1 x(1+2x)^7 \, dx$

$u = 1 + 2x$

STEP 2: Find $\frac{du}{dx}$ and rearrange

$\frac{du}{dx} = 2$

$\frac{1}{2}du = dx$

STEP 3: Change limits from x values to u values

$x = 0, \quad u = 1 + 2(0) = 1$

$x = 1, \quad u = 1 + 2(1) = 3$

STEP 4: Rewrite the integral, find x in terms of u

$I = \int_1^3 \frac{1}{2}(u-1)u^7 \times \frac{1}{2}du = \frac{1}{4}\int_1^3 (u^8 - u^7) \, du$

x in terms of u
$u = 1 + 2x$
$\therefore x = \frac{1}{2}(u-1)$

STEP 5: Integrate and evaluate

$I = \frac{1}{4}\left[\frac{u^9}{9} - \frac{u^8}{8}\right]_1^3$

$I = \frac{1}{4}\left[\left(\frac{3^9}{9} - \frac{3^8}{8}\right) - \left(\frac{1^9}{9} - \frac{1^8}{8}\right)\right]$

$\therefore I = \frac{6151}{18}$

Integration by Parts

What is integration by parts?

- Integration by parts is generally used to integrate the product of two functions
  - however reverse chain rule and/or substitution should be considered first

    e.g. $\int 2x\cos(x^2)\ \mathrm{d}x$ can be solved using reverse chain rule or the substitution

    $u = x^2$
  - Integration by parts is essentially 'reverse product rule'

    whilst every product can be differentiated, not every product can be integrated (analytically)

What is the formula for integration by parts?

- $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x$
- This is given in the **formula booklet** alongside its alternative form $\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u$

How do I use integration by parts?

- For a given integral u and $\frac{\mathrm{d}v}{\mathrm{d}x}$ (rather than u and v) are assigned functions of x
- Generally, the function that becomes simpler when differentiated should be assigned to u
- There are various stages of integrating in this method
  - only one overall constant of integration ("+c") is required
  - put this in at the last stage of working
  - if it is a definite integral then "+c" is not required at all

STEP 1

Name the integral if it doesn't have one already!

This saves having to rewrite it several times – I is often used for this purpose.

e.g. $I = \int x\sin x\ \mathrm{d}x$

STEP 2

Assign u and $\frac{\mathrm{d}v}{\mathrm{d}x}$.

Differentiate u to find $\frac{\mathrm{d}u}{\mathrm{d}x}$ and integrate $\frac{\mathrm{d}v}{\mathrm{d}x}$ to find v

e.g. $u = x \qquad v = -\cos x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$

STEP 3

Apply the integration by parts formula

eg $I = -x\cos x - \int -\cos x \, \mathrm{d}x$

STEP 4

Work out the second integral, $\int v \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$

Now include a "+c" (unless definite integration)
e.g. $I = -x\cos x + \sin x + c$

STEP 5
Simplify the answer if possible or apply the limits for definite integration
e.g. $I = \sin x - x\cos x + c$

- In trickier problems other rules of differentiation and integration may be needed
  - chain, product or quotient rule
  - reverse chain rule, substitution

Can integration by parts be used when there is only a single function?

- Some single functions (non-products) are awkward to integrate directly
  - e.g. $y = \ln x$, $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$
- These can be integrated using parts however
  - rewrite as the product '$1 \times f(x)$' and choose $u = f(x)$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$
  - 1 is easy to integrate and the functions above have standard derivatives listed in the formula booklet

Exam Tip

- If $\ln x$ or one of the inverse trig functions are one of the functions involved in the product then these should be assigned to "u" when applying parts
  - They are (realtively) easy to differentiate (to find u') but are awkward to integrate

Worked Example

a) Find $\int 5xe^{3x}\,dx$.

STEP 1: Name the integral

$$I = \int 5xe^{3x}\,dx = 5\int xe^{3x}\,dx$$

STEP 2: Assign u and v'

Find u' and v

$u = x$ $\quad v = \frac{1}{3}e^{3x}$ (reverse chain rule)

$u' = 1$ $\quad v' = e^{3x}$

x becomes simpler when differentiated

STEP 3: Apply the integration by parts formula

$$I = 5\left[\frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,dx\right]$$

STEP 4: Work out the second integral

$$I = 5\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c\right]$$

← include "+c" at last working stage

reverse chain rule

STEP 5: Simplify

$$I = \frac{5}{9}e^{3x}(3x-1) + c$$

b) Show that $\int 8x\ln x\,dx = 2x^2(1 + \ln x^2) + c$.

STEP 1: Name the integral

$$I = \int 8x \ln x \, dx$$

STEP 2: Assign u and v' – as ln is involved, $u = \ln x$

Find u' and v

$u = \ln x$ $\quad v = 4x^2$

$u' = \frac{1}{x}$ $\quad v' = 8x$

STEP 3: Apply the integration by parts formula

$$I = 4x^2 \ln x - \int 4x^2 \times \frac{1}{x} \, dx = 4x^2 \ln x - \int 4x \, dx$$

STEP 4: Work out the second integral, include "+c" at this stage

$$I = 4x^2 \ln x - 2x^2 + c$$

STEP 5: Simplify

$$I = 2x^2(2\ln x - 1) + c$$

$$\therefore I = 2x^2(\ln x^2 - 1) + c$$

Repeated Integration by Parts

When will I have to repeat integration by parts?

- In some problems, applying integration by parts still leaves the second integral as a product of two functions of x
  - integration by parts will need to be applied again to the second integral
- This occurs when one of the functions takes more than one derivative to become simple enough to make the second integral straightforward
  - These functions usually have the form $x^2 g(x)$

How do I apply integration by parts more than once?

STEP 1
Name the integral if it doesn't have one already!

STEP 2
Assign u and $\frac{\mathrm{d}v}{\mathrm{d}x}$. Find $\frac{\mathrm{d}u}{\mathrm{d}x}$ and v

STEP 3
Apply the integration by parts formula

STEP 4
Repeat STEPS 2 and 3 for the second integral

STEP 5
Work out the second integral and include a "+c" if necessary

STEP 6
Simplify the answer or apply limits

What if neither function never becomes simpler when differentiating?

- It is possible that integration by parts will end up in a seemingly endless loop
  - consider the product $\mathrm{e}^x \sin x$
  - the derivative of e^x is e^x
    - no matter how many times a function involving e^x is differentiated, it will still involve e^x
  - the derivative of $\sin x$ is $\cos x$
    - $\cos x$ would then have derivative $-\sin x$, and so on
    - no matter how many times a function involving $\sin x$ or $\cos x$ is differentiated, it will still involve $\sin x$ or $\cos x$
- This loop can be trapped by spotting when the second integral becomes identical to (or a multiple of) the original integral
  - naming the original integral (I) at the start helps
  - I then appears twice in integration by parts
    - e.g. $I = g(x) - I$
    - where $g(x)$ are parts of the integral not requiring further work
  - It is then straightforward to rearrange and solve the problem

- e.g. $2I = g(x) + c$
  $I = \frac{1}{2}g(x) + c$

Worked Example

a) Find $\int x^2 \cos x \, dx$.

STEP 1: Name the integral

$I = \int x^2 \cos x \, dx$

STEP 2: Assign u and v'

Find u' and v

$u = x^2 \qquad v = \sin x$

$u' = 2x \qquad v' = \cos x$

x^2 becomes 'simpler' when differentiated

STEP 3: Apply the integration by parts formula

$I = x^2 \sin x - 2\int x \sin x \, dx$

STEP 4: Repeat STEPS 2 and 3 for the second integral

$u = x \qquad v = -\cos x$

$u' = 1 \qquad v' = \sin x$

$I = x^2 \sin x - 2\left[-x\cos x - \int -\cos x \, dx\right]$

STEP 5: Work out the second integral now it is straightforward

$I = x^2 \sin x + 2x\cos x - 2\sin x + c$

STEP 6: Simplify

$I = (x^2 - 2)\sin x + 2x\cos x + c$

b) Find $\int e^x \sin x \, dx$.

STEP 1: Name the integral

$I = \int e^x \sin x \, dx$

STEP 2: Assign u and v'. Neither function becomes simpler when differentiated. Find u' and v.

$u = e^x \qquad v = -\cos x$

$u' = e^x \qquad v' = \sin x$

STEP 3: Apply the integration by parts formula

$I = -e^x \cos x - \int -e^x \cos x \, dx = -e^x \cos x + \int e^x \cos x \, dx$

STEP 4: Repeat STEPS 2 and 3 for the second integral

$u = e^x \qquad v = \sin x$

$u' = e^x \qquad v' = \cos x$

$I = -e^x \cos x + \left[e^x \sin x - \int e^x \sin x \, dx \right]$

Spot that this is the same as the original question, i.e. I

STEP 5: "Work out" the second integral, include "+c" at this stage

$I = e^x \sin x - e^x \cos x - I + c$

STEP 6: Simplify

$2I = e^x (\sin x - \cos x) + c$

$\therefore I = \frac{1}{2} e^x (\sin x - \cos x) + c_1 \qquad \left(\text{where } c_1 = \frac{1}{2}c\right)$

5.9.3 Integrating with Partial Fractions

Integrating with Partial Fractions

What are partial fractions?

- **Partial fractions** arise when a quotient is rewritten as the **sum** of fractions
  - The process is the opposite of adding or subtracting fractions
- Each partial fraction has a denominator which is a **linear factor** of the quotient's denominator
  - e.g. A quotient with a denominator of $x^2 + 4x + 3$
    factorises to $(x+1)(x+3)$
    so the quotient will split into two partial fractions
    one with the (linear) denominator $(x+1)$
    one with the (linear) denominator $(x+3)$

How do I know when to use partial fractions in integration?

- For this course, the denominators of the quotient will be of quadratic form
  - i.e. $f(x) = ax^2 + bx + c$
  - check to see if the quotient can be written in the form $\frac{f'(x)}{f(x)}$
    in this case, reverse chain rule applies
- If the denominator does not factorise then the **inverse trigonometric functions** are involved

How do I integrate using partial fractions?

STEP 1
Write the quotient in the integrand as the sum of partial fractions
This involves factorising the denominator, writing it as an identity of two partial fractions and using values of x to find their numerators

e.g. $I = \int \frac{1}{x^2+4x+3}\,dx = \int \frac{1}{(x+1)(x+3)}\,dx = \frac{1}{2}\int \left(\frac{1}{x+1} - \frac{1}{x+3}\right) dx$

STEP 2
Integrate each partial fraction leading to an expression involving the sum of natural logarithms

e.g. $I = \frac{1}{2}\int \left(\frac{1}{x+1} - \frac{1}{x+3}\right) dx = \frac{1}{2}[\ln|x+1| - \ln|x+3|] + c$

STEP 3
Use the laws of logarithms to simplify the expression and/or apply the limits
(Simplifying first may make applying the limits easier)

e.g. $I = \frac{1}{2}\ln\left|\frac{x+1}{x+3}\right| + c$

- By rewriting the constant of integration as a logarithm ($c = \ln k$, say) it is possible to write the final answer as a single term e.g. $I = \frac{1}{2}\ln\left|\frac{x+1}{x+3}\right| + \ln k = \ln\sqrt{\left|\frac{x+1}{x+3}\right|} + \ln k = \ln\left(k\sqrt{\left|\frac{x+1}{x+3}\right|}\right)$

Exam Tip

- Always check to see if the numerator can be written as the derivative of the denominator
  - If so then it is reverse chain rule, not partial fractions
  - Use the number of marks a question is worth to help judge how much work should be involved

Worked Example

Find $\int \frac{3x+1}{x^2+3x-10}\,dx$.

The integrand is NOT of the form $\frac{f'(x)}{f(x)}$ but the denominator does factorise

STEP 1: Write the quotient as partial fractions

$$\frac{3x+1}{x^2+3x-10} \equiv \frac{A}{x+5} + \frac{B}{x-2}$$

$$3x+1 \equiv A(x-2) + B(x+5)$$

Let $x=2$, $7=7B$, $B=1$

Let $x=-5$, $-14=-7A$, $A=2$

$$\therefore I = \int \frac{3x+1}{x^2+3x-10}\,dx = \int \left(\frac{2}{x+5} + \frac{1}{x-2}\right) dx$$

STEP 2: Integrate the partial fractions

$$I = 2\ln|x+5| + \ln|x-2| + c$$

STEP 3: Simplify using laws of logarithms

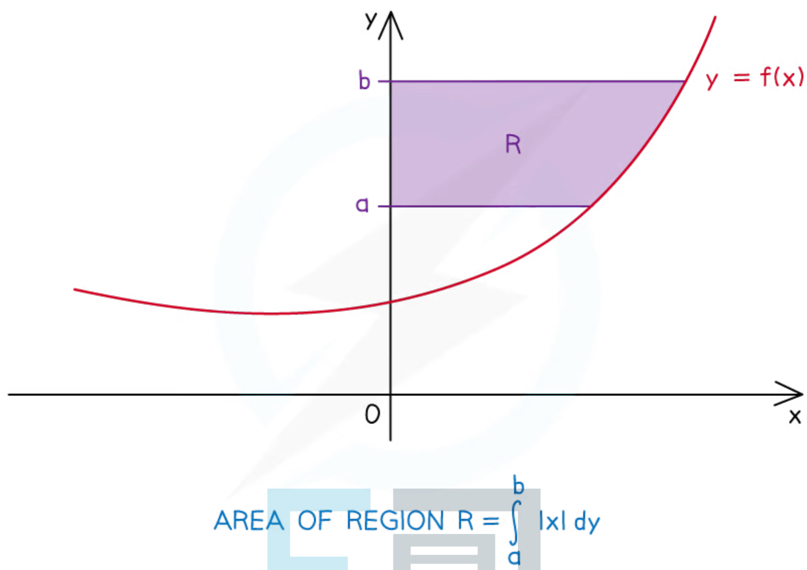
$$I = \ln(x+5)^2 + \ln|x-2| + c$$

$$\therefore I = \ln\left|(x+5)^2(x-2)\right| + c$$

5.9.4 Advanced Applications of Integration

Area Between Curve & y-axis

What is meant by the area between a curve and the y-axis?



- The area referred to is the region bounded by
  - the graph of $y = f(x)$
  - the y-axis
  - the horizontal line $y = a$
  - the horizontal line $y = b$
- The exact area can be found by evaluating a definite integral
- The graph of $y = f(x)$ could be a straight line
  - using basic shape area formulae may be easier than integration
    - e.g. area of a trapezium: $A = \frac{1}{2}h(a+b)$

How do I find the area between a curve and the y-axis?

- Use the formula

$$A = \int_a^b |x| \, dx$$

  - This is given in the **formula booklet**
  - The function is normally given in the form $y = f(x)$
    - so will need rearranging into the form $x = g(y)$
  - a and b may not be given directly as could involve the x-axis ($y = 0$) and/or a root of $x = g(y)$

use a GDC to plot the curve, sketch it and highlight the area to help

STEP 1
Identify the limits a and b
Sketch the graph of $y = f(x)$ or use a GDC to do so, especially if a and b are not given directly in the question

STEP 2
Rearrange $y = f(x)$ into the form $x = g(y)$
This is similar to finding the inverse function $f^{-1}(x)$

STEP 3
Evaluate the formula to evaluate the integral and find the area required
If using a GDC remember to include the modulus (| ... |) symbols around x

- In trickier problems some (or all) of the area may be 'negative'
  - this will be any area that is left of the y-axis (negative x-values)
  - $|x|$ makes such areas 'positive'
    - a GDC will apply '$|x|$' automatically as long as the | ... | are included
    - otherwise, to apply '$|x|$', split the integral into positive and negative parts; write an integral and evaluate each part separately and add the modulus of each part together to give the total area

Exam Tip

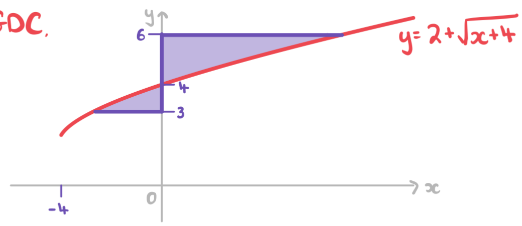
- Longer problems may require you to rotate an area around both the x-axis and the y-aixs
- Sketch and/or use your GDC to help visualise what the problem looks like

Worked Example

Find the area enclosed by the curve with equation $y = 2 + \sqrt{x+4}$, the y-axis and the horizontal lines with equations $y = 3$ and $y = 6$.

STEP 1: Identify limits, sketch graph/use GDC

From GDC,



STEP 2: Rearrange $y = f(x)$ into $x = g(y)$

$y = 2 + \sqrt{x+4}$

$x = (y-2)^2 - 4 = y^2 - 4y + 4 - 4$

$x = y^2 - 4y$

STEP 3: Evaluate integral to find area

As some area 'is' negative, split the integral

$$A = -\int_3^4 (y^2 - 4y)\, dy + \int_4^6 (y^2 - 4y)\, dy$$

↑ From GDC/sketch this area is 'negative'

If using GDC you can still do this in one go: $\int_3^6 |y^2 - 4y|\, dy$

$$\therefore A = \left[\frac{y^3}{3} - 2y^2\right]_4^6 - \left[\frac{y^3}{3} - 2y^2\right]_3^4$$

$$A = \left[(72 - 72) - \left(\frac{64}{3} - 32\right)\right] - \left[\left(\frac{64}{3} - 32\right) - (9 - 18)\right]$$

$$A = \frac{32}{3} - -\frac{5}{3}$$

$$\therefore A = \frac{37}{3} \text{ square units}$$

Volumes of Revolution Around x-axis

What is a volume of revolution around the x-axis?

- A **solid** of **revolution** is formed when an **area** bounded by a function $y = f(x)$ (and other boundary equations) is rotated 2π radians $(360°)$ around the x-axis
- The **volume** of **revolution** is the volume of this solid

- Be careful – the 'front' and 'back' of this solid are flat
  - they were created from straight (vertical) lines
  - 3D sketches can be misleading

How do I solve problems involving the volume of revolution around x-axis?

- Use the formula

$$V = \pi \int_a^b y^2 \, dx$$

- This is given in the **formula booklet**
- y is a function of x
- $x = a$ and $x = b$ are the equations of the (vertical) lines bounding the area
  - If $x = a$ and $x = b$ are not stated in a question, the boundaries could involve the y-axis ($x = 0$) and/or a root of $y = f(x)$
  - Use a GDC to plot the curve, sketch it and highlight the area to help
- Visualising the solid created is helpful
  - Try sketching some functions and their solids of revolution to help

STEP 1
Identify the limits a and b
Sketching the graph of $y = f(x)$ or using a GDC to do so is helpful, especially when a and b are not given directly in the question

STEP 2
Square y

STEP 3
Use the formula to evaluate the integral and find the volume of revolution
An answer may be required in exact form

Exam Tip

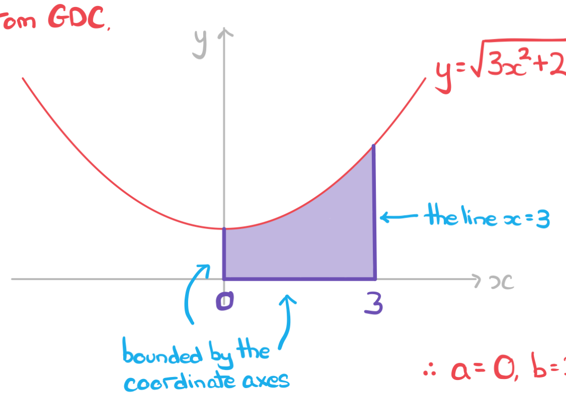
- If the given function involves a square root(s), problems can seem quite daunting
  - However, this is often deliberate, as the square root will be squared when applying the Volume of Revolution formula, and should leave the integrand as something more manageable
- Whether a diagram is given or not, using your GDC to plot the curve, limits, etc (where possible) can help you to visualise and make progress with problems

Worked Example

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of $y = \sqrt{3x^2 + 2}$, the coordinate axes and the line $x = 3$ by 2π radians around the x-axis. Give your answer as an exact multiple of π.

STEP 1: Identify limits, sketch graph/use GDC

From GDC,



$\therefore a = 0,\ b = 3$

STEP 2: Square y

$y^2 = \left(\sqrt{3x^2+2}\right)^2 = 3x^2 + 2$

STEP 3: Find the volume

$$V = \pi\int_0^3 (3x^2+2)\,dx = \pi\left[x^3 + 2x\right]_0^3$$
$$= \pi(27 + 6)$$

$\therefore V = 33\pi$ cubic units

Volumes of Revolution Around y-axis

What is a volume of revolution around the y-axis?

- Very similar to above, this is a **solid** of **revolution** which is formed when an **area** bounded by a function $y = f(x)$ (and other boundary equations) is rotated 2π radians $(360°)$ around the y-axis
- The **volume** of **revolution** is the volume of this solid

How do I solve problems involving the volume of revolution around y-axis?

- Use the formula

$$V = \pi \int_a^b x^2 \, dy$$

- This is given in the **formula booklet**
- The function is usually given in the form $y = f(x)$
  - so will need rearranging into the form $x = g(y)$
- a and b may not be given directly as could involve the x-axis ($y = 0$) and/or a root of $x = g(y)$
  - Use a GDC to plot the curve, sketch it and highlight the area to help
- Visualising the solid created is helpful

STEP 1
Identify the limits a and b
Sketching the graph of $y = f(x)$ or using a GDC to do so is helpful, especially if a and b are not given directly in the question

STEP 2
Rearrange $y = f(x)$ into the form $x = g(y)$
This is similar to finding the inverse function $f^{-1}(x)$

STEP 3
Square x

STEP 4
Use the formula to evaluate the integral and find the volume of revolution
An answer may be required in exact form

Exam Tip

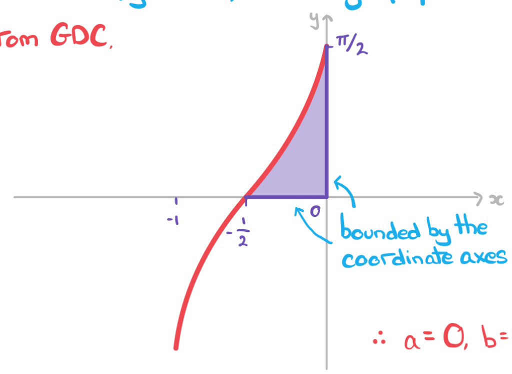
- If the given function involves a square root, problems can seem quite daunting
  - This is often deliberate, as the square root will be squared when applying the Volume of Revolution formula and the integrand will then become more manageable
- Whether a diagram is given or not, using your GDC to plot the curve, limits, etc (where possible) can help you to visualise the problem and make progress

Worked Example

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of $y = \arcsin(2x+1)$ and the coordinate axes by 2π radians around the y-axis. Give your answer to three significant figures.

STEP 1: Identify limits, sketch graph/use GDC

From GDC.



$\therefore\ a = 0,\ b = \pi/2$

STEP 2: Rearrange $y = f(x)$ into $x = g(y)$

$y = \arcsin(2x+1)$

$\sin y = 2x + 1$

$x = \frac{1}{2}(\sin y - 1)$

STEP 3: Square x

$x^2 = \frac{1}{4}(\sin y - 1)^2$

STEP 4: Find the volume

$$V = \pi \int_0^{\pi/2} \frac{1}{4}(\sin y - 1)^2 \, dy$$

As this is awkward, use your GDC but

- your GDC will expect the integrand in terms of x
- remember π !

$V = 0.279\,754\ldots$

$\therefore V = 0.280$ cubic units (3 s.f.)

The volume of the solid of revolution formed by rotating an area through 2π radians around the x-axis is $V = \pi \int_a^b y^2 \, dx$, and for the y-axis is $V = \pi \int_a^b x^2 \, dy$. These are both given in the **formula booklet**.

5.9.5 Modelling with Volumes of Revolution

Adding & Subtracting Volumes

When would volumes of revolution need to be added or subtracted?

- The 'curve' boundary of an area may consist of **more than one** function of x
  - For example
    the 'curve' boundary from $x = 0$ to $x = 3$ is $y = f(x)$
    the 'curve' boundary from $x = 3$ to $x = 6$ is $y = g(x)$
  - So the **total volume** would be $V = \pi \int_0^3 [f(x)]^2 \, dx + \pi \int_3^6 [g(x)]^2 \, dx$
- The solid of revolution may have a 'hole' in it
  - e.g. a 'toilet roll' shape would be the **difference** of two cylindrical volumes

How do I know whether to add or subtract volumes of revolution?

- When the **area** to be **rotated** around the x-axis has more than one function defining its boundary it can be trickier to tell whether to **add** or **subtract volumes** of **revolution**
  - It will depend on the **nature** of the **functions** and their **points** of **intersection**
  - With help from a GDC, sketch the graph of the functions and highlight the area required

How do I solve problems involving adding or subtracting volumes of revolution?

- Visualising the solid created becomes increasingly useful (but also trickier) for shapes generated by separate volumes of revolution
  - Continue trying to sketch the functions and their solids of revolution to help

STEP 1
Identify the functions $(y = f(x),\ y = g(x),\ \dots)$ involved in generating the volume
Determine whether the separate volumes will need to be added or subtracted
Identify the limits for each volume involved
Sketching the graphs of $y = f(x)$ and $y = g(x)$, or using a GDC to do so, is helpful, especially when the limits are not given directly in the question

STEP 2
Square y for all functions $([f(x)]^2,\ [g(x)]^2,\ \dots)$
This step is not essential if a GDC can be used to calculate integrals and an exact answer is not required.

STEP 3
Use the appropriate volume of revolution formula for each part, evaluate the definite integral and add or subtract as necessary
The answer may be required in exact form

Exam Tip

- A sketch of the graph, limits, etc is always helpful, whether one has been given in the question or not
  Use your GDC where possible

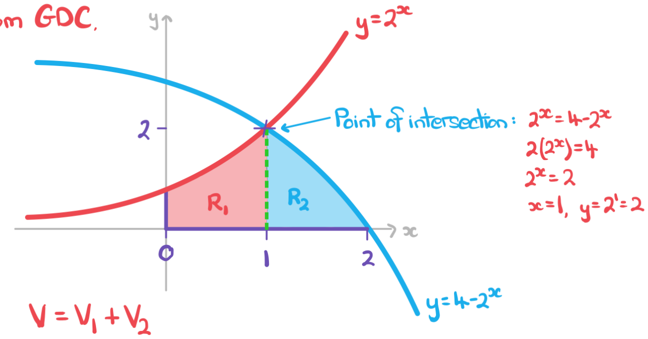
Worked Example

Find the volume of revolution of the solid formed by rotating the region enclosed by the positive coordinate axes and the graphs of $y = 2^x$ and $y = 4 - 2^x$ by 2π radians around the x-axis. Give your answer to three significant figures.

STEP 1: Identify functions, limits and whether to add or subtract

Use GDC to help sketch the graphs

From GDC,



$V = V_1 + V_2$

For R_1, $a = 0$, $b = 1$

For R_2, $a = 1$, $b = 2$

STEP 2: Square all functions - this step is not required in this question

STEP 3: Use formula for each part, evaluate and add

$$V = \pi \int_0^1 \left(2^x\right)^2 dx + \pi \int_1^2 \left(4 - 2^x\right)^2 dx$$

Use your GDC to evaluate - to avoid typing errors evaluate each integral separately, store in memory, then add

$V = 6.798\ 540... + 4.941\ 881... = 11.740...$

$\therefore V = 11.7$ cubic units (3 s.f.)

Modelling with Volumes of Revolution

What is meant by modelling volumes of revolution?

- Many everyday objects such as buckets, beakers, vases and lamp shades can be modelled as a **solid of revolution**
- The volume of revolution of the solid can then be calculated
- An object that would usually stand **upright** can be **modelled horizontally** so its **volume of revolution** can be found

What modelling assumptions are there with volumes of revolution?

- The solids formed are usually the main shape of the body of the object
  - For example, the handle on a bucket would not be included
- The thickness of the solid is negligible relative to the size of the object
  - thickness will depend on the purpose of the object and the material it is made from

How do I solve modelling problems with volumes of revolution?

- Visualising and sketching the solid formed can help with starting problems
- Familiarity with applying the volume of revolution fomulae
  - around the x-axis: $V = \int_a^b y^2 \, dx$
  - around the y-axis: $V = \int_a^b x^2 \, dy$
- The volume of revolution may involve adding or subtracting partial volumes
- Questions may ask related questions in context
  - g. A question about a bucket may ask about its **capacity**
    this would be measured in litres
    so a conversion of units may be required
    ($100 \text{ cm}^3 = 1$ litre)

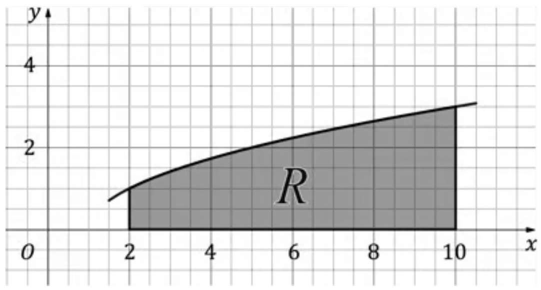
Exam Tip

- Remember to answer questions directly
  - In modelling scenarios, interpretation is often needed after finding the 'final answer'
- Modelling questions often ask about assumptions, criticisms and/or improvements
- Examples
  - it is assumed the thickness of the material an object is made from is negligible
  - a 'smooth' curve may not be a good model if the item is being made from a rough material
  - other things may significantly reduce the volume found and impact conclusions
    - e.g. Stones, plants and decorations placed in an aquarium will reduce the volume of water needed to fill it - and hence the number/size/type of fish it can accommodate may be impacted

Worked Example

The diagram below shows the region *R*, which is bounded by the function $y = \sqrt{x-1}$, the lines $x = 2$ and $x = 10$, and the x-axis.

Dimensions are in centimetres.



A mathematical model for a miniature vase is produced by rotating the region *R* through 2π radians around the x-axis.

Find the volume of the miniature vase, giving your answer in litres to three significant figures.

STEP 1 Identify limits

$a = 2$

$b = 10$

STEP 2 Square y

$y^2 = (\sqrt{x-1})^2 = x - 1$

STEP 3 Evaluate the integral

$$V = \pi \int_2^{10} (x-1)\, dx = \pi \left[0.5x^2 - x \right]_2^{10}$$

$$= \pi \left[(50 - 10) - (2 - 2) \right]$$

$$= 40\pi$$

Now we need to interpret this in the context of the miniature vase

$V = 40\pi \text{ cm}^3$

$V = \frac{40\pi}{1000}$ litres $1000 \text{cm}^3 = 1$ litre

$V = 0.125\,663\ldots$

Volume of the miniature vase is 0.126 litres (3 s.f.)

5.4 Further Integration

5.4.1 Integrating Special Functions

Integrating Trig Functions

How do I integrate sin, cos and sec^2?

- The **antiderivatives** for **sine** and **cosine** are

$$\int \sin x \; dx = -\cos x + c$$

$$\int \cos x \; dx = \sin x + c$$

where c is the **constant** of **integration**

- Also, from the **derivative** of $\tan x$

$$\int \sec^2 x \; dx = \tan x + c$$

- The **derivatives** of $\sin x$, $\cos x$ and $\tan x$ are in the **formula booklet**
  - so these **antiderivatives** can be easily deduced
- For the **linear** function $ax + b$, where a and b are constants,

$$\int \sin (ax + b) \; dx = -\frac{1}{a} \cos (ax + b) + c$$

$$\int \cos (ax + b) \; dx = \frac{1}{a} \sin (ax + b) + c$$

$$\int \sec^2 (ax + b) \; dx = \frac{1}{a} \tan (ax + b) + c$$

- For **calculus** with **trigonometric** functions **angles must** be **measured** in **radians**
  - Ensure you know how to change the angle mode on your GDC

Exam Tip

- The formula booklet can be used to find antiderivatives from the derivatives
  - Make sure you have the page with the section of standard derivatives open
  - Use these backwards to find any antiderivatives you need
  - Remember to add 'c', the constant of integration, for any indefinite integrals

Worked Example

a)

Find, in the form $\mathrm{F}(x)+c$, an expression for each integral

i. $\int \cos x \, \mathrm{d}x$

ii. $\int \sec^2\left(3x-\frac{\pi}{3}\right)\mathrm{d}x$

i. $\int \cos x \, dx = \sin x + c$

ii. $\int \sec^2\left(3x-\frac{\pi}{3}\right)dx = \frac{1}{3}\tan\left(3x-\frac{\pi}{3}\right)+c$

(Linear function $ax+b$)

b) A curve has equation $y=\int 2\sin\left(2x+\frac{\pi}{6}\right)\mathrm{d}x$.

The curve passes through the point with coordinates $\left(\frac{\pi}{3}, \sqrt{3}\right)$.

Find an expression for y.

$y = 2\int \sin\left(2x+\frac{\pi}{6}\right) dx$

$y = 2\left[-\frac{1}{2}\cos\left(2x+\frac{\pi}{6}\right)\right]+c$

At $x=\frac{\pi}{3}, y=\sqrt{3}$, $\sqrt{3} = -\cos\left(\frac{2\pi}{3}+\frac{\pi}{6}\right)+c$

$c = \cos\left(\frac{5\pi}{6}\right)+\sqrt{3}$

$c = \frac{\sqrt{3}}{2}$

$\therefore y = \frac{\sqrt{3}}{2} - \cos\left(2x+\frac{\pi}{6}\right)$

Integrating e^x & 1/x

How do I integrate exponentials and 1/x?

- The **antiderivatives** involving $\mathbf{e}^{x}$ and $\ln x$ are

$$\int e^{x}\,dx = e^{x} + c$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

where c is the **constant** of **integration**

  - These are given in the **formula booklet**
- For the **linear** function $(ax+b)$, where a and b are constants,

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

- It follows from the last result that

$$\int \frac{a}{ax+b}\,dx = \ln|ax+b| + c$$

  - which can be deduced using **Reverse Chain Rule**
- With **ln,** it can be useful to write the constant of integration, c, as a logarithm
  - using the laws of logarithms, the answer can be written as a single term
  - $\int \frac{1}{x}\,dx = \ln|x| + \ln k = \ln k|x|$ where k is a constant
  - This is similar to the special case of **differentiating** $\ln(ax+b)$ when $b=0$

Exam Tip

- Make sure you have a copy of the formula booklet during revision but don't try to remember everything in the formula booklet
  - However, do be familiar with the **layout** of the formula booklet
    - You'll be able to quickly locate whatever you are after
    - You do not want to be searching every line of every page!
  - For formulae you think you have remembered, use the booklet to double-check

Worked Example

A curve has the gradient function $f'(x) = \dfrac{3}{3x+2} + e^{4-x}$.

Given the exact value of $f(1)$ is $\ln 10 - e^3$ find an expression for $f(x)$.

$$f(x) = \int \left(\frac{3}{3x+2} + e^{4-x} \right) dx$$

$$f(x) = 3\int \frac{1}{3x+2}\, dx + \int e^{4-x}\, dx$$

$$= 3\left[\frac{1}{3} \ln |3x+2| \right] - e^{4-x} + c$$

$f(1) = \ln 10 - e^3$, $\quad \ln|3 \times 1 + 2| - e^{4-1} + c = \ln 10 - e^3$

$$\therefore c = \ln 10 - \ln 5$$

$$c = \ln\left(\frac{10}{5}\right) = \ln 2$$

$$\therefore f(x) = \ln|3x+2| - e^{4-x} + \ln 2$$

$$\left(= \ln 2|3x+2| - e^{4-x} \right)$$

5.5 Optimisation

5.5.1 Modelling with Differentiation

Modelling with Differentiation

What can be modelled with differentiation?

- Recall that **differentiation** is about the **rate** of **change** of a function and provides a way of finding **minimum** and **maximum** values of a function
- Anything that involves **maximising** or **minimising** a quantity can be modelled using differentiation; for example
  - minimising the cost of raw materials in manufacturing a product
  - the maximum height a football could reach when kicked
- These are called **optimisation** problems

What modelling assumptions are used in optimisation problems?

- The quantity being optimised needs to be dependent on a **single** variable
  - If other variables are initially involved, **constraints** or **assumptions** about them will need to be made; for example
    - minimising the cost of the **main** raw material – timber in manufacturing furniture say
    - the cost of screws, glue, varnish, etc can be fixed or considered **negligible**
- Other **modelling assumptions** may have to be made too; for example
  - ignoring air resistance and wind when modelling the path of a kicked football

How do I solve optimisation problems?

- In **optimisation** problems, letters other than ***x***, ***y*** and ***f*** are often used including capital letters
  - ***V*** is often used for volume, ***S*** for surface area
  - ***r*** for radius if a circle, cylinder or sphere is involved
- **Derivatives** can still be found but be clear about which letter is representing the independent (x) variable and which letter is representing the dependent (y) variable
  - A GDC may always use x and y but ensure you use the correct letters throughout your working and final answer
- Problems often start by **linking** two connected quantities together – for example **volume** and **surface area**
  - Where more than one variable is involved, **constraints** will be given such that the quantity of interest can be rewritten in terms of **one** variable
- Once the quantity of interest is written as a function of a single variable, **differentiation** can be used to **maximise** or **minimise** the quantity as required

STEP 1
Rewrite the quantity to be optimised in terms of a single variable, using any constraints given in the question

STEP 2

Differentiate and solve the derivative equal to zero to find the “x"-coordinate(s) of any stationary points

STEP 3
If there is more than one stationary point, or the requirement to justify the nature of the stationary point, differentiate again

STEP 4
Use the second derivative to determine the nature of each stationary point and select the maximum or minimum point as necessary

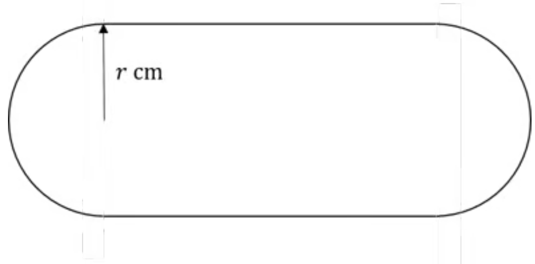
STEP 5
Interpret the answer in the context of the question

Exam Tip

- The first part of rewriting a quantity as a single variable is often a “show that” question – this means you may still be able to access later parts of the question even if you can’t do this bit
- Even when an algebraic solution is required you can still use your GDC to check answers and help you get an idea of what you are aiming for

Worked Example

A large allotment bed is being designed as a rectangle with a semicircle on each end, as shown in the diagram below.



The total area of the bed is to be $100\pi \text{ m}^2$.

a)
Show that the perimeter of the bed is given by the formula

$$P = \pi\left(r + \frac{100}{r}\right)$$

STEP 1: The width of the rectangle is $2r$ m and its length L m
The AREA of the bed, 100π m^2 is given by

$$\frac{1}{2}\pi r^2 + 2rL + \frac{1}{2}\pi r^2 = 100\pi$$

semi-circle (first term), rectangle (second term), semi-circle (third term), total area (constraint)

$$\therefore \pi r^2 + 2rL = 100\pi$$
$$2rL = 100\pi - \pi r^2$$
$$L = \frac{50\pi}{r} - \frac{\pi}{2}r$$

Write L in terms of r

The PERIMETER of the bed is

$$P = \pi r + \pi r + 2L$$

semi-circular arcs ($\pi r + \pi r$), two straight lengths ($2L$)

Use L from the area constraint to write P in terms of r only

$$P = 2\pi r + 2\left(\frac{50\pi}{r} - \frac{\pi}{2}r\right)$$

$$P = \pi r + \frac{100\pi}{r}$$

$$\therefore P = \pi\left(r + \frac{100}{r}\right)$$

b) Find $\frac{dP}{dr}$.

STEP 1: Rewrite P as powers of r

$$P = \pi(r + 100r^{-1})$$

STEP 2: $$\frac{dP}{dr} = \pi(1 - 100r^{-2})$$

$$\therefore \frac{dP}{dr} = \pi\left(1 - \frac{100}{r^2}\right)$$

c)

Find the value of r that minimises the perimeter.

STEP 2: $$\pi\left(1 - \frac{100}{r^2}\right) = 0$$

$$r^2 - 100 = 0$$

$r = 10$ (reject -10 as r is a length)

This is the only stationary point so we can assume it is minimal.

$\therefore r = 10$ m minimises the perimeter

d)

Hence find the minimum perimeter.

STEP 5: Interpret answer in context

Minimum perimeter is when $r = 10$

$$\therefore P = \pi\left(10 + \frac{100}{10}\right) = 20\pi$$

Minimum perimeter is 20π m

Use your GDC to check

e)
Justify that this is the minimum perimeter.

STEP 4: Use second derivative

$$\frac{d^2P}{dr^2} = \pi\left(200r^{-3}\right)$$

at $r = 10$, $\frac{d^2P}{dr^2} = \frac{\pi}{5} > 0$ ∴ minimum

∴ 20π is the minimum perimeter

5.3 Integration

5.3.2 Applications of Integration

Finding the Constant of Integration

What is the constant of integration?

- When finding an **anti-derivative** there is a constant term to consider
  - this constant term, usually called c, is the **constant** of **integration**
- In terms of **graphing** an **anti-derivative**, there are endless possibilities
  - collectively these may be referred to as the **family** of **antiderivatives** or **family** of **curves**
  - the constant of integration is determined by the **exact** location of the curve
    - if a **point** on the **curve** is **known,** the **constant** of **integration** can be found

How do I find the constant of integration?

- For $\mathrm{F}(x) + c = \int f(x)\, \mathrm{d}x$, the **constant** of **integration**, c – and so the particular **antiderivative** - can be found if a point the graph of $y = \mathrm{F}(x) + c$ passes through is known

STEP 1
If need be, rewrite $f(x)$ into an integrable form
Each term needs to be a power of x (or a constant)

STEP 2
Integrate each term of $f'(x)$, remembering the constant of integration, "$+c$"
(Increase power by 1 and divide by new power)

STEP 3
Substitute the x and y coordinates of a given point in to $\mathrm{F}(x) + c$ to form an equation in c
Solve the equation to find c

Exam Tip

- If a constant of integration can be found then the question will need to give you some extra information
  - If this is given then make sure you use it to find the value of c

Worked Example

The graph of passes through the point . The gradient function of is given by $f'(x)$.

Find $f(x)$.

STEP 1 $f'(x)$ is already in an integrable form

$f'(x) = 3x^2 - 4x - 4$

STEP 2 Integrate, remembering "+c"

$f(x) = \frac{3x^3}{3} - \frac{4x^2}{2} - 4x + c$

$f(x) = x^3 - 2x^2 - 4x + c$

STEP 3 Substitute x and y coordinates to find c

$f(3) = -4$

$\therefore (3)^3 - 2(3)^2 - 4(3) + c = -4$

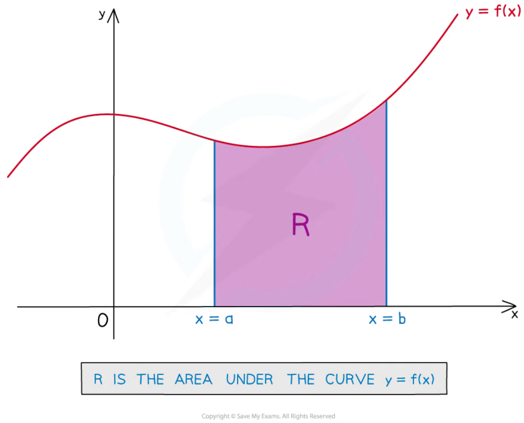
$27 - 18 - 12 + c = -4$

$c = -1$

$\therefore f(x) = x^3 - 2x^2 - 4x - 1$

Area Under a Curve Basics

What is meant by the area under a curve?



- The phrase "**area under a curve**" refers to the area bounded by
  - the graph of $y = f(x)$
  - the x-axis
  - the **vertical** line $x = a$
  - the **vertical** line $x = b$
- The **exact area under a curve** is found by evaluating a **definite integral**
- The graph of $y = f(x)$ could be a **straight line**
  - the use of **integration** described below would still apply

    but the shape created would be a **trapezoid**

    so it is easier to use "$A = \frac{1}{2}h(a+b)$"

What is a definite integral?

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- This is known as the **Fundamental Theorem of Calculus**
- **a** and **b** are called limits
  - **a** is the **lower** limit
  - **b** is the **upper** limit
- $f(x)$ is the **integrand**
- $F(x)$ is an **antiderivative** of $f(x)$
- The **constant** of **integration** ("$+c$") is not needed in **definite integration**
  - "$+c$" would appear alongside both **F(a)** and **F(b)**
  - subtracting means the "$+c$"'s cancel

How do I form a definite integral to find the area under a curve?

- The graph of $y = f(x)$ and the x-axis should be obvious boundaries for the area so the key here is in finding a and b - the **lower** and **upper** limits of the **integral**

STEP 1

Use the given sketch to help locate the limits
You may prefer to plot the graph on your GDC and find the limits from there

STEP 2
Look carefully where the 'left' and 'right' boundaries of the area lie
If the boundaries are vertical lines, the limits will come directly from their equations
Look out for the y-axis being one of the (vertical) boudnaries – in this case the limit (x) will be 0
One, or both, of the limits, could be a root of the equation $f(x)=0$
i.e. where the graph of $y=f(x)$ crosses the x-axis
In this case solve the equation $f(x)=0$ to find the limit(s)
A GDC will solve this equation, either from the graphing screen or the equation solver

STEP 3
The definite integral for finding the area can now be set up in the form

$$A=\int_a^b f(x)\,\mathrm{d}x$$

Exam Tip

- Look out for questions that ask you to find an **indefinite** integral in one part (so "**+c**" needed), then in a later part use the same integral as a **definite** integral (where "**+c**" is not needed)
- Add information to any diagram provided in the question, as well as axes intercepts and values of limits
  - Mark and shade the area you're trying to find, and if no diagram is provided, **sketch** one!

Definite Integrals using GDC

Does my calculator/GDC do definite integrals?

- Modern graphic calculators (and some 'advanced' scientific calculators) have the functionality to evaluate **definite integrals**
  - i.e. they can calculate the **area under a curve** (see above)
- If a calculator has a button for evalutaing definite integrals it will look something like

$$\int_{\square}^{\square} \square$$

- This may be a physical button or accessed via an on-screen menu
- Some GDCs may have the ability to find the area under a curve from the graphing screen
- Be careful with **any** calculator/GDC, they may not produce an **exact** answer

How do I use my GDC to find definite integrals?

Without graphing first ...

- Once you know the **definite integral** function your calculator will need three things in order to evaluate it
  - The function to be integrated (**integrand**) ($f(x)$)
  - The **lower** limit (a from $x = a$)
  - The **upper** limit (b from $x = b$)
- Have a play with the order in which your calculator expects these to be entered – some do not always work left to right as it appears on screen!

With graphing first ...

- Plot the graph of $y = f(x)$
  - You may also wish to plot the vertical lines $x = a$ and $x = b$
    make sure your GDC is expecting an "$x =$ " style equation
  - Once you have plotted the graph you need to look for an option regarding "area" or a physical button
    it may appear as the integral symbol (e.g. $\int \mathrm{d}x$)
    your GDC may allow you to select the lower and upper limits by moving a cursor along the curve - however this may not be very accurate
    your GDC may allow you to type the exact limits required from the keypad
      the lower limit would be typed in first
      read any information that appears on screen carefully to make sure

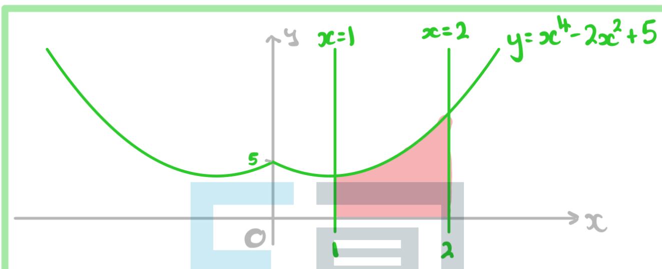
Exam Tip

- When revising for your exams always use your GDC to check any definite integrals you have carried out by hand
  - This will ensure you are confident using the calculator you plan to take into the exam and should also get you into the habit of using you GDC to check your work, something you should do if possible

Worked Example

a)
Using your GDC to help, or otherwise, sketch the graphs of
$y = x^4 - 2x^2 + 5$,
$x = 1$ and
$x = 2$ on the same diagram

Use the 'graph' menu on your GDC to plot $y = x^4 - 2x^2 + 5$.
You may then need to change the 'input type' to '$x =$'
to enter $x = 1$ and $x = 2$.
Plot the graph on your GDC and sketch the result, ensuring to include all the main properties of each graph.



b)
The area enclosed by the three graphs from part (a) and the x-axis is to be found.
Write down an integral that would find this area.

$$\int_1^2 (x^4 - 2x^2 + 5)\, dx$$

c)
Using your GDC, or otherwise, find the exact area described in part (b).

Give your answer in the form $\frac{a}{b}$ where a and b are integers.

$$\text{Area} = \int_1^2 (x^4 - 2x^2 + 5)\, dx = \frac{98}{15} \text{ square units}$$

From the graphing screen on our GDC the integral value was given as 6.53333333 - not exact!

5.4 Further Integration

5.4.2 Techniques of Integration

Integrating Composite Functions (ax+b)

What is a composite function?

- A **composite function** involves one function being applied after another
- A composite function may be described as a "function of a function"
- This Revision Note focuses on one of the functions being **linear** – i.e. of the form $\boldsymbol{ax+b}$

How do I integrate linear (*ax+b*) functions?

- A **linear function** (of x) is of the form $ax+b$
- The special cases for **trigonometric functions** and **exponential** and **logarithm functions** are
  - $\int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b)+c$
  - $\int \cos(ax+b)\,\mathrm{d}x = \frac{1}{a}\sin(ax+b)+c$
  - $\int \mathrm{e}^{ax+b}\,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax+b}+c$
  - $\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b|+c$
- There is one more special case
  - $\int (ax+b)^n\,\mathrm{d}x = \frac{1}{a(n+1)}(ax+b)^{n+1}+c$ where $n\in\mathbb{Q},\ n\neq -1$
- c, in all cases, is the **constant** of **integration**
- All the above can be deduced using **reverse chain rule**
  - However, spotting them can make solutions more efficient

Exam Tip

- Although the specific formulae in this revision note are NOT in the **formula booklet**
  - almost all of the information you will need to apply reverse chain rule is provided
  - make sure you have the formula booklet open at the right page(s) and practice using it

Worked Example

Find the following integrals

a) $\int 3(7-2x)^{\frac{5}{3}}\ dx$

$I = \int 3(7-2x)^{5/3}\ dx = 3\int (-2x+7)^{5/3}\ dx$

Using $\int (ax+b)^n\ dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$,

$I = 3\left[\frac{1}{-2 \times \frac{8}{3}}(-2x+7)^{8/3}\right] + c$ ($\frac{8}{3} = \frac{5}{3}+1$)

$\therefore I = -\frac{9}{16}(7-2x)^{8/3} + c$

b) $\int \frac{1}{2}\cos(3x-2)\ dx$

$I = \int \frac{1}{2}\cos(3x-2)\ dx = \frac{1}{2}\int \cos(3x-2)\ dx$

Using $\int \cos(ax+b)\ dx = \frac{1}{a}\sin(ax+b) + c$

$I = \frac{1}{2}\left[\frac{1}{3}\sin(3x-2)\right] + c$

$\therefore I = \frac{1}{6}\sin(3x-2) + c$

Reverse Chain Rule

What is reverse chain rule?

- The **Chain Rule** is a way of differentiating two (or more) functions
- **Reverse Chain Rule** (RCR) refers to **integrating** by **inspection**
    - spotting that chain rule would be used in the reverse (differentiating) process

How do I know when to use reverse chain rule?

- **Reverse chain rule** is used when we have the **product** of a **composite function** and the **derivative** of its **second function**
- Integration is trickier than differentiation; many of the shortcuts do not work
    - For example, in general $\int e^{f(x)}\,dx \neq \frac{1}{f'(x)}e^{f(x)}$
    - However, this result **is true** if $f(x)$ is linear $(ax+b)$
- Formally, in **function notation**, **reverse chain rule** is used for **integrands** of the form

$$I=\int g'(x)f\big(g(x)\big)\,dx$$

- this does not have to be strictly true, but '*algebraically*' it should be
  if **coefficients** do not match '**adjust** and **compensate'** can be used
  e.g. $5x^2$ is not quite the derivative of $4x^3$
  the *algebraic* part (x^2) is '*correct*'
  but the coefficient 5 is '*wrong*'
  use '**adjust** and **compensate'** to '*correct*' it
- A particularly useful instance of reverse chain rule to recognise is

$$I=\int\frac{f'(x)}{f(x)}\,dx=\ln|f(x)|+c$$

- i.e. the **numerator** is (almost) the **derivative** of the **denominator**
- '**adjust** and **compensate**' may need to be used to deal with any coefficients
  e.g.

$$I=\int\frac{x^2+1}{x^3+3x}\,dx=\frac{1}{3}\int 3\frac{x^2+1}{x^3+3x}\,dx=\frac{1}{3}\int\frac{3x^2+3}{x^3+3x}\,dx=\frac{1}{3}\ln|x^3+3x|+c$$

How do I integrate using reverse chain rule?

- If the product **can** be identified, the **integration** can be done "by **inspection**"
    - there may be some "**adjusting** and **compensating**" to do
- Notice a lot of the "**adjust** and **compensate** method" happens mentally
    - this is indicated in the steps below by quote marks

STEP 1

Spot the 'main' function

e.g. $I=\int x(5x^2-2)^6\,dx$

"the main function is $(\ldots)^6$ which would come from $(\ldots)^7$"

STEP 2

'**Adjust'** and '**compensate'** any coefficients required in the integral

e.g. " $(\ \dots\ )^7$ would differentiate to $7(\ \dots\ )^6$ "

"chain rule says multiply by the derivative of $5x^2 - 2$, which is $10x$"

"there is no '7' or '10' in the integrand so adjust and compensate"

$$I = \frac{1}{7} \times \frac{1}{10} \times \int 7 \times 10 \times x(5x^2 - 2)^6 \,\mathrm{d}x$$

STEP 3

Integrate and simplify

e.g. $I = \frac{1}{7} \times \frac{1}{10} \times (5x^2 - 2)^7 + c$

$$I = \frac{1}{70}(5x^2 - 2)^7 + c$$

- Differentiation can be used as a means of checking the final answer
- After some practice, you may find Step 2 is not needed
  - Do use it on more awkward questions (negatives and fractions!)
- If the product **cannot** easily be identified, use **substitution**

Exam Tip

- Before the exam, practice this until you are confident with the pattern and do not need to worry about the formula or steps anymore
  - This will save time in the exam
- You can always check your work by differentiating, if you have time

Worked Example

A curve has the gradient function $f'(x) = 5x^2\sin(2x^3)$.

Given that the curve passes through the point $(0,\ 1)$, find an expression for $f(x)$.

$$f(x) = \int 5x^2 \sin(2x^3)\,dx$$

$$f(x) = 5\int x^2 \sin(2x^3)\,dx$$ Take 5 out as a factor

This is a product, almost in the form $g'(x)f(g(x))$

STEP 1: Spot the 'main' function

"the main function is sin (...) which would come from cos (...)"

STEP 2: 'Adjust and compensate' coefficients

"cos (...) would differentiate to −sin (...)"

"$2x^3$ would differentiate to $6x^2$"

$$f(x) = 5 \times - \times \frac{1}{6} \times \int - \times 6 \times x^2 \sin(2x^3)\,dx$$

compensate (− and $\frac{1}{6}$), adjust (− and 6)

STEP 3: Integrate and simplify

$$f(x) = -\frac{5}{6}\cos(2x^3) + c$$

Substitution: Reverse Chain Rule

What is integration by substitution?

- When reverse chain rule is difficult to spot or awkward to use then **integration** by **substitution** can be used
  - **substitution** simplifies the integral by defining an alternative variable (usually u) in terms of the original variable (usually x)
  - **everything** (including "$\mathrm{d}x$" and **limits** for **definite integrals**) is then substituted which makes the integration much easier

How do I integrate using substitution?

STEP 1
Identify the substitution to be used – it will be the secondary function in the composite function

So $g(x)$ in $f(g(x))$ and $u = g(x)$

STEP 2
Differentiate the substitution and rearrange

$\frac{\mathrm{d}u}{\mathrm{d}x}$ can be treated like a fraction
(i.e. "multiply by $\mathrm{d}x$" to get rid of fractions)

STEP 3
Replace all parts of the integral
All x terms should be replaced with equivalent u terms, including $\mathrm{d}x$
If finding a **definite integral** change the limits from x-values to u-values too

STEP 4
Integrate and either
substitute x back in
or
evaluate the definte integral using the u limits (either using a GDC or manually)

STEP 5
Find c, the constant of integration, if needed

- For **definite integrals**, a GDC should be able to process the integral without the need for a substitution
  - be clear about whether working is required or not in a question

Exam Tip

- Use your GDC to check the value of a definite integral, even in cases where working needs to be shown

Worked Example

a)

Find the integral

$$\int \frac{6x+5}{(3x^2+5x-1)^3}\,dx$$

STEP 1: Identify the substitution

The composite function is $(3x^2+5x-1)^3$

The secondary function of this is $3x^2+5x-1$

$\therefore$ Let $u = 3x^2+5x-1$

STEP 2: Differentiate u and rearrange

$$\frac{du}{dx} = 6x+5$$

$$\therefore du = (6x+5)\,dx$$

STEP 3: Replace all parts of the integral

$$I = \int \frac{6x+5}{(3x^2+5x-1)^2}\,dx = \int \frac{du}{u^3}$$

$$= \int u^{-3}\,du$$

STEP 4: Integrate and substitute x back in

(STEP 5 not needed, evaluating c is not required)

$$I = \frac{u^{-2}}{-2} + c$$

$$I = -\frac{1}{2}(3x^2+5x-1)^{-2} + c$$

$$\therefore I = \frac{-1}{2(3x^2+5x-1)^2} + c$$

b)

Evaluate the integral

$$\int_1^2 \frac{6x+5}{(3x^2+5x-1)^3}\,dx$$

giving your answer as an exact fraction in its simplest terms.

Note that you could use your GDC for this part
Certainly use it to check your answer!

From STEP 3 above, $I = \int_{x=1}^{x=2} u^{-3}\, du$

Change limits too, $x=1,\ u = 3(1)^2 + 5(1) - 1 = 7$

$x=2,\ u = 3(2)^2 + 5(2) - 1 = 21$

STEP 4: Integrate and evaluate

$I = \left[-\frac{1}{2}u^{-2}\right]_7^{21} = \left[-\frac{1}{2}(21)^{-2}\right] - \left[-\frac{1}{2}(7)^{-2}\right]$

$\therefore I = \dfrac{4}{441}$

5.6 Kinematics

5.6.2 Calculus for Kinematics

Differentiation for Kinematics

How is differentiation used in kinematics?

- **Displacement**, **velocity** and **acceleration** are related by calculus
- In terms of differentiation and derivatives
  - **velocity** is the **rate** of **change** of **displacement**

    $v = \frac{ds}{dt}$ or $v(t) = s'(t)$
  - **acceleration** is the **rate** of **change** of **velocity**

    $a = \frac{dv}{dt}$ or $a(t) = v'(t)$
  - so **acceleration** is also the **second derivative** of **displacement**

    $a = \frac{d^2 s}{dt^2}$ or $a(t) = s''(t)$
- If a graph is not given you can use your GDC to draw one
  - you can then use your GDC's graphing features to find **gradients**

    velocity is the **gradient** on a **displacement** (-time) graph

    acceleration is the **gradient** on a **velocity** (-time) graph

Worked Example

The displacement, s m, of a particle at t seconds, is modelled by
$s(t) = 2t^3 - 27t^2 + 84t$

i. Find $v(t)$ and $a(t)$.
ii. Find the times at which the particle is at rest.

i. $v(t) = s'(t) = 6t^2 - 54t + 84 = 6(t^2 - 9t + 14)$
$a(t) = v'(t) = 12t - 54 = 6(2t - 9)$

$\therefore$ $v(t) = 6(t-7)(t-2)$
$a(t) = 6(2t-9)$

It's not essential to factorise the final answers

ii. The particle is at rest when $v(t) = 0$
$6(t-7)(t-2) = 0$
$t = 7, \ t = 2$

$\therefore$ The particle is at rest at 2 seconds and 7 seconds

Integration for Kinematics

How is integration used in kinematics?

- Since **velocity** is the **derivative** of **displacement** ($v = \frac{ds}{dt}$) it follows that

$$s = \int v \, dt$$

- Similarly, **velocity** will be an **antiderivative** of **acceleration**

$$v = \int a \, dt$$

How would I find the constant of integration in kinematics problems?

- A **boundary** or **initial** condition would need to be known
  - phrases involving the word **"initial"**, or **"initially"** are referring to **time** being **zero,** i.e. $t = 0$
  - you might also be given information about the object at some other time (this is called a **boundary condition**)
  - **substituting** the values in from the **initial or boundary condition** would allow the **constant** of **integration** to be found

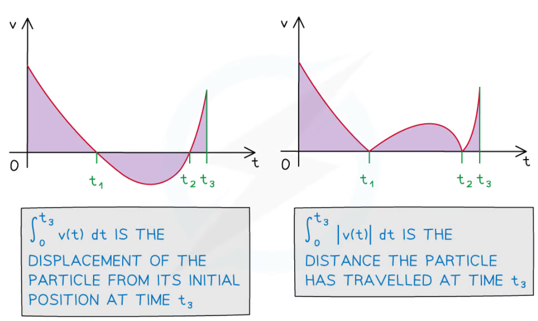
How are definite integrals used in kinematics?

- Definite integrals can be used to find the displacement of a particle between two points in time
  - $\int_{t_1}^{t_2} v(t) \, dt$ would give the **displacement** of the particle **between** the times $t = t_1$ and $t = t_2$

    This can be found using a velocity-time graph by **subtracting** the **total area below** the horizontal axis from the **total area above**
  - $\int_{t_1}^{t_2} |v(t)| \, dt$ gives the **distance** a particle has **travelled** between the times $t = t_1$ and $t = t_2$

    This can be found using a velocity velocity-time graph by **adding** the **total area below** the horizontal axis to the **total area above**

    Use a GDC to plot the modulus graph $y = |v(t)|$



Exam Tip

- Sketching the velocity-time graph can help you visualise the distances travelled using areas between the graph and the horizontal axis

Worked Example

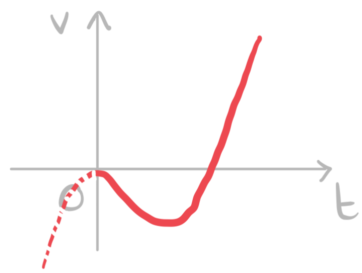
A particle moving in a straight horizontal line has velocity (v m s^{-1}) at time t seconds modelled by $v(t) = 8t^3 - 12t^2 - 2t$.

i. Given that the initial position of the particle is at the origin, find an expression for its displacement from the origin at time t seconds.

ii. Find the displacement of the particle from the origin in the first five seconds of its motion.

iii. Find the distance travelled by the particle in the first five seconds of its motion.

Use your GDC to sketch a velocity(-time) graph and use it to check to see if your answers are sensible.



i. "initial" – $t=0$, "origin" – $s=0$

$s(t) = \int v(t)\,dt = \int (8t^3 - 12t^2 - 2t)\,dt$

$s(t) = 2t^4 - 4t^3 - t^2 + c$ where c is a constant

at $t=0$, $s=0$, $\therefore c = 0$

$\therefore s(t) = 2t^4 - 4t^3 - t^2$

ii. "first five seconds" – $t_1 = 0$, $t_2 = 5$

Using a GDC this would be

$s = \int_0^5 (8t^3 - 12t^2 - 2t)\,dt$

$s = 725$ m

iii. Using a GDC this would be

$d = \int_0^5 |8t^3 - 12t^2 - 2t|\,dt$ d for distance

$d = 736.734\ 020\ldots$

$\therefore d = 737$ m (3 s.f.)

5.8 Advanced Differentiation

5.8.2 Applications of Chain Rule

Related Rates of Change

What is meant by rates of change?

- A rate of change is a measure of how a quantity is changing with respect to another quantity
- Mathematically rates of change are derivatives
  - $\frac{\mathrm{d}V}{\mathrm{d}r}$ could be the rate at which the volume of a sphere changes relative to how its radius is changing
- Context is important when interpreting positive and negative rates of change
  - A positive rate of change would indicate an increase
    - e.g. the change in volume of water as a bathtub fills
  - A negative rate of change would indicate a decrease
    - e.g. the change in volume of water in a leaking bucket

What is meant by related rates of change?

- Related rates of change are connected by a linking variable or parameter
  - this is often time, represented by t
  - seconds is the standard unit for time but this will depend on context
- e.g. Water running into a large hemi-spherical bowl
  - both the height and volume of water in the bowl are changing with time
    - time is the linking parameter between the rate of change of height and the rate of change of volume

How do I solve problems involving related rates of change?

- Use of chain rule and product rule are common in such problems
- Be clear about which variables are representing which quantities

STEP 1
Write down any variables and derivatives involved in the problem
e.g. x, y, t, $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}x}{\mathrm{d}t}$, $\frac{\mathrm{d}y}{\mathrm{d}t}$

STEP 2
Use an appropriate differentiation rule to set up an equation linking 'rates of change'
e.g. Chain rule: $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$

STEP 3
Substitute in known values
e.g. If, when $t = 3$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 8$, then $8 = \frac{\mathrm{d}y}{\mathrm{d}x} \times 2$

STEP 4

Solve the problem and interpret the answer in context if required

e.g. $\frac{dy}{dx} = \frac{8}{2} = 4$ 'when $t = 3$, y changes at a rate of 4, with respect to x'

Exam Tip

- If you struggle to determine which rate to use then you can look at the units to help
  - e.g. A rate of 5 **cm^3** *per* **second** implies **volume** per **time** so the rate would be $\frac{dV}{dt}$

Worked Example

In a manufacturing process a metal component is heated such that it's cross-sectional area expands but always retains the shape of a right-angled triangle. At time t seconds the triangle has base b cm and height h cm.

At the time when the component's cross-sectional area is changing at 4 cm s^{-1}, the base of the triangle is 3 cm and its height is 6 cm. Also at this time, the rate of change of the height is twice the rate of change of the base.

Find the rate of change of the base at this point of time.

STEP 1: List variables and derivatives

$A, b, h, t, \frac{dA}{dt}, \frac{db}{dt}, \frac{dh}{dt}$

$A = \frac{1}{2}bh$

STEP 2: Use a differentiation rule to link 'rates of change'

$A = \frac{1}{2}bh$ is a product – so use product rule

$$\frac{dA}{dt} = \frac{1}{2}\left[b\frac{dh}{dt} + h\frac{db}{dt}\right]$$

STEP 3: Substitute known values

$$4 = \frac{1}{2}\left[3\left(2\frac{db}{dt}\right) + 6\frac{db}{dt}\right]$$

$\frac{dh}{dt} = 2\frac{db}{dt}$ in question

STEP 4: Solve and interpret

$$8 = 12\frac{db}{dt}$$

$\therefore \frac{db}{dt} = \frac{2}{3}$ cm s^{-1}

The rate of change of the base is $\frac{2}{3}$ cm per second.

Differentiating Inverse Functions

What is meant by an inverse function?

- Some functions are easier to process with x (rather than y) as the subject
    - i.e. in the form $x = f(y)$
- This is particularly true when dealing with inverse functions
    - e.g. If $y = f(x)$ the inverse would be written as $y = f^{-1}(x)$
      finding $f^{-1}(x)$ can be awkward
      so write $x = f(y)$ instead

How do I differentiate inverse functions?

- Since $x = f(y)$ it is easier to differentiate "x with respect to y" rather than "y with respect to x"
    - i.e. find $\frac{dx}{dy}$ rather than $\frac{dy}{dx}$
    - Note that $\frac{dx}{dy}$ will be in terms of y but can be substituted

STEP 1
For the function $y = f(x)$, the inverse will be $y = f^{-1}(x)$

Rewrite this as $x = f(y)$

STEP 2

From $x = f(y)$ find $\frac{dx}{dy}$

STEP 3

Find $\frac{dy}{dx}$ using $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ - this will usually be in terms of y

- If an algebraic solution in terms of x is required substitute $f(x)$ for y in $\frac{dy}{dx}$
- If a numerical derivative (e.g. a gradient) is required then use the y-coordinate
    - If the y-coordinate is not given, you should be able to work it out from the orginal function and x-coordinate

Exam Tip

- With x's and y's everywhere this can soon get confusing!
  - Be clear of the key information and steps - and set your wokring out accordingly
    - The orginal function, $y = f(x)$
    - Its inverse, $y = f^{-1}(x)$
    - Rewriting the inverse, $x = f(y)$
    - Finding $\frac{dx}{dy}$ first, then finding its reciprocal for $\frac{dy}{dx}$
- Your GDC can help when numerical derivatives (gradients) are required

Worked Example

a)

Find the gradient of the curve at the point where $y = 3$ on the graph of $y = f^{-1}(x)$ where $f(x) = \sqrt{(5x+1)^3}$.

STEP 1: Rewrite inverse as $x = f(y)$

$f(x) = \sqrt{(5x+1)^3}$

$\therefore$ For $y = f^{-1}(x)$, $x = f(y)$

$x = \sqrt{(5y+1)^3}$

STEP 2: Find $\frac{dx}{dy}$

$x = (5y+1)^{3/2}$ — Write as powers

$\frac{dx}{dy} = \frac{3}{2}(5y+1)^{1/2} \times 5$ — Using chain rule

$\frac{dx}{dy} = \frac{15}{2}\sqrt{5y+1}$

STEP 3: Find $\frac{dy}{dx}$

A gradient is required - substitute $y = 3$

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{15}{2}\sqrt{5y+1}} = \frac{2}{15\sqrt{5y+1}}$

At $y = 3$, $\frac{dy}{dx} = \frac{2}{15\sqrt{5(3)+1}}$

$\therefore$ Gradient, at $y = 3$, on the graph of $y = f^{-1}(x)$ is $\frac{1}{30}$.

b) Given that $y = e^x$ show that the derivative of $y = \ln x$ is $\frac{1}{x}$.

The key to this question is realising that e^x and $\ln x$ are inverses

$y = e^x$ so $y = \ln x$ will be its inverse

STEP 1: $y = e^x$ "$y = f(x)$"

$\therefore$ The inverse will be $x = e^y$ "$y = f^{-1}(x)$, $x = f(y)$"

STEP 2: $\frac{dx}{dy} = e^y$

STEP 3: $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$ $e^{\ln x} = x$ since e^x and $\ln x$ are inverses

$\therefore$ If $y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$

5.4 Further Integration

5.4.3 Definite Integrals

Definite Integrals

What is a definite integral?

$$\int_a^b f(x)\ dx = [F(x)]_a^b = F(b) - F(a)$$

- This is known as the **Fundamental Theorem of Calculus**
- ***a*** and ***b*** are called limits
  - ***a*** is the lower limit
  - ***b*** is the upper limit
- $f(x)$ is the **integrand**
- $F(x)$ is an **antiderivative** of $f(x)$
- The **constant** of **integration** (“**+c**”) is not needed in **definite integration**
  - “+c” would appear alongside both **F(*a*)** and **F(*b*)**
  - subtracting means the “+c”’s cancel

How do I find definite integrals analytically (manually)?

STEP 1

Give the integral a name to save having to rewrite the whole integral every time

If need be, rewrite the integral into an integrable form

$$I = \int_a^b f(x)\ dx$$

STEP 2

Integrate without applying the limits; you will not need “+c”

Notation: use square brackets [ ] with limits placed at the end bracket

STEP 3

Substitute the limits into the function and evaluate

Exam Tip

- If a question does not state that you can use your GDC then you must show all of your working clearly, however it is always good practice to check you answer by using your GDC if you have it in the exam

Worked Example

a)

Show that

$$\int_2^4 3x(x^2-2)\,dx = 144$$

STEP 1: Name the integral and rewrite into an integratable form

$$I = \int_2^4 (3x^3 - 6x)\,dx$$

STEP 2: Integrate

$$I = \left[\frac{3}{4}x^4 - 3x^2\right]_2^4$$

STEP 3: Evaluate

$$I = \left[\frac{3}{4}(4)^4 - 3(4)^2\right] - \left[\frac{3}{4}(2)^4 - 3(2)^2\right]$$

$$I = 144 - 0$$

$$\therefore \int_2^4 3x(x^2-2) = 144$$

b)

Use your GDC to evaluate

$$\int_0^1 3e^{x^2 \sin x}\,dx$$

giving your answer to three significant figures.

Using GDC,

$$\int_0^1 3e^{x^2 \sin x}\,dx = 3.872957\ldots$$

$$\therefore \int_0^1 3e^{x^2 \sin x}\,dx = 3.87 \quad (3 \text{ s.f.})$$

Properties of Definite Integrals

Fundamental Theorem of Calculus

$$\int_a^b f(x)\ dx = [F(x)]_a^b = F(b) - F(a)$$

- Formally,
  - $f(x)$ is **continuous** in the interval $a \le x \le b$
  - $F(x)$ is an **antiderivative** of $f(x)$

What are the properties of definite integrals?

- Some of these have been encountered already and some may seem obvious ...
  - taking **constant** factors outside the integral

    $\int_a^b k f(x)\ dx = k\int_a^b f(x)\ dx$ where k is a constant

    useful when fractional and/or negative values involved
  - integrating term by term

    $\int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$

    the above works for subtraction of terms/functions too
  - equal upper and lower limits

    $\int_a^a f(x)\ dx = 0$

    on evaluating, this would be a value, subtract itself !
  - swapping limits gives the same, but negative, result

    $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

    compare 8 subtract 5 say, with 5 subtract 8 ...
  - splitting the interval

    $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ where $a \le c \le b$

    this is particularly useful for areas under multiple curves or areas under the x-axis
  - horizontal translations

    $\int_a^b f(x)\ dx = \int_{a-k}^{b-k} f(x+k)\ dx$ where k is a constant

    the graph of $y = f(x \pm k)$ is a horizontal translation of the graph of $y = f(x)$
    ($f(x+k)$ translates left, $f(x-k)$ translates right)

Exam Tip

- Learning the properties of definite integrals can help to save time in the exam

Worked Example

$f(x)$ is a continuous function in the interval $5 \leq x \leq 15$.

It is known that $\int_5^{10} f(x)\,dx = 12$ and that $\int_{10}^{15} f(x)\,dx = 5$.

a)
Write down the values of

i)

$$\int_7^7 f(x)\,dx$$

ii)

$$\int_{10}^{5} f(x)\,dx$$

i. $\int_7^7 f(x)\,dx = 0$

"equal limits"
$\int_a^a f(x)\,dx = 0$

ii. $\int_{10}^5 f(x)\,dx = -12$

"swapped limits"
$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

b)
Find the values of

i)

$$\int_5^{15} f(x)\,dx$$

ii)

$$\int_5^{10} 6f(x+5)\,dx$$

i. $I = \int_5^{15} f(x)\,dx = \int_5^{10} f(x)\,dx + \int_{10}^{15} f(x)\,dx = 12 + 5 = 17$

"split limits"

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^a f(x)\,dx$

$\therefore \int_5^{15} f(x)\,dx = 17$

ii. $I = \int_5^{10} 6f(x+5) = 6\int_5^{10} f(x+5)\,dx$

"factors"

$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

$I = 6\int_{5+5}^{10+5} f(x)\,dx$

"horizontal translation"

$\int_a^b f(x+k)\,dx = \int_{a+k}^{b+k} f(x)\,dx$

$I = 6\int_{10}^{15} f(x)\,dx = 6 \times 5 = 30$

$\therefore \int_5^{10} 6f(x+5)\,dx = 30$

5.8 Advanced Differentiation

5.8.3 Implicit Differentiation

Implicit Differentiation

What is implicit differentiation?

- An equation connecting x and y is not always easy to write **explicitly** in the form $y = f(x)$ or $x = f(y)$
  - In such cases the equation is written **implicitly**
    - as a function of x and y
    - in the form $f(x, y) = 0$
- Such equations can be **differentiated implicitly** using the **chain rule**

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(y)] = f'(y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

- A shortcut way of thinking about this is that 'y is a function of a x'
  - when differentiating a function of y chain rule says "differentiate with respect to y, then multiply by the derivative of y" (which is $\frac{\mathrm{d}y}{\mathrm{d}x}$)

Applications of Implicit Differentiation

What type of problems could involve implicit differentiation?

- Broadly speaking there are three types of problem that could involve implicit differentiation
    - algebraic problems involving graphs, derivatives, tangents, normals, etc
        - where it is not practical to write y explicitly in terms of x
        - usually in such cases, $\frac{dy}{dx}$ will be in terms of x and y
    - **optimisation** problems that involve **time derivatives**
        - more than one variable may be involved too
        - e.g. Volume of a cylinder, $V = \pi r^2 h$
        - e.g. The side length and (so) area of a square increase over time
    - any problem that involves differentiating with respect to an extraneous variable
        - e.g. $y = f(x)$ but the derivative $\frac{dy}{d\theta}$ is required (rather than $\frac{dy}{dx}$)

How do I apply implicit differentiation to algebraic problems?

- Algebraic problems revolve around values of the derivative (gradient) $\left(\frac{dy}{dx}\right)$
    - if not required to find this value it will either be given or implied
- Particular problems focus on special case tangent values
    - horizontal tangents
        - also referred to as tangents parallel to the x-axis
        - this is when $\frac{dy}{dx} = 0$
    - vertical tangents
        - also referred to as tangents parallel to the y-axis
        - this is when $\frac{dx}{dy} = 0$
        - In such cases it may appear that $\frac{1}{\frac{dy}{dx}} = 0$ but this has no solutions; this occurs when for nearby values of x, $\frac{dy}{dx} \to \pm\infty$
        - (i.e. very steep gradients, near vertical)
- Other problems may involve finding equations of (other) tangents and/or normals
- For problems that involve finding the coordinates of points on a curve with a specified gradient the method below can be used

STEP 1
Differentiate the equation of the curve implicitly

STEP 2

Substitute the given or implied value of $\frac{dy}{dx}$ to create an equation linking x and y

STEP 3

There are now two equations

- the original equation
- the linking equation

Solve them simultaneously to find the x and y coordinates as required

Exam Tip

- After some rearranging, $\frac{dy}{dx}$ will be in terms of both x and y
  - There is usually no need (unless asked to by the question) to write $\frac{dy}{dx}$ in terms of x (or y) only
- If evalutaing derivatives, you'll need both x and y coordinates, so one may have to be found from the other using the original function

Worked Example

The curve C has equation $x^2 + 2y^2 = 16$.

a)
Find the exact coordinates of the points where the normal to curve C has gradient 2.

STEP 1: Differentiate implicitly

$$2x + 4y\frac{dy}{dx} = 0$$

STEP 2: Substitute value of $\frac{dy}{dx}$ in

Gradient of normal is 2

$\therefore$ gradient of tangent $\left(\frac{dy}{dx}\right)$ is $-\frac{1}{2}$

(Normal and tangent are perpendicular, so product of their gradients is -1)

$\therefore 2x + 4y\left(-\frac{1}{2}\right) = 0$

$2x - 2y = 0$

$x = y$ — Equation linking x and y

STEP 3: Solve simultaneously, obtain coordinates

$x^2 + 2y^2 = 16$

$y = x$

$\therefore x^2 + 2x^2 = 16$

$3x^2 = 16$ $\quad x = \pm\frac{4}{\sqrt{3}} = \pm\frac{4}{3}\sqrt{3}$ $\quad \therefore y = \pm\frac{4}{3}\sqrt{3}$

$\therefore$ Coordinates are $\left(\frac{4}{3}\sqrt{3}, \frac{4}{3}\sqrt{3}\right)$ and $\left(-\frac{4}{3}\sqrt{3}, -\frac{4}{3}\sqrt{3}\right)$

b)
Find the equations of the tangents to the curve that are

(i) parallel to the x-axis

(ii) parallel to the y-axis.

These are special cases,

in part (i), $\frac{dy}{dx} = 0$ (parallel to x-axis)

in part (ii), $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = 0$ (parallel to y-axis)

$x^2 + 2y^2 = 16$

$2x + 4y\frac{dy}{dx} = 0$

(i) $\frac{dy}{dx} = 0$ $\therefore$ $2x = 0$

$x = 0$ $\therefore$ $2y^2 = 16$

$y = \pm\sqrt{8} = \pm 2\sqrt{2}$

(ii) $\frac{dy}{dx} = \frac{-2x}{4y} = \frac{-x}{2y}$

$\therefore \frac{dx}{dy} = \frac{2y}{-x}$

$\frac{dx}{dy} = 0$ $\therefore$ $2y = 0$

$y = 0$ $\therefore$ $x^2 = 16$

$x = \pm 4$

$\therefore$ Tangents (i) parallel to x-axis are $y = 2\sqrt{2}$ and $y = -2\sqrt{2}$ and (ii) parallel to y-axis are $x = 4$ and $x = -4$

How do I apply implicit differentiation to optimisation problems?

- For a single variable use chain rule to differentiate implicitly
  - e.g. A square with side length changing over time, $A = x^2$

$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

- For more than one variable use product rule (and chain rule) to differentiate implicitly
  - e.g. A square-based pyramid with base length and height changing over time,

$$V = \frac{1}{3}x^2 h$$

$$\frac{dV}{dt} = \frac{1}{3}\left[x^2\frac{dh}{dt} + 2x\frac{dx}{dt}h\right] = \frac{1}{3}x\left(x\frac{dh}{dt} + 2h\frac{dx}{dt}\right)$$

- After differentiating implicitly the rest of the question should be similar to any other optimisation problem
  - be aware of phrasing

    "the rate of change of the height of the pyramid" (over time) is $\frac{\mathrm{d}h}{\mathrm{d}t}$
  - when finding the location of minimum and maximum problems

    there is not necessarily a turning point

    the minimum or maximum could be at the start or end of a given or appropriate interval

Exam Tip

- If you are struggling to tell which derivative is needed for a question, writing all possibilities down may help you
  - You don't need to work them out at this stage but if you conisder them it may nudge you to the next stage of the solution
  - e.g. For $V = \pi r^2 h$, possible derivatives are $\frac{dV}{dr}$, $\frac{dV}{dh}$ and $\frac{dV}{dt}$

Worked Example

The radius, r cm, and height, h cm, of a cylinder are increasing with time. The volume, V cm^3, of the cylinder at time t seconds is given by $V = \pi r^2 h$.

a) Find an expression for $\frac{\mathrm{d}V}{\mathrm{d}t}$.

Using implicit differentiation with product rule

$$\frac{dV}{dt} = \pi\left[2r\frac{dr}{dt}h + \frac{dh}{dt}r^2\right]$$

$$\therefore \frac{dV}{dt} = \pi r\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right)$$

b)

At time T seconds, the radius of the cylinder is 4 cm, expanding at a rate of 2 cm s^{-1}. At the same time, the height of the cylinder is 10 cm, expanding at a rate of 3 cm s^{-1}.

Find the rate at which the volume is expanding at time T seconds.

At time T, $r = 4$, $\frac{dr}{dt} = 2$

$h = 10$, $\frac{dh}{dt} = 3$

$$\therefore \frac{dV}{dt} = \pi(4)(2 \times 10 \times 2 + 4 \times 3)$$

∴ At time T seconds, the volume is expanding at a rate of 208π cm^3 s^{-1}

5.4 Further Integration

5.4.4 Further Applications of Integration

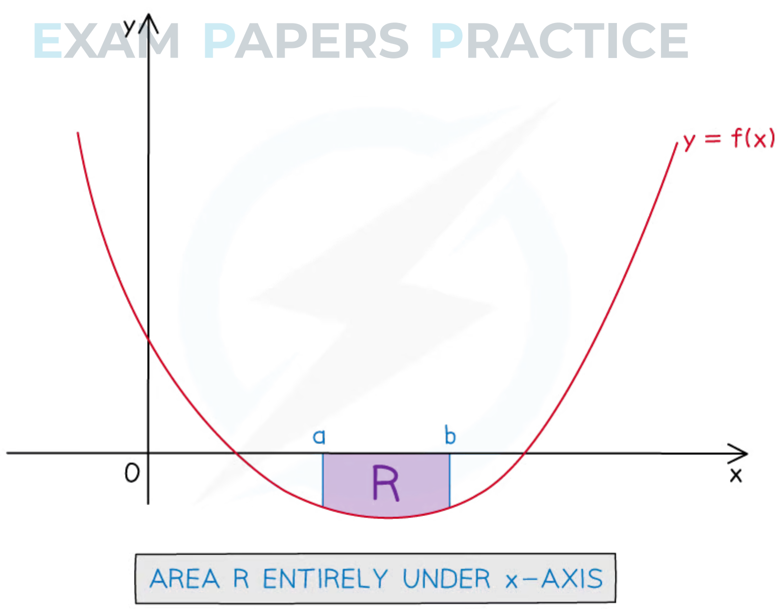
Negative Integrals

- The area under a curve may appear **fully** or **partially** under the x-axis
  - This occurs when the function $f(x)$ takes **negative** values within the boundaries of the area
- The **definite integrals** used to find such **areas**
  - will be **negative** if the area is **fully** under the x-axis
  - possibly **negative** if the area is **partially** under the y-axis
    - this occurs if the negative area(s) is/are greater than the positive area(s), their **sum** will be **negative**
- When using a GDC use the modulus (absolute value) function so that all definite integrals have a positive value

$$A = \int_a^b |y| \, dx$$

  - This is given in the **formula booklet**

How do I find the area under a curve when the curve is fully under the x-axis?



STEP 1
Write the expression for the definite integral to find the area as usual
This may involve finding the lower and upper limits from a graph sketch or GDC and f(x) may need to be rewritten in an integrable form

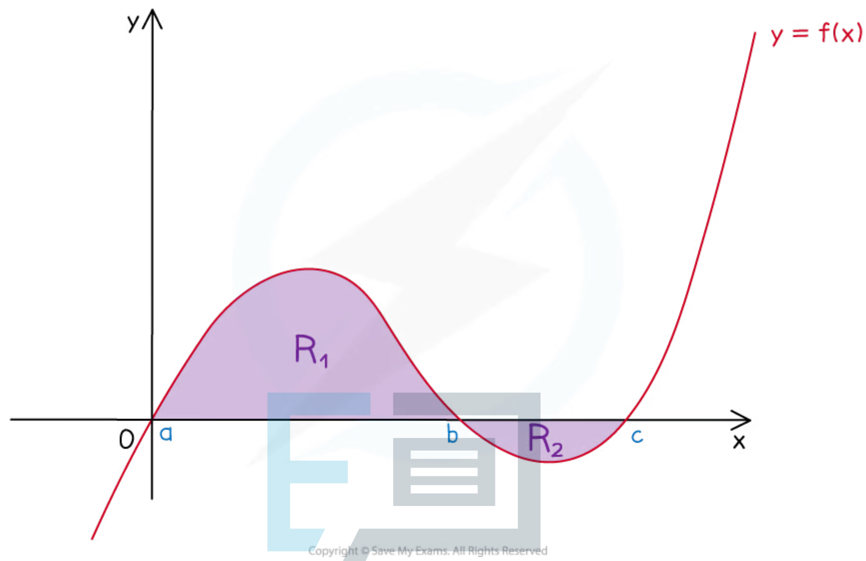
STEP 2

The answer to the definite integral will be negative

Area must always be positive so take the modulus (absolute value) of it

e.g. If $I = -36$ then the area would be 36 (square units)

How do I find the area under a curve when the curve is partially under the x-axis?



- For questions that allow the use of a GDC you can still use

$$A = \int_a^c |f(x)|\, dx$$

- To find the area analytically (manually) use the following method

STEP 1

Split the area into parts - the area(s) that are above the x-axis and the area(s) that are below the x-axis

STEP 2

Write the expression for the definite integral for each part (give each part a name, I_1, I_2, etc)

This may involve finding the lower and upper limits of each part from a graph sketch or a GDC, finding the roots of the function (i.e. where $f(x) = 0$) and rewriting $f(x)$ in an integrable form

STEP 3

Find the value of each definite integral separately

STEP 4

Find the area by summing the modulus (absolute values) of each integral

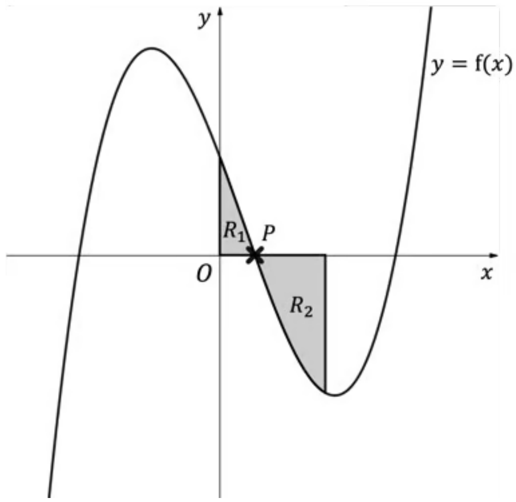
(Mathematically this would be written $A = |I_1| + |I_2| + |I_3| + \ldots$)

Exam Tip

- If no diagram is provided, quickly sketch one so that you can see where the curve is above and below the x - axis and split up your integrals accordingly

Worked Example

The diagram below shows the graph of $y = f(x)$ where $f(x) = (x+4)(x-1)(x-5)$.



The region R_1 is bounded by the curve $y = f(x)$, the x-axis and the y-axis.

The region R_2 is bounded by the curve $y = f(x)$, the x-axis and the line $x = 3$.

a)

Determine the coordinates of the point labelled P.

a) The x-coordinate of P is a root of $f(x)$

$f(x) = 0$

$(x+4)(x-1)(x-5) = 0$

$x = -4,\ x = 1,\ x = 5$

Clearly from the graph, $x = 1$ at point P

∴ P(1, 0)

b)

i)

Find a definite integral that would help find the area of the shaded region R_2 and briefly explain why this would **not** give the area of the region R_2.

ii)

Find the exact area of the shaded region R_2.

b) i)

$I_2 = \int_1^3 (x+4)(x-1)(x-5)\ dx$

R_2 is underneath the x-axis so the value of the definite integral will be negative. Area cannot be negative.

ii) STEP 1:

$I_2 = \int_1^3 (x+4)(x-1)(x-5)\ dx$

$I_2 = \int_1^3 (x^2+3x-4)(x-5)\ dx$ — Rewrite in an integrable form

$I_2 = \int_1^3 (x^3-2x^2-19x+20)\ dx$

$I_2 = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{19x^2}{2} + 20x\right]_1^3$ — Integrate (no need for "+c")

$I_2 = \left(\frac{3^4}{4} - \frac{2(3)^3}{3} - \frac{19(3)^2}{2} + 20(3)\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{19}{2} + 20\right)$

$I_2 = -\frac{93}{4} - \frac{121}{12}$

$I_2 = -\frac{100}{3}$

STEP 2: $\therefore$ Area of R_2, $A_2 = \frac{100}{3}$ square units

c)

Find the exact total area of the shaded regions, R_1 and R_2.

c) STEP 1, 2: $A_1 = I_1 = \int_0^1 (x^3-2x^2-19x+20)\ dx$ — Use the relevant results from b) ii)

$I_1 = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{19x^2}{2} + 20x\right]_0^1$

STEP 3: $I_1 = \frac{121}{12} - 0$

STEP 4: $\therefore A_1 + A_2 = \frac{121}{12} + \frac{100}{3} = \frac{521}{12}$

$\therefore$ Total area shaded $= \frac{521}{12}$ square units

You can check the final answer using your GDC and the formula (in booklet) $A = \int_a^b |y|\ dx$.

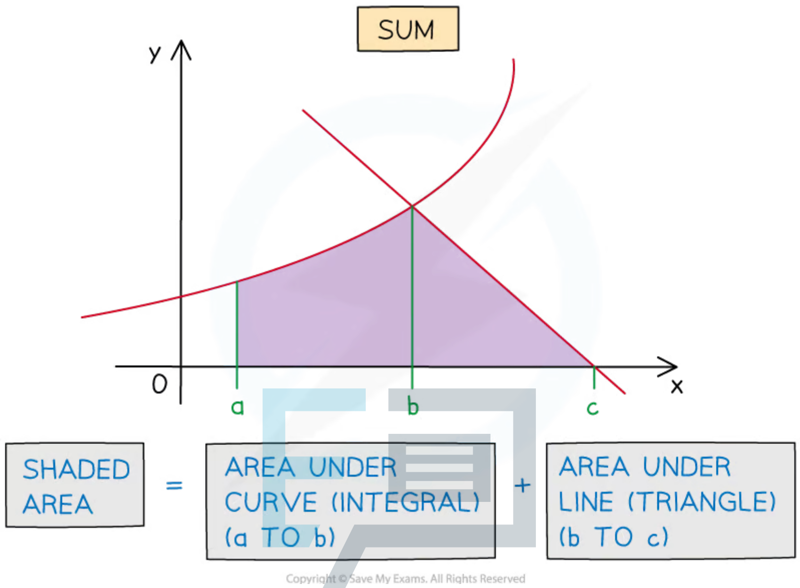
Here, $A = \int_0^3 |(x+4)(x-1)(x-5)|\ dx$

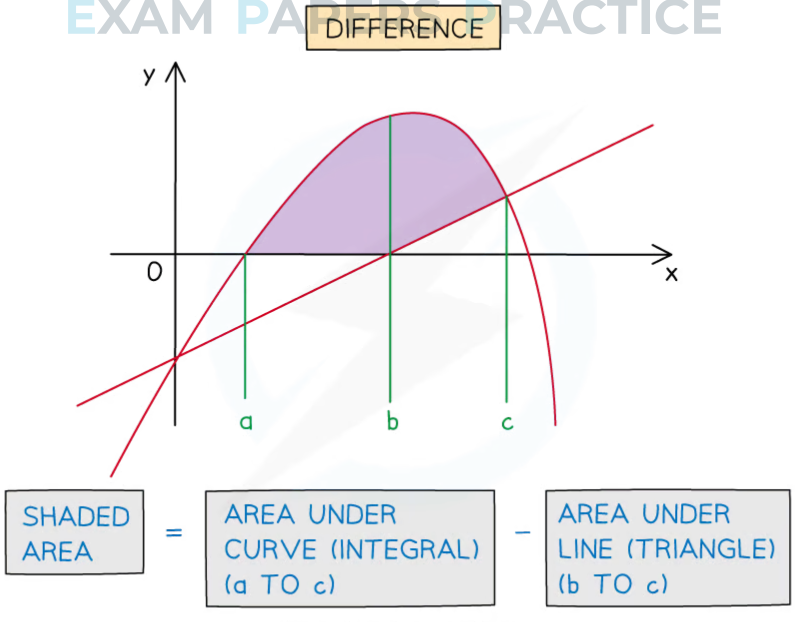
$A = 43.41666...$

(Note that our GDC was not able to produce the exact answer...)

Area Between a Curve and a Line

- **Areas** whose boundaries include a **curve** and a (non-vertical) **straight line** can be found using integration
  - For an **area** under a **curve** a **definite integral** will be needed
  - For an **area** under a **line** the shape formed will be a **trapezium** or **triangle**
    - **basic area formulae** can be used rather than a definite integral (although a definite integral would still work)
- The **area** required could be the **sum** or **difference** of areas under the curve and line





How do I find the area between a curve and a line?

STEP 1

If not given, sketch the graphs of the curve and line on the same diagram
Use a GDC to help with this step

STEP 2
Find the intersections of the curve and the line
If no diagram is given this will help identify the area(s) to be found

STEP 3
Determine whether the area required is the sum or difference of the area under the curve and the area under the line
Calculate the area under a curve using a integral of the form

$$\int_a^b y \, \mathrm{d}x$$

Calculate the area under a line using either $A = \frac{1}{2}bh$ for a triangle or $A = \frac{1}{2}h(a+b)$ for a trapezium (y-coordinates will be needed)

STEP 4
Evaluate the definite integrals and find their sum or difference as necessary to obtain the area required

Exam Tip

- Add information to any diagram provided
- Add axes intercepts, as well as intercepts between lines and curves
- Mark and shade the area you're trying to find
- If no diagram is provided, **sketch** one!

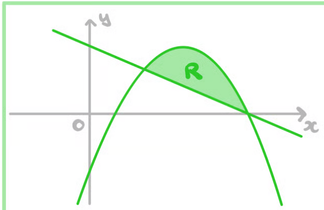
Worked Example

The region R is bounded by the curve with equation $y = 10x - x^2 - 16$ and the line with equation $y = 8 - x$.

R lies entirely in the first quadrant.

a)
Using your GDC, or otherwise, sketch the graphs of the curve and the line on the same diagram.
Identify and label the region R on your sketch and use your GDC to find the x-coordinates of the points of intersection between the curve and the line.

STEP 1:



STEP 2: Points of intersection are $(3, 5)$ and $(8, 0)$

b)
i)
Write down an integral that would find the area of the region R.
ii)
Find the area of the region R.

i) STEP 3: Curve is 'upper' boundary of R

$\therefore\ y_1 = 10x - x^2 - 16$

$y_2 = 8 - x$

$y_1 - y_2 = 10x - x^2 - 16 - (8 - x) = 11x - x^2 - 24$

$\therefore$ Area of R, $A_R = \int_3^8 (11x - x^2 - 24)\, dx$

ii) STEP 4:

$A_R = \int_3^8 (11x - x^2 - 24)\, dx$

$A_R = \left[\frac{11x^2}{2} - \frac{x^3}{3} - 24x \right]_3^8$

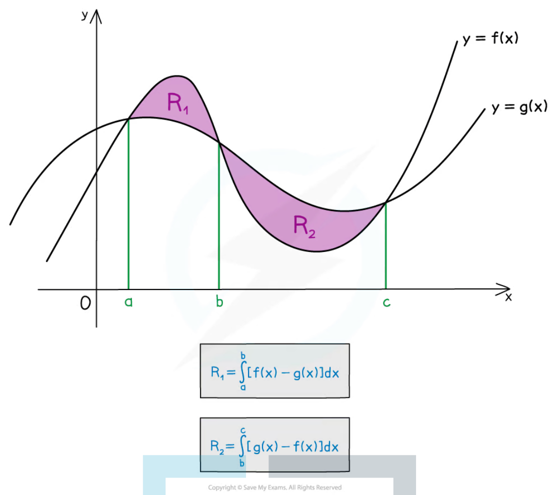
$A_R = \left[\frac{11(8)^2}{2} - \frac{(8)^3}{3} - 24(8) \right] - \left[\frac{11(3)^2}{2} - \frac{(3)^3}{3} - 24(3) \right]$

$A_R = -\frac{32}{3} - -\frac{63}{2}$

$\therefore$ Area of region R is $\frac{125}{6}$ square units

Area Between 2 Curves

- **Areas** whose boundaries include **two curves** can be found by integration
  - The **area between two curves** will be the **difference** of the areas under the two curves
    - both areas will require a **definite integral**
  - Finding points of intersection may involve a more awkward equation than solving for a curve and a line



How do I find the area between two curves?

STEP 1
If not given, sketch the graphs of both curves on the same diagram
Use a GDC to help with this step

STEP 2
Find the intersections of the two curves
If no diagram is given this will help identify the area(s) to be found

STEP 3
For each area (there may only be one) determine which curve is the 'upper' boundary
For each area, write a definite integral of the form

$$\int_a^b (y_1 - y_2)\,\mathrm{d}x$$

where y_1 is the function for the 'upper' boundary and y_2 is the function for the 'lower' boundary
Be careful when there is more than one region – the 'upper' and 'lower' boundaries will swap

STEP 4
Evaluate the definite integrals and sum them up to find the total area
(Step 3 means no definite integral will have a negative value)

Exam Tip

- If no diagram is provided sketch one, even if the curves are not accurate
- Add information to any given diagram as you work through a question
- Maximise use of your GDC to save time and maintain accuracy:
  - Use it to sketch the graphs and help you visualise the problem
  - Use it to find definite integrals

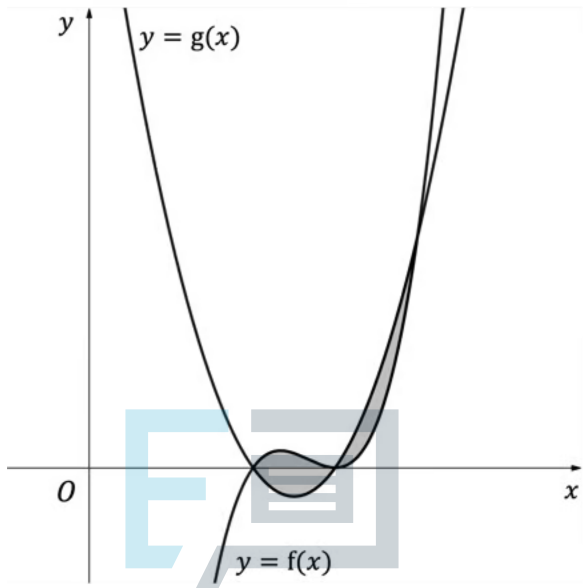
Worked Example

The diagram below shows the curves with equations $y = \mathrm{f}(x)$ and $y = g(x)$ where

$\mathrm{f}(x) = (x-2)(x-3)^2$

$$\mathrm{g}(x) = x^2 - 5x + 6$$

Find the area of the shaded region.



STEP 1: Sketch of graph given

STEP 2: Two intersections are the roots of $f(x)$

$f(x) = (x-2)(x-3)^2 = 0$ at $x = 2$, $(y=0)$

and $x = 3$ $(y=0)$

Solve $f(x) = g(x)$ to find the other intersection

$(x-2)(x-3)^2 = x^2 - 5x + 6$

$(x-2)(x-3)^2 = (x-2)(x-3)$

$x - 3 = 1$

$x = 4$, $y = (4-2)(4-3) = 2$

STEP 3: The area, A_1, of the first region is given by

$A_1 = \int_2^3 \left[(x-2)(x-3)^2 - (x^2 - 5x + 6)\right] dx$

$A_1 = \int_2^3 (x-2)(x-3)\left[(x-3) - 1\right] dx$ Factorise $(x-2)(x-3)$

$A_1 = \int_2^3 (x^2 - 5x + 6)(x-4) \, dx$

$A_1 = \int_2^3 (x^3 - 9x^2 + 26x - 24) \, dx$

$A_1 = \left[\frac{x^4}{4} - 3x^3 + 13x^2 - 24x\right]_2^3$

$A_1 = \left(\frac{(3)^4}{4} - 3(3)^3 + 13(3)^2 - 24(3)\right) - \left(\frac{(2)^4}{4} - 3(2)^3 + 13(2)^2 - 24(2)\right)$

$A_1 = -\frac{63}{4} - -16 = \frac{1}{4}$

For A_2, the 'upper' and 'lower' boundaries swap

$A_2 = \int_3^4 \left[(x^2 - 5x + 6) - (x-2)(x-3)^2\right] dx$

$A_2 = \int_3^4 (x-2)(x-3)\left[1 - (x-3)\right] dx$

$A_2 = \int_3^4 (x^2 - 5x + 6)(4 - x) \, dx$

$A_2 = \int_3^4 (-x^3 + 9x^2 - 26x + 24) \, dx$

$A_2 = \left[-\frac{x^4}{4} + 3x^3 - 13x^2 + 24x\right]_3^4$

$A_2 = \left(-\frac{(4)^4}{4} + 3(4)^3 - 13(4)^2 + 24(4)\right) - \left(-\frac{(3)^4}{4} + 3(3)^3 - 13(3)^2 + 24(3)\right)$

$A_2 = 16 - \frac{63}{4} = \frac{1}{4}$

Total area is $A_1 + A_2$

∴ **Area of shaded region is $\frac{1}{2}$ square unit**

5.8 Advanced Differentiation

5.8.4 Differentiation Further Functions

Differentiating Reciprocal Trigonometric Functions

What are the reciprocal trigonometric functions?

- **Secant**, **cosecant** and **cotangent** and abbreviated and defined as

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

- Remember that for calculus, angles need to be measured in **radians**
  - θ may be used instead of x
- $\operatorname{cosec} x$ is sometimes further abbreviated to $\csc x$

What are the derivatives of the reciprocal trigonometric functions?

- $f(x) = \sec x$
  - $f'(x) = \sec x \tan x$
- $f(x) = \operatorname{cosec} x$
  - $f'(x) = -\operatorname{cosec} x \cot x$
- $f(x) = \cot x$
  - $f'(x) = -\operatorname{cosec}^2 x$
- These are given in the **formula booklet**

How do I show or prove the derivatives of the reciprocal trigonometric functions?

- For $y = \sec x$
  - Rewrite, $y = \dfrac{1}{\cos x}$
  - Use quotient rule, $\dfrac{dy}{dx} = \dfrac{\cos x(0) - (1)(-\sin x)}{\cos^2 x}$
  - Rearrange, $\dfrac{dy}{dx} = \dfrac{\sin x}{\cos^2 x}$
  - Separate, $\dfrac{dy}{dx} = \dfrac{1}{\cos x} \times \dfrac{\sin x}{\cos x}$
  - Rewrite, $\dfrac{dy}{dx} = \sec x \tan x$
- Similarly, for $y = \operatorname{cosec} x$
  - $y = \dfrac{1}{\sin x}$
  - $\dfrac{dy}{dx} = \dfrac{\sin x(0) - (1)\cos x}{\sin^2 x}$
  - $\dfrac{dy}{dx} = \dfrac{-\cos x}{\sin^2 x}$
  - $\dfrac{dy}{dx} = -\dfrac{1}{\sin x} \times \dfrac{\cos x}{\sin x}$
  - $\dfrac{dy}{dx} = -\operatorname{cosec} x \cot x$

What do the derivatives of reciprocal trig look like with a linear functions of x?

- For linear functions of the form ax+b
  - $f(x) = \sec(ax + b)$
    $f'(x) = a \sec(ax + b) \tan(ax + b)$
  - $f(x) = \text{cosec}(ax + b)$
    $f'(x) = -a \,\text{cosec}(ax + b) \cot(ax + b)$
  - $f(x) = \cot(ax + b)$
    $f'(x) = -a \,\text{cosec}^2(ax + b)$
  - These are not given in the formula booklet
    they can be derived from chain rule
    they are not essential to remember

Exam Tip

- Even if you think you have remembered these derivatives, always use the formula booklet to double check
  - those squares and negatives are easy to get muddled up!
- Where two trig functions are involved in the derivative be careful with the angle multiple; $x, 2x, 3x$, etc
  - An example of a common mistake is differentiating $y = cosec\, 3x$
    $\frac{dy}{dx} = -3\ cosec\ x\ cot\ 3x$ instead of $\frac{dy}{dx} = -3\ cosec\, 3x\ cot\, 3x$

Worked Example

Curve C has equation $y = 2\cot\left(3x - \frac{\pi}{8}\right)$.

a)
Show that the derivative of $\cot x$ is $-\text{cosec}^2 x$.

$y = \cot x = \frac{\cos x}{\sin x}$

Quotient rule

$u = \cos x \quad v = \sin x$

$u' = -\sin x \quad v' = \cos x$

$\therefore \frac{dy}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$

$\frac{dy}{dx} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$\sin^2 x + \cos^2 x \equiv 1$

$\therefore \frac{dy}{dx} = -\frac{1}{\sin^2 x}$

$\frac{dy}{dx} = -\text{cosec}^2 x$

b) Find $\frac{dy}{dx}$ for curve C.

Chain rule/Linear function of x

$\frac{dy}{dx} = -2\text{cosec}^2\left(3x - \frac{\pi}{8}\right) \times 3$

$\therefore \frac{dy}{dx} = -6\text{cosec}^2\left(3x - \frac{\pi}{8}\right)$

c) Find the gradient of curve C at the point where $x = \frac{7\pi}{24}$.

When $x = \frac{7\pi}{24}$

$$\frac{dy}{dx} = -6\operatorname{cosec}^2\left(\frac{\overset{7}{\cancel{21}}\pi}{\underset{8}{\cancel{24}}} - \frac{\pi}{8}\right)$$

$$\frac{dy}{dx} = \frac{-6}{\sin^2\left(\frac{3\pi}{4}\right)} = \frac{-6}{\left(\frac{\sqrt{2}}{2}\right)^2}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=\frac{7\pi}{24}} = -12$$

Your GDC may be able to do this directly

Differentiating Inverse Trigonometric Functions

What are the inverse trigonometric functions?

- **arcsin**, **arccos** and **arctan** are functions defined as the inverse functions of sine, cosine and tangent respectively
  - $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ which is equivalent to $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
  - $\arctan(-1) = \frac{3\pi}{4}$ which is equivalent to $\tan\left(\frac{3\pi}{4}\right) = -1$

What are the derivatives of the inverse trigonometric functions?

- $f(x) = \arcsin x$
  - $f'(x) = \frac{1}{\sqrt{1-x^2}}$
- $f(x) = \arccos x$
  - $f'(x) = -\frac{1}{\sqrt{1-x^2}}$
- $f(x) = \arctan x$
  - $f'(x) = \frac{1}{1+x^2}$
- Unlike other derivatives these look completely unrelated at first
  - their derivation involves use of the identity $\cos^2 x + \sin^2 x \equiv 1$
  - hence the squares and square roots!
- All three are given in the **formula booklet**
- Note with the derivative of $\arctan x$ that $(1+x^2)$ is the same as (x^2+1)

How do I show or prove the derivatives of the inverse trigonometric functions?

- For $y = \arcsin x$
  - Rewrite, $\sin y = x$
  - Differentiate implicitly, $\cos y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$
  - Rearrange, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y}$
  - Using the identity $\cos^2 y \equiv 1 - \sin^2 y$ rewrite, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-\sin^2 y}}$
  - Since, $\sin y = x$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}$
- Similarly, for $y = \arccos x$
  - $\cos y = x$
  - $-\sin y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$
  - $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin y}$

- $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sqrt{1-\cos^2 y}}$
- $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sqrt{1-x^2}}$

- Notice how the derivative of $y = \arcsin x$ is positive but is negative for $y = \arccos x$
  - This subtle but crucial difference can be seen in their graphs
    - $y = \arcsin x$ has a positive gradient for all values of x in its domain
    - $y = \arccos x$ has a negative gradient for all values of x in its domain

What do the derivative of inverse trig look like with a linear function of x?

- For linear functions of the form $ax + b$
- $f(x) = \arcsin(ax + b)$
  - $f'(x) = \frac{a}{\sqrt{1-(ax+b)^2}}$
- $f(x) = \arccos(ax + b)$
  - $f'(x) = \frac{a}{\sqrt{1-(ax+b)^2}}$
- $f(x) = \arctan(ax + b)$
  - $f'(x) = \frac{a}{1+(ax+b)^2}$
- These are **not** in the formula booklet
  - they can be derived from chain rule
  - they are not essential to remember
  - they are not commonly used

Exam Tip

- For $f(x) = \arctan x$ the terms on the denominator can be reversed (as they are being added rather than subtracted)
  - $f'(x) = \frac{1}{1+x^2} = \frac{1}{x^2+1}$
  - Don't be fooled by this, it sounds obvious but on awkward "show that" questions it can be off-putting!

Worked Example

a) Show that the derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$

$y = \arctan x$

$\tan y = x$

Differentiate implicitly

$\sec^2 y \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$

Using the identity $\tan^2 y + 1 \equiv \sec^2 y$

$\dfrac{dy}{dx} = \dfrac{1}{\tan^2 y + 1}$

Since $\tan y = x$

$\dfrac{dy}{dx} = \dfrac{1}{x^2+1}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

b)

Find the derivative of $\arctan(5x^3 - 2x)$.

$5x^3 - 2x$ is not a linear function of x, use chain rule

$y = \arctan(5x^3 - 2x)$

$\dfrac{dy}{dx} = \dfrac{1}{1+(5x^3-2x)^2} \times (15x^2 - 2)$

$\therefore \dfrac{dy}{dx} = \dfrac{15x^2-2}{1+(5x^3-2x)^2}$

Differentiating Exponential & Logarithmic Functions

What are exponential and logarithmic functions?

- Exponential functions have term(s) where the variable (x) is the power (exponent)
  - In general, these would be of the form $y = a^x$
    The special case of this is when $a = e$, i.e. $y = e^x$
- Logarithmic functions have term(s) where the logarithms of the variable (x) are involved
  - In general, these would be of the form $y = \log_a x$
    The special case of this is when $a = e$, i.e. $y = \log_e x = \ln x$

What are the derivatives of exponential functions?

- The first two results, of the special cases above, have been met before
  - $f(x) = e^x, \quad f'(x) = e^x$
  - $f(x) = \ln x, \quad f'(x) = \frac{1}{x}$
  - These are given in the formula booklet
- For the general forms of exponentials and logarithms
  - $f(x) = a^x$
    $f'(x) = a^x(\ln a)$
  - $f(x) = \log_a x$
    $f'(x) = \frac{1}{x \ln a}$
  - These are also given in the **formula booklet**

How do I show or prove the derivatives of exponential and logarithmic functions?

- For $y = a^x$
  - Take natural logarithms of both sides, $\ln y = x \ln a$
  - Use the laws of logarithms, $\ln y = x \ln a$
  - Differentiate, implicitly, $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \ln a$
  - Rearrange, $\frac{\mathrm{d}y}{\mathrm{d}x} = y \ln a$
  - Substitute for y, $\frac{\mathrm{d}y}{\mathrm{d}x} = a^x \ln a$
- For $y = \log_a x$
  - Rewrite, $x = a^y$
  - Differentiate x with respect to y, using the above result, $\frac{\mathrm{d}x}{\mathrm{d}y} = a^y \ln a$
  - Using $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a^y \ln a}$

- Substitute for y, $\frac{dy}{dx} = \frac{1}{a^{\log_a x} \ln a}$
- Simplify, $\frac{dy}{dx} = \frac{1}{x \ln a}$

What do the derivatives of exponentials and logarithms look like with a linear functions of x?

- For linear functions of the form $px + q$
  - $f(x) = a^{px+q}$
    $f'(x) = pa^{px+q}(\ln a)$
  - $f(x) = \log_a(px + q)$
    $f'(x) = \frac{p}{(px + q)\ln a}$
  - These are **not** in the formula booklet
    they can be derived from chain rule
    they are not essential to remember

Exam Tip

- For questions that require the derivative in a particular format, you may need to use the laws of logarithms
  - With ln appearing in denominators be careful with the division law
    $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
    but $\frac{\ln a}{\ln b}$ cannot be simplified (unless there is some numerical connection between a and b)

Worked Example

a)

Find the derivative of a^{3x-2}.

Chain rule or 'px+q shortcut' is required

$$\frac{d}{dx}\left[a^{3x-2}\right] = a^{3x-2}\ln a \times 3$$

∴ The derivative of a^{3x-2} is $3a^{3x-2}\ln a$

b) Find an expression for $\frac{dy}{dx}$ given that $y = \log_5(2x^3)$

Chain rule is needed.

$$\frac{dy}{dx} = \frac{1}{2x^3 \ln 5} \times 6x^2$$

$$\therefore \frac{dy}{dx} = \frac{3}{x \ln 5}$$

5.10 Differential Equations

5.10.1 Numberical Solutions to Differential Equations

First Order Differential Equations

What is a differential equation?

- A **differential equation** is simply an equation that contains derivatives
  - For example $\frac{dy}{dx} = 12xy^2$ is a differential equation
  - And so is $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 7x = 5\sin t$

What is a first order differential equation?

- A **first order differential equation** is a differential equation that contains first derivatives but no second (or higher) derivatives
  - For example $\frac{dy}{dx} = 12xy^2$ is a first order differential equation
  - But $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 7x = 5\sin t$ is **not** a first order differential equation, because it contains the second derivative $\frac{d^2x}{dt^2}$

Wait – haven't I seen first order differential equations before?

- Yes you have!
  - For example $\frac{dy}{dx} = 3x^2$ is also a first order differential equation, because it contains a first derivative and no second (or higher) derivatives
  - But for that equation you can just integrate to find the solution $y = x^3 + c$ (where c is a constant of integration)
- In this section of the course you learn how to solve differential equations that can't just be solved right away by integrating

Euler's Method: First Order

What is Euler's method?

- **Euler's method** is a numerical method for finding approximate solutions to differential equations
- It treats the derivatives in the equation as being constant over short 'steps'
- The accuracy of the Euler's Method approximation can be improved by making the step sizes smaller

How do I use Euler's method with a first order differential equation?

- STEP 1: Make sure your differential equation is in $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ form
- STEP 2: Write down the recursion equations using the formulae $y_{n+1} = y_n + h \times f(x_n, y_n)$ and $x_{n+1} = x_n + h$ from the exam formula booklet
  - *h* in those equations is the **step size**
  - the exam question will usually tell you the correct value of *h* to use
- STEP 3: Use the recursion feature on your GDC to calculate the Euler's method approximation over the correct number of steps
  - the values for x_0 and y_0 will come from the boundary conditions given in the question

Exam Tip

- Be careful with letters – in the equations in the exam, and in your GDC's recursion calculator, the variables may not be *x* and *y*
- If an exam question asks you how to improve an Euler's method approximation, the answer will almost always have to do with decreasing the step size!

Worked Example

Consider the differential equation $\frac{dy}{dx} + y = x + 1$ with the boundary condition $y(0) = 0.5$.

a)
Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 1$.

Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$	h is a constant (step length)

} from formula booklet

STEP 1: $\frac{dy}{dx} = \underbrace{x - y + 1}_{f(x,y)}$

STEP 2: $y_{n+1} = y_n + \underbrace{0.2}_{h \text{ (from question)}} \times \underbrace{(x_n - y_n + 1)}_{f(x_n, y_n)}$ $\quad x_{n+1} = x_n + 0.2$

STEP 3: We need to get x from 0 to 1, so we will need $\frac{1-0}{0.2} = 5$ steps.

n	x_n	y_n
0	0	0.5
1	0.2	0.6
2	0.4	0.72
3	0.6	0.856
4	0.8	1.0048
5	1	1.16384

$y(0) = 0.5$; from GDC

$y(1) = 1.16$ (3 s.f.)

b)
Explain how the accuracy of the approximation in part (a) could be improved.

Make the step size smaller.

5.10.2 Analytical Solutions to Differential Equations

Separation of Variables

What is separation of variables?

- **Separation of variables** can be used to solve certain types of first order differential equations
- Look out for equations of the form $\frac{dy}{dx} = g(x)h(y)$
  - i.e. $\frac{dy}{dx}$ is a function of x multiplied by a function of y
  - be careful – the 'function of x' $g(x)$ may just be a constant!
    For example in $\frac{dy}{dx} = 6y$, $g(x) = 6$ and $h(y) = y$
- If the equation is in that form you can use separation of variables to try to solve it
- If the equation is not in that form you will need to use another solution method

How do I solve a differential equation using separation of variables?

- STEP 1: Rearrange the equation into the form $\left(\frac{1}{h(y)}\right)\frac{dy}{dx} = g(x)$
- STEP 2: Take the integral of both sides to change the equation into the form

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx$$

  - You can think of this step as 'multiplying the dx across and integrating both sides'
    Mathematically that's not *quite* what is actually happening, but it will get you the right answer here!
- STEP 3: Work out the integrals on both sides of the equation to find the **general solution** to the differential equation
  - Don't forget to include a constant of integration
    Although there are two integrals, you only need to include one constant of integration
  - Look out for integrals that require you to use **partial fractions** to solve them
    See 'Integrating with Partial Fractions' in 5.9 Advanced Integration
- STEP 4: Use any boundary or initial conditions in the question to work out the value of the integration constant
- STEP 5: If necessary, rearrange the solution into the form required by the question

Exam Tip

- Be careful with letters – the equation on an exam may not use x and y as the variables
- Unless the question asks for it, you don't have to change your solution into $y = f(x)$ form – sometimes it might be more convenient to leave your solution in another form

Worked Example

For each of the following differential equations, either (i) solve the equation by using separation of variables giving your answer in the form $y = f(x)$, or (ii) state why the equation may not be solved using separation of variables.

a) $\frac{dy}{dx} = \frac{e^x + 4x}{3y^2}$.

STEP 1: $3y^2 \frac{dy}{dx} = e^x + 4x$ $g(x) = e^x + 4x$ $h(y) = \frac{1}{3y^2}$

STEP 2: $\int 3y^2 \, dy = \int (e^x + 4x) \, dx$

STEP 3: $y^3 = e^x + 2x^2 + c$ ← Don't forget constant of integration

STEP 4: No boundary conditions given, so skip step

STEP 5: $y = \sqrt[3]{e^x + 2x^2 + c}$ $y = f(x)$

b) $\frac{dy}{dx} = 4xy - 2\ln x$.

$4xy - 2\ln x$ is not of the form $g(x)h(y)$, so it may not be solved using separation of variables.

c) $\frac{dy}{dx} = 2y^2 + 2y$, given that $y = 2$ when $x = 0$.

STEP 1 : $\frac{1}{y^2+y}\frac{dy}{dx} = 2$ $g(x) = 2$ $h(y) = y^2 + y$

STEP 2 : $\int \frac{1}{y^2+y}\, dy = \int 2\, dx$

partial fractions

STEP 3 : $\int \frac{1}{y^2+y}\, dy = \int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy = \ln\left|\frac{y}{y+1}\right| + c$

$\ln\left|\frac{y}{y+1}\right| = 2x + c$ ← Don't forget constant of integration

y = 2 when x = 0

STEP 4 : $\ln\left|\frac{2}{2+1}\right| = 2(0) + c \implies c = \ln\left(\frac{2}{3}\right)$

STEP 5 : For the boundary condition $y = 2$, $\frac{y}{y+1} > 0$.
Therefore we can drop the modulus sign from $\left|\frac{y}{y+1}\right|$.

$\frac{y}{y+1} = e^{2x+\ln(2/3)} = \left(e^{2x}\right)\left(e^{\ln(2/3)}\right) = \frac{2}{3}e^{2x}$

$\implies y = \frac{2e^{2x}}{3-2e^{2x}}$ $y = f(x)$

Homogeneous Differential Equations

What is a homogeneous first order differential equation?

- If a first order differential equation can be written in the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$ then it is said to be **homogeneous**

How do I solve a homogeneous first order differential equation?

- These equations can be solved using the substitution $v = \frac{y}{x} \Leftrightarrow y = vx$
- STEP 1: If necessary, rearrange the equation into the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$
- STEP 2: Replace all instances of $\frac{y}{x}$ in your equation with *v*
- STEP 3: Use the product rule and implicit differentiation to replace $\frac{\mathrm{d}y}{\mathrm{d}x}$ in your equation with $v + x\frac{\mathrm{d}v}{\mathrm{d}x}$
  - This is because $y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(vx) = v\frac{\mathrm{d}}{\mathrm{d}x}(x) + x\frac{\mathrm{d}}{\mathrm{d}x}(v) = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$
- STEP 4: Solve your new differential equation to find the solution in terms of *v* and *x*
  - You may need to use other methods for differential equations, such as **separation of variables**, at this stage
- STEP 5: Substitute $v = \frac{y}{x}$ into the solution from Step 4, in order to find the solution in terms of *y* and *x*

What else should I know about solving homogeneous first order differential equations?

- After finding the solution in terms of *y* and *x* you may be asked to do other things with the solution
  - For example you may be asked to find the solution corresponding to certain initial or boundary conditions
  - Or you may be asked to express your answer in a particular form, such as $y = f(x)$
- It is sometimes possible to solve differential equations that are *not* homogeneous by using the substitution $v = \frac{y}{x}$
  - For such a situation in an exam question, you would be told explicitly to use the substitution
  - You would not be expected to know that you could use the substitution in a case where the differential equation was not homogeneous

Exam Tip

- Unless the question asks for it, you don't have to change your solution into $y = f(x)$ form – sometimes it might be more convenient to leave your solution in another form

Worked Example

Consider the differential equation $xy\frac{dy}{dx} = y^2 - x^2$ where $y = 3$ when $x = 1$.

a)
Show that the differential equation is homogeneous.

$xy\frac{dy}{dx} = y^2 - x^2$

$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{xy} = \frac{y}{x} - \frac{x}{y} = (y/x) - \frac{1}{(y/x)} = f\left(\frac{y}{x}\right)$

$\Rightarrow$ The equation is homogeneous.

b)

Use the substitution $v = \frac{y}{x}$ to solve the differential equation with the given boundary condition.

STEP 1: $\frac{dy}{dx} = (y/x) - \frac{1}{(y/x)}$ } from part (a)

STEP 2: Let $v = \frac{y}{x}$. Then $\frac{dy}{dx} = v - \frac{1}{v}$.

STEP 3: And $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

So $v + x\frac{dv}{dx} = v - \frac{1}{v} \Rightarrow x\frac{dv}{dx} = -\frac{1}{v}$

STEP 4: $\int 2v\,dv = \int \frac{-2}{x}\,dx$ Separation of variables

$v^2 = -2\ln|x| + c = c - \ln(x^2)$ Don't forget constant of integration

STEP 5: $\left(\frac{y}{x}\right)^2 = c - \ln(x^2) \Rightarrow y^2 = x^2(c - \ln(x^2))$

Now use the boundary condition

And $y(1) = 3$, so $(3)^2 = (1)^2(c - \ln(1^2)) \Rightarrow c = 9$

$$y^2 = x^2(9 - \ln(x^2))$$

Question doesn't ask for solution to be in $y=f(x)$ form, so it's easiest just to keep it like this.

Integrating Factor

What is an integrating factor?

- An **integrating factor** can be used to solve a differential equation that can be written in the **standard form** $\frac{dy}{dx} + p(x)y = q(x)$
    - Be careful – the 'functions of *x*' *p*(*x*) and *q*(*x*) may just be constants!
      For example in $\frac{dy}{dx} + 6y = e^{-2x}$, $p(x) = 6$ and $q(x) = e^{-2x}$
      While in $\frac{dy}{dx} + \frac{y}{2x} = 12$, $p(x) = \frac{1}{2x}$ and $q(x) = 12$
- For an equation in standard form, the integrating factor is $e^{\int p(x)\ dx}$

How do I use an integrating factor to solve a differential equation?

- STEP 1: If necessary, rearrange the differential equation into standard form
- STEP 2: Find the integrating factor
    - Note that you don't need to include a constant of integration here when you integrate $\int p(x)\ dx$
- STEP 3: Multiply both sides of the differential equation by the integrating factor
- This will turn the equation into an **exact differential equation** of the form
  $\frac{d}{dx}\left(ye^{\int p(x)\ dx}\right) = q(x)e^{\int p(x)\ dx}$
- STEP 4: Integrate both sides of the equation with respect to *x*
    - The left side will automatically integrate to $ye^{\int p(x)\ dx}$
    - For the right side, integrate $\int q(x)e^{\int p(x) dx} dx$ using your usual techniques for integration
    - Don't forget to include a constant of integration
      Although there are two integrals, you only need to include one constant of integration
- STEP 5: Rearrange your solution to get it in the form *y* = *f*(*x*)

What else should I know about using an integrating factor to solve differential equations?

- After finding the general solution using the steps above you may be asked to do other things with the solution
    - For example you may be asked to find the solution corresponding to certain initial or boundary conditions

Worked Example

Consider the differential equation $\frac{dy}{dx} = 2xy + 5e^{x^2}$ where $y = 7$ when $x = 0$.

Use an integrating factor to find the solution to the differential equation with the given boundary condition.

Integrating factor for $y' + P(x)y = Q(x)$	$e^{\int P(x)dx}$

} from formula booklet

STEP 1 : $\frac{dy}{dx} - 2xy = 5e^{x^2}$ $\quad p(x) = -2x \quad q(x) = 5e^{x^2}$

STEP 2 : $e^{\int -2x\,dx} = e^{-x^2}$

STEP 3 : $\left(\frac{dy}{dx} - 2xy\right)e^{-x^2} = \left(5e^{x^2}\right)e^{-x^2}$

$$e^{-x^2}\frac{dy}{dx} - 2xye^{-x^2} = 5$$

$$\frac{d}{dx}\left(ye^{-x^2}\right) = 5$$ Exact differential equation

STEP 4 : $ye^{-x^2} = \int 5\,dx = 5x + c$ $\quad$ Don't forget constant of integration

STEP 5 : $y = e^{x^2}(5x + c)$

Now use the boundary condition

$$7 = e^0(5(0) + c) \implies c = 7$$

$$\boxed{y = e^{x^2}(5x + 7)}$$

5.10.3 Modelling with Differential Equations

Modelling with Differential Equations

Why are differential equations used to model real-world situations?

- A **differential equation** is an equation that contains one or more derivatives
- Derivatives deal with rates of change, and with the way that variables change with respect to one another
- Therefore differential equations are a natural way to model real-world situations involving change
  - Most frequently in real-world situations we are interested in how things change over time, so the derivatives used will usually be with respect to time t

How do I set up a differential equation to model a situation?

- An exam question may require you to create a differential equation from information provided
- The question will provide a context from which the differential equation is to be created
- Most often this will involve the rate of change of a variable being proportional to some function of the variable
  - For example, the rate of change of a population of bacteria, P, at a particular time may be proportional to the size of the population at that time
- The expression 'rate of' ('rate of change of…', 'rate of growth of…', etc.) in a modelling question is a strong hint that a differential equation is needed, involving derivatives with respect to time t
  - So with the bacteria example above, the equation will involve the derivative $\frac{dP}{dt}$
- Recall the basic equation of proportionality
  - If y is proportional to x, then $y = kx$ for some **constant of proportionality** k
    - So for the bacteria example above the differential equation needed would be $\frac{dP}{dt} = kP$
  - The precise value of k will generally not be known at the start, but will need to be found as part of the process of solving the differential equation
  - It can often be useful to assume that $k > 0$ when setting up your equation
    - In this case, $-k$ will be used in the differential equation in situations where the rate of change is expected to be negative
    - So in the bacteria example, if it were known that the population of bacteria was decreasing, then the equation could instead be written $\frac{dP}{dt} = -kP$

Worked Example

a)

In a particular pond, the rate of change of the area covered by algae, *A*, at any time *t* is directly proportional to the square root of the area covered by algae at that time. Write down a differential equation to model this situation.

$$\frac{dA}{dt} = k\sqrt{A}$$ (where k is a constant of proportionality)

b)

Newton's Law of Cooling states that the rate of change of the temperature of an object, *T*, at any time *t* is proportional to the difference between the temperature of the object and the ambient temperature of its surroundings, T_a, at that time. Assuming that the object starts off warmer than its surroundings, write down the differential equation implied by Newton's Law of Cooling.

The object is assumed to be warmer than its surroundings, so $T - T_a > 0$

$$\frac{dT}{dt} = -k(T - T_a)$$

(where $k > 0$ is a constant of proportionality)

We expect the temperature to be decreasing, so $-k$ in the equation combined with $k > 0$ assures that $\frac{dT}{dt}$ is negative.

The Logistic Equation

What is the logistic equation?

- The differential equation $\frac{\mathrm{d}N}{\mathrm{d}t} = kN$ is a very simple example of a model in which the rate of change of a population at any moment in time is dependent on the size of the population (N) at that time
  - The solution is $N = A\mathrm{e}^{kt}$ (where $A > 0$ is a constant)
  - If $k > 0$, this represents unlimited exponential growth of the variable N
- In many real-world contexts (for example when considering populations of living organisms), unlimited growth is not a realistic modelling assumption
  - For reproducing populations it is logical to assume that the rate of change of the population will be dependent on the size of the population (more rabbits means more production of baby rabbits!)
  - But there are generally limiting factors on populations that prevent them from growing without limits
    - For example, availability of food or other resources, or the presence of predators or other threats, may limit the population that can exist in a given area
- A **logistic equation** incorporates such limiting factors into the model, and therefore can provide a more realistic model for real-world populations
- The **standard logistic equation** is of the form

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN(a - N)$$

  - t represents the time (since the moment defined as $t = 0$) that the population has been growing
  - N represents the size of the population at time t
  - $k \in \mathbb{R}$ is a constant determining the relative rate of population growth
    - For the models dealt with here it is most common to have $k > 0$, with a larger value of k representing a faster rate of change
  - $a \in \mathbb{R}$ is a constant that places a limit on the maximum size to which the population N can grow
    - For a population model it can be assumed that $a > 0$
    - For $k > 0$ and an initial population N_0 such that $0 < N_0 < a$, the population N will grow and will converge to the value a as time t increases
    - For $k > 0$ and an initial population N_0 such that $0 < N_0 < a$, the population N will shrink and will converge to the value a as time t increases
- There are other forms of logistic equation
  - The exact form of the logistic equation you are to use will always be given in an exam question

How do I solve problems that involve a logistic equation model?

- Solving the differential equation will generally involve the technique of **separation of variables**
  - Usually this will also involve rearranging one of the integrals using **partial fractions** (see the worked example below for an example)

- You will usually be given 'boundary conditions' specific to the context of the problem
  - For example, you may be told the initial population at time $t = 0$
  - These conditions will allow you to work out the exact value of any integrating constants that occur while solving the differential equation
- You will need to take account of the context of the question in answering the question or in commenting on the model used

Worked Example

A group of ecologists are studying a population of rabbits on a particular island. The population of rabbits, *N*, on the island is modelled by the logistic equation

$$\frac{dN}{dt} = 0.0012N(1500 - N)$$

where *t* represents the time in years since the ecologists began their study. At the time the study begins there are 300 rabbits on the island.

a)

Show that the population of rabbits at time *t* years is given by $N = \frac{1500e^{1.8t}}{4 + e^{1.8t}}$.

$$\int \frac{1}{N(1500-N)}\,dN = 0.0012\int dt$$ Separation of variables

$$\int \left(\frac{1}{N} - \frac{1}{N-1500}\right)dN = 1.8\int dt$$ Partial fractions

$$\ln\left|\frac{N}{N-1500}\right| = 1.8t + c$$ Don't forget constant of integration

$$\frac{N}{1500-N} = Ae^{1.8t}$$ $A = e^c$

$0 < N < 1500$, so $\left|\frac{N}{N-1500}\right| = -\frac{N}{N-1500} = \frac{N}{1500-N}$

$$N(0) = 300, \text{ so } \frac{300}{1500-300} = Ae^{1.8(0)} \Rightarrow A = \frac{1}{4}$$ Use initial condition

$$\Rightarrow \frac{N}{1500-N} = \frac{1}{4}e^{1.8t} \Rightarrow \frac{4N}{1500-N} = e^{1.8t}$$

$$\Rightarrow N = \frac{1500e^{1.8t}}{4 + e^{1.8t}}$$ Rearrange

b)

Find the population of rabbits that the model predicts will be on the island two years after the beginning of the study.

$$N(2) = \frac{1500e^{1.8(2)}}{4 + e^{1.8(2)}} = \frac{1500e^{3.6}}{4 + e^{3.6}} = 1352.210\ldots$$

1352 rabbits

Round to nearest rabbit

c)

Determine the maximum size that the model predicts the population of rabbits can grow to. Justify your answer by an appropriate analysis of the equation in part (a).

Take limit to determine long-term behaviour :

$$\lim_{t\to\infty} \frac{1500e^{1.8t}}{4 + e^{1.8t}} = \lim_{t\to\infty} \frac{1500}{\frac{4}{e^{1.8t}} + 1} = \frac{1500}{0+1} = 1500$$

1500 rabbits is the maximum population predicted by the model.

5.11 MacLaurin Series

5.11.1 Maclaurin Series

Maclaurin Series of Standard Functions

What is a Maclaurin Series?

- A Maclaurin series is a way of representing a function as an infinite sum of increasing integer powers of x (x^1, x^2, x^3, etc.)
  - If all of the infinite number of terms are included, then the Maclaurin series is exactly equal to the original function
  - If we **truncate** (i.e., shorten) the Maclaurin series by stopping at some particular power of x, then the Maclaurin series is only an approximation of the original function
- A truncated Maclaurin series will always be exactly equal to the original function for $x = 0$
- In general, the approximation from a truncated Maclaurin series becomes less accurate as the value of x moves further away from zero
- The accuracy of a truncated Maclaurin series approximation can be improved by including more terms from the complete infinite series
  - So, for example, a series truncated at the x^7 term will give a more accurate approximation than a series truncated at the x^3 term

How do I find the Maclaurin series of a function 'from first principles'?

- Use the **general Maclaurin series formula**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

- This formula is in your exam formula booklet
- STEP 1: Find the values of $f(0)$, $f'(0)$, $f''(0)$, etc. for the function
  - An exam question will specify how many terms of the series you need to calculate (for example, "up to and including the term in x^4")
  - You may be able to use your GDC to find these values directly without actually having to find all the necessary derivatives of the function first
- STEP 2: Put the values from Step 1 into the general Maclaurin series formula
- STEP 3: Simplify the coefficients as far as possible for each of the powers of x

Is there an easier way to find the Maclaurin series for standard functions?

- Yes there is!
- The following Maclaurin series expansions of standard functions are contained in your exam formula booklet:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

- Unless a question specifically asks you to derive a Maclaurin series using the general Maclaurin series formula, you can use those standard formulae from the exam formula booklet in your working

Is there a connection Maclaurin series expansions and binomial theorem series expansions?

- Yes there is!
- For a function like $(1+x)^n$ the binomial theorem series expansion is **exactly the same** as the Maclaurin series expansion for the same function
  - So unless a question specifically tells you to use the general Maclaurin series formula, you can use the binomial theorem to find the Maclaurin series for functions of that type
  - Or if you've forgotten the binomial series expansion formula for $(1+x)^n$ where n is not a positive integer, you can find the binomial theorem expansion by using the general Maclaurin series formula to find the Maclaurin series expansion

Worked Example

a)

Use the Maclaurin series formula to find the Maclaurin series for $f(x) = \sqrt{1+2x}$ up to and including the term in x^4.

$$f(x) = \sqrt{1+2x} = (1+2x)^{\frac{1}{2}}$$

STEP 1: $f(0) = 1 \quad f'(0) = 1 \quad f''(0) = -1$

$f'''(0) = 3 \quad f^{(4)}(0) = -15$

STEP 2: $f(x) = 1 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(3) + \frac{x^4}{4!}(-15) + \ldots$

STEP 3: Up to the x^4 term,

$$\sqrt{1+2x} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4$$

Note: This is the same as the binomial theorem expansion of $(1+2x)^{\frac{1}{2}}$

b)

Use your answer from part (a) to find an approximation for the value of $\sqrt{1.02}$, and compare the approximation found to the actual value of the square root.

Up to the x^4 term,

$$\sqrt{1+2x} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4$$

} from part (a)

Let $x = 0.01$. Then $\sqrt{1+2x} = \sqrt{1+2(0.01)} = \sqrt{1.02}$.

So

$$\sqrt{1.02} \approx 1 + (0.01) - \frac{1}{2}(0.01)^2 + \frac{1}{2}(0.01)^3 - \frac{5}{8}(0.01)^4$$

$$\sqrt{1.02} \approx 1.00995049375$$

The exact value of the square root is

$$\sqrt{1.02} = 1.00995049383 6\ldots$$

The approximation is accurate to 10 d.p. or 11 s.f.

Maclaurin Series of Composites & Products

How can I find the Maclaurin series for a composite function?

- A **composite function** is a 'function of a function' or a 'function within a function'
  - For example sin(2*x*) is a composite function, with 2*x* as the 'inside function' which has been put into the simpler 'outside function' sin *x*
  - Similarly e^{x^2} is a composite function, with x^2 as the 'inside function' and e^x as the 'outside function'
- To find the Maclaurin series for a composite function:
  - STEP 1: Start with the Maclaurin series for the basic 'outside function'
    - Usually this will be one of the 'standard functions' whose Maclaurin series are given in the exam formula booklet
  - STEP 2: Substitute the 'inside function' every place that *x* appears in the Maclaurin series for the 'outside function'
    - So for sin(2*x*), for example, you would substitute 2*x* everywhere that *x* appears in the Maclaurin series for sin *x*
  - STEP 3: Expand the brackets and simplify the coefficients for the powers of *x* in the resultant Maclaurin series
- This method can theoretically be used for quite complicated 'inside' and 'outside' functions
  - On your exam, however, the 'inside function' will usually not be more complicated than something like kx (for some constant *k*) or x^n (for some constant power *n*)

How can I find the Maclaurin series for a product of two functions?

- To find the Maclaurin series for a product of two functions:
  - STEP 1: Start with the Maclaurin series of the individual functions
    - For each of these Maclaurin series you should only use terms up to an appropriately chosen power of *x* (see the worked example below to see how this is done!)
  - STEP 2: Put each of the series into brackets and multiply them together
    - Only keep terms in powers of *x* up to the power you are interested in
  - STEP 3: Collect terms and simplify coefficients for the powers of *x* in the resultant Maclaurin series

Worked Example

a)

Find the Maclaurin series for the function $f(x) = \ln(1 + 3x)$, up to and including the term in x^4.

Maclaurin series for special functions	$e^x = 1 + x + \frac{x^2}{2!} + \ldots$	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$
	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$	

from exam formula booklet

STEP 1: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

STEP 2: $\ln(1+3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \ldots$

STEP 3: $\ln(1+3x) = 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \ldots$

b)

Find the Maclaurin series for the function $g(x) = e^x \sin x$, up to and including the term in x^4.

Maclaurin series for special functions	$e^x = 1 + x + \frac{x^2}{2!} + \ldots$ $\quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ $\quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

from exam formula booklet

STEP 1 : $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ ← Higher powers of x here will give powers higher than 4 when multiplied by the sin x series.

$\sin x = x - \frac{x^3}{6} + \ldots$ ← Don't need terms in powers of x higher than 4

STEP 2 : $e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(x - \frac{x^3}{6} + \ldots\right)$

$= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^3}{6} - \frac{x^4}{6} - \frac{x^5}{12} - \frac{x^6}{36}$

Note that the x^4 terms cancel out

Discard terms for powers higher than 4

STEP 3 : $e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$

Differentiating & Integrating Maclaurin Series

How can I use differentiation to find Maclaurin Series?

- If you differentiate the Maclaurin series for a function $f(x)$ term by term, you get the Maclaurin series for the function's derivative $f'(x)$
- You can use this to find new Maclaurin series from existing ones
  - For example, the derivative of $\sin x$ is $\cos x$
  - So if you differentiate the Maclaurin series for $\sin x$ term by term you will get the Maclaurin series for $\cos x$

How can I use integration to find Maclaurin series?

- If you integrate the Maclaurin series for a derivative $f'(x)$, you get the Maclaurin series for the function $f(x)$
  - Be careful however, as you will have a constant of integration to deal with
  - The value of the constant of integration will have to be chosen so that the series produces the correct value for $f(0)$
- You can use this to find new Maclaurin series from existing ones
  - For example, the derivative of $\sin x$ is $\cos x$
  - So if you integrate the Maclaurin series for $\cos x$ (and correctly deal with the constant of integration) you will get the Maclaurin series for $\sin x$

Worked Example

a)

(i)

Write down the derivative of $\arctan x$.

(ii)

Hence use the Maclaurin series for $\arctan x$ to derive the Maclaurin series for $\frac{1}{1+x^2}$.

Standard derivatives	
$\arctan x$	$f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$

Maclaurin series for special functions		
	$e^x = 1 + x + \frac{x^2}{2!} + \ldots$	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$
	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$	

from exam formula booklet

(i) $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

(ii) $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

$\Rightarrow \frac{1}{1+x^2} = \frac{d}{dx}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)$

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$

Note: This is the same as the binomial theorem expansion of $(1+x^2)^{-1}$

b)

(i)

Write down the derivative of $-\sin x$.

(ii)

Hence derive the Maclaurin series for $\cos x$, being sure to justify your method.

Maclaurin series for special functions	$e^x = 1 + x + \frac{x^2}{2!} + \ldots$	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$
	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$	

from exam formula booklet

(i) $-\sin x = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots$

(ii) $-\sin x$ is the derivative of $\cos x$, so we can integrate the Maclaurin series for $-\sin x$ to find the Maclaurin series for $\cos x$.

$$\cos x = \int \left(-x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots\right) dx$$

constant of integration

$$= c - \frac{1}{2}x^2 + \frac{1}{4}\cdot\frac{x^4}{3!} - \frac{1}{6}\cdot\frac{x^6}{5!} + \frac{1}{8}\cdot\frac{x^8}{7!} - \ldots$$

$$= c - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

And $\cos(0) = 1$, so $c - \frac{0^2}{2!} + \frac{0^4}{4!} - \frac{0^6}{6!} + \frac{0^8}{8!} - \ldots = 1 \implies c = 1$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

5.11.2 Maclaurin Series from Differential Equations

Maclaurin Series for Differential Equations

Can I apply Maclaurin Series to solving differential equations?

- If you have a differential equation of the form $\frac{dy}{dx} = g(x, y)$ along with the value of $y(0)$ it is possible to build up the Maclaurin series of the solution $y = f(x)$ term by term
  - This does not necessarily tell you the explicit function of x that corresponds to the Maclaurin series you are finding
  - But the Maclaurin series you find is the exact Maclaurin series for the solution to the differential equation
- The Maclaurin series can be used to approximate the value of the solution $y = f(x)$ for different values of x
  - You can increase the accuracy of this approximation by calculating additional terms of the Maclaurin series for higher powers of x

How can I find the Maclaurin Series for the solution to a differential equation?

- STEP 1: Use **implicit differentiation** to find expressions for y'', y''' etc., in terms of x, y and lower-order derivatives of y
  - The number of derivatives you need to find depends on how many terms of the Maclaurin series you want to find
  - For example, if you want the Maclaurin series up to the term, then you will need to find derivatives up to $y^{(4)}$ (the fourth derivative of y)
- STEP 2: Using the given initial value for $y(0)$, find the values of $y'(0)$, $y''(0)$, $y'''(0)$, etc., one by one
  - Each value you find will then allow you to find the value for the next higher derivative
- STEP 3: Put the values found in STEP 2 into the **general Maclaurin series formula**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

  - This formula is in your exam formula booklet
  - $y = f(x)$ is the solution to the differential equation, so $y(0)$ corresponds to $f(0)$ in the formula, $y'(0)$ corresponds to $f'(0)$, and so on
- STEP 4: Simplify the coefficients for each of the powers of x in the resultant Maclaurin series

Worked Example

Consider the differential equation $y' = y^2 - x$ with the initial condition $y(0) = 2$.

a)
Use implicit differentiation to find expressions for y'', y''' and $y^{(4)}$.

STEP 1:

$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(y^2 - x) = 2yy' - 1$

$$y'' = 2yy' - 1$$

$y''' = \frac{d}{dx}(y'') = \frac{d}{dx}(2yy' - 1) = 2yy'' + 2(y')^2$

$$y''' = 2yy'' + 2(y')^2$$

$y^{(4)} = \frac{d}{dx}(y''') = \frac{d}{dx}(2yy'' + 2(y')^2)$

$= 2y'y'' + 2yy''' + 4y'y''$

$$y^{(4)} = 6y'y'' + 2yy'''$$

b)
Use the given initial condition to find the values of $y'(0)$, $y''(0)$, $y'''(0)$ and $y^{(4)} = 0$.

STEP 2:

$y(0) = 2$, so $y'(0) = 2^2 - 0 = 4$ $\qquad y' = y^2 - x$

Then $y''(0) = 2(2)(4) - 1 = 15$ $\qquad y'' = 2yy' - 1$

$y'''(0) = 2(2)(15) + 2(4)^2 = 92$ $\qquad y''' = 2yy'' + 2(y')^2$

$y^{(4)}(0) = 6(4)(15) + 2(2)(92) = 728$ $\qquad y^{(4)} = 6y'y'' + 2yy'''$

$$y'(0) = 4 \quad y''(0) = 15$$
$$y'''(0) = 92 \quad y^{(4)}(0) = 728$$

Let $y = f(x)$ be the solution to the differential equation with the given initial condition.

c)
Find the first five terms of the Maclaurin series for $f(x)$.

Maclaurin series	$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \ldots$

from exam formula booklet

STEP 3: $f(x) = 2 + x(4) + \frac{x^2}{2!}(15) + \frac{x^3}{3!}(92) + \frac{x^4}{4!}(728) + \ldots$

STEP 4:

$$f(x) = 2 + 4x + \frac{15}{2}x^2 + \frac{46}{3}x^3 + \frac{91}{3}x^4 + \ldots$$

5.12 Further Limits (inc I'Hopital's Rules)

5.12.1 Further Limits

I'Hôpital's Rule

What is l'Hôpital's Rule?

- **l'Hôpital's rule** is a method involving calculus that allows us to find the value of certain limits
- Specifically, it allows us to attempt to evaluate the limit of a quotient $\frac{f(x)}{g(x)}$ for which our usual limit evaluation techniques would return one of the **indeterminate forms** $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

How do I evaluate a limit using l'Hôpital's Rule?

- STEP 1: Check that the limit of the quotient results in one of the indeterminate forms given above
  - I.e., check that $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$
- STEP 2: Find the derivatives of the numerator and denominator of the quotient
- STEP 3: Check whether the limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists
- STEP 4: If that limit does exist, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$
- STEP 5: If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ then you may repeat the process by considering $\lim_{x \to a} \frac{f''(x)}{g''(x)}$ (and possibly higher order derivatives after that)
  - As long as the limits continue giving indeterminate forms you may continue applying l'Hôpital's rule
  - Each time this happens find the next set of derivatives and consider the limit again

Exam Tip

- Some limits of an indeterminate form can also be evaluated using the **Maclaurin series** for the numerator and denominator
- If an exam question does not specify a method to use, then you are free to use whichever method you prefer

Worked Example

Use l'Hôpital's rule to evaluate each of the following limits:

a) $\lim_{x \to 0} \frac{\sin x}{e^x - 1}$.

STEP 1: $\lim_{x \to 0} \frac{\sin x}{e^x - 1} = \frac{\sin(0)}{e^0 - 1} = \frac{0}{0}$ ← indeterminate form

($\sin x$ is $f(x)$, $e^x - 1$ is $g(x)$)

STEP 2: $\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(e^x - 1) = e^x$

STEP 3: $\lim_{x \to 0} \frac{\cos x}{e^x} = \frac{\cos(0)}{e^0} = \frac{1}{1} = 1$ ← limit exists

STEP 4: $\lim_{x \to 0} \frac{\sin x}{e^x - 1} = 1$

b) $\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x}$.

STEP 1: $\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x} = \frac{0^3}{-2(0) + \sin 0} = \frac{0}{0}$ ← indeterminate form

STEP 2: $\frac{d}{dx}(x^3) = 3x^2 \qquad \frac{d}{dx}(-2x + \sin 2x) = -2 + 2\cos 2x$

STEP 3: $\lim_{x \to 0} \frac{3x^2}{-2 + 2\cos 2x} = \frac{3(0)^2}{-2 + 2\cos 0} = \frac{0}{0}$ ← indeterminate form

STEP 4: That limit is still an indeterminate form, so proceed to STEP 5.

STEP 5: $\lim_{x \to 0} \frac{6x}{-4\sin 2x} = \frac{6(0)}{-4\sin 0} = \frac{0}{0}$

$\frac{d}{dx}(3x^2) = 6x$

$\frac{d}{dx}(-2 + 2\cos 2x) = -4\sin 2x$

That's still an indeterminate form, so repeat again:

$\lim_{x \to 0} \frac{6}{-8\cos 2x} = \frac{6}{-8\cos 0} = -\frac{3}{4}$

$\frac{d}{dx}(6x) = 6$

$\frac{d}{dx}(-4\sin 2x) = -8\cos 2x$

And that limit exists, so

$\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x} = -\frac{3}{4}$

Limits Using a Maclaurin Series

How do I evaluate a limit using Maclaurin series?

- Limits of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ or $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ may sometimes be evaluated by using Maclaurin series
- Usually this will be in a situation where attempting to evaluate the limit in the usual way returns an **indeterminate form** $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.
- In such a case:
  - STEP 1: Find the Maclaurin series for $f(x)$ and $g(x)$
  - STEP 2: Rewrite $\frac{f(x)}{g(x)}$ using the Maclaurin series in the numerator and denominator
  - STEP 3: Use algebra to simplify your new expression for $\frac{f(x)}{g(x)}$ as far as possible
  - STEP 4: Evaluate the limit using your simplified form of the expression

Exam Tip

- Some limits of an indeterminate form can also be evaluated using **l'Hôpital's Rule**
- If an exam question does not specify a method to use, then you are free to use whichever method you prefer

Worked Example

Use Maclaurin series to evaluate the limit

$$\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x}$$

Maclaurin series for special functions		
	$e^x = 1 + x + \frac{x^2}{2!} + \dots$	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$
	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$	

from exam formula booklet

$$\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x} = \frac{0^3}{-2(0) + \sin 0} = \frac{0}{0}$$ ← indeterminate form

STEP 1: $-2x + \sin 2x = -2x + \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \dots\right)$

$$= -\frac{4}{3}x^3 + \frac{4}{15}x^5 - \dots$$

STEP 2: $\frac{x^3}{-2x + \sin 2x} = \frac{x^3}{-\frac{4}{3}x^3 + \frac{4}{15}x^5 - \dots}$

STEP 3: $\frac{x^3}{-\frac{4}{3}x^3 + \frac{4}{15}x^5 - \dots} \times \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \frac{1}{-\frac{4}{3} + \frac{4}{15}x^2 - \dots}$

STEP 4: $\lim_{x \to 0} \frac{1}{-\frac{4}{3} + \frac{4}{15}x^2 - \dots} = \frac{1}{-\frac{4}{3} + \frac{4}{15}(0)^2 - \dots} = \frac{1}{-\frac{4}{3}} = -\frac{3}{4}$

higher powers of x will also be zero

$$\lim_{x \to 0} \frac{x^3}{-2x + \sin 2x} = -\frac{3}{4}$$