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3. Geometry & Trigonometry

3.9 Vector Properties

MATHS

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3. Geometry & Trigonometry

CONTENTS

3.1 Geometry of 3D Shapes

3.1.1 3D Coordinate Geometry

3D Coordinate Geometry

How does the 3D coordinate system work?

- In three-dimensional space we can label where any object is using the *x*-*y*-*z* coordinate system
- In the 3D cartesian system, the *x*- and *y*- axes usually represent lateral space (length and width) and the *z*-axis represents vertical height

What can we do with 3D coordinates?

- If we have two points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) then we should be able to find:
  - The **midpoint** of the two points
  - The **distance** between the two points
- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]

How do I find the midpoint of two points in 3D?

- The midpoint is the **average (middle) point**
  - It can be found by finding the middle of the *x*-coordinates and the middle of the *y*-coordinates
- The coordinates of the midpoint will be

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

  - This is given in the formula booklet, you do not need to remember it

How do I find the distance between two points in 3D?

- The distance between two points with coordinates $((x_1, y_1, z_1)$ and (x_2, y_2, z_2) can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

  - This is given in the formula booklet, you do not need to remember it

Worked Example

The points A and B have coordinates (-2, 1, 5) and (4, -3, 2) respectively.

i)

Calculate the distance of the line segment AB.

Formula for the distance of a line segment:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$ ← in formula booklet

A : (−2, 1, 5) with x_1, y_1, z_1 B : (4, −3, 2) with x_2, y_2, z_2

Substitute:

$$d = \sqrt{(-2-4)^2 + (1-(-3))^2 + (5-2)^2}$$

$$= \sqrt{(-6)^2 + 4^2 + 3^2}$$

$$= \sqrt{36 + 16 + 9}$$

$$= \sqrt{61}$$

$d = 7.81$ units (3 sf)

ii)

Find the midpoint of [AB].

Formula for the midpoint of a line segment:

$$MP = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$ ← in formula booklet

A : (−2, 1, 5) with x_1, y_1, z_1 B : (4, −3, 2) with x_2, y_2, z_2

Substitute:

$$MP = \left(\frac{-2+4}{2}, \frac{1+(-3)}{2}, \frac{5+2}{2}\right)$$

$$= \left(\frac{2}{2}, -\frac{2}{2}, \frac{7}{2}\right)$$

MP = (1, −1, 3.5)

3.1.2 Volume & Surface Area

Volume of 3D Shapes

What is volume?

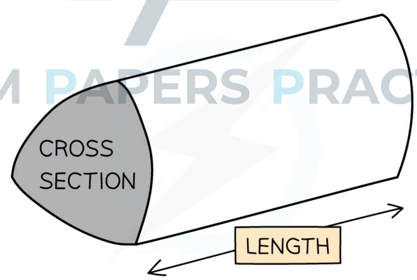
- The volume of a 3D shape is a measure of how much 3D space it takes up
  - A 3D shape is also called a **solid**
- You need to be able to calculate the volume of a number of common shapes

How do I find the volume of cuboids, prisms and cylinders?

- A prism is a 3–D shape that has two identical **base** shapes connected by parallel **edges**
  - A prism has the same base shape all the way through
  - A **prism** takes its name from its base
- To find the **volume** of any prism use the formula:

$$\text{Volume of a prism} = Ah$$

  - Where $\boldsymbol{A}$ is the area of the cross section and $\boldsymbol{h}$ is the base height
    - h could also be the length of the prism, depending on how it is oriented
  - This is in the formula booklet in the **prior learning** section at the beginning
  - The base could be any shape so as long as you know its area and length you can calculate the volume of any prism



- Note two special cases:
  - To find the volume of a cuboid use the formula:

$$\text{Volume of a cuboid} = \text{length} \times \text{width} \times \text{height}$$
$$V = lwh$$

  - The volume of a **cylinder** can be found in the same way as a prism using the formula:

$$\text{Volume of a cylinder} = \pi r^2 h$$

  - where r is the radius, h is the height (or length, depending on the orientation
  - Note that a cylinder is technically not a prism as its base is not a polygon, however the method for finding its volume is the same
- Both of these are **in the formula booklet** in the **prior learning** section

How do I find the volume of pyramids and cones?

- In a **right-pyramid** the apex (the joining point of the triangular faces) is vertically above the centre of the base
  - The base can be any shape but is usually a square, rectangle or triangle
- To calculate the volume of a **right-pyramid** use the formula

$$V = \frac{1}{3} Ah$$

  - Where *A* is the area of the base, *h* is the height
  - Note that the height must be **vertical to the base**

- **A right cone** is a circular-based pyramid with the vertical height joining the apex to the centre of the circular base
- To calculate the volume of a **right-cone** use the formula

$$V = \frac{1}{3} \pi r^2 h$$

  - Where r is the radius, h is the height

- These formulae are both given in the formula booklet

How do I find the volume of a sphere?

- To calculate the volume of a **sphere** use the formula

$$V = \frac{4}{3} \pi r^3$$

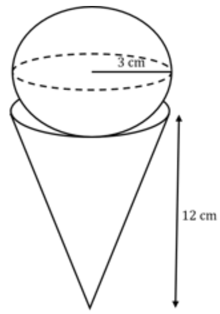
  - Where *r* is the radius
    - the line segment from the centre of the sphere to the surface
  - This formula is given in the formula booklet

Exam Tip

- Remember to make use of the formula booklet in the exam as all the volume formulae you need will be here
  - Formulae for basic 3D objects (cuboid, cylinder and prism) are in the **prior learning** section
  - Formulae for other 3D objects (pyramid, cone and sphere) are in the **Topic 3: Geometry** section

Worked Example

A dessert can be modelled as a right-cone of radius 3 cm and height 12 cm and a scoop of ice-cream in the shape of a sphere of radius 3 cm. Find the total volume of the ice-cream and cone.



Volume of a sphere: $V = \frac{4}{3}\pi r^3$ (In formula booklet)

Substitute: $r = 3 \Rightarrow V = \frac{4}{3}\pi \times 3^3$

$= 36\pi$

Volume of a right cone: $V = \frac{1}{3}\pi r^2 h$ (In formula booklet)

Substitute: $r = 3$, $h = 12 \Rightarrow V = \frac{1}{3}\pi(3)^2(12)$

$= 36\pi$

Total Volume $= 72\pi \text{ cm}^3$

Total Volume = 226 cm^3 (3sf)

Surface Area of 3D Shapes

What is surface area?

- The surface area of a 3D shape is the sum of the areas of all the **faces** that make up a shape
  - A **face** is one of the flat or curved surfaces that make up a 3D shape
  - It often helps to consider a 3D shape in the form of its 2D net

How do I find the surface area of cuboids, pyramids and prisms?

- Any prisms and pyramids that have polygons as their bases have only flat faces
  - The surface area is simply found by adding up the areas of these flat faces
  - Drawing a 2D net will help to see which faces the 3D shape is made up of

How do I find the surface area of cylinders, cones and spheres?

- Cones, cylinders and spheres all have curved faces so it is not always as easy to see their shape
  - The net of a **cylinder** is made up of two identical circles and a rectangle
  - The rectangle is the curved surface area and is harder to identify
  - The length of the rectangle is the same as the circumference of the circle
  - The area of the **curved surface area** is

$$A = 2\pi rh$$

    - where *r* is the radius, *h* is the height
  - This is given in the formula book in the prior learning section
  - The area of the **total surface area of a cylinder** is

$$A = 2\pi rh + 2\pi r^2$$

  - This is **not** given in the formula book, however it is easy to put together as both the area of a circle and the area of the curved surface area are given

- The net of a **cone** consists of the circular base along with the curved surface area
  - The area of the **curved surface area** is

$$A = \pi rl$$

    - Where *r* is the radius and *l* is the **slant height**
  - This is **given in the formula book**
    - Be careful not to confuse the slant height, *l*, with the vertical height, *h*
    - Note that *r*, *h* and *l* will create a **right-triangle** with *l* as the hypotenuse
  - The area of the **total surface area of a cone** is

$$A = \pi rl + \pi r^2$$

  - This is **not** given in the formula book, however it is easy to put together as both the area of a circle and the area of the curved surface area are given

- To find the surface area of a **sphere** use the formula

$$A = 4\pi r^2$$

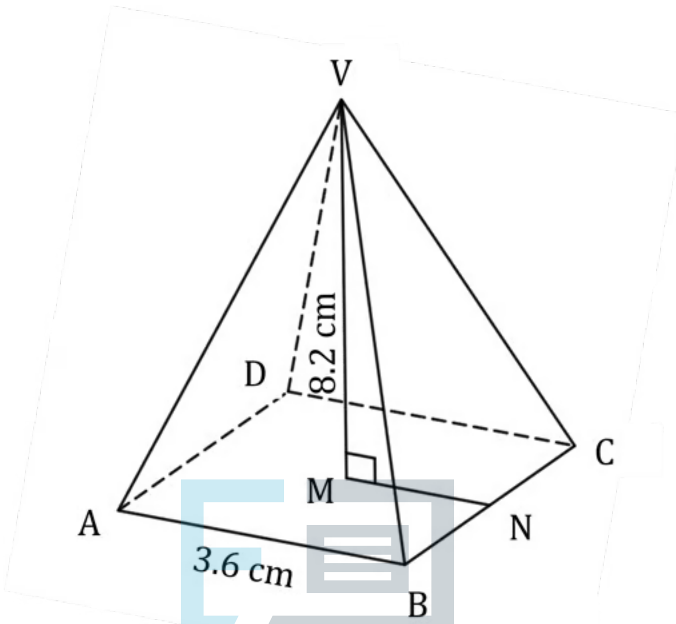
    - where r is the radius (line segment from the centre to the surface)
  - This is given in the formula booklet, you do not have to remember it

Exam Tip

- Remember to make use of the formula booklet in the exam as all the area formulae you need will be here
  - Formulae for basic 2D shapes (parallelogram, triangle, trapezoid, circle, curved surface of a cylinder) are in the **prior learning** section
  - Formulae for other 2D shapes (curved surface area of a cone and surface area of a sphere) are in the **Topic 3: Geometry** section

Worked Example

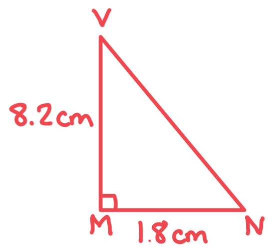
In the diagram below $ABCD$ is the square base of a right pyramid with vertex V. The centre of the base is M. The sides of the square base are 3.6 cm and the vertical height is 8.2 cm.



i)

Use the Pythagorean Theorem to find the distance VN.

Sketch the triangle MNV:



M is the midpoint
so MN = 3.6 ÷ 2

By the Pythagorean Theorem:

$$VN^2 = \sqrt{VM^2 + MN^2}$$

$$= \sqrt{8.2^2 + 1.8^2}$$

$$= \sqrt{70.48}$$

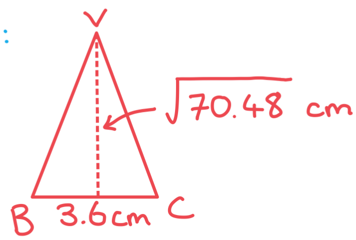
$VN = 8.40$ cm (3sf)

ii)

Calculate the area of the triangle ABV.

Area $\triangle ABV$ = area $\triangle BCV$

Sketch $\triangle BCV$:



Area of a triangle $= \frac{1}{2}bh$

Substitute $b = 3.6$, $h = \sqrt{70.48}$

$$\text{Area} = \frac{1}{2}(3.6)(\sqrt{70.48})$$

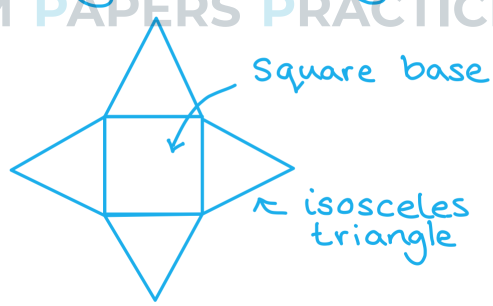
$$= 15.111\ldots \text{ cm}^2$$

Area $\triangle ABV = 15.1 \text{ cm}^2$

iii)

Find the surface area of the right pyramid.

Considering the net may help:



Surface area = area square + 4(area triangle)

$$SA = 3.6^2 + 4(15.111\ldots)$$

$$= 73.405\ldots \text{ cm}^2$$

$SA = 73.4 \text{ cm}^2$ (3sf)

3.2 Geometry Toolkit

3.2.1 Coordinate Geometry

Basic Coordinate Geometry

What are cartesian coordinates?

- **Cartesian** coordinates are basically the *x*-*y* coordinate system
  - They allow us to label where things are in a two-dimensional plane
- In the 2D cartesian system, the horizontal axis is labelled *x* and the vertical axis is labelled *y*

What can we do with coordinates?

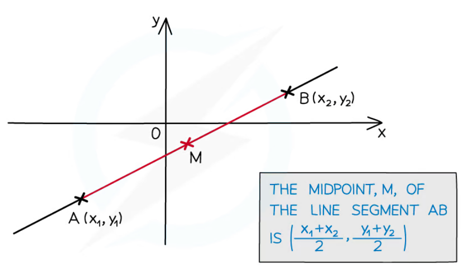
- If we have two points with coordinates (x_1, y_1) and (x_2, y_2) then we should be able to find
  - The **midpoint** of the two points
  - The **distance** between the two points
  - The **gradient** of the line between them

How do I find the midpoint of two points?

- The midpoint is the **average (middle) point**
  - It can be found by finding the middle of the *x*-coordinates and the middle of the *y*-coordinates
- The coordinates of the midpoint will be

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

  - This is given in the formula booklet under the prior learning section at the beginning



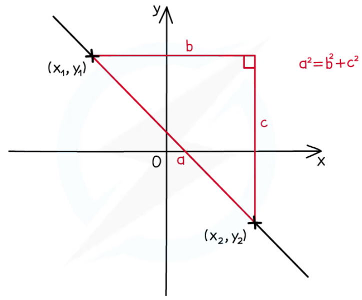
How do I find the distance between two points?

- The distance between two points with coordinates (x_1, y_1) and (x_2, y_2) can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

  - This is given in the formula booklet in the *prior learning* section at the beginning
- Pythagoras' Theorem $a^2 = b^2 + c^2$ is used to find the length of a line between two coordinates

- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]



How do I find the gradient of the line between two points?

- The gradient of a line between two points with coordinates (x_1, y_1) and (x_2, y_2) can be found using the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  - This is given in the formula booklet under section 2.1 Gradient formula
  - This is usually known as $m = \frac{\text{rise}}{\text{run}}$

Worked Example

Point A has coordinates (3, -4) and point B has coordinates (-5, 2).

i)

Calculate the distance of the line segment AB.

A: (3, −4), B: (−5, 2)

$x_1 = 3$, $y_1 = -4$, $x_2 = -5$, $y_2 = 2$

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Sub coordinates for A and B into the formula:

$$d = \sqrt{(3-(-5))^2 + (-4-2)^2}$$

$$= \sqrt{8^2 + (-6)^2} = \sqrt{100}$$

$d = 10$ units

ii)

Find the gradient of the line connecting points A and B.

A: (3, −4), B: (−5, 2)

$x_1 = 3$, $y_1 = -4$, $x_2 = -5$, $y_2 = 2$

Formula for gradient of a line segment:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Sub coordinates for A and B into the formula:

$$m = \frac{2 - -4}{-5 - 3} = \frac{6}{-8} = -\frac{3}{4}$$

$m = -\frac{3}{4}$

iii)

Find the midpoint of $[AB]$.

A: (3, −4) B: (−5, 2)

x_1, y_1, x_2, y_2

Formula for the midpoint of two coordinates:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Sub values in:

$$\text{Midpoint} = \left(\frac{3 + (-5)}{2}, \frac{-4 + 2}{2}\right) = (-1, -1)$$

Midpoint = (−1, −1)

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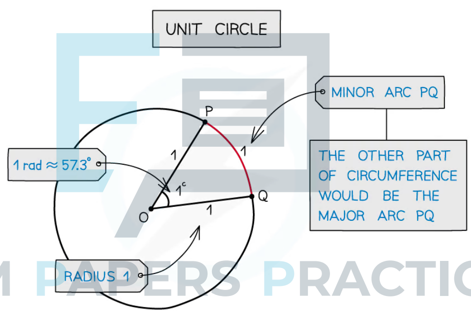
3.2.2 Radian Measure

Radian Measure

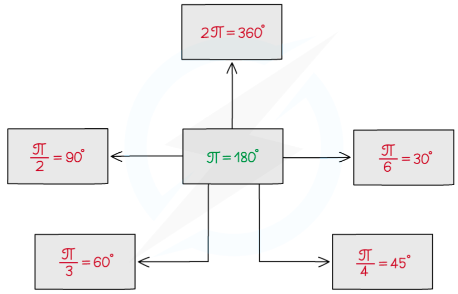
What are radians?

- Radians are an alternative to degrees for measuring angles
- 1 radian is the angle in a **sector** of radius 1 and arc length 1
  - A circle with radius 1 is called a **unit circle**
- Radians are normally quoted in terms of π
  - 2π radians = 360°
  - π radians = 180°
- The symbol for radians is ^{c} but it is more usual to see **rad**
  - Often, when π is involved, no symbol is given as it is obvious it is in radians
  - Whilst it is okay to omit the symbol for radians, you should never omit the symbol for degrees
- In the exam you should use radians unless otherwise indicated



How do I convert between radians and degrees?

- Use $\pi^c = 180°$ to convert between radians and degrees
  - To convert from radians to degrees multiply by $\frac{180}{\pi}$
  - To convert from degrees to radians multiply by $\frac{\pi}{180}$
- Some of the common conversions are:
  - $2\pi^c = 360°$
  - $\pi^c = 180°$
  - $\frac{\pi}{2}^c = 90°$
  - $\frac{\pi}{3}^c = 60°$
  - $\frac{\pi}{4}^c = 45°$
  - $\frac{\pi}{6}^c = 30°$
- It is a good idea to remember some of these and use them to work out other conversions
- Your GDC will be able to work with both radians and degrees



Exam Tip

- Sometimes an exam question will specify whether you should be using degrees or radians and sometimes it will not, if it doesn't it is expected that you will work in radians
- If the question involves π then working in radians is useful as there will likely be opportunities where you can cancel out π
- Make sure that your calculator is in the correct mode for the type of angle you are working with

Worked Example

i)

Convert 43.8° to radians.

43.8° $\div 180^\circ$ ($\pi^c = 180^\circ$)

$\frac{73}{300}$ $\times \pi^c$

$\frac{73\pi}{300}^c$

$43.8^\circ = 0.764^c$ (3 s.f.)

ii) Convert $\frac{5\pi}{4}$ to degrees.

$\frac{5\pi}{4}$ $\div \pi^c$ ($\pi^c = 180^\circ$)

$\frac{5}{4}$ $\times 180^\circ$

225°

$\frac{5\pi}{4} = 225^\circ$

3.2.3 Arcs & Sectors

Length of an Arc

What is an arc?

- An arc is a part of the **circumference** of a circle
  - It is easiest to think of it as the crust of a single slice of pizza
- The length of an arc depends of the size of the angle at the centre of the circle
- If the angle at the centre is **less than 180°** then the arc is known as a **minor arc**
  - This could be considered as the crust of a single slice of pizza
- If the angle at the centre is **more than 180°** then the arc is known as a **major arc**
  - This could be considered as the crust of the remaining pizza after a slice has been taken away

How do I find the length of an arc?

- The length of an arc is simply a fraction of the circumference of a circle
  - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the length, l, of an arc is

$$l = \frac{\theta}{360} \times 2\pi r$$

  - Where θ is the angle measured in degrees
  - r is the radius
  - This is **in the formula booklet for radian measure only**
    - Remember 2π radians = 360°

Exam Tip

- Make sure that you read the question carefully to determine if you need to calculate the arc length of a sector, the perimeter or something else that incorporates the arc length!

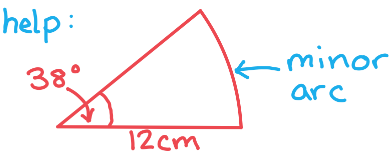
Worked Example

A circular pizza has had a slice cut from it, the angle of the slice that was cut was 38 °. The radius of the pizza is 12 cm. Find

i)

the length of the outside crust of the slice of pizza (the minor arc),

A diagram will help:



Formula for the length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

$$l = \frac{38}{360} \times 2\pi(12)$$

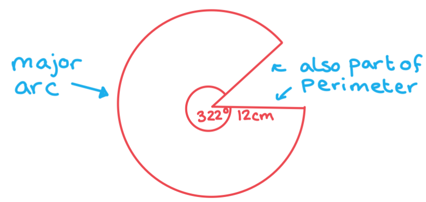
$$= \frac{38\pi}{15} = 7.9587\ldots \text{ cm}$$

Length of crust = 7.96 cm (3s.f)

ii)

the perimeter of the remaining pizza.

A diagram will help:



Formula for the length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

$$l = \frac{322}{360} \times 2\pi(12)$$

$$= \frac{322\pi}{15}$$ ← Length of major arc

Find perimeter:

P = major arc + radius + radius

$$= \frac{322\pi}{15} + 12 + 12 = 91.4395\ldots \text{ cm}$$

Perimeter = 91.4 cm (3s.f)

Area of a Sector

What is a sector?

- A sector is a part of a circle enclosed by two radii (radiuses) and an arc
  - It is easier to think of this as the shape of a single slice of pizza
- The area of a sector depends of the size of the angle at the centre of the sector
- If the angle at the centre is **less than 180°** then the sector is known as a **minor sector**
  - This could be considered as the shape of a single slice of pizza
- If the angle at the centre is **more than 180°** then the sector is known as a **major sector**
  - This could be considered as the shape of the remaining pizza after a slice has been taken away

How do I find the area of a sector?

- The area of a sector is simply a fraction of the area of the whole circle
  - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the area, A, of a sector is

$$A = \frac{\theta}{360} \times \pi r^2$$

  - Where θ is the angle measured in degrees
  - r is the radius
  - This is **in the formula booklet for radian measure only**
    - Remember 2π radians = 360°

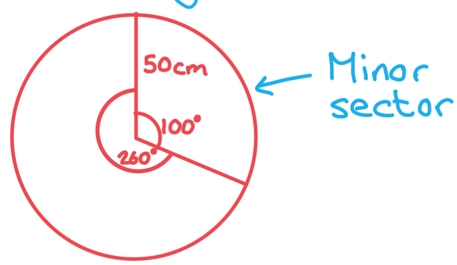
Worked Example

Jamie has divided a circle of radius 50 cm into two sectors; a minor sector of angle 100° and a major sector of angle 260°. He is going to paint the minor sector blue and the major sector yellow. Find

i)

the area Jamie will paint blue,

Start with a diagram:



Formula for the area of a sector:

$$A = \frac{\theta}{360°} \times \pi r^2$$

Substitute: $A = \frac{100}{360} \times \pi \times 50^2$

$$= \frac{6250}{9}\pi$$

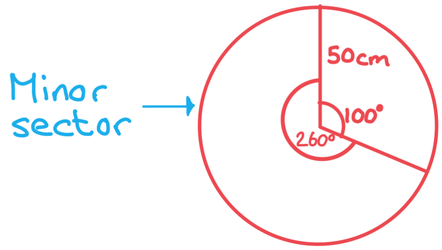
$$= 2181.66\ldots \text{ cm}^2$$

Blue area = 2180 cm^2 (3sf)

ii)

the area Jamie will paint yellow.

Start with a diagram:



Formula for the area of a sector:

$$A = \frac{\theta}{360°} \times \pi r^2$$

Substitute: $A = \frac{260}{360} \times \pi \times 50^2$

$= \frac{16250}{9}\pi$

$= 5672.32... \text{ cm}^2$

Yellow area = 5670 cm² (3sf)

Arcs & Sectors Using Radians

How do I use radians to find the length of an arc?

- As the radian measure for a **full turn** is 2π, the fraction of the circle becomes $\frac{\theta}{2\pi}$
- Working in radians, the formula for the length of an arc will become

$$l = \frac{\theta}{2\pi} \times 2\pi r$$

- Simplifying, the formula for the length, l, of an arc is

$$l = r\theta$$

  - θ is the angle measured in **radians**
  - r is the radius
  - This is **given in the formula booklet,** you do not need to remember it

How do I use radians to find the area of a sector?

- As the radian measure for a **full turn** is 2π, the fraction of the circle becomes $\frac{\theta}{2\pi}$
- Working in radians, the formula for the area of a sector will become

$$A = \frac{\theta}{2\pi} \times \pi r^2$$

- Simplifying, the formula for the area, A, of a sector is

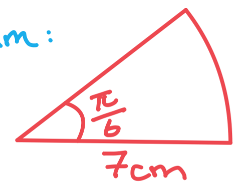
$$A = \frac{1}{2} r^2 \theta$$

  - θ is the angle measured in **radians**
  - r is the radius
  - This is **given in the formula booklet,** you do not need to remember it

Worked Example

A slice of cake forms a sector of a circle with an angle of $\frac{\pi}{6}$ radians and radius of 7 cm. Find the area of the surface of the slice of cake and its perimeter.

Draw a diagram:



Area of a sector: $A = \frac{1}{2}r^2\theta$

Substitute: $r = 7, \ \theta = \frac{\pi}{6}$

$$A = \frac{1}{2}(7)^2\left(\frac{\pi}{6}\right) = \frac{49\pi}{12}$$

Area = 12.8 cm² (3 s.f.)

Perimeter = arc length + 2(radius)

Length of an arc: $l = r\theta$

$$P = 7\left(\frac{\pi}{6}\right) + 2(7)$$

Perimeter = 17.7 cm (3 s.f.)

3.3 Trigonometry Toolkit

3.3.1 Pythagoras & Right- Angled Trigonometry

Pythagoras

What is the Pythagorean theorem?

- Pythagoras' theorem is a formula that works for **right-angled triangles** only
- It states that for any right-angled triangle, the **square of the hypotenuse is equal to the sum of the squares of the two shorter sides**
  - The **hypotenuse** is the **longest side** in a right-angled triangle
    - It will always be **opposite** the right angle
  - If we label the hypotenuse c, and label the other two sides *a* and *b*, then Pythagoras' theorem tells us that

$$a^2 + b^2 = c^2$$

- The formula for Pythagoras' theorem is assumed prior knowledge and is **not in the formula booklet**
  - You will need to remember it

How can we use Pythagoras' theorem?

- If you know two sides of any right-angled triangle you can use Pythagoras' theorem to find the length of the third side
  - Substitute the values you have into the formula and either solve or rearrange
- To find the length of the **hypotenuse** you can use:

$$c = \sqrt{a^2 + b^2}$$

- To find the length of **one of the other sides** you can use:

$$a = \sqrt{c^2 - b^2} \text{ or } b = \sqrt{c^2 - a^2}$$

- Note that when finding the **hypotenuse** you should **add** inside the square root and when finding **one of the other sides** you should **subtract** inside the square root
- Always **check** your answer carefully to make sure that the hypotenuse is the longest side
- Note that Pythagoras' theorem questions will rarely be standalone questions and will often be 'hidden' in other geometry questions

What is the converse of the Pythagorean theorem?

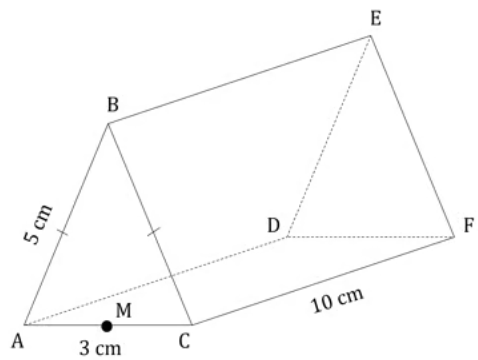
- The converse of the Pythagorean theorem states that if $a^2 + b^2 = c^2$ is true then the triangle must be a right-angled triangle
  - This is a very useful way of determining whether a triangle is right-angled
- If a diagram in a question does not clearly show that something is right-angled, you may need to use Pythagoras' theorem to check

Exam Tip

- Pythagoras' theorem pops up in lots of exam questions so bear it in mind whenever you see a right-angled triangle in an exam question!

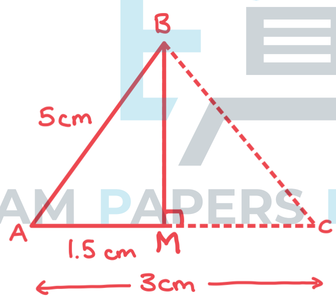
Worked Example

ABCDEF is a chocolate bar in the shape of a triangular prism. The end of the chocolate bar is an isosceles triangle where AC = 3 cm and AB = BC = 5 cm. M is the midpoint of AC. This information is shown in the diagram below.



Calculate the length BM.

Sketch the triangle ABM:



By the Pythagorean Theorem:

$$BM^2 = \sqrt{AB^2 - AM^2}$$

(BM: shorter side; AB: hypotenuse; AM: shorter side)

$$= \sqrt{5^2 - 1.5^2}$$

$$= \sqrt{22.75}$$

BM = 4.77 cm (3sf)

Right-Angled Trigonometry

What is Trigonometry?

- Trigonometry is the mathematics of angles in triangles
- It looks at the relationship between side lengths and angles of triangles
- It comes from the Greek words *trigonon* meaning 'triangle' and *metron* meaning 'measure'

What are Sin, Cos and Tan?

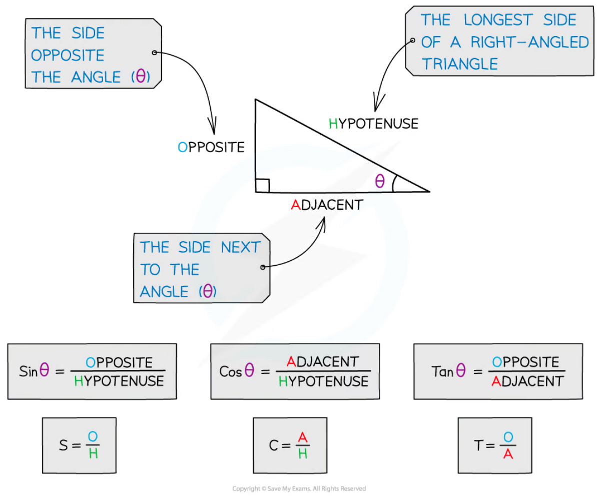
- The three trigonometric functions Sine, Cosine and Tangent come from ratios of side lengths in right-angled triangles
- To see how the ratios work you must first label the sides of a right-angled triangle in relation to a chosen angle
  - The **hypotenuse, *H*,** is the **longest side** in a right-angled triangle
    - It will always be **opposite** the right angle
  - If we label one of the other angles θ, the side opposite θ will be labelled **opposite**, ***O*,** and the side next to θ will be labelled **adjacent**, ***A***
- The functions Sine, Cosine and Tangent are the ratios of the lengths of these sides as follows

$$\text{Sin } \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H}$$

$$\text{Cos } \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{A}}{\text{H}}$$

$$\text{Tan } \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{O}}{\text{A}}$$

  - These are **not in the formula book**, you must remember them
- The mnemonic **SOHCAHTOA** is often used as a way of remembering which ratio is which
  - **S**in is **O**pposite over **H**ypotenuse
  - **C**os is **A**djacent over **H**ypotenuse
  - **T**an is **O**pposite over **A**djacent



How can we use SOHCAHTOA to find missing lengths?

- If you know the length of one of the sides of any right-angled triangle and one of the angles you can use SOHCAHTOA to find the length of the other sides
  - Always start by **labelling the sides** of the triangle with H, O and A
  - Choose the correct ratio by looking only at the values that you have and that you want
    - For example if you know the angle and the side opposite it (O) and you want to find the hypotenuse (H) you should use the sine ratio
  - Substitute the values into the ratio
  - Use your calculator to find the solution

How can we use SOHCAHTOA to find missing angles?

- If you know two sides of any right-angled triangle you can use SOHCAHTOA to find the size of one of the angles
- Missing angles are found using the **inverse functions**:

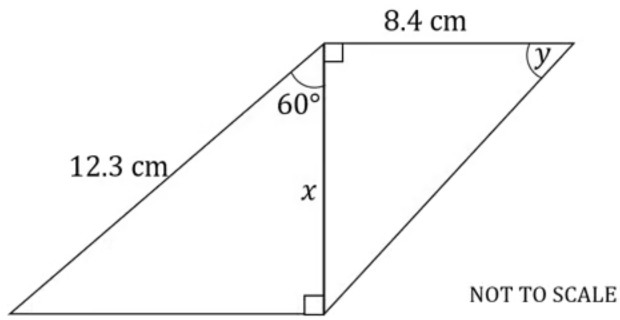
$$\theta = \text{Sin}^{-1}\frac{O}{H}\ ,\ \theta = \text{Cos}^{-1}\frac{\text{A}}{\text{H}}\ ,\ \theta = \text{Tan}^{-1}\frac{\text{O}}{\text{A}}$$

- After choosing the correct ratio and substituting the values use the inverse trigonometric functions on your calculator to find the correct answer

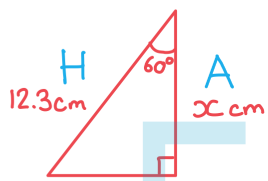
> **Exam Tip**
>
> - You need to remember the sides involved in the different trig ratios as they are not given to you in the exam

Worked Example

Find the values of x and y in the following diagram. Give your answers to 3 significant figures.



Start by labelling the sides of the triangle:



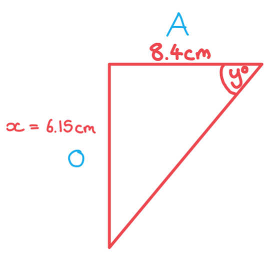
SOHCAHTOA

We know H and we want to find A so we need to use $\cos\theta = \frac{A}{H}$

$\cos 60° = \frac{x}{12.3}$

$x = 12.3\cos 60°$

$x = 6.15$ cm



SOHCAHTOA

$\tan y° = \frac{O}{A}$

$\tan y° = \frac{6.15}{8.4}$

$y° = \tan^{-1}\left(\frac{6.15}{8.4}\right)$

$y° = 36.2°$ (3 s.f.)

3D Problems

How does Pythagoras work in 3D?

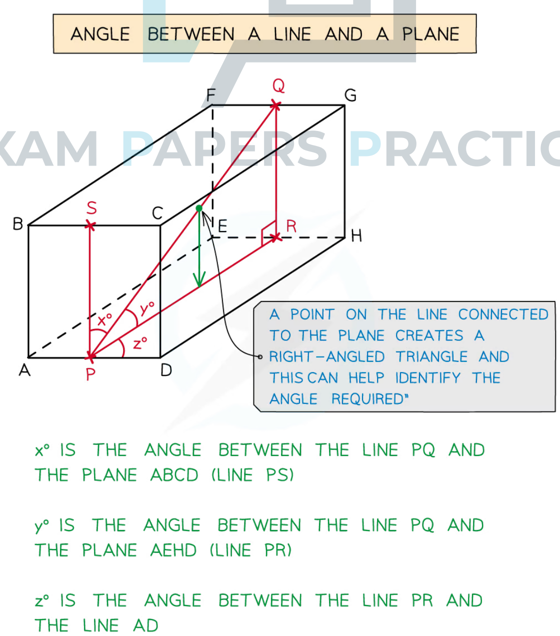
- 3D shapes can often be broken down into several 2D shapes
- With Pythagoras' Theorem you will be specifically looking for right-angled triangles
    - The right-angled triangles you need will have two known sides and one unknown side
    - Look for perpendicular lines to help you spot right-angled triangles
- There is a 3D version of the Pythagorean theorem formula:

$$d^2 = x^2 + y^2 + z^2$$

    - However it is usually easier to see a problem by breaking it down into two or more 2D problems

How does SOHCAHTOA work in 3D?

- Again look for a combination of right-angled triangles that would lead to the missing angle or side
- The angle you are working with can be awkward in 3D
    - The angle between a line and a plane is not always obvious
    - If unsure put a point on the line and draw a new line to the plane
        - This should create a right-angled triangle



Exam Tip

- Annotate diagrams that are given to you with values that you have calculated
- It can be useful to make additional sketches of parts of any diagrams that are given to you, especially if there are multiple lengths/angles that you are asked to find
- If you are not given a diagram, sketch a nice, big, clear one!

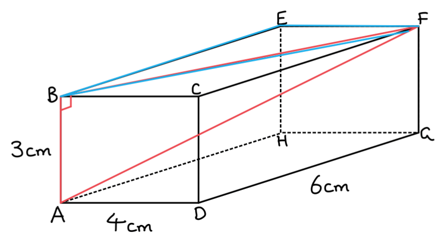
Worked Example

A pencil is being put into a cuboid shaped box which has dimensions 3 cm by 4 cm by 6 cm. Find:

a)

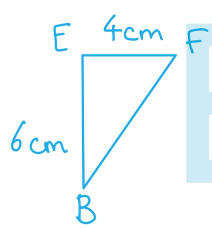
the length of the longest pencil that could fit inside the box,

Draw a diagram:



The longest pencil could fit on any of the diagonals, e.g. AF.

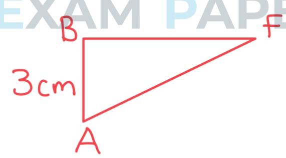
To find AF we must first find BF:



$BF^2 = 4^2 + 6^2$

$BF^2 = 16 + 36$

$BF^2 = 52$ ← can leave as BF^2 for now.



$AF^2 = 3^2 + BF^2$

$= 9 + 52$

$AF^2 = 61$

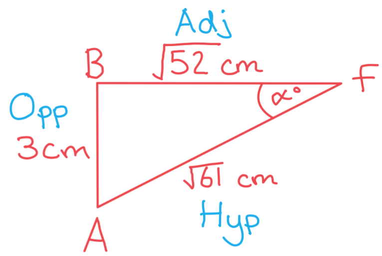
$AF = \sqrt{61} = 7.8102...$

7.81 cm (3 s.f.)

b)

the angle that the pencil would make with the top of the box.

Find $A\hat{F}B$:



All three sides are known so can use any of the trig ratios.

S^{O}H C^{A}H T^{O}A

Choose $\tan\alpha = \frac{opp}{adj}$

$$\tan\alpha = \frac{3}{\sqrt{52}}$$

$$\alpha = \tan^{-1}\left(\frac{3}{\sqrt{52}}\right)$$

$$= 22.588...$$

$$A\hat{F}B = 22.6° \text{ (3 s.f.)}$$

3.3.2 Non Right- Angled Trigonometry

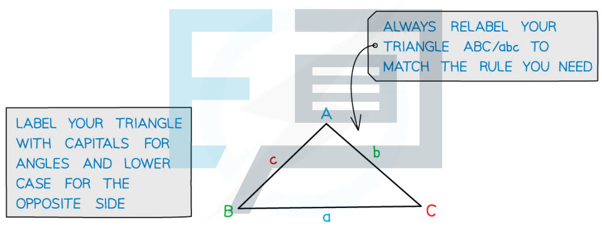
Sine Rule

What is the sine rule?

- The sine rule allows us to find missing side lengths or angles in **non-right-angled triangles**
- It states that for any triangle with angles *A*, *B* and *C*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

  - Where
    - *a* is the side **opposite** angle *A*
    - *b* is the side **opposite** angle *B*
    - *c* is the side **opposite** angle *C*
- This formula **is in the formula booklet**, you do not need to remember it
- Sin 90° = 1 so if one of the angles is 90° this becomes SOH from **SOHCAHTOA**



How can we use the sine rule to find missing side lengths or angles?

- The sine rule can be used when you have any opposite pairs of sides and angles
- Always **start by labelling your triangle** with the angles and sides
  - Remember the sides with the lower-case letters are **opposite** the angles with the equivalent upper-case letters
- Use the formula in the formula booklet to find the **length of a side**
- To find a missing angle you can rearrange the formula and use the form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

  - This is **not in the formula booklet** but can easily be found by rearranging the one given

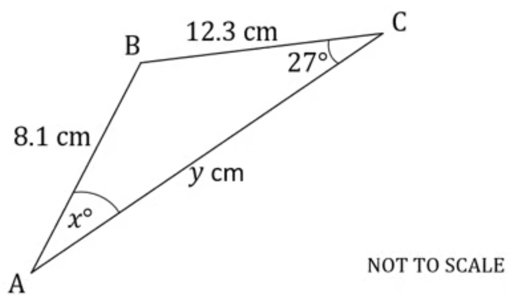
- Substitute the values you have into the formula and solve

Exam Tip

- If you're using a calculator make sure that it is in the correct mode (degrees/radians)
- Remember to give your answers as exact values if you are asked too

Worked Example

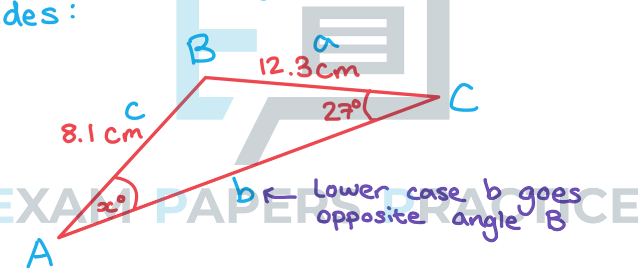
The following diagram shows triangle ABC. $AB = 8.1$ cm, $AC = 12.3$ cm, $B\widehat{C}A = 27°$.



Use the sine rule to calculate the value of:

i)
x,

Sketch the diagram and label the sides:



Using the sine rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

← We are looking for an angle so this version is easier.

$$\frac{\sin x}{12.3} = \frac{\sin 27}{8.1}$$

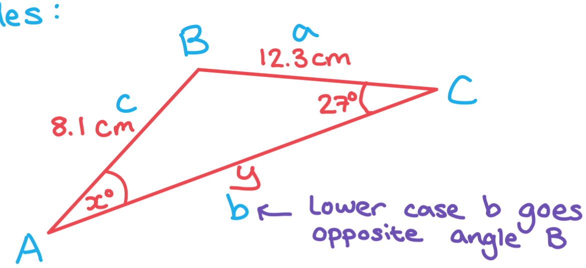
$$\sin x = \frac{12.3 \sin 27}{8.1}$$

$$x = \sin^{-1}\left(\frac{12.3 \sin 27}{8.1}\right)$$

$$x = 43.6° \text{ (3 s.f.)}$$

ii)

Sketch the diagram and label the sides:



Find $A\hat{B}C$: $180-(27+43.582...)$

$A\hat{B}C = 109.417...$

Using the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

← We are looking for a side so this version is easier.

$$\frac{y}{\sin(109.417...)} = \frac{8.1}{\sin 27}$$

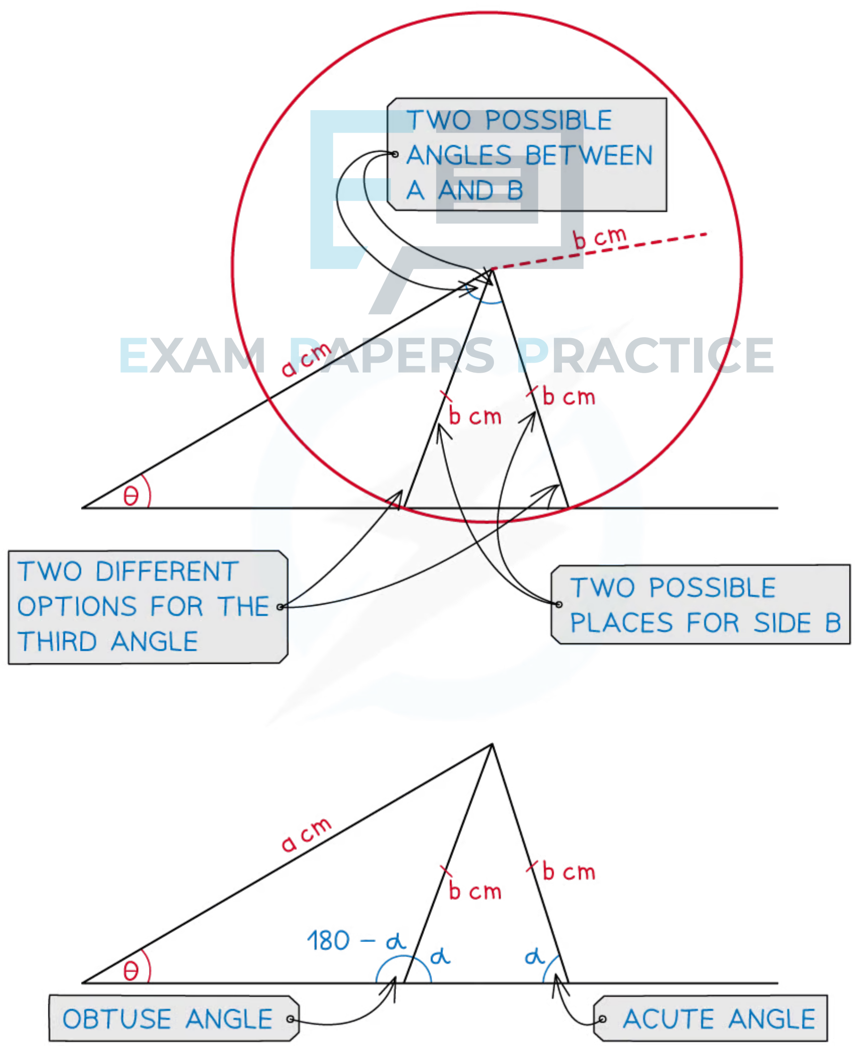
$$y = \frac{8.1\sin(109.417...)}{\sin 27}$$

$y = 16.8\text{cm}$ (3 s.f.)

Ambiguous Sine Rule

What is the ambiguous case of the sine rule?

- If the sine rule is used in a triangle **given two sides and an angle which is not the angle between them** there **may** be more than one possible triangle which could be drawn
- The side **opposite** the given angle could be in two possible positions
- This will create two possible values for each of the missing angles and two possible lengths for the missing side
- The two angles found **opposite** the given side (not the ambiguous side) will **add up to 180°**
  - In IB the question will usually tell you whether the angle you are looking for is **acute** or **obtuse**
  - The sine rule will always give you the acute option but you can **subtract from 180°** to find the obtuse angle
  - Sometimes the obtuse angle will not be valid
    - It could cause the sum of the three interior angles of the triangle to exceed 180°



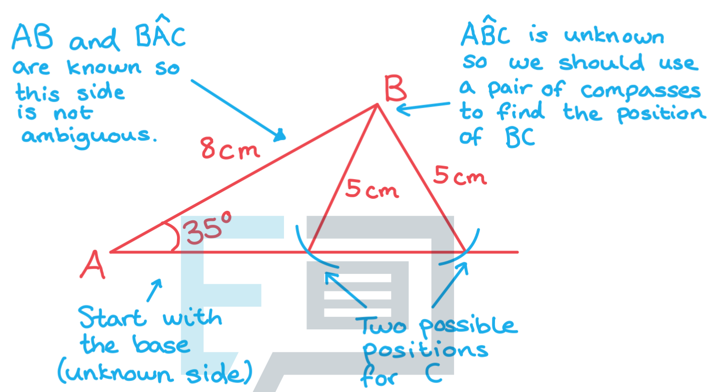
Exam Tip

- Make sure that you are clear which of the two answers is the one that is required and make sure that you communicate this clearly to the examiner by writing it on the answer line!

Worked Example

Given triangle ABC, $\mathrm{AB} = 8\ \mathrm{cm}$, $\mathrm{BC} = 5\ \mathrm{cm}$, $\mathrm{B\widehat{A}C} = 35°$. Find the two possible options for $\mathrm{A\widehat{C}B}$, giving both answers to 1 decimal place.

There are two ways triangle ABC can be drawn:



Find $\mathrm{A\widehat{C}B}$: $\dfrac{\sin 35°}{5} = \dfrac{\sin C}{8}$

$$C = \sin^{-1}\left(\frac{8\sin 35°}{5}\right)$$

$$= 66.59...$$

$\mathrm{A\widehat{C}B} = 66.6°$ or $113.4°$ (1dp)

Cosine Rule

What is the cosine rule?

- The cosine rule allows us to find missing side lengths or angles in **non-right-angled triangles**
- It states that for any triangle

$$c^2 = a^2 + b^2 - 2ab\cos C \; ; \; \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

  - Where
    - *c* is the side **opposite** angle C
    - *a* and *b* are the other two sides
- Both of these formulae **are in the formula booklet**, you do not need to remember them
  - The first version is used to find a missing side
  - The second version is a rearrangement of this and can be used to find a missing angle
- Cos 90° = 0 so if C = 90° this becomes **Pythagoras' Theorem**

How can we use the cosine rule to find missing side lengths or angles?

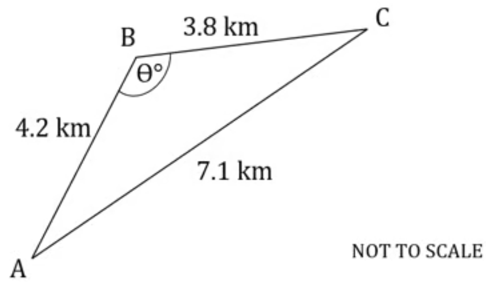
- The cosine rule can be used when you have two sides and the angle between them or all three sides
- Always **start by labelling your triangle** with the angles and sides
  - Remember the sides with the lower-case letters are **opposite** the angles with the equivalent upper-case letters
- Use the formula $c^2 = a^2 + b^2 - 2ab\cos C$ to find an unknown side
- Use the formula $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ to find an unknown angle
  - C is the angle **between** sides *a* and *b*
- Substitute the values you have into the formula and solve

Exam Tip

- If you're using a calculator make sure that it is in the correct mode (degrees/radians)
- Remember to give your answers as exact values if you are asked too

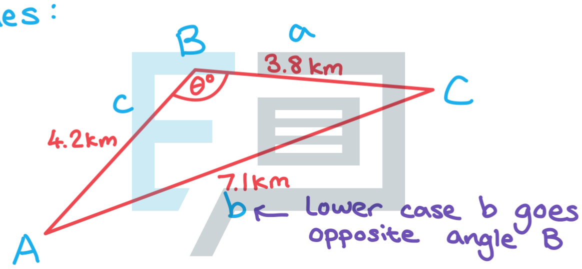
Worked Example

The following diagram shows triangle ABC. $AB = 4.2$ km, $BC = 3.8$ km, $AC = 7.1$ km.



Calculate the value of $A\hat{B}C$.

Sketch the diagram and label the sides:



Using the cosine rule:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

← We are looking for an angle so this version is easier.

$$\cos \theta = \frac{4.2^2 + 3.8^2 - 7.1^2}{2(4.2)(3.8)}$$

$$\theta = \cos^{-1}\left(\frac{4.2^2 + 3.8^2 - 7.1^2}{2(4.2)(3.8)}\right)$$

$$\theta = 128° \text{ (3 s.f.)}$$

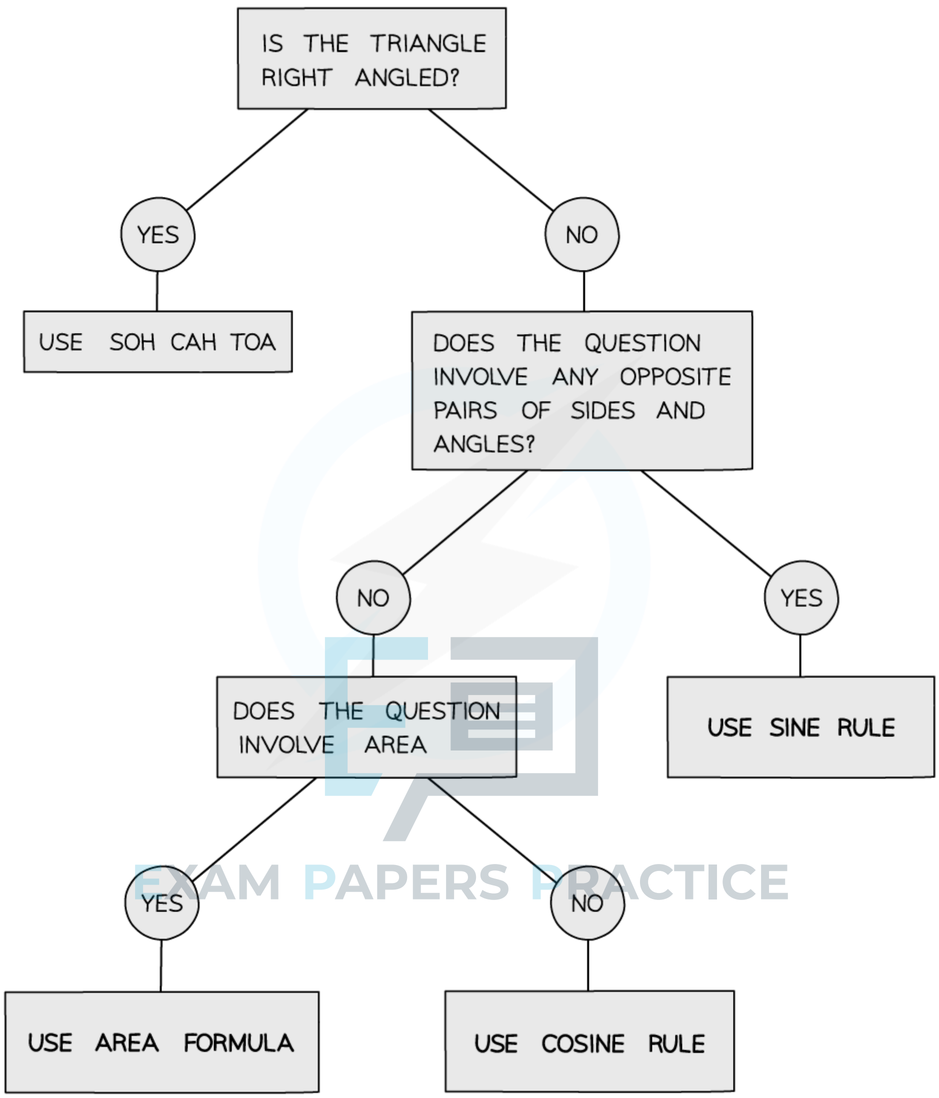
Area of a Triangle

How do I find the area of a non-right triangle?

- The area of **any triangle** can be found using the formula

$$A = \frac{1}{2}ab\sin C$$

  - Where C is the angle between sides *a* and *b*
  - This formula **is in the formula booklet**, you do not need to remember it
- Be careful to label your triangle correctly so that C is always the angle **between** the two sides
- Sin 90° = 1 so if C = 90° this becomes Area = ½ × base × height

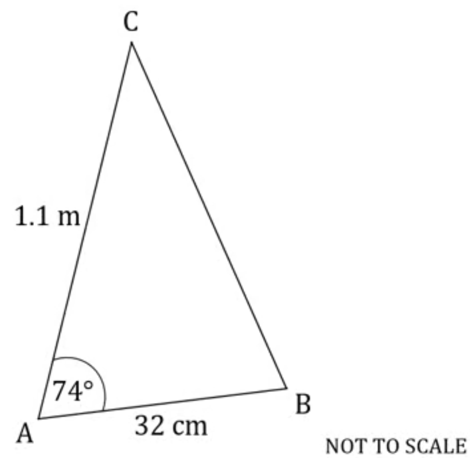


Exam Tip

- If you're using a calculator make sure that it is in the correct mode (degrees/radians)
- Remember to give your answers as exact values if you are asked too

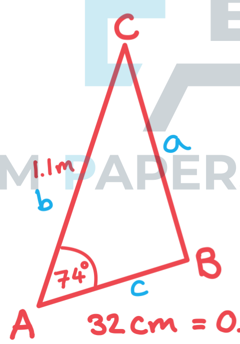
Worked Example

The following diagram shows triangle ABC. AB = 32 cm, AC = 1.1 m, $B\widehat{A}C = 74°$.



Calculate the area of triangle .

Label the sides of the triangle:



32 cm = 0.32 m

↖ change all units to be the same

Area of a triangle: $A = \frac{1}{2}ab\sin C$

$A = \frac{1}{2}(1.1)(0.32)\sin 74°$

$A = 0.169 \text{ m}^2$

3.3.3 Applications of Trigonometry & Pythagoras

Bearings

What are bearings?

- **Bearings** are a way of describing and using **directions** as **angles**
- They are specifically defined for use in navigation because they give a precise **location** and/or **direction**

How are bearings defined?

- There are **three rules** which must be followed every time a bearing is defined
  - They are **measured** from the **North** direction
    - An arrow showing the North line should be included on the diagram
  - They are **measured clockwise**
  - The angle is always written in **3 figures**
    - If the angle is less than 100° the first digit will be a zero

What are bearings used for?

- Bearings questions will normally involve the use of Pythagoras or trigonometry to find missing distances (lengths) and directions (angles) within navigation questions
  - You should always **draw a diagram**
- There may be a scale given or you may need to consider using a scale
  - However normally in IB you will be using triangle calculations to find the distances
- Some questions may also involve the use of angle facts to find the missing directions
- To answer a question involving **drawing bearings** the following steps may help:
  - STEP 1: Draw a diagram adding in any points and distances you have been given
  - STEP 2: Draw a North line (arrow pointing vertically up) at the point you wish to measure the bearing **from**
    - If you are given the bearing **from A to B** draw the North line at **A**
  - STEP 3: Measure the angle of the bearing given **from the North line** in the **clockwise direction**
  - STEP 4: Draw a line and add the point B at the given distance
- You will likely then need to use trigonometry to calculate the shortest distance or another given distance

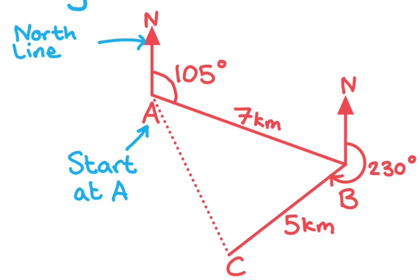
Exam Tip

- **Always** draw a big, clear diagram and annotate it, be especially careful to label the angles in the correct places!

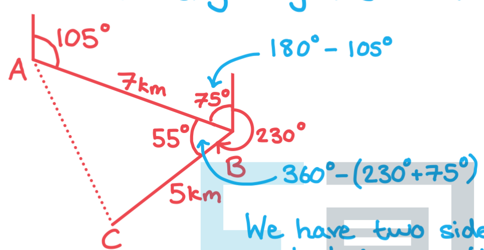
Worked Example

The point B is 7 km from A on a bearing of 105°. The distance from B to C is 5 km and the bearing from B to C is 230°. Find the distance from A to C.

Always start with a diagram:



Fill in the angles you can on the diagram



We have two sides and the angle between them so we can use the cosine rule for the third side

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$AC^2 = 7^2 + 5^2 - 2(7)(5)\cos(55°)$$

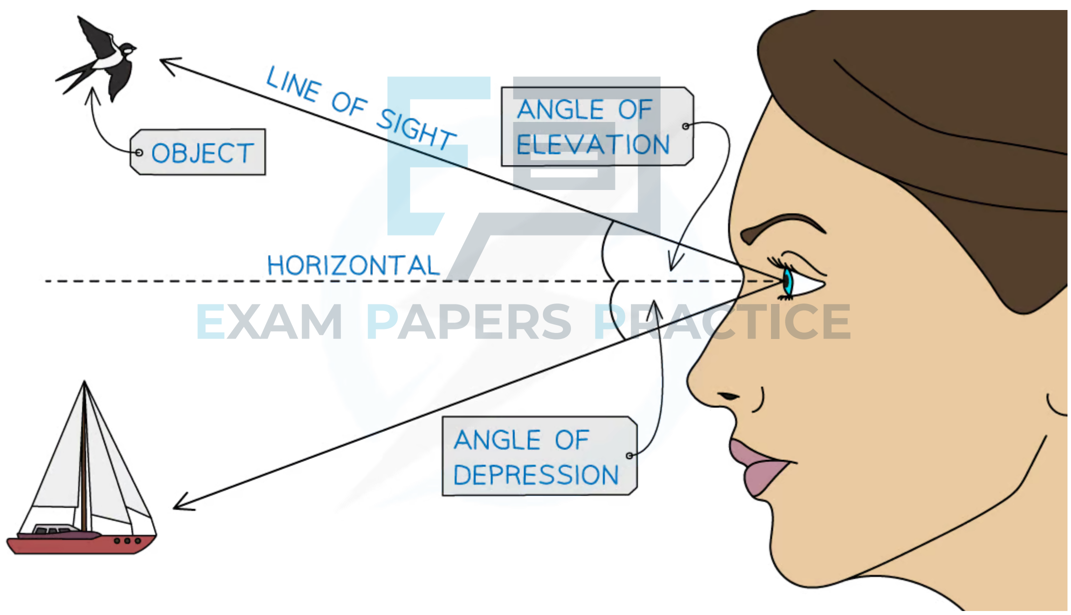
$$= 33.849\ldots$$

$AC = 5.82$ km (3 s.f.)

Elevation & Depression

What are the angles of elevation and depression?

- If a person looks at an **object** that is not on the same horizontal line as their eye-level they will be looking at either an angle of **elevation** or **depression**
  - If a person looks **up** at an object their line of sight will be at an **angle of elevation** with the horizontal
  - If a person looks **down** at an object their line of sight will be at an **angle of depression** with the horizontal
- Angles of elevation and depression are measured **from the horizontal**
- **Right-angled trigonometry** can be used to find an angle of elevation or depression or a missing distance
- Tan is often used in real-life scenarios with angles of elevation and depression
  - For example if we know the distance we are standing from a tree and the angle of elevation of the top of the tree we can use Tan to find its height
  - Or if we are looking at a boat at to sea and we know our height above sea level and the angle of depression we can find how far away the boat is



Exam Tip

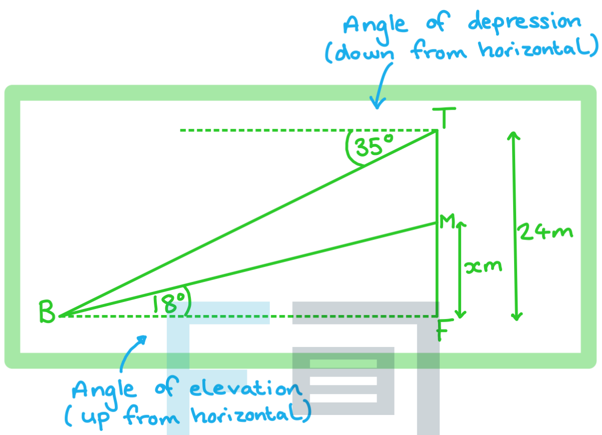
- It may be useful to draw more than one diagram if the triangles that you are interested in overlap one another

Worked Example

A cliff is perpendicular to the sea and the top of the cliff stands 24 m above the level of the sea. The angle of depression from the cliff to a boat at sea is 35°. At a point x m up the cliff is a flag marker and the angle of elevation from the boat to the flag marker is 18°.

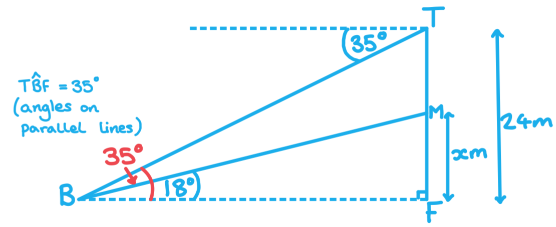
a)

Draw and label a diagram to show the top of the cliff, T, the foot of the cliff, F, the flag marker, M, and the boat, B, labelling all the angles and distances given above.

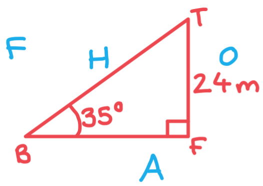


b)

Find the distance from the boat to the foot of the cliff.



Consider triangle TBF



SOHCAHTOA

we have opposite and adjacent so use Tan

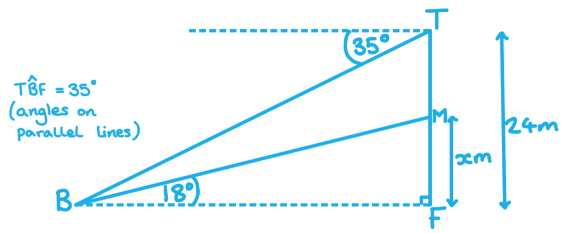
$$\tan 35° = \frac{24}{BF}$$

$$BF = \frac{24}{\tan 35°}$$

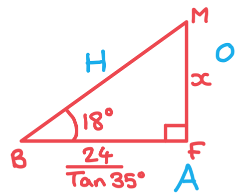
BF = 34.3m (3s.f.)

c)

Find the value of x.



Consider triangle FBM



SOHCAHTOA
we have opposite and adjacent so use Tan

$$\text{Tan } 18° = \frac{x}{\left(\frac{24}{\text{Tan } 35°}\right)}$$

$$x = \text{Tan } 18° \times \left(\frac{24}{\text{Tan } 35°}\right)$$

$$= 11.136...$$

$$x = 11.1\text{m (3s.f.)}$$

Constructing Diagrams

What diagrams will I need to construct?

- In IB you will be expected to construct diagrams based on information given
- The information will include **compass directions, bearings, angles**
  - Look out for the **plane** the diagram should be drawn in
  - It will either be **horizontal** (something occurring at sea or on the ground)
  - Or it will be **vertical** (Including height)
- Work through the statements given in the instructions systematically

What do I need to know?

- Your diagrams will be sketches, they do not need to be accurate or to scale
  - However the more accurate your diagram is the easier it is to work with
- Read the full set of instructions once before beginning to draw the diagram so you have a rough idea of where each object is
- Make sure you know your **compass directions**
  - **Due east** means on a **bearing of 090°**
    - Draw the line directly to the right
  - **Due south** means on a **bearing of 180°**
    - Draw the line vertically downwards
  - **Due west** means on a **bearing of 270°**
    - Draw the line directly to the left
  - **Due north** means on a **bearing of 360° (or 000°)**
    - Draw the line vertically upwards
- Using the above bearings for compass directions will help you to estimate angles for other bearings on your diagram

Exam Tip

- Draw your diagrams in pencil so that you can easily erase any errors

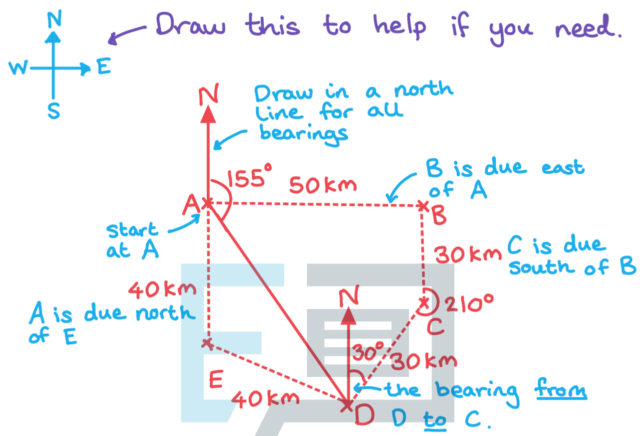
Worked Example

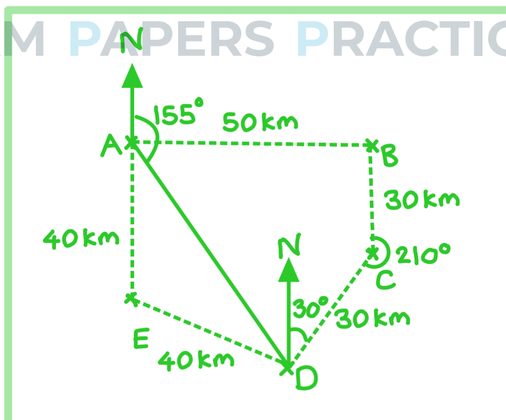
A city at B is due east of a city at A and A is due north of a city at E. A city at C is due south of B.

The bearing from A to D is 155° and the bearing from D to C is 30°.

The distance AB = 50 km, the distances BC = CD = 30 km and the distances DE = AE = 40 km.

Draw and label a diagram to show the cities A, B, C, D and E and clearly mark the bearings and distances given.





3.4 Trigonometry

3.4.1 The Unit Circle

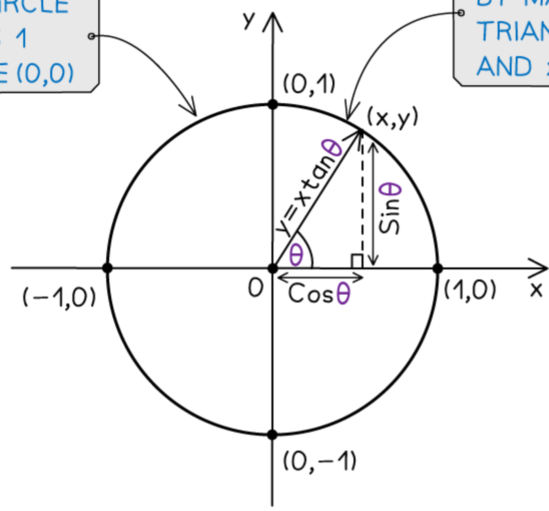
Defining Sin, Cos and Tan

What is the unit circle?

- The unit circle is a circle with radius 1 and centre (0, 0)
- Angles are always measured from the positive x-axis and turn:
  - **anticlockwise** for **positive** angles
  - **clockwise** for **negative** angles
- It can be used to calculate trig values as a coordinate point (x, y) on the circle
  - Trig values can be found by making a right triangle with the radius as the hypotenuse
  - Where θ is the angle measured anticlockwise from the positive *x*-axis
  - The *x*-axis will always be adjacent to the angle, θ
- SOHCAHTOA can be used to find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ easily
- As the radius is 1 unit
  - the ***x* coordinate** gives the value of **cosθ**
  - the ***y* coordinate** gives the value of **sinθ**
- As the origin is one of the end points - dividing the y coordinate by the x coordinate gives the gradient
  - the **gradient** of the line gives the value of **tanθ**
- It allows us to calculate sin, cos and tan for angles greater than 90° ($\frac{\pi}{2}$ rad)

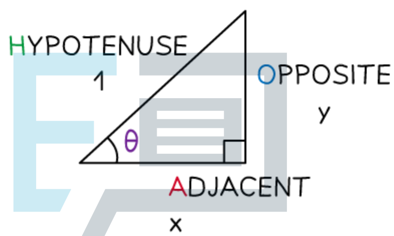
THE UNIT CIRCLE HAS RADIUS 1 AND CENTRE (0,0)

TRIG VALUES CAN BE FOUND BY MAKING A RIGHT ANGLED TRIANGLE WITH THE RADIUS AND x-AXIS



THE ANGLE IS MEASURED ANTI-CLOCKWISE FROM THE X-AXIS IN EITHER RADIANS OR DEGREES

ANY POINT (x,y) ON THE UNIT CIRCLE CAN BE FOUND USING (cosθ, sinθ)



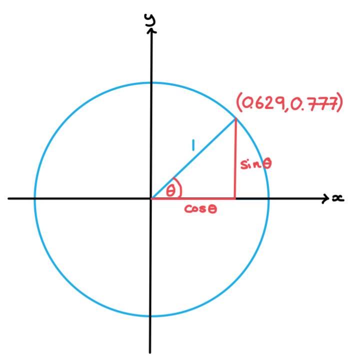
$\text{Cos}\theta = \frac{A}{H} = \frac{x}{1} = x$

$\text{Sin}\theta = \frac{O}{H} = \frac{y}{1} = y$

$\text{Tan}\theta = \frac{O}{A} = \frac{y}{x}$

Worked Example

The coordinates of a point on a unit circle, to 3 significant figures, are (0.629, 0.777).
Find $\theta°$ to the nearest degree.



We know $(x, y) = (\cos\theta, \sin\theta)$

so,

$\cos\theta = 0.629$

$\sin\theta = 0.777$

Using either ratio :

$\theta = \cos^{-1}(0.629)$

$= 51.023\ldots$

$\theta = 51°$ (nearest degree)

Using The Unit Circle

What are the properties of the unit circle?

- The unit circle can be split into four **quadrants** at every 90° ($\frac{\pi}{2}$ rad)
  - The first quadrant is for angles between 0 and 90°
    - **A**ll three of Sinθ, Cosθ and Tanθ are positive in this quadrant
  - The second quadrant is for angles between 90° and 180° ($\frac{\pi}{2}$ rad and π rad)
    - **S**inθ is positive in this quadrant
  - The third quadrant is for angles between 180° and 270° (π rad and $\frac{3\pi}{2}$)
    - **T**anθ is positive in this quadrant
  - The fourth quadrant is for angles between 270° and 360° ($\frac{3\pi}{2}$ rad and 2π)
    - **C**osθ is positive in this quadrant
  - Starting from the **fourth** quadrant (on the bottom right) and working anti-clockwise the positive trig functions spell out **CAST**
    - This is why it is often thought of as the **CAST** diagram
    - You may have your own way of remembering this
    - A popular one starting from the first quadrant is **All Students Take Calculus**
  - To help picture this better try sketching all three trig graphs on one set of axes and look at which graphs are positive in each 90° section

How is the unit circle used to find secondary solutions?

- Trigonometric functions have more than one input to each output
  - For example sin 30° = sin 150° = 0.5
  - This means that trigonometric equations have more than one solution
  - For example both 30° and 150° satisfy the equation sin x = 0.5
- The unit circle can be used to find all solutions to trigonometric equations in a given interval
  - Your calculator will only give you the first solution to a problem such as $x = \sin^{-1}(0.5)$
    - This solution is called the **primary value**
  - However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
    - Further solutions are called the **secondary values**
  - This is why you will be given a **domain** in which your solutions should be found
    - This could either be in degrees or in radians
    - If you see π or some multiple of π then you must work in radians
- The following steps may help you use the unit circle to find **secondary values**

STEP 1: Draw the angle into the first quadrant using the x or y coordinate to help you

- If you are working with sin $x = k$, draw the line from the origin to the circumference of the circle at the point where the **y coordinate** is k
- If you are working with cos $x = k$, draw the line from the origin to the circumference of the circle at the point where the **x coordinate** is k
- If you are working with tan $x = k$, draw the line from the origin to the circumference of the circle such that the gradient of the line is k

  - This will give you the angle which should be measured from the **positive x-axis…**
    … anticlockwise for a positive angle
    … clockwise for a negative angle

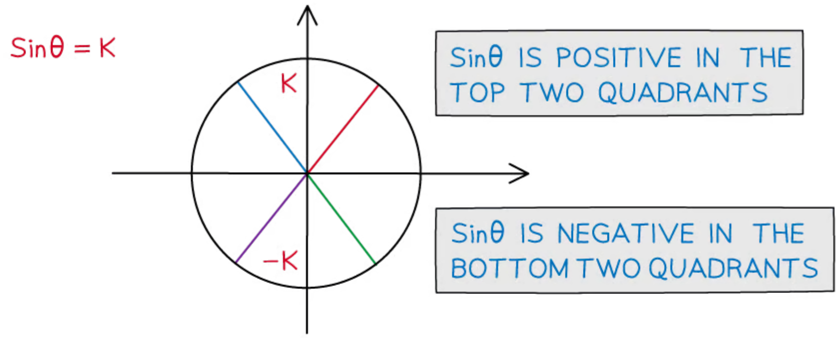
STEP 2: Draw the radius in the other quadrant which has the same...

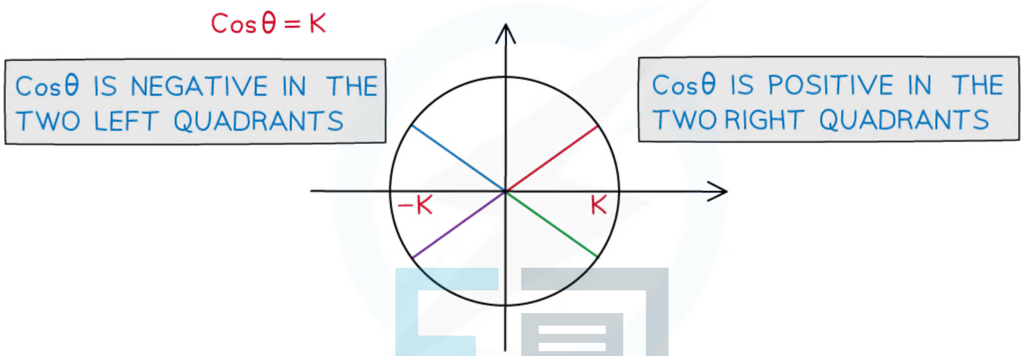
- ... x-coordinate if solving $\cos x = k$
  - This will be the quadrant which is vertical to the original quadrant
- ... y-coordinate if solving $\sin x = k$
  - This will be the quadrant which is horizontal to the original quadrant
- ... gradient if solving $\tan x = k$
  - This will be the quadrant diagonal to the original quadrant

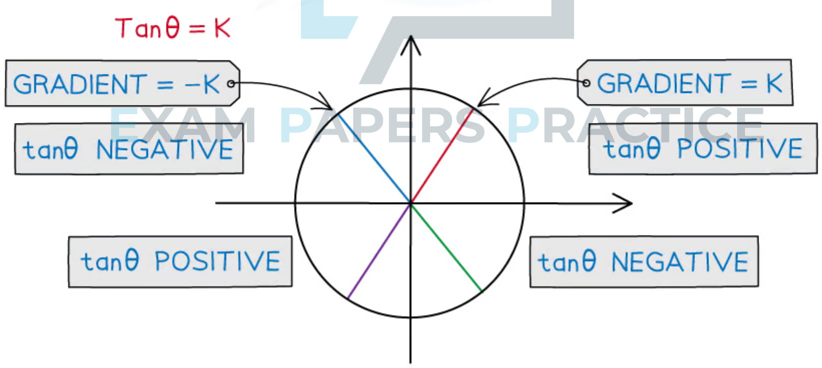
STEP 3: Work out the size of the second angle, measuring from the positive x-axis

- … anticlockwise for a positive angle
- … clockwise for a negative angle
  - You should look at the given range of values to decide whether you need the negative or positive angle

STEP 4: Add or subtract either 360° or 2π radians to both values until you have all solutions in the required range





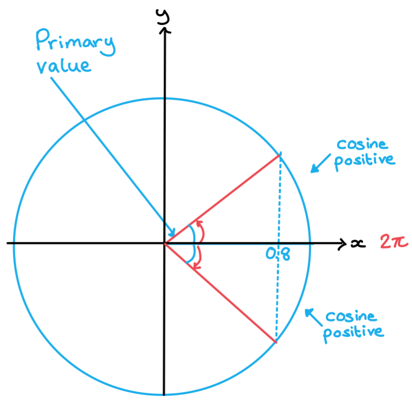


Exam Tip

- Being able to sketch out the unit circle and remembering CAST can help you to find all solutions to a problem in an exam question

Worked Example

Given that one solution of $\cos\theta = 0.8$ is $\theta = 0.6435$ radians correct to 4 decimal places, find all other solutions in the range $-2\pi \le \theta \le 2\pi$. Give your answers correct to 3 significant figures.



Cosine is positive in the first and fourth quadrants so draw the angle from the horizontal axis in both quadrants.

Primary value $= 0.6435$

Using diagram, secondary value $= -0.6435$

Therefore all values are: $0.6435 \pm 2\pi n$

and $-0.6435 \pm 2\pi n$

Within given domain: $-2\pi \le \theta \le 2\pi$

$\theta = -5.64^c, -0.644^c, 0.644^c, 5.64^c$

3.4.2 Exact Values

Trigonometry Exact Values

What are exact values in trigonometry?

- For certain angles the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ can be written **exactly**
  - This means using fractions and surds
  - You should be familiar with these values and be able to derive the values using geometry
- You are expected to know the exact values of sin, cos and tan for angles of 0°, 30°, 45°, 60°, 90°, 180° and their multiples
  - In **radians** this is $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and their multiples
- The exact values you are expected to know are here:

DEGREES	0°	30°	45°	60°	90°	180°	360°
RADIANS	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	UNDEFINED	0	0

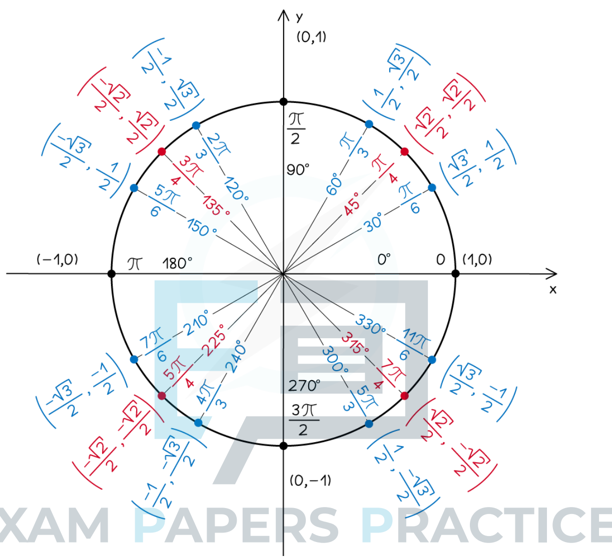
How do I find the exact values of other angles?

- The exact values for sin and cos can be seen on the **unit circle** as the *y* and *x* coordinates respectively
  - If using the coordinates on the unit circle to memorise the exact values, remember that cos comes **before** sin
- The **unit circle** can also be used to find exact values of other angles using symmetry
- If you know the exact value for an angle in the first quadrant you can draw the same angle from the x-axis in any other quadrant to find other angles
- Remember that the angles are **measured anticlockwise** from the positive x-axis
- For example if you know that the exact value for is 0.5
  - draw the angle 30° from the horizontal in the three other quadrants
  - measuring from the positive x-axis you have the angles of 150°, 210° and 330°
    - sin is positive in the second quadrant so sin150° = 0.5

sin is negative in the third quadrant so sin210° = - 0.5
sin is negative in the fourth quadrant so sin330° = - 0.5

- It is also possible to find the **negative** angles by measuring **clockwise** from the positive x-axis
  - draw the angle 30° from the horizontal in the three other quadrants
  - measuring **clockwise** from the positive x-axis you have the angles of -30°, -150°, -210° and -330°

sin is negative in the fourth quadrant so sin(-30°) = - 0.5
sin is negative in the third quadrant so sin(-150°) = - 0.5
sin is positive in the second quadrant so sin(-210°) = 0.5
sin is positive in the fourth quadrant so sin(-330°) = 0.5



How are exact values in trigonometry derived?

- There are two special **right-triangles** that can be used to derive all of the exact values you need to know
- Consider a **right-triangle** with a hypotenuse of 2 units and a shorter side length of 1 unit
  - Using Pythagoras' theorem the third side will be $\sqrt{3}$
  - The angles will be $\frac{\pi}{2}$ radians (90°), $\frac{\pi}{3}$ radians (60°) and $\frac{\pi}{6}$ radians (30°)
  - Using SOHCAHTOA gives...

$\text{Sin}\ \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\quad$ $\text{Sin}\ \frac{\pi}{6} = \frac{1}{2}$

$\text{Cos}\ \frac{\pi}{3} = \frac{1}{2}$ $\quad$ $\text{Cos}\ \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\text{Tan}\ \frac{\pi}{3} = \sqrt{3}$ $\quad$ $\text{Tan}\ \frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

- Consider an **isosceles triangle** with two equal side lengths (the opposite and adjacent) of 1 unit
  - Using Pythagoras' theorem it will have a hypotenuse of $\sqrt{2}$

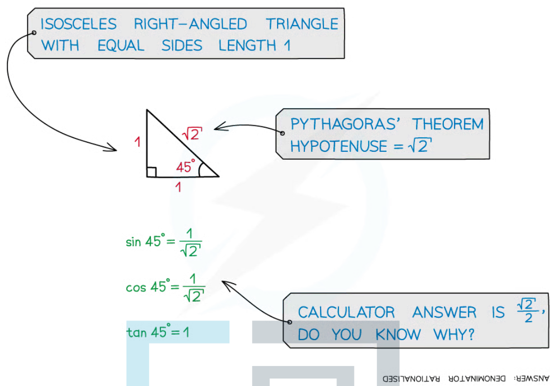
EXAM PAPERS PRACTICE

- The two equal angles will be $\frac{\pi}{4}$ radians (45°)
- Using SOHCAHTOA gives…

$$\text{Sin}\ \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\text{Cos}\ \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\text{Tan}\ \frac{\pi}{4} = 1$$

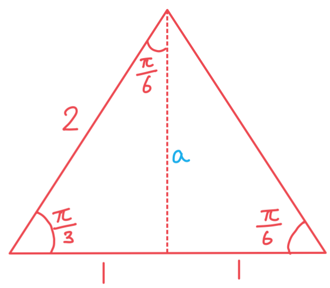


Exam Tip

- You will be expected to be comfortable using exact trig values for certain angles but it can be easy to muddle them up if you just try to remember them from a list, sketch the triangles and trig graphs on your paper so that you can use them as many times as you need to during the exam!
  - sketch the triangles for the key angles $45°/\frac{\pi}{4}$, $30°/\frac{\pi}{6}$, $60°/\frac{\pi}{3}$
  - sketch the trig graphs for the key angles $0°$, $90°/\frac{\pi}{2}$, $180°/\pi$, $270°/\frac{3\pi}{2}$, $360°/2\pi$

Worked Example

Using an equilateral triangle of side length 2 units, derive the exact values for the sine, cosine and tangent of $\frac{\pi}{6}$ and $\frac{\pi}{3}$.



Use Pythagoras' Theorem to find a:

$$a^2 = \sqrt{2^2 - 1^2}$$
$$= \sqrt{3}$$

Using SOHCAH TOA:

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \qquad \tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{1} = \sqrt{3}$$

3.5 Trigonometric Functions & Graphs

3.5.1 Graphs of Trigonometric Functions

Graphs of Trigonometric Functions

What are the graphs of trigonometric functions?

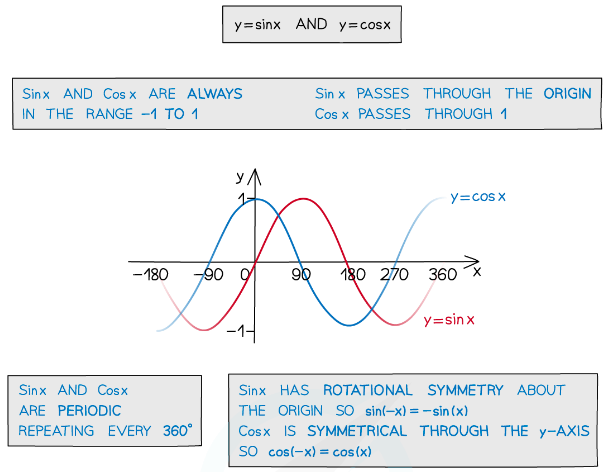
- The trigonometric functions sin, cos and tan all have special **periodic graphs**
- You'll need to know their properties and how to sketch them for a given domain in either **degrees** or **radians**
- Sketching the trigonometric graphs can help to
  - Solve trigonometric equations and find all solutions
  - Understand transformations of trigonometric functions

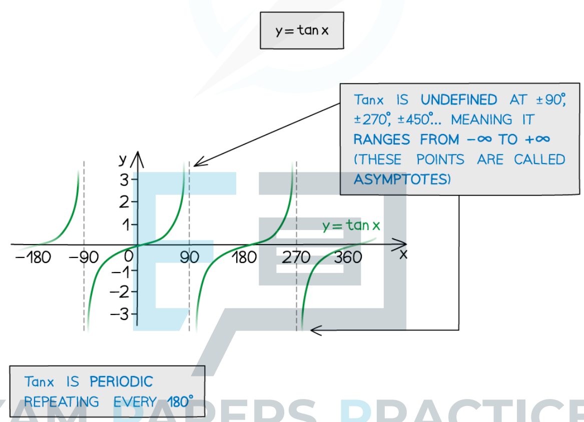
What are the properties of the graphs of sin x and cos x?

- The graphs of sin x and cos x are both **periodic**
  - They **repeat every 360°** (2π radians)
  - The angle will always be on the x-axis
    - Either in degrees or radians
- The graphs of sin x and cos x are always in the **range −1 ≤ y ≤ 1**
  - **Domain:** $\{\quad \mid \boldsymbol{x} \in \mathbb{R}\}$
  - **Range:** $\{\boldsymbol{y} \mid -1 \le \boldsymbol{y} \le 1\}$
  - The graphs of sin x and cos x are identical however one is a **translation** of the other
    - sin x passes through the origin
    - cos x passes through (0, 1)
- The **amplitude** of the graphs of sin x and cos x is 1

What are the properties of the graph of tan x?

- The graph of tan x is **periodic**
  - It **repeats every 180°** (π radians)
  - The angle will always be on the x-axis
    - Either in degrees or radians
- The graph of tan x is **undefined** at the points ± 90°, ± 270° etc
  - There are **asymptotes** at these points on the graph
  - In radians this is at the points $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ etc
- The range of the graph of tan x is
  - **Domain:** $\left\{\boldsymbol{x} \mid \boldsymbol{x} \neq \frac{\pi}{2} + k\pi,\ \ k \in \mathbb{Z}\right\}$
  - **Range:** $\{\boldsymbol{y} \mid \boldsymbol{y} \in \mathbb{R}\}$





How do I sketch trigonometric graphs?

- You may need to sketch a trigonometric graph so you will need to remember the key features of each one
- The following steps may help you sketch a trigonometric graph
  - STEP 1: Check whether you should be working in degrees or radians
    - You should check the domain given for this
    - If you see π in the given domain then you should work in radians
  - STEP 2: Label the x-axis in multiples of 90°
    - This will be multiples of $\frac{\pi}{2}$ if you are working in radians
    - Make sure you cover the whole domain on the x-axis
  - STEP 3: Label the y-axis
    - The range for the y-axis will be $-1 \le y \le 1$ for sin or cos
    - For tan you will not need any specific points on the y-axis
  - STEP 4: Draw the graph
    - Knowing exact values will help with this, such as remembering that sin(0) = 0 and cos(0) = 1
    - Mark the important points on the axis first
    - If you are drawing the graph of tan x put the asymptotes in first

If you are drawing sin x or cos x mark in where the maximum and minimum points will be

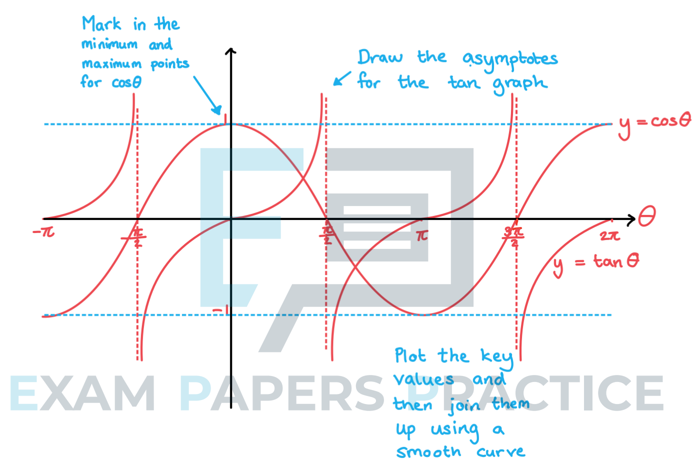
Try to keep the symmetry and rotational symmetry as you sketch, as this will help when using the graph to find solutions

Exam Tip

- Sketch all three trig graphs on your exam paper so you can refer to them as many times as you need to!

Worked Example

Sketch the graphs of y = cosθ and y = tanθ on the same set of axes in the interval -π ≤ θ ≤ 2π. Clearly mark the key features of both graphs.



Using Trigonometric Graphs

How can I use a trigonometric graph to find extra solutions?

- Your calculator will only give you the first solution to a problem such as $\sin^{-1}(0.5)$
  - This solution is called the **primary value**
- However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
  - Further solutions are called the **secondary values**
- This is why you will be given a **domain** (interval) in which your solutions should be found
  - This could either be in degrees or in radians
    - If you see π or some multiple of π then you must work in radians
- The following steps will help you use the **trigonometric graphs** to find **secondary values**
  - STEP 1: Sketch the graph for the given function and interval
    - Check whether you should be working in degrees or radians and label the axes with the key values
  - STEP 2: Draw a horizontal line going through the y-axis at the point you are trying to find the values for
    - For example if you are looking for the solutions to $\sin^{-1}(-0.5)$ then draw the horizontal line going through the y-axis at -0.5
    - The number of times this line cuts the graph is the number of solutions within the given interval
  - STEP 3: Find the primary value and mark it on the graph
    - This will either be an exact value and you should know it
    - Or you will be able to use your calculator to find it
  - STEP 4: Use the symmetry of the graph to find all the solutions in the interval by adding or subtracting from the key values on the graph

What patterns can be seen from the graphs of trigonometric functions?

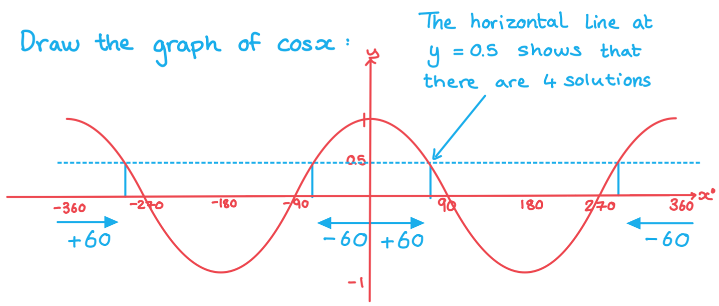
- The graph of sin x has rotational symmetry about the origin
  - So $\sin(-x) = -\sin(x)$
  - $\sin(x) = \sin(180° - x)$ or $\sin(\pi - x)$
- The graph of cos x has reflectional symmetry about the y-axis
  - So $\cos(-x) = \cos(x)$
  - $\cos(x) = \cos(360° - x)$ or $\cos(2\pi - x)$
- The graph of tan x repeats every 180° (π radians)
  - So $\tan(x) = \tan(x \pm 180°)$ or $\tan(x \pm \pi)$
- The graphs of sin x and cos x repeat every 360° (2π radians)
  - So $\sin(x) = \sin(x \pm 360°)$ or $\sin(x \pm 2\pi)$
  - $\cos(x) = \cos(x \pm 360°)$ or $\cos(x \pm 2\pi)$

Exam Tip

- Take care to always check what the **interval** for the angle is that the question is focused on

Worked Example

One solution to cos x = 0.5 is 60°. Find all the other solutions in the range -360° ≤ x ≤ 360°.



Solutions are : 60°, 360° − 60°, −60°, −360° + 60°

−60°, −300°, 60°, 300°

3.5.2 Transformations of Trigonometric Functions

Transformations of Trigonometric Functions

What transformations of trigonometric functions do I need to know?

- As with other graphs of functions, trigonometric graphs can be transformed through **translations**, **stretches** and **reflections**
- **Translations** can be either horizontal (parallel to the x-axis) or vertical (parallel to the y-axis)
  - For the function **y = sin (x)**
    - A **vertical** translation of ***a* units** in the **positive direction** (up) is denoted by **y = sin (x) + *a***
    - A **vertical** translation of ***a* units** in the **negative direction** (down) is denoted by **y = sin (x) – *a***
    - A **horizontal** translation in the **positive direction** (right) is denoted by **y = sin (x – *a*)**
    - A **horizontal** translation in the **negative direction** (left) is denoted by **y = sin (x + *a*)**
- **Stretches** can be either horizontal (parallel to the x-axis) or vertical (parallel to the y-axis)
  - For the function **y = sin (x)**
    - A **vertical** stretch of a **factor *a* units** is denoted by **y = *a* sin (x)**
    - A **horizontal** stretch of a **factor *a* units** is denoted by **y = sin** $\left(\frac{x}{a}\right)$
- **Reflections** can be either across the x-axis or across the y-axis
  - For the function **y = sin (x)**
    - A **reflection** across the x-axis is denoted by **y = – sin (x)**
    - A **reflection** across the y-axis is denoted by **y = sin (–x)**

What combined transformations are there?

- **Stretches** in the horizontal and vertical direction are often combined
- The functions ***a* sin(*bx*)** and ***a* cos(*bx*)** have the following properties:
  - The **amplitude** of the graph is |*a*|
  - The **period** of the graph is $\frac{360}{b}$° (or $\frac{2\pi}{b}$ rad)
- **Translations** in both directions could also be combined with the stretches
- The functions ***a* sin(*b*(*x* – c)) + *d*** and ***a* cos(*b*(*x* – c)) + *d*** have the following properties:
  - The **amplitude** of the graph is |*a*|
  - The **period** of the graph is $\frac{360}{b}$° (or $\frac{2\pi}{b}$)
  - The translation in the horizontal direction is c
  - The translation in the vertical direction is *d*
    - *d* represents the **principal axis** (the line that the function fluctuates about)
- The function ***a* tan(*b*(*x* – c)) + *d*** has the following properties:
  - The **amplitude** of the graph does not exist
  - The **period** of the graph is $\frac{180}{b}$° (or $\frac{2\pi}{b}$)
  - The translation in the horizontal direction is c
  - The translation in the vertical direction (principal axis) is *d*

How do I sketch transformations of trigonometric functions?

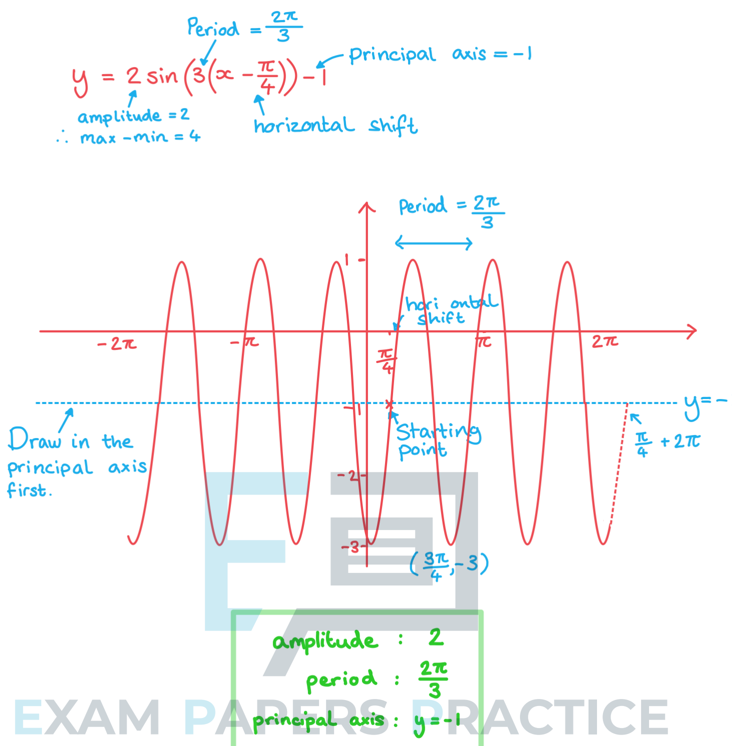
- Sketch the graph of the original function first
- Carry out each transformation separately
  - The **order** in which you carry out the transformations is important
  - Given the form $y = a\sin(b(x - c)) + d$ carry out any **stretches** first, **translations** next and **reflections** last
    - If the function is written in the form $y = a\sin(bx - bc) + d$ factorise out the coefficient of x before carrying out any transformations
  - Use a very light pencil to mark where the graph has moved for each transformation
- It is a good idea to mark in the principal axis the lines corresponding to the maximum and minimum points first
  - The **principal axis** will be the line $y = d$
  - The **maximum points** will be on the line $y = d + a$
  - The **minimum points** will be on the line $y = d - a$
- Sketch in the new transformed graph
- Check it is correct by looking at some key points from the **exact values**

Exam Tip

- Be sure to apply transformations in the correct order – applying them in the wrong order can produce an incorrect transformation
- When you sketch a transformed graph, indicate the new coordinates of any points that are marked on the original graph
- Try to indicate the coordinates of points where the transformed graph intersects the coordinate axes (although if you don't have the equation of the original function this may not be possible)
- If the graph has asymptotes, don't forget to sketch the asymptotes of the transformed graph as well

Worked Example

Sketch the graph of $y = 2\sin\left(3\left(x - \frac{\pi}{4}\right)\right) - 1$ for the interval $-2\pi \le x \le 2\pi$. State the amplitude, period and principal axis of the function.



3.5.3 Modelling with Trigonometric Functions

Modelling with Trigonometric Functions

What can be modelled with trigonometric functions?

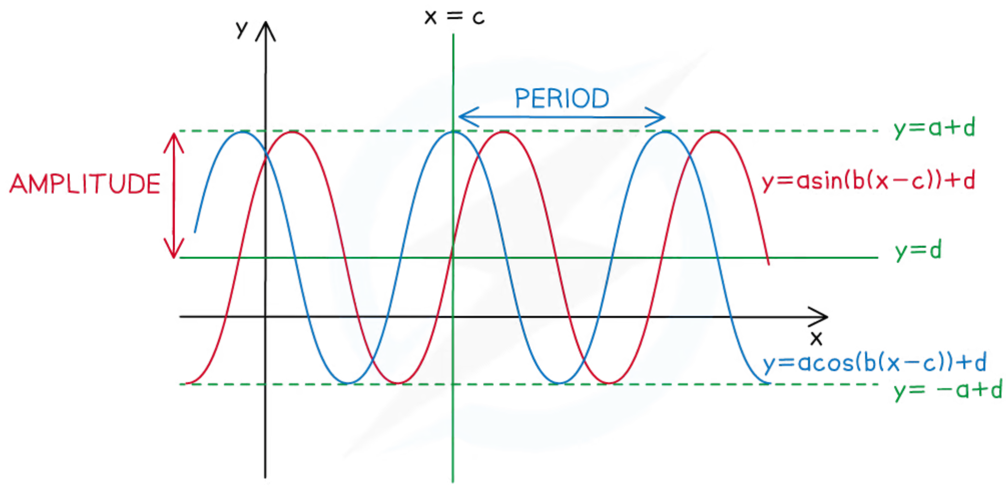
- Anything that oscillates (fluctuates periodically) can be modelled using a trigonometric function
  - Normally some transformation of the sine or cosine function
- Examples include:
  - $D(t)$ is the depth of water at a shore *t* hours after midnight
  - $T(d)$ is the temperature of a city *d* days after the 1st January
  - $H(t)$ is vertical height above ground of a person *t* seconds after entering a Ferris wheel
- Notice that the x-axis will not always contain an angle
  - In the examples above time or number of days would be on the x-axis
  - Depth of the water, temperature or vertical height would be on the y-axis

What are the parameters of trigonometric models?

- A **trigonometric model** could be of the form
  - $f(x) = a\sin(b(x-c)) + d$
  - $f(x) = a\cos(b(x-c)) + d$
  - $f(x) = a\tan(b(x-c)) + d$
- The *a* represents the **amplitude** of the function
  - The bigger the value of *a* the bigger the **range** of values of the function
  - For the function $a\tan(b(x-c)) + d$ the amplitude is undefined
- The *b* determines the **period** of the function
  - Period = $\frac{360°}{b} = \frac{2\pi}{b}$
  - The bigger the value of *b* the quicker the function repeats a cycle
- The c represents the **horizontal shift**
- The *d* represents the **vertical shift**
  - This is the **principal axis**

What are possible limitations of a trigonometric model?

- The amplitude is the same for each cycle
  - In real-life this might not be the case
  - The function might get closer to the value of *d* over time
- The period is the same for each cycle
  - In real-life this might not be the case
  - The time to complete a cycle might change over time



Exam Tip

- The variable in these questions is often ***t*** for time.
- Read the question carefully to make sure you know what you are being asked to solve.

Worked Example

The water depth, *D*, in metres, at a port can be modelled by the function

$$D(t) = 3\sin\left(15°(t-2)\right) + 12, \qquad 0 \le t < 24$$

where *t* is the elapsed time, in hours, since midnight.

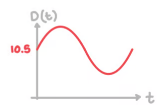
a)
Write down the depth of the water at midnight.

Substitute $t = 0$ for midnight:

$D(0) = 3\sin(15(0-2)) + 12$

$= 3\sin(-30) + 12$

$= 3(-\tfrac{1}{2}) + 12 \quad = 10.5$



$D = 10.5$ m

b)
Find the minimum water depth and the number of hours after midnight that this depth occurs.

$D(t) = 3\sin(15(t-2)) + 12$

amplitude (3) ; principal axis (12)

Principal axis is at $y = 12$

amplitude is 3 minimum $= 12 - 3 = 9$

Let $D(t) = 9$

$$3\sin(15(t-2)) + 12 = 9$$

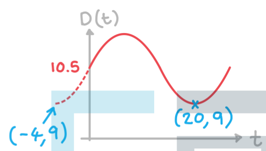
$$3\sin(15(t-2)) = -3$$

$$\sin(15(t-2)) = -1$$

$$15(t-2) = -90$$

$$t = -4 + 24n$$

($24n$: cycle repeats every 24 hours)



Minimum = 9m
20 hours after midnight

c)

Calculate how long the water depth is at least 13.5 m each day.

Let $D(t) = 13.5$

$$3\sin(15(t-2)) + 12 = 13.5$$

$$3\sin(15(t-2)) = 1.5$$

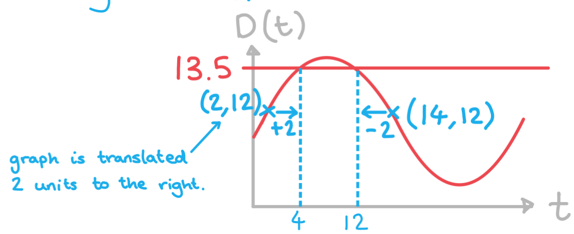
$$\sin(15(t-2)) = 0.5$$

$$15(t-2) = 30$$

$$t = 4 + 24n$$

cycle repeats every 24 hours

Use symmetry and properties of the graph to find secondary value of t:



$t = 4$ and $t = 12$

Find the difference between the times.

$12 - 4 = 8$

8 hours

3.6 Trigonometric Equations & Identities

3.6.1 Simple Identities

Simple Identities

What is a trigonometric identity?

- Trigonometric identities are statements that are true for all values of x or θ
- They are used to help simplify trigonometric equations before solving them
- Sometimes you may see identities written with the symbol $\equiv$
  - This means 'identical to'

What trigonometric identities do I need to know?

- The two trigonometric identities you must know are
  - $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
    - This is the identity for $\tan\theta$
  - $\sin^2\theta + \cos^2\theta = 1$
    - This is the Pythagorean identity
    - Note that the notation $\sin^2\theta$ is the same as $(\sin\theta)^2$
- Both identities can be found **in the formula booklet**
- Rearranging the second identity often makes it easier to work with
  - $\sin^2\theta = 1 - \cos^2\theta$
  - $\cos^2\theta = 1 - \sin^2\theta$

Where do the trigonometric identities come from?

- You do not need to know the proof for these identities but it is a good idea to know where they come from
- From SOHCAHTOA we know that
  - $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{O}{H}$
  - $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{A}{H}$
  - $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{O}{A}$
- The identity for $\tan\theta$ can be seen by diving $\sin\theta$ by $\cos\theta$?
  - $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{O}{H}}{\frac{A}{H}} = \dfrac{O}{A} = \tan\theta$
- The Pythagorean identity can be seen by considering a right-triangle with a hypotenuse of 1
  - Then $(\text{opposite})^2 + (\text{adjacent})^2 = 1$
  - Therefore $\sin^2\theta + \cos^2\theta = 1$
- Considering the equation of the unit circle also shows the Pythagorean identity
  - The equation of the unit circle is $x^2 + y^2 = 1$
  - The coordinates on the unit circle are $(\cos\theta, \sin\theta)$

- Therefore the equation of the unit circle could be written $\cos^2\theta + \sin^2\theta = 1$
- A third very useful identity is $\sin\theta = \cos(90° - \theta)$ or $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$
  - This is not included in the formula booklet but is useful to remember

How are the trigonometric identities used?

- Most commonly trigonometric identities are used to change an equation into a form that allows it to be solved
- They can also be used to prove further identities such as the **double angle formulae**

Exam Tip

- If you are asked to show that one thing is identical (≡) to another, look at what parts are missing – for example, if tan x has gone it must have been substituted

Worked Example

Show that the equation $2\sin^2 x - \cos x = 0$ can be written in the form $a\cos^2 x + b\cos x + c = 0$, where a, b and c are integers to be found.

$2\sin^2 x - \cos x = 0$

Equation has both $\sin x$ and $\cos x$ so will need changing before it can be solved.

Use the identity $\sin^2 x = 1 - \cos^2 x$

Substitute : $2(1 - \cos^2 x) - \cos x = 0$

Expand : $2 - 2\cos^2 x - \cos x = 0$

Rearrange : $2\cos^2 x + \cos x - 2 = 0$

$a = 2, \quad b = 1, \quad c = -2$

3.6.2 Compound Angle Formulae

Compound Angle Formulae

What are the compound angle formulae?

- There are six **compound angle formulae** (also known as **addition formulae**), two each for **sin**, **cos** and **tan**:
- For **sin** the +/- sign on the left-hand side **matches** the one on the right-hand side
  - sin(A+B)≡sinAcosB + cosAsinB
  - sin(A-B)≡sinAcosB - cosAsinB

- For **cos** the +/- sign on the left-hand side is **opposite to** the one on the right-hand side
  - cos(A+B)≡cosAcosB - sinAsinB
  - cos(A-B)≡cosAcosB + sinAsinB

- For **tan** the +/- sign on the left-hand side **matches** the one in the **numerator** on the right-hand side, and is **opposite to** the one in the **denominator**
  - $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
  - $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

- The compound angle formulae can all the found in the formula booklet, you do not need to remember them

When are the compound angle formulae used?

- The compound angle formulae are particularly useful when finding the values of trigonometric ratios without the use of a calculator
  - For example to find the value of sin15° rewrite it as sin (45 – 30)° and then
    - apply the compound formula for sin(A – B)
    - use your knowledge of exact values to calculate the answer
- The compound angle formulae are also used…
  - … to derive further multiple angle trig identities such as the double angle formulae
  - … in trigonometric proof
  - … to simplify complicated trigonometric equations before solving

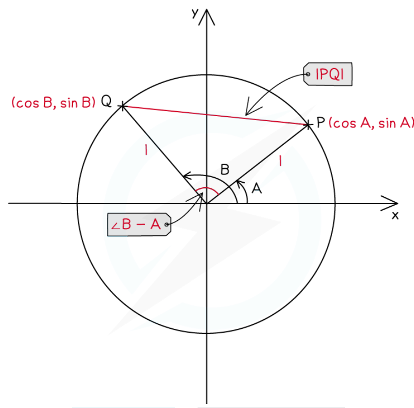
How are the compound angle formulae for cosine proved?

- The proof for the compound angle identity cos (*A* – *B*) = cos *A* cos *B* + sin *A* sin *B* can be seen by considering two coordinates on a unit circle, *P* (cos *A*, sin *A*) and Q (cos *B*, sin *B*)
  - The angle between the positive *x*- axis and the point *P* is *A*
  - The angle between the positive *x*- axis and the point Q is *B*
  - The angle between *P* and Q is *B* – *A*

- Using the distance formula (Pythagoras) the distance *PQ* can be given as
  - $|PQ|^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$
- Using the cosine rule the distance *PQ* can be given as

- $|PQ|^2 = 1^2 + 1^2 - 2(1)(1)\cos(B-A) = 2 - 2\cos(B-A)$

- Equating these two formulae, expanding and rearranging gives
  - $2 - 2\cos(B-A) = \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos A\cos B - 2\sin A\sin B$
  - $2 - 2\cos(B-A) = 2 - 2(\cos A\cos B + \sin A\sin B)$
- Therefore $\cos(B-A) = \cos A\cos B + \sin A\sin B$
- Changing $-A$ for A in this identity and rearranging proves the identity for $\cos(A+B)$
  - $\cos(B-(-A)) = \cos(-A)\cos B + \sin(-A)\sin B = \cos A\cos B - \sin A\sin B$



How are the compound angle formulae for sine proved?

- The proof for the compound angle identity $\sin(A+B)$ can be seen by using the above proof for $\cos(B-A)$ and
  - Considering $\cos(\pi/2 - (A+B)) = \cos(\pi/2)\cos(A+B) + \sin(\pi/2)\sin(A+B)$
  - Therefore $\cos(\pi/2 - (A+B)) = \sin(A+B)$
  - Rewriting $\cos(\pi/2 - (A+B))$ as $\cos((\pi/2 - A) + B)$ gives

    $\cos(\pi/2 - (A+B)) = \cos(\pi/2 - A)\cos B + \sin(\pi/2 - A)\sin B$
  - Using $\cos(\pi/2 - A) = \sin A$ and $\sin(\pi/2 - A) = \cos A$ and equating gives

    $\sin(A+B) = \sin A\cos B + \cos A\cos B$
- Substituting B for $-B$ proves the result for $\sin(A-B)$

How are the compound angle formulae for tan proved?

- The proof for the compound angle identities $\tan(A \pm B)$ can be seen by
  - Rewriting $\tan(A \pm B)$ as $\frac{\sin(A \pm B)}{\cos(A \pm B)}$
  - Substituting the compound angle formulae in
  - Dividing the numerator and denominator by $\cos A\cos B$

Exam Tip

- All these formulae are in the **Topic 3: Geometry and Trigonometry** section of the formula booklet – make sure that you use them correctly paying particular attention to any negative/positive signs

Worked Example

a)

Show that $\tan\left(x+\frac{\pi}{4}\right)-\tan\left(x-\frac{\pi}{4}\right)=\frac{2(\tan^2 x+1)}{1-\tan^2 x}$

Use the compound angle formula for tan:

$$\tan\left(x+\frac{\pi}{4}\right)=\frac{\tan x+\tan\frac{\pi}{4}}{1-\tan x\tan\frac{\pi}{4}}=\frac{\tan x+1}{1-\tan x}$$

$$\tan\left(x-\frac{\pi}{4}\right)=\frac{\tan x-\tan\frac{\pi}{4}}{1+\tan x\tan\frac{\pi}{4}}=\frac{\tan x-1}{1+\tan x}$$

Put together and simplify:

$$\frac{\tan x+1}{1-\tan x}-\frac{\tan x-1}{1+\tan x}=\frac{(\tan x+1)(1+\tan x)-(\tan x-1)(1-\tan x)}{(1-\tan x)(1+\tan x)}$$

$$=\frac{\tan^2 x+2\tan x+1-(-\tan^2 x+2\tan x-1)}{(1-\tan x)(1+\tan x)}$$

$$=\frac{2\tan^2 x+2}{1-\tan^2 x}$$

$$\boxed{\frac{2(\tan^2 x+1)}{1-\tan^2 x}}$$

b)

Hence, solve $\tan\left(x+\frac{\pi}{4}\right)-\tan\left(x-\frac{\pi}{4}\right)=-4$ for $0\le x\le\frac{\pi}{2}$

Use the answer found in (a) to write a new equation:

$$\frac{2(\tan^2 x+1)}{1-\tan^2 x}=-4$$

Rearrange and bring all terms in $\tan x$ to one side:

$$2(\tan^2 x+1)=-4(1-\tan^2 x)$$

$$2\tan^2 x+2=-4+4\tan^2 x$$

$$2\tan^2 x-6=0$$

$$\tan^2 x=3$$

$$\tan x=\pm\sqrt{3}$$

$$x=\frac{\pi}{3},\ -\frac{\pi}{3}$$ ← outside of given range

$$\boxed{x=\frac{\pi}{3}}$$

EXAM PAPERS PRACTICE

3.6.3 Double Angle Formulae

Double Angle Formulae

What are the double angle formulae?

- The **double angle formulae** for **sine** and **cosine** are:
  - $\sin 2\theta = 2\sin\theta\cos\theta$
  - $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
  - $\tan 2\theta \equiv \dfrac{2\tan\theta}{1-\tan^2\theta}$
- These can be found in the formula booklet
  - The formulae for sin and cos can be found in the SL section
  - The formula for tan can be found in the HL section

How are the double angle formulae derived?

- The double angle formulae can be derived from the compound angle formulae
- Simply replace *B* for *A* in each of the formulae and simplify
- For example
  - Sin 2*A* = sin (*A* + *A*) = sin*A*cos*A* + sin*A*cos*A* = 2sin*A*cos*A*

How are the double angle formulae used?

- Double angle formulae will often be used with...
  - ... trigonometry exact values
  - ... graphs of trigonometric functions
  - ... relationships between trigonometric ratios
- To help solve trigonometric equations which contain $\sin\theta\cos\theta$:
  - Substitute $\sin\theta\cos\theta$ for $\frac{1}{2}\sin 2\theta$
  - Solve for 2θ, finding all values in the range for 2θ
    - The range will need adapting for 2θ
  - Find the solutions for θ
- To help solve trigonometric equations which contain $\sin 2\theta$ and $\sin\theta$ or $\cos\theta$
  - Substitute $\sin 2\theta$ for $2\sin\theta\cos\theta$
  - Isolate all terms in θ
  - Factorise or use another identity to write the equation in a form which can be solved
- To help solve trigonometric equations which contain $\cos 2\theta$ and $\sin\theta$ or $\cos\theta$
  - Substitute $\cos 2\theta$ for either $2\cos^2\theta - 1$ or $1 - 2\sin^2\theta$
    - Choose the trigonometric ratio that is already in the equation
  - Isolate all terms in θ
  - Solve
    - The equation will most likely be in the form of a quadratic
- To help solve trigonometric equations which contain tan 2θ
  - Substitute tan 2θ for the double angle identity
  - Rearrange, often this will lead to a quadratic equation in terms of tan θ
  - Solve
- Double angle formulae can be used in proving other trigonometric identities

Exam Tip

- All these formulae are in the **Topic 3: Geometry and Trigonometry** section of the formula booklet
- If you are asked to show that one thing is identical (≡) to another, look at what parts are missing – for example, if $\sin\theta$ has disappeared you may want to choose the equivalent expression for $\cos 2\theta$ that does not include $\sin\theta$

Worked Example

Without using a calculator, solve the equation $\sin 2\theta = \sin \theta$ for $0° \leq \theta \leq 360°$.
Show all working clearly.

Double angle identity: $\sin 2\theta = 2\sin\theta\cos\theta$

$2\sin\theta\cos\theta = \sin\theta$

Bring both identities to one side:

$2\sin\theta\cos\theta - \sin\theta = 0$

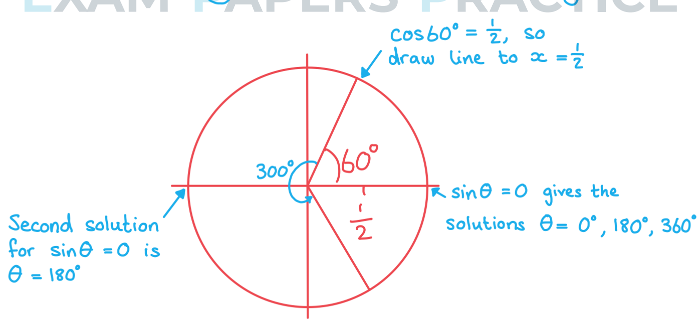
Factorise: $\sin\theta(2\cos\theta - 1) = 0$

Find solutions: $\sin\theta = 0$ $\quad$ $2\cos\theta - 1 = 0$

$\theta = 0$ $\quad$ $\cos\theta = \frac{1}{2}$

$\theta = 60°$

Find secondary values within range:



$\theta = 0°, 60°, 180°, 300°, 360°$

3.6.4 Relationship Between Trigonometric Ratios

Relationship Between Trigonometric Ratios

What relationships between trigonometric ratios should I know?

- If you know a value for one trig ratio you can often use this to work out the value for the others without needing to find θ
- If you know that $\sin\theta = \frac{a}{b}$, where $a, b \in \mathbb{N}$, you can:
  - Sketch a right-triangle with *a* opposite θ and *b* on the hypotenuse
  - Use Pythagoras' theorem to find the value of the adjacent side
  - Use SOHCAHTOA to find the values of cos θ and tan θ
- If you know a value for sin θ or cos θ you can use the Pythagorean relationship
  - $\sin^2\theta + \cos^2\theta = 1$
  - to find the value of the other
- If you know a value for sin θ or cos θ you can use the double angle formulae to find the value of sin 2θ or cos 2θ
- If you know a value for tan θ you can use the double angle formulae to find the value of tan 2θ
- If you know two out of the three values for sin θ, cos θ or tan θ you can use the identity in tan
  - $\tan\theta = \frac{\sin\theta}{\cos\theta}$
  - to find the value of the third ratio

How do we determine whether a trigonometric ratio will be positive or negative?

- It is possible to determine whether a trigonometric ratio will be positive or negative by looking at the size of the angle and considering the **unit circle**
  - Angles in the range $0° < \theta° < 90°$ will be positive for all three ratios
  - Angles in the range $90° < \theta° < 180°$ will be positive for sin and negative for cos and tan
  - Angles in the range $180° < \theta° < 270°$ will be positive for tan and negative for sin and cos
  - Angles in the range $270° < \theta° < 360°$ will be positive for cos and negative for sin and tan
- The ratios for angles of 0°, 90°, 180°, 270° and 360° are either 0, 1, -1 or undefined
  - You should know these ratios or know how to derive them without a calculator

> **Exam Tip**
>
> - Being able to sketch out the unit circle and remembering CAST can help you to find all solutions to a problem in an exam question

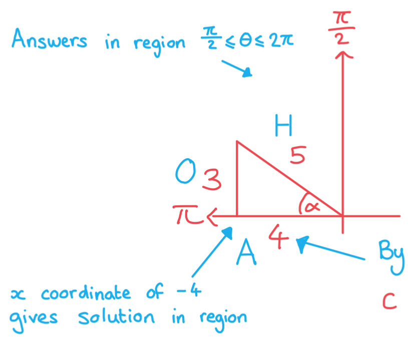
Worked Example

The value of $\sin\alpha = \frac{3}{5}$ for $\frac{\pi}{2} \leq \alpha \leq \pi$. Find:

i)
$\cos\alpha$

Method 1: Use right-triangle:

$\frac{\pi}{2} \leq \alpha \leq \pi$



$\sin\alpha = \frac{3}{5}$

$\sin\alpha = \frac{O}{H}$

By Pythagoras' theorem:

$c = \sqrt{5^2 - 3^2} = 4$

$\cos\alpha = \frac{A}{H} = -\frac{4}{5}$

$\cos\alpha = -\frac{4}{5}$

Method 2: Use Pythagorean identity:

$\cos^2\alpha = 1 - \sin^2\alpha = 1 - \left(\frac{3}{5}\right)^2$

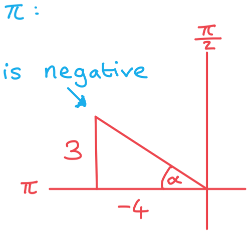
$\cos\alpha = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$

Check which solution is in range.

ii)
$\tan\alpha$

Use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4}$

Check if $\tan\alpha = -\frac{3}{4}$ is in the correct range for $\frac{\pi}{2} \le \alpha \le \pi$:



$\tan\alpha = -\frac{3}{4}$

iii)
$\sin 2\alpha$

Double angle identity: $\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)$

$= -\frac{24}{25}$

$\sin\alpha = \frac{3}{5}$
$\cos\alpha = -\frac{4}{5}$

$\sin 2\alpha = -\frac{24}{25}$

iv)
$\cos 2\alpha$

Double angle identity: $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\cos 2\alpha = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$

$\cos 2\alpha = \frac{7}{25}$

v)
$\tan 2\alpha$

Using identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}$$

$$\tan 2\alpha = -\frac{24}{7}$$

3.6.5 Linear Trigonometric Equations

Trigonometric Equations: sinx = k

How are trigonometric equations solved?

- Trigonometric equations can have an infinite number of solutions
  - For an equation in sin or cos you can add 360° or 2π to each solution to find more solutions
  - For an equation in tan you can add 180° or π to each solution
- When solving a trigonometric equation you will be given a range of values within which you should find all the values
- Solving the equation normally and using the inverse function on your calculator or your knowledge of **exact values** will give you the **primary value**
- The **secondary values** can be found with the help of:
  - The **unit circle**
  - The **graphs of trigonometric functions**

How are trigonometric equations of the form sin x = k solved?

- It is a good idea to sketch the graph of the trigonometric function first
  - Use the given range of values as the domain for your graph
  - The intersections of the graph of the function and the line y = k will show you
    - The location of the solutions
    - The number of solutions
  - You will be able to use the symmetry properties of the graph to find all secondary values within the given range of values
- The method for finding secondary values are:
  - For the equation sin x = k the primary value is $x_1 = \sin^{-1}k$
    - A secondary value is $x_2 = 180° - \sin^{-1}k$
    - Then all values within the range can be found using $x_1 \pm 360n$ and $x_2 \pm 360n$ where $n \in \mathbb{N}$
  - For the equation cos x = k the primary value is $x_1 = \cos^{-1}k$
    - A secondary value is $x_2 = -\cos^{-1}k$
    - Then all values within the range can be found using $x_1 \pm 360n$ and $x_2 \pm 360n$ where $n \in \mathbb{N}$
  - For the equation tan x = k the primary value is $x = \tan^{-1}k$
    - All secondary values within the range can be found using $x \pm 180n$ where $n \in \mathbb{N}$

Exam Tip

- If you are using your GDC it will only give you the principal value and you need to find all other solutions for the given interval
- Sketch out the CAST diagram and the trig graphs on your exam paper to refer back to as many times as you need to

Worked Example

Solve the equation $2\cos x = -1$, finding all solutions in the range $-\pi \leq x \leq \pi$.

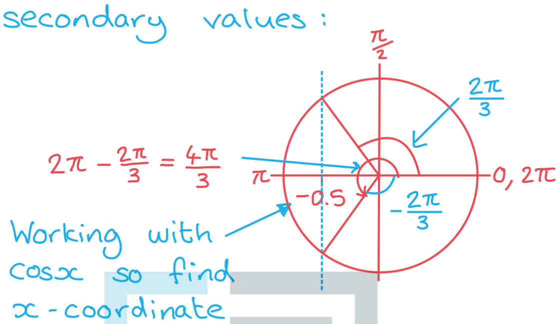
Isolate $\cos x$: $\cos x = \frac{-1}{2}$

$x = \cos^{-1}\left(-\frac{1}{2}\right)$

Use GDC or knowledge of exact values

$= \frac{2\pi}{3}$ ← Primary value

Find secondary values :



$\frac{2\pi}{3} \pm 2\pi n$ and $\frac{4\pi}{3} \pm 2\pi n$

Find all answers in range $-\pi \leq x \leq 3\pi$

$-\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$

Trigonometric Equations: sin(ax + b) = k

How can I solve equations with transformations of trig functions?

- Trigonometric equations in the form $\sin(ax + b)$ can be solved in more than one way
- The easiest method is to consider the transformation of the angle as a substitution
  - For example let $u = ax + b$
- Transform the given interval for the solutions in the same way as the angle
  - For example if the given interval is $0° \le x \le 360°$ the new interval will be
  - $(a(0°) + b) \le u \le (a(360°) + b)$
- Solve the function to find the primary value for u
- Use either the unit circle or sketch the graph to find all the other solutions in the range for u
- Undo the substitution to convert all of the solutions back into the corresponding solutions for x
- Another method would be to sketch the transformation of the function
  - If you use this method then you will not need to use a substitution for the range of values

Exam Tip

- If you transform the interval, remember to convert the found angles back to the final values at the end!
- If you are using your GDC it will only give you the principal value and you need to find all other solutions for the given interval
- Sketch out the CAST diagram and the trig graphs on your exam paper to refer back to as many times as you need to

Worked Example

Solve the equation $2\cos(2x - 30°) = -1$, finding all solutions in the range $-360° \le x \le 360°$.

$2\cos(2x - 30°) = -1$ $\quad -360° \le x \le 360°$

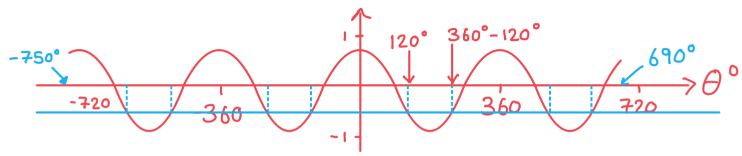
Start by changing the range: $-750° \le 2x - 30 \le 690°$

Substitute $\theta = 2x - 30$:

$2\cos\theta = -1$ $\quad -750° \le \theta \le 690°$

$\cos\theta = -\frac{1}{2}$

$\theta = \cos^{-1}(-\frac{1}{2}) = 120°$ ← primary value



From the sketch you can see there are 8 solutions:

$\theta = 120° \pm 360°$ and $\theta = 240° \pm 360°$

$\theta = -600°, -480°, -240°, -120°, 120°, 240°, 480°, 600°$

Solve for x: $x = \frac{\theta° + 30}{2}$

$x = -285°, -225°, -105°, -45°,$
$75°, 135°, 255°, 315°$

3.6.6 Quadratic Trigonometric Equations

Quadratic Trigonometric Equations

How are quadratic trigonometric equations solved?

- A quadratic trigonometric equation is one that includes either $\sin^2\theta$, $\cos^2\theta$ or $\tan^2\theta$
- Often the **identity** $\sin^2\theta + \cos^2\theta = 1$ can be used to rearrange the equation into a form that is possible to solve
  - If the equation involves both sine and cosine then the **Pythagorean identity** should be used to write the equation in terms of just one of these functions
- Solve the **quadratic equation** using your GDC, the quadratic equation or factorisation
  - This can be made easier by changing the function to a single letter
    Such as changing $2\cos^2\theta - 3\cos\theta - 1 = 0$ to $2c^2 - 3c - 1 = 0$
- A quadratic can give up to two solutions
  - You must consider both solutions to see whether a real value exists
  - Remember that solutions for sin θ = k and cos θ = k only exist for -1 ≤ k ≤ 1
  - Solutions for tan θ = k exist for all values of k
- Find all solutions within the given interval
  - There will often be more than two solutions for one quadratic equation
  - The best way to check the number of solutions is to sketch the graph of the function

Exam Tip

- Sketch the trig graphs on your exam paper to refer back to as many times as you need to!
- Be careful to make sure you have found **all** of the solutions in the given interval, being super-careful if you get a negative solution but have a positive interval

Worked Example

Solve the equation $11\sin x - 7 = 5\cos^2 x$, finding all solutions in the range $0 \le x \le 2\pi$.

Use the identity $\cos^2 x = 1 - \sin^2 x$ to write equation in terms of $\sin x$:

$$11\sin x - 7 = 5(1 - \sin^2 x)$$ in formula booklet.

$$= 5 - 5\sin^2 x$$

Move all terms to one side:

$$11\sin x - 7 - (5 - 5\sin^2 x) = 0$$

Spot the hidden quadratic:

$$11\sin x - 7 - 5 + 5\sin^2 x = 0$$

$$5\sin^2 x + 11\sin x - 12 = 0$$

$$\sin x = \frac{4}{5} \quad \text{or} \quad \sin x = -3$$

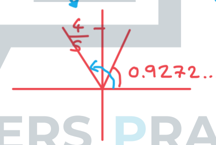
$\sin x = -3$: no solution so disregard.

$\sin x$ so use y coordinate.

$$\sin x = \frac{4}{5}$$

$$x = 0.9272...$$

primary solution



$$x = \pi - 0.9272...$$

$$= 2.214...$$

secondary solution

$$x = 0.927,\ 2.21 \quad (3\text{ s.f.})$$

3.7 Inverse & Reciprocal Trig Functions

3.7.1 Reciprocal Trig Functions

Reciprocal Trig Functions

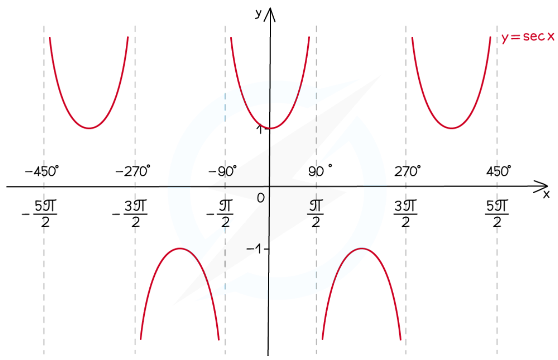
What are the reciprocal trig functions?

- There are three reciprocal trig functions that each correspond to either sin, cos or tan
  - Secant (sec *x*)
    $$\sec x = \frac{1}{\cos x}$$
  - Cosecant (cosec *x*)
    $$\text{cosec } x = \frac{1}{\sin x}$$
  - Cotangent (cot *x*)
    $$\cot x = \frac{1}{\tan x}$$
  - The identities above for sec *x* and cosec *x* are given in the formula booklet
  - The identity for cot *x* is **not given**, you will need to remember it
  - A good way to remember which function is which is to look at the **third** letter in each of the reciprocal trig functions
    - co**t** *x* is 1 over **t**an *x* etc
- Each of the reciprocal trig functions are undefined for certain values of *x*
  - sec *x* is undefined for values of *x* for which cos *x* = 0
  - cosec *x* is undefined for values of *x* for which sin *x* = 0
  - cot *x* is undefined for values of *x* for which tan *x* = 0
    - When tan *x* is undefined, cot *x* = 0
- Rearranging the identity $\tan x = \frac{\sin x}{\cos x}$ gives
  - $\cot x = \frac{\cos x}{\sin x}$
    - This is not in the formula booklet but is easily derived
- Be careful not to confuse the reciprocal trig functions with the inverse trig functions
  - $\sin^{-1} x \neq \frac{1}{\sin x}$

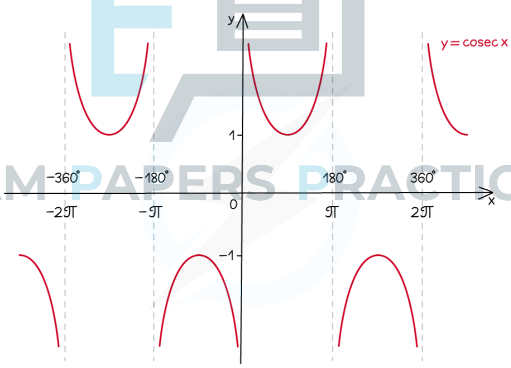
What do the graphs of the reciprocal trig functions look like?

- The graph of ***y* = sec*x*** has the following properties:
  - *The* ***y*-axis** is a **line of symmetry**
  - It has a **period** of **360°** (**2π** radians)
  - There are vertical **asymptotes** wherever **cos *x* = 0**
    - **If drawing the graph without the help of a GDC it is a good idea to sketch cos *x* first and draw these in**
  - **The domain** is all *x* **except odd multiples of 90°** (90°, -90°, 270°, -270°, etc.)
    - in **radians** this is all *x* **except odd multiples of π/2** (π/2, - π/2, 3π/2, -3π/2, etc.)

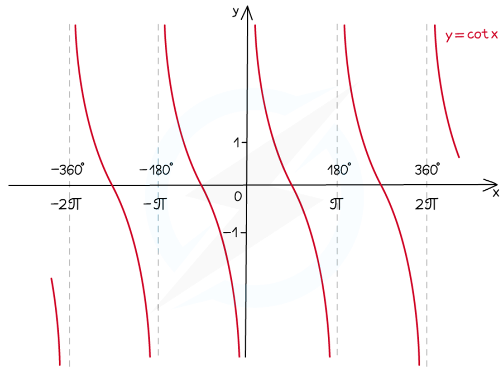
- **The range** is ***y* ≤ −1 or y ≥ 1**



- The graph of ***y* = cosec *x*** has the following properties:
  - It has a **period** of **360°** (**2π** radians)
  - There are vertical **asymptotes** wherever **sin *x* = 0**
    - **If drawing the graph it is a good idea to sketch sin *x* first and draw these in**
  - The **domain** is all *x* **except multiples of 180°** (0°, 180°, -180°, 360°, -360°, etc.)
    - in radians this is all *x* **except multiples of π** (0, π, - π, 2π, -2π, etc.)
  - The range is *y* ≤ -1 or y ≥ 1



- The graph of ***y* = cot *x*** has the following properties
  - It has a **period** of **180°** or **π** radians
  - There are vertical **asymptotes** wherever **tan *x* = 0**
  - The **domain** is all *x* **except multiples of 180°** (0°, 180°, -180°, 360°, -360°, etc.)
    - In radians this is all *x* **except multiples of π** (0, π, - π, 2π, -2π, etc.)
  - The **range** is $y \in \mathbb{R}$ (i.e. cot can take *any* real number value)



Exam Tip

- To solve equations with the reciprocal trig functions, convert them into the regular trig functions and solve in the usual way
- Don't forget that both **tan** and **cot** can be written in terms of **sin** and **cos**
- You will sometimes see **csc** instead of **cosec** for cosecant

Worked Example

Without the use of a calculator, find the values of

a)

$\sec \frac{\pi}{6}$

$\sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)}$

the third letter is c so sec is related to cos

$\cos\left(\frac{\pi}{6}\right)$ is an exact value you should know.

$= \frac{1}{\frac{\sqrt{3}}{2}}$

$\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$

b)
cot 45°

$\cot 45° = \frac{1}{\tan 45°}$

the third letter is t so cot is related to tan

tan 45° is an exact value you should know.

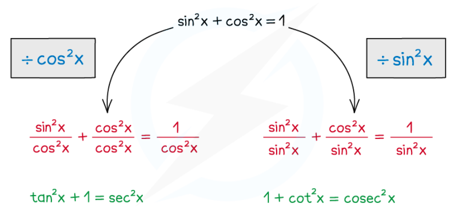
$= \frac{1}{1}$

$\cot 45° = 1$

Pythagorean Identities

What are the Pythagorean Identities?

- Aside from the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ there are two further Pythagorean identities you will need to learn
  - $1 + \tan^2\theta = \sec^2\theta$
  - $1 + \cot^2\theta = \mathrm{cosec}^2\theta$
  - Both can be found in the formula booklet
- Both of these identities can be derived from $\sin^2 x + \cos^2 x = 1$
  - To derive the identity for $\mathbf{sec^2 x}$ divide $\sin^2 x + \cos^2 x = 1$ by $\mathbf{cos^2 x}$
  - To derive the identity for $\mathbf{cosec^2 x}$ divide $\sin^2 x + \cos^2 x = 1$ by $\mathbf{sin^2 x}$



> **Exam Tip**
>
> All the Pythagorean identities can be found in the **Topic 3: Geometry and Trigonometry** section of the formula booklet

Worked Example

Solve the equation $9\sec^2\theta - 11 = 3\tan\theta$ in the interval $0 \le \theta \le 2\pi$.

$9\sec^2\theta - 11 = 3\tan\theta, \quad 0 \le \theta \le 2\pi$

Consider how this could be changed to use $\tan^2 + 1 = \sec^2$

Range is given in terms of π so work in radians

$(9\sec^2\theta - 9) - 2 = 3\tan\theta$

$9(\sec^2\theta - 1) - 2 = 3\tan\theta$

$9\tan^2\theta - 3\tan\theta - 2 = 0$

$(3\tan\theta - 2)(3\tan\theta + 1) = 0$

$\tan\theta = \frac{2}{3} \Rightarrow \theta = 0.5880...$

or $\theta = \pi + 0.5880... = 3.729...$

or $\tan\theta = -\frac{1}{3} \Rightarrow \theta = -0.3217...$

or $\theta = \pi + (-0.3217...) = 2.819...$

and $\theta = 2\pi + (-0.3217...) = 5.961...$

$\theta = 0.588, 2.82, 3.73, 5.96$ (3 s.f.)

3.7.2 Inverse Trig Functions

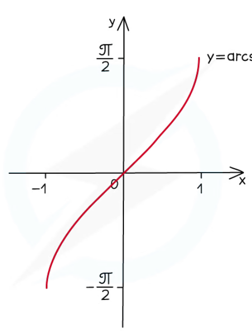
Inverse Trig Functions

What are the inverse trig functions?

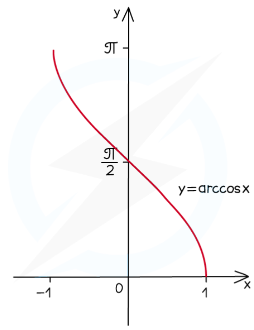
- The functions **arcsin**, **arccos** and **arctan** are the **inverse functions** of **sin**, **cos** and **tan** respectively when their domains are restricted
  - $\sin(\arcsin x) = x$ for $-1 \le x \le 1$
  - $\cos(\arccos x) = x$ for $-1 \le x \le 1$
  - $\tan(\arctan x) = x$ for all x
- You will have seen and used the inverse trig **operations** many times already
  - Arcsin is the operation $\mathbf{sin^{-1}}$
  - Arccos is to the operation $\mathbf{cos^{-1}}$
  - Arctan is the operation $\mathbf{tan^{-1}}$
- The domains of **sin**, **cos**, and **tan** must first be restricted to make them **one-to-one functions**
  - A function can only have an inverse if it is a one-to-one function
- The domain of **sin *x*** is restricted to $-\pi/2 \le x \le \pi/2$ ($-90° \le x \le 90°$)
- The domain of **cos *x*** is restricted to $0 \le x \le \pi$ ($0° \le x \le 180°$)
- The domain of **tan *x*** is restricted to $-\pi/2 < x < \pi/2$ ($-90° < x < 90°$)
- Be aware that $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$ are **not** the same as the reciprocal trig functions
  - They are used to solve trig equations such as $\sin x = 0.5$ for all values of x
  - $\arcsin x$ is the same as $\sin^{-1}x$ but not the same as $(\sin x)^{-1}$

What do the graphs of the inverse trig functions look like?

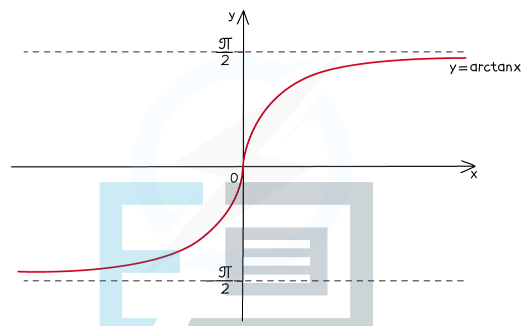
- The graphs of **arcsin**, **arccos** and **arctan** are the **reflections** of the graphs of **sin**, **cos** and **tan** (after their domains have been restricted) in the line $y = x$
  - The **domains** of $\arcsin x$ and $\arccos x$ are both $-1 \le x \le 1$
  - The **range** of $\arcsin x$ is $-\pi/2 \le y \le \pi/2$



  - The **range** of $\arccos x$ is $0 \le y \le \pi$



- The **domain** of arctan x is $x \in \mathbb{R}$
- The **range** of arctan x is $-\pi/2 < y < \pi/2$
  - Note that there are horizontal asymptotes at $\pi/2$ and $-\pi/2$



How are the inverse trig functions used?

- The functions **arcsin**, **arccos** and **arctan** are used to evaluate trigonometric equations such as $\sin x = 0.5$
  - If $\sin x = 0.5$ then arcsin $0.5 = x$ for values of x between $-\pi/2 \leq x \leq \pi/2$
    - You can then use symmetries of the trig function to find solutions over other intervals
- The inverse trig functions are also used to help evaluate algebraic expressions
  - From $\sin(\arcsin x) = x$ we can also say that $\sin^n(\arcsin x) = x^n$ for $-1 \leq x \leq 1$
  - If using an inverse trig function to evaluate an algebraic expression then remember to consider the domain and range of the function
    - $\arcsin(\sin x) = x$ only for $-\pi/2 \leq x \leq \pi/2$
    - $\arccos(\cos x) = x$ only for $0 \leq x \leq \pi$
    - $\arctan(\tan x) = x$ only for $-\pi/2 < x < \pi/2$
  - The symmetries of the trig functions can be used when values lie outside of the domain or range
    - Using $\sin(x) = \sin(\pi - x)$ you get $\arcsin(\sin(2\pi/3)) = \arcsin(\sin(\pi/3)) = \pi/3$

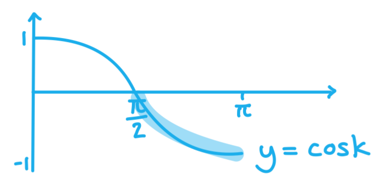
Exam Tip

- Make sure you know the shapes of the graphs for **sin**, **cos** and **tan** so that you can easily reflect them in the line $y = x$ and hence sketch the graphs of **arcsin**, **arccos** and **arctan**

Worked Example

Given that x satisfies the equation $\arccos x = k$ where $\frac{\pi}{2} < k < \pi$, state the range of possible values of x.

If $\arccos x = k$, then $x = \cos k$ $(\cos(\arccos x) = x)$



For $\frac{\pi}{2} < k < \pi$, $-1 < \cos k < 0$

$-1 < x < 0$

3.8 Further Trigonometry

3.8.1 Trigonometric Proof

Trigonometric Proof

How do I prove new trigonometric identities?

- You can use trigonometric identities you already know to prove new identities
- Make sure you know how to find all of the trig identities in the formula booklet
  - The identity for tan, simple Pythagorean identity and the double angle identities for in and cos are in the SL section

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cos^2\theta + \sin^2\theta = 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

  - The reciprocal trigonometric identities for sec and cosec, further Pythagorean identities, compound angle identities and the double angle formula for tan

$$\sec\theta = \frac{1}{\cos\theta}$$
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$
$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

  - The identity for cot is **not in the formula booklet**, you will need to remember it

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

- To prove an identity start on one side and proceed step by step until you get to the other side
  - It is more common to start on the left hand side but you can start a proof from either end
  - Occasionally it is easier to show that one side subtracted from the other is zero
  - You should not work on both sides simultaneously

What should I look out for when proving new trigonometric identities?

- Look for anything that could be a part of one of the above identities on either side
  - For example if you see $\sin 2\theta$ you can replace it with $2\sin\theta\cos\theta$
  - If you see $2\sin\theta\cos\theta$ you can replace it with $\sin 2\theta$

- Look for ways of reducing the number of different trigonometric functions there are within the identity
  - For example if the identity contains $\tan\theta$, $\cot\theta$ and $\operatorname{cosec}\theta$ you could try
    - using the identities $\tan\theta = 1/\cot\theta$ and $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ to write it all in terms of $\cot\theta$
    - or rewriting it all in terms of $\sin\theta$ and $\cos\theta$ and simplifying
- Often you may need to trial a few different methods before finding the correct one
- Clever substitution into the **compound angle formulae** can be a useful tool for proving identities
  - For example rewriting $\cos\frac{\theta}{2}$ as $\cos\left(\theta - \frac{\theta}{2}\right)$ doesn't change the ratio but could make an identity easier to prove
- You will most likely need to be able to work with fractions and fractions-within-fractions
- Always keep an eye on the 'target' expression – this can help suggest what identities to use

Exam Tip

- Don't forget that you can start a proof from either end – sometimes it might be easier to start from the left-hand side and sometimes it may be easier to start from the right-hand side
- Make sure you use the formula booklet as all of the relevant trigonometric identities are given to you
- Look out for special angles (0°, 90°, etc) as you may be able to quickly simplify or cancel parts of an expression (e.g. $\cos 90° = 0$)

Worked Example

Prove that $8\cos^4\theta - 8\cos^2\theta + 1 = \cos 4\theta$.

It is easiest to start on the right-hand side and apply the double angle formula for $\cos 2\theta$.

$$8\cos^4\theta - 8\cos^2\theta + 1 = \cos 4\theta$$

The form of the left-hand side suggests that the identity $\cos 2A = 2\cos^2 A - 1$ would be more useful than the other options.

$$\cos 4\theta = 2\cos^2 2\theta - 1$$
$$= 2(2\cos^2\theta - 1)^2 - 1$$
$$= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1$$
$$= 8\cos^4\theta - 8\cos^2\theta + 2 - 1$$
$$\therefore \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

EXAM PAPERS PRACTICE

3.8.2 Strategy for Trigonometric Equations

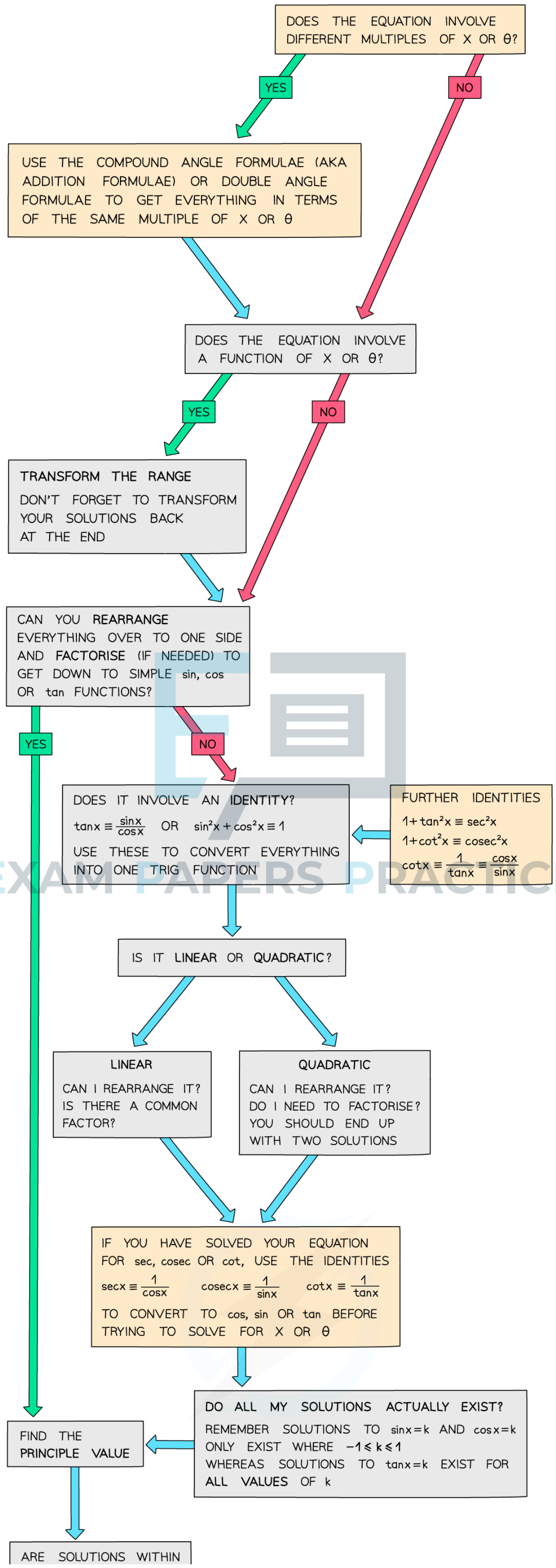
Strategy for Trigonometric Equations

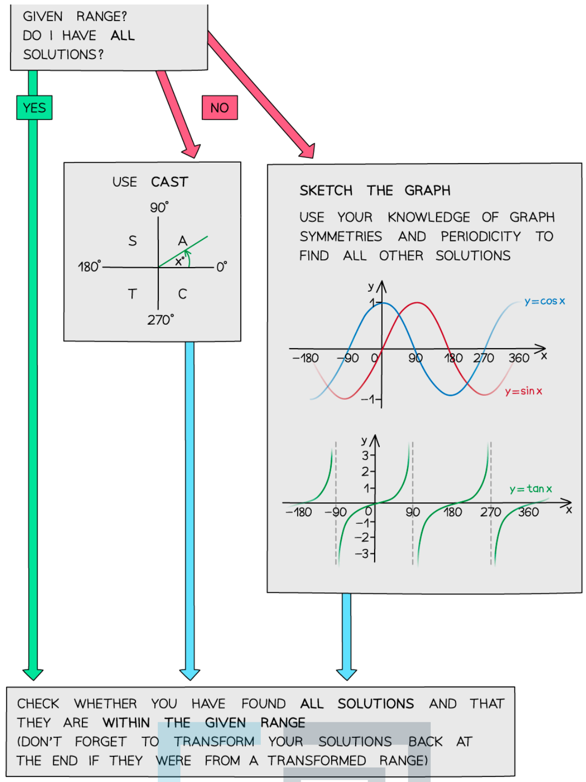
How do I approach solving trig equations?

- You can solve trig equations in a variety of different ways
  - **Sketching a graph**
    - If you have your GDC it is always worth sketching the graph and using this to analyse its features
  - Using **trigonometric** identities**, Pythagorean** identities**,** the **compound** or **double angle** identities
    - Almost all of these are in the formula booklet, make sure you have it open at the right page
  - Using the **unit circle**
  - Factorising **quadratic** trig equations
    - Look out for quadratics such as $5\tan^2 x - 3\tan x - 4 = 0$
- The final rearranged equation you solve will involve **sin**, **cos** or **tan**
  - Don't try to solve an equation with **cosec**, **sec**, or **cot** directly

What should I look for when solving trig equations?

- Check the value of x or θ
  - If it is just x or θ you can begin solving
  - If there are **different multiples** of x or θ you will need to use the **double angle formulae** to get everything in terms of the same multiple of x or θ
  - If it is a **function** of x or θ, e.g. $2x - 15$, you will need to **transform the range** first
    - You must remember to transform your solutions back again at the end
- Does it involve more than one trigonometric function?
  - If it does, try to **rearrange** everything to bring it to one side, you may need to **factorise**
  - If not, can you use an identity to reduce the number of different trigonometric functions?
    - You should be able to use identities to reduce everything to just one simple trig function (either sin, cos or tan)
- Is it **linear** or **quadratic**?
  - If it is linear you should be able to rearrange and solve it
  - If it is quadratic you may need to factorise first
    - You will most likely get two solutions, consider whether they both **exist**
    - Remember solutions to $\sin x = k$ and $\cos x = k$ only exist for $-1 \le k \le 1$ whereas solutions to $\tan x = k$ exist for all values of k
- Are my solutions within the given range and do I need to find more solutions?
  - Be extra careful if your solutions are negative but the given range is positive only
  - Use a sketch of the graph or the unit circle to find the other solutions within the range
  - If you have a function of x or θ make sure you are finding the solutions within the **transformed range**
    - Don't forget to transform the solutions back so that they are in the required range at the end


DOES THE EQUATION INVOLVE DIFFERENT MULTIPLES OF X OR θ?
YES
NO
USE THE COMPOUND ANGLE FORMULAE (AKA ADDITION FORMULAE) OR DOUBLE ANGLE FORMULAE TO GET EVERYTHING IN TERMS OF THE SAME MULTIPLE OF X OR θ
DOES THE EQUATION INVOLVE A FUNCTION OF X OR θ?
YES
NO
TRANSFORM THE RANGE
DON'T FORGET TO TRANSFORM YOUR SOLUTIONS BACK AT THE END
CAN YOU REARRANGE EVERYTHING OVER TO ONE SIDE AND FACTORISE (IF NEEDED) TO GET DOWN TO SIMPLE sin, cos OR tan FUNCTIONS?
YES
NO
DOES IT INVOLVE AN IDENTITY?
$\tan x \equiv \frac{\sin x}{\cos x}$ OR $\sin^2 x + \cos^2 x \equiv 1$
USE THESE TO CONVERT EVERYTHING INTO ONE TRIG FUNCTION
FURTHER IDENTITIES
$1 + \tan^2 x \equiv \sec^2 x$
$1 + \cot^2 x \equiv \operatorname{cosec}^2 x$
$\cot x \equiv \frac{1}{\tan x} \equiv \frac{\cos x}{\sin x}$
IS IT LINEAR OR QUADRATIC?
LINEAR
CAN I REARRANGE IT? IS THERE A COMMON FACTOR?
QUADRATIC
CAN I REARRANGE IT? DO I NEED TO FACTORISE? YOU SHOULD END UP WITH TWO SOLUTIONS
IF YOU HAVE SOLVED YOUR EQUATION FOR sec, cosec OR cot, USE THE IDENTITIES
$\sec x \equiv \frac{1}{\cos x}$ $\operatorname{cosec} x \equiv \frac{1}{\sin x}$ $\cot x \equiv \frac{1}{\tan x}$
TO CONVERT TO cos, sin OR tan BEFORE TRYING TO SOLVE FOR X OR θ
DO ALL MY SOLUTIONS ACTUALLY EXIST?
REMEMBER SOLUTIONS TO $\sin x = k$ AND $\cos x = k$ ONLY EXIST WHERE $-1 \leq k \leq 1$ WHEREAS SOLUTIONS TO $\tan x = k$ EXIST FOR ALL VALUES OF k
FIND THE PRINCIPLE VALUE
ARE SOLUTIONS WITHIN



Exam Tip

- Try to use identities and formulas to reduce the equation into its simplest terms.
- Don't forget to check the function range and ensure you have included all possible solutions.
- If the question involves a function of x or θ ensure you transform the range first (and ensure you transform your solutions back again at the end!).

Worked Example

Find the solutions of the equation $(1+\cot^2 2\theta)(5\cos^2\theta - 1) = \cot^2 2\theta$ in the interval $0 \le \theta \le 2\pi$.

Move equivalent trig functions to the same sides:

$$5\cos^2\theta - 1 = \frac{\cot^2 2\theta}{1+\cot^2 2\theta}$$ divide both sides by $1+\cot^2 2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\therefore \cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$

$1+\cot^2\theta = \text{cosec}^2\theta$

$$5\left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right) - 1 = \frac{\cot^2 2\theta}{\text{cosec}^2 2\theta}$$

$$\frac{5}{2}\cos 2\theta + \frac{3}{2} = \frac{\frac{\cos^2 2\theta}{\sin^2 2\theta}}{\frac{1}{\sin^2 2\theta}}$$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\text{cosec}\theta = \frac{1}{\sin\theta}$

$$\frac{1}{2}(5\cos 2\theta + 3) = \cos^2 2\theta$$ Rearrange to form a quadratic in $\cos 2\theta$

$$2\cos^2 2\theta - 5\cos 2\theta - 3 = 0$$

$$(2\cos 2\theta + 1)(\cos 2\theta - 3) = 0$$

$\cos 2\theta = -\frac{1}{2}$ or $\cos 2\theta = 3$ ← no solutions

We are solving the equation for 2θ so we must transform the range first: $0 \le 2\theta \le 4\pi$

$$2\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \quad \text{(primary value)}$$

so $2\theta = \frac{2\pi}{3}$, $2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$, $\frac{2\pi}{3} + 2\pi = \frac{8\pi}{3}$, $\frac{4\pi}{3} + 2\pi = \frac{10\pi}{3}$

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

3.9 Vector Properties

3.9.1 Introduction to Vectors

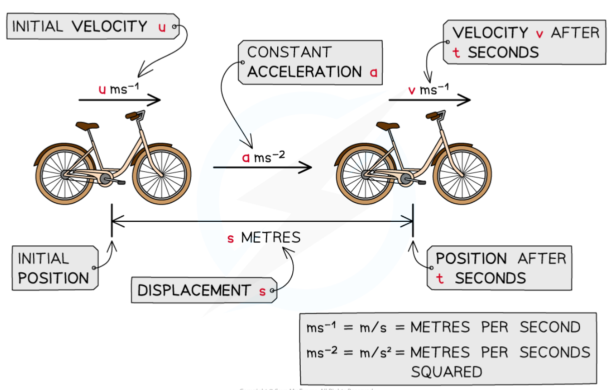
Scalars & Vectors

What are scalars?

- **Scalars** are quantities without **direction**
  - They have only a **size (magnitude)**
  - For example: **speed, distance, time, mass**
- **Most scalar quantities** can never be **negative**
  - You cannot have a negative speed or distance

What are vectors?

- **Vectors** are quantities which also have a **direction**, this is what makes them more than just a scalar
  - For example: two objects with **velocities** of 7 m/s and -7 m/s are travelling at the **same speed** but in **opposite directions**
- A **vector quantity** is described by both its **magnitude** and **direction**
- A vector has **components** in the direction of the *x*- , *y*-, and *z*- axes
  - Vector quantities can have **positive** or **negative** components
- Some examples of vector quantities you may come across are **displacement**, **velocity**, **acceleration**, **force/weight**, **momentum**
  - **Displacement** is the position of an object from a starting point
  - **Velocity** is a speed in a given direction (displacement over time)
  - **Acceleration** is the change in velocity over time
- Vectors may be given in either 2– or 3– dimensions



> **Exam Tip**
>
> - Make sure you fully understand the definitions of all the words in this section so that you can be clear about what your exam question is asking of you

Worked Example

State whether each of the following is a scalar or a vector quantity.

a)

A speed boat travels at 3 m/s on a bearing of 052°

Speed with a given direction → velocity

Vector

b)

A garden is 1.7 m wide

Length with no direction

Scalar

c)

A car accelerates forwards at 5.4 ms^{-2}

Acceleration has direction

Vector

d)

A film lasts 2 hours 17 minutes

Time has no direction

Scalar

e)

An athlete runs at an average speed of 10.44 ms^{-1}

Speed with no direction is a scalar

Scalar

f)

A ball rolls forwards 60 cm before stopping

Displacement has direction

Vector

EXAM PAPERS PRACTICE

Vector Notation

How are vectors represented?

- **Vectors** are usually represented using an arrow in the direction of movement
  - The length of the arrow represents its magnitude
- They are written as lowercase letters either in **bold** or underlined
  - For example a vector from the point O to A will be written **a** or *a*
    The vector from the point A to O will be written **-a** or -*a*
- If the start and end point of the vector is known, it is written using these points as capital letters with an arrow showing the direction of movement
  - For example: $\overrightarrow{AB}$ or $\overrightarrow{BA}$
- Two vectors are equal only if their corresponding components are equal
- Numerically, vectors are either represented using **column vectors** or **base vectors**
  - Unless otherwise indicated, you may carry out all working and write your answers in either of these two types of vector notation

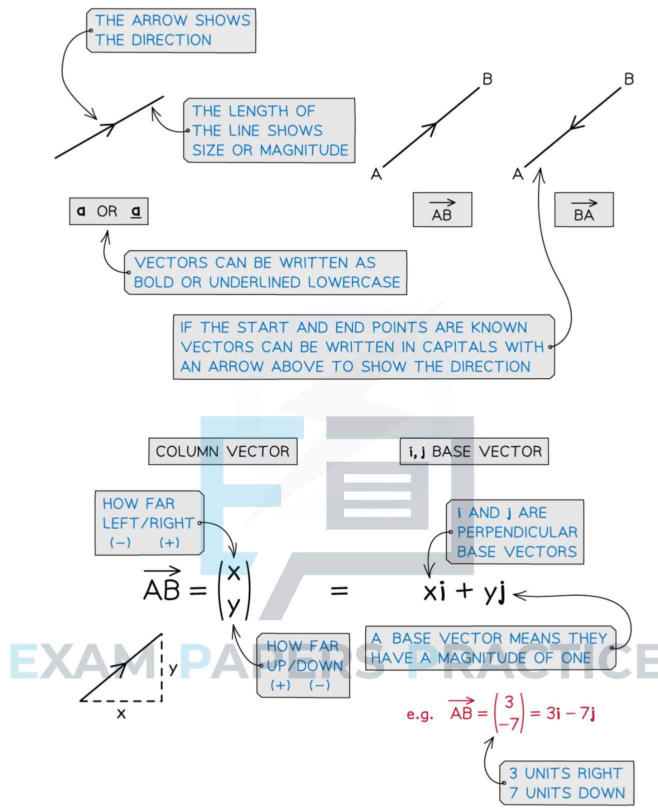
What are column vectors?

- **Column vectors** are where one number is written above the other enclosed in brackets
- **In 2-dimensions the top number represents movement in the horizontal direction (right/left) and the bottom number represents movement in the vertical direction (up/down)**
- **A positive value represents movement in the positive direction (right/up) and a negative value represents movement in the negative direction (left/down)**
  - For example: The column vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ represents **3 units** in the **positive horizontal** (*x*) direction (i.e., **right**) and **2 units** in the **negative vertical** (*y*) direction (i.e., **down**)
- **In 3-dimensions the top number represents the movement in the *x* direction (length), the middle number represents movement in the *y* direction (width) and the bottom number represents the movement in the *z* direction (depth)**
  - For example: The column vector $\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$ represents **3 units** in the **positive *x* direction**, **4 units** in the **negative *y* direction** and **2 units** in the **positive *z* direction**

What are base vectors?

- **Base vectors use i**, **j** and **k** notation **where i**, **j** and **k** are **unit vectors** in the positive *x*, *y*, and *z* directions respectively
  - **This is sometimes also called unit vector notation**
  - **A unit vector has a magnitude of 1**
- **In 2-dimensions i represents movement in the horizontal direction (right/left) and j represents the movement in the vertical direction (up/down)**
  - **For example:** The vector (-4**i** + 3**j**) would mean **4 units** in the **negative horizontal** (*x*) direction (i.e., **left**) and **3 units** in the **positive vertical** (*y*) direction (i.e., **up**)

- **In 3-dimensions i represents movement in the *x* direction (length), j represents movement in the *y* direction (width) and k represents the movement in the *z* direction (depth)**
  - **For example:** The vector (-4**i** + 3**j** - **k**) would mean **4 units** in the **negative *x* direction**, **3 units** in the **positive *y* direction** and **1 unit** in the **negative *z* direction**
- As they are vectors, **i, j and k** are displayed in **bold** in textbooks and online but in handwriting they would be underlined (i, j and k)



Exam Tip

- Practice working with all types of vector notation so that you are prepared for whatever comes up in the exam
  - Your working and answer in the exam can be in any form unless told otherwise
  - It is generally best to write your final answer in the same form as given in the question, however you will not lose marks for not doing this unless it is specified in the question
- Vectors appear in **bold** (non-italic) font in textbooks and on exam papers, etc (i.e. ***F***, ***a***) but in handwriting should be underlined (i.e. $\underline{F}$, $\underline{a}$)

Worked Example

a)

Write the vector $\begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$ using **base vector** notation.

$$\begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix} = -4\underline{i} + 0\underline{j} + 5\underline{k}$$

$0\underline{j}$ is not needed when giving answer in base vector form.

$$5\underline{k} - 4\underline{i}$$

b)

Write the vector $\mathbf{k} - 2\mathbf{j}$ using **column vector** notation.

$$\underline{k} - 2\underline{j} = 0\underline{i} - 2\underline{j} + 1\underline{k}$$

Be careful with negative components and missing terms when working with base vectors

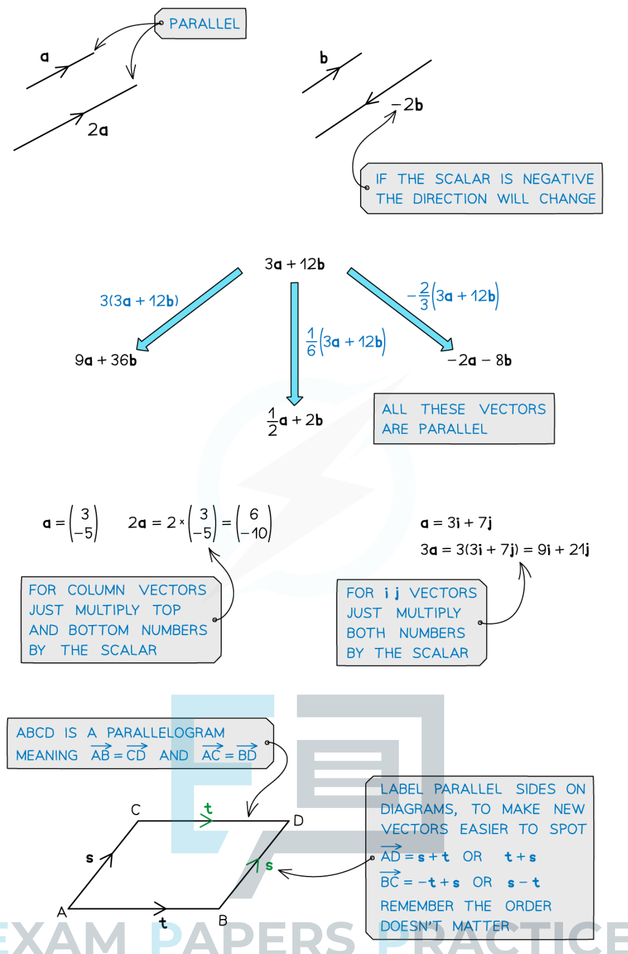
$$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

The zero term is needed when using column vector notation

Parallel Vectors

How do you know if two vectors are parallel?

- Two vectors are parallel if one is a **scalar multiple** of the other
  - This means that all components of the vector have been multiplied by a **common constant (scalar)**
- Multiplying every component in a vector by a **scalar** will change the **magnitude** of the vector but not the **direction**
  - For example: the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{b} = 2\mathbf{a} = 2\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$ will have the **same direction** but the vector **b** will have twice the magnitude of **a**
    - They are **parallel**
- If a vector can be factorised by a **scalar** then it is parallel to any **scalar multiple** of the factorised vector
  - For example: The vector $9\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ can be factorised by the scalar 3 to $3(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ so the vector $9\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ is parallel to any **scalar multiple** of $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
- If a vector is multiplied by a **negative scalar** its direction will be **reversed**
  - It will still be **parallel** to the original vector
- Two vectors are **parallel** if they have the same or reverse **direction** and **equal** if they have the same **size and direction**



Exam Tip

- It is easiest to spot that two vectors are parallel when they are in column vector notation
  - in your exam by writing vectors in column vector form and looking for a scalar multiple you will be able to quickly determine whether they are parallel or not

Worked Example

Show that the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$ and $\mathbf{b} = 6\mathbf{k} - 3\mathbf{i}$ are parallel and find the scalar multiple that maps **a** onto **b**.

Convert both vectors into the same form and then look for a value of k such that $\underline{a} = k\underline{b}$, where k is a scalar.

$\underline{a} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

$\underline{b} = 6\underline{k} - 3\underline{i} = -3\underline{i} + 0\underline{j} + 6\underline{k}$

$= \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

$= -\frac{3}{2}\underline{a}$

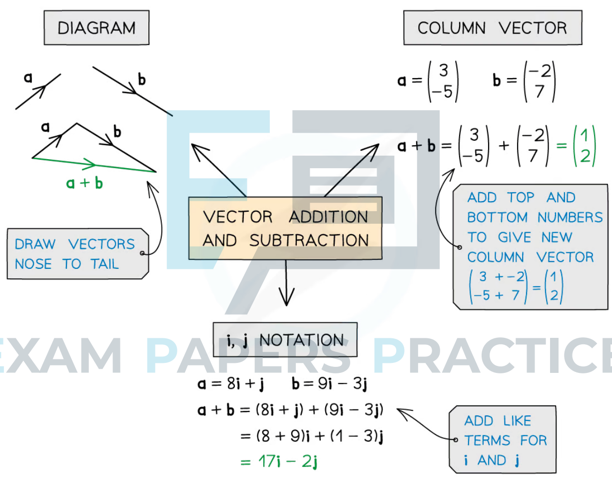
$\underline{b} = -\frac{3}{2}\underline{a}$, $k = -\frac{3}{2}$

3.9.2 Position & Displacement Vectors

Adding & Subtracting Vectors

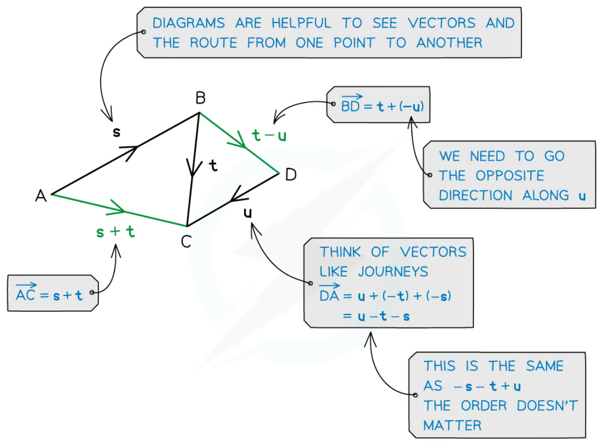
How are vectors added and subtracted numerically?

- To **add** or **subtract** vectors numerically simply add or subtract each of the corresponding components
- In **column vector** notation just add the top, middle and bottom parts together
  - For example: $\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -8 \end{pmatrix}$
- In **base vector** notation add each of the **i**, **j**, and **k** components together separately
  - For example: (2**i** + **j** – 5**k**) – (**i** + 4**j** + 3**k**) = (**i** – 3**j** – 8**k**)



How are vectors added and subtracted geometrically?

- Vectors can be **added** geometrically by joining the end of one vector to the start of the next one
- The **resultant** vector will be the shortest route from the start of the first vector to the end of the second
  - A **resultant** vector is a vector that results from **adding** or **subtracting** two or more vectors
- If the two vectors have the same **starting position**, the second vector can be **translated** to the end of the first vector to find the resultant vector
  - This results in a **parallelogram** with the resultant vector as the diagonal
- To **subtract** vectors, consider this as **adding on the negative vector**
  - For example: **a** – **b** = **a** + (–**b**)
  - The end of the **resultant vector a – b** will not be anywhere near the end of the vector **b**
    Instead, it will be at the point where the end of the vector **–b** would be



Exam Tip

- Working in column vectors tends to be easiest when adding and subtracting
  - in your exam, it can help to convert any vectors into column vectors before carrying out calculations with them
- If there is no diagram, drawing one can be helpful to help you visualise the problem

Worked Example

Find the resultant of the vectors $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$.

$\underline{a} = 5\underline{i} - 2\underline{j} + 0\underline{k} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

Writing as a column vector makes adding and subtracting easier.

$\underline{a} + \underline{b} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

Resultant vector $= 2\underline{i} - \underline{j} + 2\underline{k}$

Position Vectors

What is a position vector?

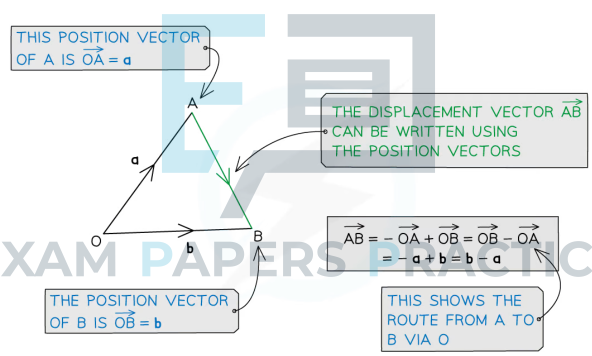
- A position vector describes the **position** of a point in relation to the **origin**
  - It describes the **direction** and the **distance** from the point O: 0**i** + 0**j** + 0**k** or $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
  - It is different to a **displacement vector** which describes the direction and distance between any two points
- The position vector of point A is written with the notation **a** = $\overrightarrow{OA}$
  - The origin is always denoted O
- The individual components of a position vector are the coordinates of its end point
  - For example the point with coordinates (3, -2, -1) has position vector 3**i** – 2**j** – **k**

> **?** Worked Example
>
> Determine the position vector of the point with coordinates (4, -1, 8).
>
> $4\underline{i} - \underline{j} + 8\underline{k}$

Displacement Vectors

What is a displacement vector?

- A **displacement vector** describes the shortest route between any two points
  - It describes the **direction** and the **distance** between any two points
  - It is different to a **position vector** which describes the direction and distance from the point O: 0**i** + 0**j** or $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- The displacement vector of point B from the point A is written with the notation $\overrightarrow{AB}$
- A displacement vector between two points can be written in terms of the displacement vectors of a third point
  - $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$
- A displacement vector can be written in terms of its position vectors
  - For example the displacement vector $\overrightarrow{AB}$ can be written in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$
  - $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$
  - For position vector **a** = $\overrightarrow{OA}$ and **b** = $\overrightarrow{OB}$ the displacement vector $\overrightarrow{AB}$ can be written **b** – **a**



Exam Tip

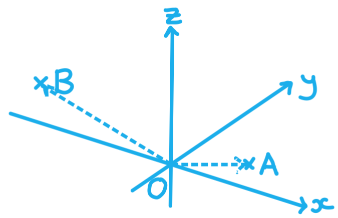
- In an exam, sketching a quick diagram can help to make working out a displacement vector easier

Worked Example

The point A has coordinates (3, 0, -1) and the point B has coordinates (-2, -5, 7).

Find the displacement vector $\overrightarrow{AB}$.

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \qquad \overrightarrow{OB} = \begin{pmatrix} -2 \\ -5 \\ 7 \end{pmatrix}$$



$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$= -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} -2 \\ -5 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

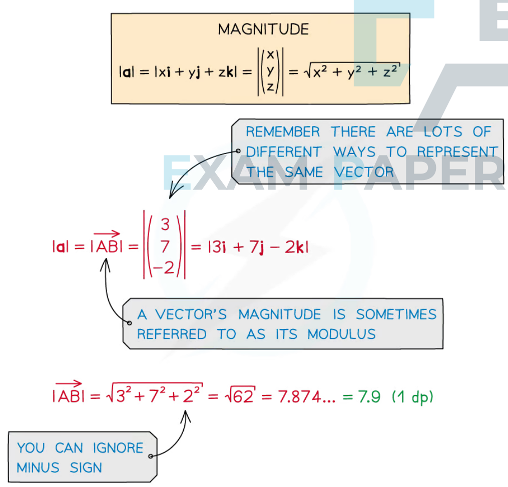
$$\overrightarrow{AB} = \begin{pmatrix} -5 \\ -5 \\ 8 \end{pmatrix}$$

3.9.3 Magnitude of a Vector

Magnitude of a Vector

How do you find the magnitude of a vector?

- The **magnitude** of a vector tells us its **size** or **length**
  - For a **displacement** vector it tells us the **distance** between the two points
  - For a **position** vector it tells us the **distance** of the point from the **origin**
- The magnitude of the vector $\overrightarrow{AB}$ is denoted $\left|\overrightarrow{AB}\right|$
  - The magnitude of the vector **a** is denoted |**a**|
- The magnitude of a vector can be found using **Pythagoras' Theroem**
- The magnitude of a vector $= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is found using
  - $|\boldsymbol{v}| = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}}$
  - where $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
  - This is **given in the formula booklet**



How do I find the distance between two points?

- **Vectors** can be used to find the distance (or displacement) between two points
  - It is the **magnitude** of the vector between them
- Given the **position vectors** of two points:
  - Find the displacement vector between them
  - Find the magnitude of the displacement vector between them

Exam Tip

- Finding the magnitude of a vector is the same as finding the distance between two coordinates, it is a useful formula to commit to memory in order to save time in the exam, however it is in your formula booklet if you need it

Worked Example

Find the magnitude of the vector AB = 4**i** – **j** + 2**k**.

Magnitude of a vector	$\lvert \boldsymbol{v} \rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$|\overrightarrow{AB}| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$$

$$|\overrightarrow{AB}| = \sqrt{21}$$

Unit Vectors

What is a unit vector?

- A **unit vector** has a **magnitude** of 1
- It can be found by dividing a vector by its **magnitude**
  - This will result in a vector with a size of 1 unit in the direction of the original vector
- A unit vector in the direction of **a** is denoted $\frac{\mathbf{a}}{|\mathbf{a}|}$
  - For example a unit vector in the direction 3**i** – 4**j** is $\frac{(3\mathbf{i} - 4\mathbf{j})}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

Exam Tip

- Finding the unit vector will not be a question on its own but will be a useful skill for further vectors problems so it is important to be confident with it

Worked Example

Find the unit vector in the direction 2**i** – 2**j** + **k**.

Let $\underline{a} = 2\underline{i} - 2\underline{j} + \underline{k}$

Find the magnitude of $\underline{a}$

Magnitude of a vector	$\lvert v \rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$, where $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$|\underline{a}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

Divide $\underline{a}$ by its magnitude:

Unit vector $= \frac{\underline{a}}{|\underline{a}|} = \frac{2\underline{i} - 2\underline{j} + \underline{k}}{3}$

$\frac{2}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{1}{3}\underline{k}$

3.9.4 The Scalar Product

The Scalar ('Dot') Product

What is the scalar product?

- The scalar product (also known as the dot product) is one form in which two vectors can be combined together
- The scalar product between two vectors **a** and **b** is denoted $\mathbf{a} \cdot \mathbf{b}$
- The result of taking the scalar product of two vectors is a **real number**
  - i.e. a scalar
- The scalar product of two vectors gives information about the angle between the two vectors
  - If the scalar product is **positive** then the angle between the two vectors is **acute** (less than 90°)
  - If the scalar product is **negative** then the angle between the two vectors is **obtuse** (between 90° and 180°)
  - If the scalar product is **zero** then the angle between the two vectors is **90°** (the two vectors are **perpendicular**)

How is the scalar product calculated?

- There are **two methods** for calculating the scalar product
- The most common method used to find the scalar product between the two vectors ***v*** and ***w*** is to find the **sum of the product of each component** in the two vectors
  - $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$
  - Where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
  - This is **given in the formula booklet**
- The scalar product is also equal to the **product of the magnitudes** of the two vectors and the **cosine of the angle between them**
  - $\boldsymbol{v} \cdot \boldsymbol{w} = |v||w|\cos\theta$
  - Where θ is the angle between ***v*** and ***w***

The two vectors ***v*** and ***w*** are joined at the start and pointing away from each other

- The scalar product can be used in the second formula to find the angle between the two vectors

What properties of the scalar product do I need to know?

- The order of the vectors doesn't change the result of the scalar product (it is **commutative**)
  - · = ·
- The **distributive law** over addition can be used to 'expand brackets'
  - $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$
- The scalar product is **associative** with respect to multiplication by a scalar
  - $(k\boldsymbol{v}) \cdot (\boldsymbol{w}) = k(\boldsymbol{v} \cdot \boldsymbol{w})$
- The scalar product between a vector and itself is equal to the **square of its magnitude**

- $\boldsymbol{v} \cdot \boldsymbol{v} = |\boldsymbol{v}|^2$

- If two vectors, ***v*** and ***w***, are **parallel** then the magnitude of the scalar product is equal to the **product** of the magnitudes of the vectors
  - $| \quad \cdot \quad | = | \quad || \boldsymbol{v}|$
  - This is because cos 0° = 1 and cos 180° = -1
- If two vectors are **perpendicular** the scalar product is **zero**
  - This is because cos 90° = 0

Exam Tip

- Whilst the formulae for the scalar product are given in the formula booklet, the properties of the scalar product are not, however they are important and it is likely that you will need to recall them in your exam so be sure to commit them to memory

EXAM PAPERS PRACTICE

Worked Example

Calculate the scalar product between the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ and

$\boldsymbol{w} = 3\mathbf{j} - 2\mathbf{k} - \mathbf{i}$ using:

i)
the formula $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$,

$\underline{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} = 2\underline{i} + 0\underline{j} - 5\underline{k}$

$\underline{w} = 3\underline{j} - 2\underline{k} - \underline{i} = -1\underline{i} + 3\underline{j} - 2\underline{k}$

Be aware of the order of the terms.

Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$\underline{v} \cdot \underline{w} = (2 \times -1) + (0 \times 3) + (-5 \times -2) = -2 + 10$

$\underline{v} \cdot \underline{w} = 8$

ii)
the formula $\boldsymbol{v} \cdot \boldsymbol{w} = |v||w|\cos\theta$, given that the angle between the two vectors is 66.6°.

$\underline{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} = 2\underline{i} + 0\underline{j} - 5\underline{k}$ $\quad \underline{w} = -1\underline{i} + 3\underline{j} - 2\underline{k}$

Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = |\boldsymbol{v}||\boldsymbol{w}|\cos\theta$

Find the magnitude of both vectors:

$|\underline{v}| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$ $\quad |\underline{w}| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$

$v \cdot w = \sqrt{29} \times \sqrt{14} \cos 66.6°$

$\underline{v} \cdot \underline{w} = 8$

Angle Between Two Vectors

How do I find the angle between two vectors?

- If two vectors with different directions are placed at the same starting position, they will form an angle between them
- The two formulae for the scalar product can be used together to find this angle
    - $\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{|\boldsymbol{v}||\boldsymbol{w}|}$
    - This is given in the formula booklet
- To find the angle between two vectors:
    - Calculate the scalar product between them
    - Calculate the magnitude of each vector
    - Use the formula to find $\cos\theta$
    - Use inverse trig to find θ

Exam Tip

- The formula for this is given in the formula booklet so you do not need to remember it but make sure that you can find it quickly and easily in your exam

Worked Example

Calculate the angle formed by the two vectors $\boldsymbol{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

$\underline{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, $\underline{w} = 3\underline{i} + 4\underline{j} - \underline{k} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

Start by finding the scalar product:

$\underline{v} \cdot \underline{w} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

$= (-1 \times 3) + (3 \times 4) + (2 \times -1) = 7$

Find the magnitude of both vectors:

$|\underline{v}| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}$

$|\underline{w}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}$

Angle between two vectors	$\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$

$\cos\theta = \frac{7}{\sqrt{14} \times \sqrt{26}} = 0.3668...$

$\theta = \cos^{-1}(0.3668...)$

$\theta = 68.5^\circ$ (3sf)

Perpendicular Vectors

How do I know if two vectors are perpendicular?

- If the **scalar product** of two (non-zero) vectors is **zero** then they are **perpendicular**
  - If $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ then $\boldsymbol{v}$ and $\boldsymbol{w}$ must be perpendicular to each other
- Two vectors are **perpendicular** if their **scalar product** is **zero**
  - The value of $\cos\theta = 0$ therefore $|\boldsymbol{v}||\boldsymbol{w}|\cos\theta = 0$

Worked Example

Find the value of t such that the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ t \\ 5 \end{pmatrix}$ and $\boldsymbol{w} = (t-1)\mathbf{i} - \mathbf{j} + \mathbf{k}$ are perpendicular to each other.

The two vectors $\underline{v}$ and $\underline{w}$ are perpendicular if $\underline{v} \cdot \underline{w} = 0$.

$\underline{v} = \begin{pmatrix} 2 \\ t \\ 5 \end{pmatrix}, \quad \underline{w} = \begin{pmatrix} t-1 \\ -1 \\ 1 \end{pmatrix}$

$\underline{v} \cdot \underline{w} = 2(t-1) + t(-1) + 5(1)$

$= 2t - 2 - t + 5$

Therefore $\underline{v}$ and $\underline{w}$ are perpendicular if

$t + 3 = 0$

$t = -3$

3.9.5 Geometric Proof with Vectors

Geometric Proof with Vectors

How can vectors be used to prove geometrical properties?

- If two vectors can be shown to be **parallel** then this can be used to prove parallel lines
  - If two vectors are **scalar multiples** of each other then they are **parallel**
  - To prove that two vectors are parallel simply show that one is a scalar multiple of the other
- If two vectors can be shown to be **perpendicular** then this can be used to prove perpendicular lines
  - If the **scalar product** is zero then the two vectors are **perpendicular**
- If two vectors can be shown to have equal **magnitude** then this can be used to prove two lines are the **same length**
- To prove a 2D shape is a **parallelogram** vectors can be used to
  - Show that there are two pairs of **parallel sides**
  - Show that the **opposite sides** are of **equal length**
    - The vectors opposite each other with be **equal**
  - If the angle between two of the vectors is shown to be 90° then the parallelogram is a **rectangle**
- To prove a 2D shape is a **rhombus** vectors can be used to
  - Show that there are two pairs of **parallel sides**
    - The vectors opposite each other with be **equal**
  - Show that **all four sides** are of **equal length**
  - If the angle between two of the vectors is shown to be 90° then the rhombus is a **square**

How are vectors used to follow paths through a diagram?

- In a geometric diagram the vector $\overrightarrow{AB}$ forms a path from the point A to the point B
  - This is specific to the path AB
  - If the vector $\overrightarrow{AB}$ is labelled **a** then any other vector with the same **magnitude** and **direction** as **a** could also be labelled **a**
- The vector $\overrightarrow{BA}$ would be labelled **-a**
  - It is **parallel** to **a** but pointing in the **opposite direction**
- If the point M is exactly halfway between A and B it is called the midpoint of A and the vector $\overrightarrow{AM}$ could be labelled $\frac{1}{2}$ **a**
- If there is a point X on the line AB such that $\overrightarrow{AX} = 2\overrightarrow{XB}$ then X is two-thirds of the way along the line $\overrightarrow{AB}$
  - Other ratios can be found in similar ways
  - A diagram often helps to visualise this
- If a point X divides a line segment AB into the ratio p : q then
  - $\overrightarrow{AX} = \frac{p}{p+q}\overrightarrow{AB}$

- $\overrightarrow{XB} = \frac{q}{p + q}\overrightarrow{AB}$

How can vectors be used to find the midpoint of two vectors?

- If the point A has position vector **a** and the point B has position vector **b** then the **position vector** of the **midpoint** of $\overrightarrow{AB}$ is $\frac{1}{2}(\mathbf{a} + \mathbf{b})$
  - The **displacement vector** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
  - Let **M** be the midpoint of $\overrightarrow{AB}$ then $\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB}) = \frac{1}{2}(\mathbf{b} - \mathbf{a})$
  - The **position vector** $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

How can vectors be used to prove that three points are collinear?

- Three points are collinear if they all **lie on the same line**
  - The vectors between the three points will be **scalar multiples** of each other
- The points A, B and C are collinear if $\overrightarrow{AB} = k\overrightarrow{BC}$
- If the points A, B and M are collinear and $\overrightarrow{AM} = \overrightarrow{MB}$ then M is the **midpoint** of $\overrightarrow{AB}$

Exam Tip

- Think of vectors like a journey from one place to another
  - You may have to take a detour e.g. A to B might be A to O then O to B
- Diagrams can help, if there isn't one, draw one
  - If a diagram has been given begin by labelling all known quantities and vectors

Worked Example

Use vectors to prove that the points *A*, *B*, *C* and *D* with position vectors $\mathbf{a} = (3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k})$, $\mathbf{b} = (8\mathbf{i} - 7\mathbf{j} - 5\mathbf{k})$, $\mathbf{c} = (3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$ and $\mathbf{d} = (5\mathbf{k} - 2\mathbf{i})$ are the vertices of a parallelogram.

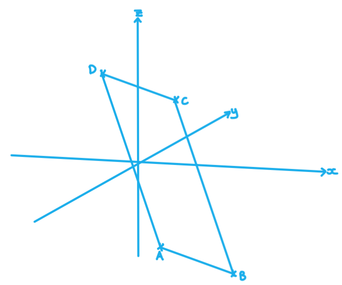
Find the displacement vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$$

$$\overrightarrow{BC} = \underline{c} - \underline{b} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 9 \end{pmatrix}$$

$$\overrightarrow{CD} = \underline{d} - \underline{c} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{DA} = \underline{a} - \underline{d} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ -9 \end{pmatrix}$$



$\overrightarrow{AB} = -\overrightarrow{CD}$ and $\overrightarrow{BC} = -\overrightarrow{DA}$ $\therefore$ ABCD must be a parallelogram

3.10 Vector Equations of Lines

3.10.1 Vector Equations of Lines

Equation of a Line in Vector Form

How do I find the vector equation of a line?

- The formula for finding the **vector equation** of a line is
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$

    Where ***r*** is the **position vector** of any point on the line

    a is the **position vector** of a known point on the line

    b is a **direction** (displacement) **vector**

    λ is a scalar
  - This is **given in the formula booklet**
  - This equation can be used for vectors in both 2– and 3– dimensions
- This formula is similar to a regular equation of a straight line in the form $y = mx + c$ but with a vector to show both a point on the line and the direction (or gradient) of the line
  - In 2D the gradient can be found from the direction vector
  - In 3D a numerical value for the direction cannot be found, it is given as a vector
- As ***a*** could be the position vector of **any** point on the line and ***b*** could be **any scalar multiple** of the direction vector there are infinite vector equations for a single line
- Given any two points on a line with position vectors ***a*** and ***b*** the **displacement** vector can be written as ***b*** - ***a***
  - So the formula **r = a** + λ(**b** - **a)** can be used to find the vector equation of the line
  - This is **not given in the formula booklet**

How do I determine whether a point lies on a line?

- Given the equation of a line $\boldsymbol{r} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix}$ the point ***c*** with position vector $\begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$ is on the line if there exists a value of λ such that
  - $\begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix}$
  - This means that there exists a single value of λ that satisfies the three equations:

    $c_1 = a_1 + \lambda b_1$

    $c_2 = a_2 + \lambda b_2$

    $c_3 = a_3 + \lambda b_3$
- A GDC can be used to solve this system of linear equations for
  - The point only lies on the line if a single value of λ exists for all three equations
- Solve one of the equations first to find a value of λ that satisfies the first equation and then check that this value also satisfies the other two equations

- If the value of λ does not satisfy all three equations, then the point **c** does not lie on the line

Exam Tip

- Remember that the vector equation of a line can take many different forms
  - This means that the answer you derive might look different from the answer in a mark scheme
- You can choose whether to write your vector equations of lines using unit vectors or as column vectors
  - Use the form that you prefer, however column vectors is generally easier to work with

Worked Example

a)

Find a vector equation of a straight line through the points with position vectors **a** = 4**i** – 5**k** and **b** = 3**i** - 3**k**

Use the position vectors to find the displacement vector between them.

$\vec{OA} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}, \quad \vec{OB} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \Rightarrow \vec{AB} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

Vector equation of a line	$r = a + \lambda b$

$r = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ (position vector of point a; direction vector) or $r = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ (position vector of point b; direction vector)

$$r = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

b)

Determine whether the point C with coordinate (2, 0, -1) lies on this line.

Let $c = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, then check to see if there exists a value of λ such that

$$\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

From the 'i' component : $4 - \lambda = 2$ ①

From the 'j' component : $0 + 0\lambda = 0$ ② (✓) Works for all λ

From the 'k' component : $-5 + 2\lambda = -1$ ③

① $\Rightarrow \lambda = 2$ sub into ③ $\Rightarrow -5 + (2 \times 2) = -5 + 4 = -1$ ✓

Point C lies on the line

Equation of a Line in Parametric Form

How do I find the vector equation of a line in parametric form?

- By considering the three separate components of a vector in the *x*, *y* and *z* directions it is possible to write the **vector equation** of a line as **three separate equations**
  - Letting $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ becomes
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

    Where $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ is a position vector and $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is a direction vector
  - This vector equation can then be split into its three separate component forms:

    $x = x_0 + \lambda l$

    $y = y_0 + \lambda m$

    $z = z_0 + \lambda n$
  - These are **given in the formula booklet**

? Worked Example

Write the parametric form of the equation of the line which passes through the point (-2, 1, 0) with direction vector $\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$.

Parametric form of the equation of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$

Use $r = a + \lambda b$ to write the equation in vector form first:

$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$

position vector of a point; direction vector

Separate the components into their 3 separate equations.

$x = -2 + 3\lambda$

$y = 1 + \lambda$

$z = -4\lambda$

Equation of a Line in Cartesian Form

- The **Cartesian** equation of a line can be found from the **vector equation of a line** by
  - Finding the vector equation of the line in parametric form
  - Eliminating λ from the parametric equations
    - λ can be eliminated by **making it the subject** of each of the parametric equations
    - For example: $x = x_0 + \lambda l$ gives $\lambda = \frac{x - x_0}{l}$
- In **2D** the **cartesian equation of a line** is a regular equation of a straight line simply given in the form
  - $y = mx + c$
  - $ax + by + d = 0$
  - $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$ by rearranging $y - y_1 = m(x - x_1)$
- In **3D** the **cartesian equation of a line** also includes *z* and is given in the form
  - $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$
  - where $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
  - This is **given in the formula booklet**
- If one of your variables **does not depend on** λ then this part can be written as a separate equation
  - For example: $m = 0 \Rightarrow y = y_0$ gives $\frac{x - x_0}{l} = \frac{z - z_0}{n}$, $y = y_0$

How do I find the vector equation of a line given the Cartesian form?

- If you are given the **Cartesian** equation of a line in the form
  - $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$
- A vector equation of the line can be found by
  - STEP 1: Set each part of the equation equal to λ individually
  - STEP 2: Rearrange each of these three equations (or two if working in 2D) to make *x*, *y*, and *z* the subjects
    - This will give you the three **parametric equations**
    - $x = x_0 + \lambda l$
    - $y = y_0 + \lambda m$
    - $z = z_0 + \lambda n$
  - STEP 3: Write this in the vector form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

- STEP 4: Set r to equal $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- If one part of the cartesian equation is given separately and is not in terms of λ then the corresponding component in the direction vector is equal to zero

Worked Example

A line has the vector equation $r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$. Find the Cartesian equation of the line.

Cartesian equations of a line	$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$

Begin by writing the equation of the line in parametric form:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \begin{aligned} x &= 1 + 4\lambda \quad ① \\ y &= -2\lambda \quad ② \\ z &= 2 + \lambda \quad ③ \end{aligned}$$

Rearrange each equation to make λ the subject:

① $\lambda = \frac{x-1}{4}$

② $\lambda = \frac{y}{-2}$

③ $\lambda = z - 2$

Set each expression for λ equal to each other:

$$\frac{x-1}{4} = \frac{y}{-2} = z - 2$$

3.10.2 Applications to Kinematics

Kinematics using Vectors

How are vectors related to kinematics?

- Vectors are often used in questions in the context of forces, acceleration or velocity
- If an object is moving in **one dimension** then its velocity, displacement and time are related using the formula $\mathbf{s} = \mathbf{v}t$
  - where s is **displacement**, v is **velocity** and t is the **time taken**
- If an object is moving in **more than one dimension** then **vectors** are needed to represent its **velocity** and **displacement**
  - Whilst **time** is a **scalar quantity**, **displacement** and **velocity** are both **vector quantities**
- For an object moving at a **constant speed** in a **straight line** its velocity, displacement and time can be related using the vector equation of a line
  - $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$
  - Letting
    - $\mathbf{r}$ be the position of the object at the time, t
    - $\mathbf{a}$ be the position vector, $\mathbf{r}_0$ at the start ($t = 0$)
    - λ represent the time, t
    - $\mathbf{b}$ be the **velocity** vector, $\mathbf{v}$
  - Then the position of the object at the time, t can be given by
    - $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$
  - The speed of the object will be the magnitude of the velocity $|\mathbf{v}|$

Exam Tip

- Kinematics questions can have a lot of information in, read them carefully and pick out the parts that are essential to the question
- Look out for where variables used are the same and/or different within vector equations, you will need to use different techniques to find these

Worked Example

A car, moving at constant speed, takes 2 minutes to drive in a straight line from point A (-4, 3) to point B (6, -5).

At time t, in minutes, the position vector ($\boldsymbol{p}$) of the car relative to the origin can be given in the form $\boldsymbol{p} = \boldsymbol{a} + t\boldsymbol{b}$.

Find the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Vector $\underline{a}$ represents the initial position and vector $\underline{b}$ represents the direction vector per minute.

Position vector $\overrightarrow{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

At $t = 0$ minutes, $\underline{p} = \underline{a}$ so $\underline{a} = \overrightarrow{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

Position vector $\overrightarrow{OB} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$

At $t = 2$ minutes, the car is at the point B and so $\overrightarrow{OB} = \underline{a} + 2\underline{b}$

$\begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + 2\underline{b}$

Direction vector $2\underline{b} = \begin{pmatrix} 6 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \end{pmatrix}$

$\underline{a} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

3.10.3 Pairs of Lines in 3D

Coincident, Parallel, Intersecting & Skew Lines

How do I tell if two lines are parallel?

- Two lines are parallel if, and only if, their **direction vectors** are **parallel**
  - This means the direction vectors will be **scalar multiples** of each other
  - For example, the lines whose equations are $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 0 \\ -8 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$ are parallel

    This is because $\begin{pmatrix} 2 \\ 0 \\ -8 \end{pmatrix} = -2 \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$

How do I tell if two lines are coincident?

- **Coincident lines** are two lines that lie directly on top of each other
  - They are indistinguishable from each other
- Two parallel lines will either **never intersect** or they are **coincident (identical)**
  - Sometimes the vector equations of the lines may look different

    for example, the lines represented by the equations $\mathbf{r} = \begin{pmatrix} 1 \\ -8 \end{pmatrix} + s \begin{pmatrix} -4 \\ 8 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are **coincident**,
  - To check whether two lines are **coincident**:

    First check that they are **parallel**

    They are because $\begin{pmatrix} -4 \\ 8 \end{pmatrix} = -4 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and so their direction vectors are parallel

    Next, determine whether **any point** on one of the lines also lies on the other

    $\begin{pmatrix} 1 \\ -8 \end{pmatrix}$ is the position vector of a point on the first line and

    $\begin{pmatrix} 1 \\ -8 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ so it also lies on the second line

    If two parallel lines share **any point**, then they share **all points** and are **coincident**

What are skew lines?

- Lines that are **not parallel** and which **do not intersect** are called **skew lines**
  - This is only possible in **3-dimensions**

How do I determine whether lines in 3 dimensions are parallel, skew, or intersecting?

- First, look to see if the direction vectors are parallel:

- if the **direction vectors are parallel**, then the **lines are parallel**
- if the **direction vectors are not parallel**, the **lines are not parallel**

- If the lines are **parallel**, check to see if the lines are **coincident**:
  - If they **share any point**, then they are **coincident**
  - If **any point** on one line is **not on the other line**, then the lines are **not coincident**
- If the lines are **not parallel**, check whether they **intersect**:
  - STEP 1: Set the vector equations of the two lines equal to each other with **different variables**
    - e.g. λ and μ, for the parameters
  - STEP 2: Write the three separate equations for the **i**, **j**, and **k** components in terms of λ and μ
  - STEP 3: **Solve** two of the equations to find a value for λ and μ
  - STEP 4: **Check** whether the values of λ and μ you have found satisfy the third equation
    - If **all three** equations are satisfied, then the lines **intersect**
    - If **not all three** equations are satisfied, then the lines are **skew**

How do I find the point of intersection of two lines?

- If a pair of lines are **not parallel** and **do intersect**, a unique point of intersection can be found
  - If the two lines intersect, there will be a single point that will lie on both lines
- Follow the steps above to find the values of λ and μ that satisfy **all three equations**
  - STEP 5: Substitute either the value of λ or the value of μ into one of the vector equations to find the position vector of the point where the lines intersect
  - It is always a good idea to **check** in the other equations as well, you should get the same point for each line

Exam Tip

- Make sure that you use different letters, e.g. λ and μ, to represent the parameters in vector equations of different lines
  - Check that the variable you are using has not already been used in the question

Worked Example

Determine whether the following pair of lines are parallel, intersect, or are skew.

$\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + s(5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = -5\mathbf{i} + 4\mathbf{j} + \mathbf{k} + t(2\mathbf{i} - \mathbf{j})$.

STEP 1: Check to see if the lines are parallel:

$$r_1 = \begin{pmatrix}4\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}5\\2\\3\end{pmatrix} \qquad r = \begin{pmatrix}-5\\4\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\0\end{pmatrix}$$

direction vectors

The lines are not parallel because there is no value of k such that $\begin{pmatrix}5\\2\\3\end{pmatrix} = k\begin{pmatrix}2\\-1\\0\end{pmatrix}$

STEP 2: Check to see if the lines intersect:

$4 + 5\lambda = -5 + 2\mu$ ①

$3 + 2\lambda = 4 - \mu$ ②

$3\lambda = 1$ ③

Set up three equations for each of the i, j and k components.

Equation ③: $\lambda = \frac{1}{3}$

Sub into ②: $3 + 2\left(\frac{1}{3}\right) = 4 - \mu$

$\frac{11}{3} = 4 - \mu$

$\mu = \frac{1}{3}$

Sub into ①: $4 + 5\left(\frac{1}{3}\right) = -5 + 2\left(\frac{1}{3}\right)$

$\frac{17}{3} \neq -\frac{13}{3}$ contradiction

There is no point of intersection.

The lines are skew

Angle Between Two Lines

How do we find the angle between two lines?

- The angle between two lines is equal to the angle between their **direction vectors**
  - It can be found using the **scalar product** of their direction vectors
- Given two lines in the form $=\boldsymbol{a}_1+\lambda\boldsymbol{b}_1$ and $= {}_2+\lambda\boldsymbol{b}_2$ use the formula
  - $\theta=\cos^{-1}\left(\frac{\boldsymbol{b}_1 \bullet \boldsymbol{b}_2}{\left|\boldsymbol{b}_1\right|\left|\boldsymbol{b}_2\right|}\right)$
- If you are given the equations of the lines in a different form or two points on a line you will need to find their direction vectors first
- To find the angle ABC the vectors BA and BC would be used, both starting from the point B
- The intersection of two lines will always create **two angles**, an acute one and an obtuse one
  - These two angles will add to 180°
  - You may need to subtract your answer from 180° to find the angle you are looking for
  - A **positive scalar product** will result in the **acute angle** and a **negative scalar product** will result in the **obtuse angle**
    - Using the absolute value of the scalar product will always result in the acute angle

Exam Tip

- In your exam read the question carefully to see if you need to find the acute or obtuse angle
  - When revising, get into the practice of double checking at the end of a question whether your angle is acute or obtuse and whether this fits the question

Worked Example

Find the acute angle, in radians between the two lines defined by the equations:

$$l_1:\ \boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \text{ and } l_2:\ \boldsymbol{b} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$$

STEP 1: Find the scalar product of the direction vectors:

$$\begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} = (1 \times -3) + (-4 \times 2) + (-3 \times 5) = -3 + (-8) + (-15) = -26$$

negative, so the angle will be the obtuse angle.

STEP 2: Find the magnitudes of the direction vectors:

$$\sqrt{(1)^2 + (-4)^2 + (-3)^2} = \sqrt{26} \qquad \sqrt{(-3)^2 + (2)^2 + (5)^2} = \sqrt{38}$$

STEP 3: Find the angle:

$$\cos\theta = \frac{|-26|}{\sqrt{26}\sqrt{38}}$$

Using the absolute value will result in the acute angle

$$\theta = \cos^{-1}\left(\frac{26}{\sqrt{26}\sqrt{38}}\right)$$

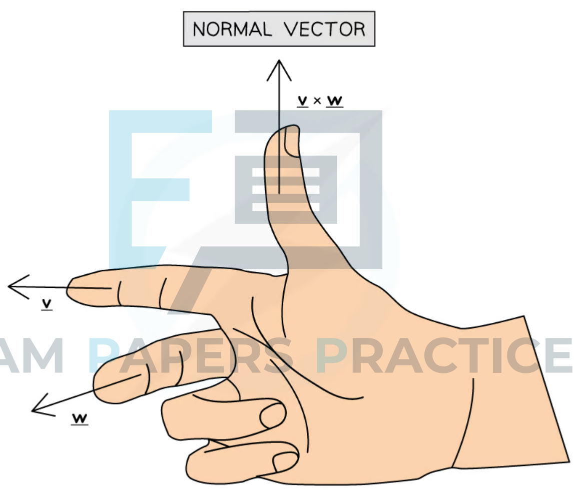
$\theta = 0.597$ radians (3sf)

3.10.4 The Vector Product

The Vector ('Cross') Product

What is the vector (cross) product?

- The **vector product** (also known as the **cross** product) is a form in which two vectors can be combined together
- The vector product between two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is denoted $\boldsymbol{v} \times \boldsymbol{w}$
- The result of taking the vector product of two vectors is a **vector**
- The **vector product** is a vector **in a plane** that is **perpendicular** to the two vectors from which it was calculated
  - This could be in either direction, depending on the angle between the two vectors
  - The **right-hand** rule helps you see which direction the vector product goes in
    - By pointing your index finger and your middle finger in the direction of the two vectors your thumb will automatically go in the direction of the vector product



How do I find the vector (cross) product?

- There are **two methods** for calculating the vector product
- The **vector product** of the two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ can be written in **component form** as follows:
  - $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$
  - Where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
  - This is **given in the formula booklet**
- The vector product can also be found in terms of its **magnitude** and **direction**

- The **magnitude of the vector product** is equal to the **product of the magnitudes** of the two vectors and the **sine of the angle between them**
  - $|\boldsymbol{v} \times \boldsymbol{w}| = |v||w|\sin\theta$
  - Where θ is the angle between ***v*** and ***w***

    The two vectors ***v*** and ***w*** are joined at the start and pointing away from each other
  - This is **given in the formula booklet**
- The **direction of the vector product** is **perpendicular** to both ***v*** and ***w***

What properties of the vector product do I need to know?

- The order of the vectors **is important** and **changes the result** of the vector product
  - $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$
  - However
  - $\boldsymbol{v} \times \quad = -\boldsymbol{w} \times \boldsymbol{v}$
- The **distributive law** can be used to 'expand brackets'
  - $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$
  - Where ***u***, ***v*** and ***w*** are all vectors
- Multiplying a **scalar** by a vector gives the result:
  - $(k\boldsymbol{v}) \times \boldsymbol{w} = \boldsymbol{v} \times (k \quad) = k(\quad \times \quad)$
- The vector product between a vector and itself is equal to **zero**
  - $\boldsymbol{v} \times \boldsymbol{v} = 0$
- If two vectors are **parallel** then the vector product is **zero**
  - This is because sin 0° = sin 180° = 0
- If $\boldsymbol{v} \times \quad = 0$ then ***v*** and ***w*** are parallel if they are non-zero
- If two vectors, ***v*** and ***w***, are **perpendicular** then the magnitude of the vector product is equal to the **product** of the magnitudes of the vectors
  - $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{w}||\boldsymbol{v}|$
  - This is because sin 90° = 1

Exam Tip

- The formulae for the vector product are given in the formula booklet, make sure you use them as this is an easy formula to get wrong
- The properties of the vector product are not given in the formula booklet, however they are important and it is likely that you will need to recall them in your exam so be sure to commit them to memory

Worked Example

Calculate the magnitude of the vector product between the two vectors

$\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ using

i)

the formula $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$,

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$

Use the formula to find the cross-product:

$\underline{v} \times \underline{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} = \begin{pmatrix} (0)(-1) - (-5)(-2) \\ (-5)(3) - (2)(-1) \\ (2)(-2) - (0)(3) \end{pmatrix} = \begin{pmatrix} -10 \\ -13 \\ -4 \end{pmatrix}$

Find the magnitude of $\underline{v} \times \underline{w}$:

$|\underline{v} \times \underline{w}| = \sqrt{(-10)^2 + (-13)^2 + (-4)^2} = \sqrt{285}$

$|\underline{v} \times \underline{w}| = 16.9 \text{ (3sf)}$

ii)

the formula , given that the angle between them is 1 radian.

Find the magnitude of $\underline{v}$ and $\underline{w}$:

$|\underline{v}| = \sqrt{2^2 + 0^2 + (-5)^2} = \sqrt{29}$

$|\underline{w}| = \sqrt{3^2 + (-2)^2 + (-1)^2} = \sqrt{14}$

$|\underline{v} \times \underline{w}| = |\underline{v}||\underline{w}| \sin\theta$

$= \sqrt{29} \times \sqrt{14} \sin(1^c)$

$|\underline{v} \times \underline{w}| = 17.0 \text{ (3sf)}$

Areas using Vector Product

How do I use the vector product to find the area of a parallelogram?

- The **area of the parallelogram** with two adjacent sides formed by the vectors ***v*** and ***w*** is equal to **the magnitude of the vector product** of two vectors ***v*** and ***w***
  - $A = |\boldsymbol{v} \times \boldsymbol{w}|$ where ***v*** and ***w*** form two **adjacent sides** of the parallelogram

    This is **given in the formula booklet**

How do I use the vector product to find the area of a triangle?

- The **area of the triangle** with two sides formed by the vectors ***v*** and ***w*** is equal to **half of the magnitude of the vector product** of two vectors ***v*** and ***w***
  - $A = \frac{1}{2}|\boldsymbol{v} \times \boldsymbol{w}|$ where ***v*** and ***w*** form two **sides** of the triangle

    This is **not** given in the formula booklet

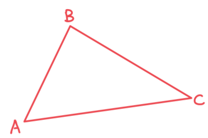
Exam Tip

- The formula for the area of the parallelogram is given in the formula booklet but the formula for the area of a triangle is not
  - Remember that the area of a triangle is half the area of a parallelogram

Worked Example

Find the area of the triangle enclosed by the coordinates (1, 0, 5), (3, -1, 2) and (2, 0, -1).

Let A be (1, 0, 5), B be (3, -1, 2) and C be (2, 0, -1)



You can use any two direction vectors moving away from any vertex.

Find the two direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix}$$

Find the cross product of the two direction vectors:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} (-1)(-6) - (-3)(0) \\ (-3)(1) - (2)(-6) \\ (2)(0) - (-1)(1) \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 1 \end{pmatrix}$$

Find the magnitude of the cross product

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{6^2 + 9^2 + 1^2} = \sqrt{118}$$

Area of the triangle is half the magnitude

$$\text{Area} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{118}$$

Area = 5.43 u² (3sf)

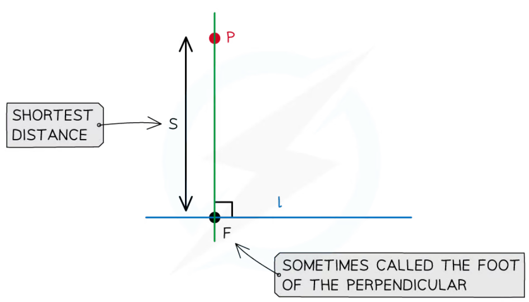
3.10.5 Shortest Distances with Lines

Shortest Distance Between a Point and a Line

How do I find the shortest distance from a point to a line?

- The shortest distance from any point to a line will always be the **perpendicular** distance
  - Given a line *l* with equation $\mathbf{r} = \quad + \lambda\mathbf{b}$ and a point *P* not on *l*
  - The **scalar product** of the direction vector, **b**, and the vector in the direction of the **shortest distance** will be zero
- The shortest distance can be found using the following steps:
  - STEP 1: Let the vector equation of the line be ***r*** and the point not on the line be *P*, then the point on the line closest to *P* will be the point *F*
    - The point *F* is sometimes called the foot of the perpendicular
  - STEP 2: Sketch a diagram showing the line *l* and the points *P* and *F*
    - The vector $\overrightarrow{FP}$ will be **perpendicular** to the line *l*
  - STEP 3: Use the equation of the line to find the position vector of the point *F* in terms of λ
  - STEP 4: Use this to find the displacement vector $\overrightarrow{FP}$ in terms of λ
  - STEP 5: The scalar product of the direction vector of the line *l* and the displacement vector $\overrightarrow{FP}$ will be zero
    - Form an equation $\overrightarrow{FP} \cdot \mathbf{b} = 0$ and solve to find λ
  - STEP 6: Substitute λ into $\overrightarrow{FP}$ and find the magnitude $\left|\overrightarrow{FP}\right|$
    - The shortest distance from the point to the line will be the magnitude of $\overrightarrow{FP}$

- Note that the shortest distance between the point and the line is sometimes referred to as the **length of the perpendicular**



How do we use the vector product to find the shortest distance from a point to a line?

- The vector product can be used to find the shortest distance from any point to a line on a 2–dimensional plane
- Given a point, P, and a line $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$
  - The shortest distance from P to the line will be $\dfrac{\left|\overrightarrow{AP} \times b\right|}{|b|}$
  - Where *A* is a point on the line
  - This is **not** given in the formula booklet

Exam Tip

- Column vectors can be easier and clearer to work with when dealing with scalar products.

Worked Example

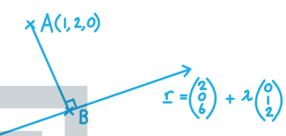
Point A has coordinates (1, 2, 0) and the line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Point B lies on the l such that $[AB]$ is perpendicular to l.

Find the shortest distance from A to the line l.

B is on l so can be written in terms of λ:

$$\vec{OB} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ \lambda \\ 6+2\lambda \end{pmatrix}$$



Find $\vec{AB}$ using $\vec{AB} = \vec{OB} - \vec{OA}$

$$\vec{AB} = \begin{pmatrix} 2 \\ \lambda \\ 6+2\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda - 2 \\ 6+2\lambda \end{pmatrix}$$

$\vec{AB}$ is perpendicular to l: $\vec{AB} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$

$$\begin{pmatrix} 1 \\ \lambda - 2 \\ 6+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$\lambda - 2 + 2(6+2\lambda) = 0$$

$$5\lambda + 10 = 0$$

$$\lambda = -2$$

Substitute back into $\vec{AB}$ and find the magnitude:

$$\vec{AB} = \begin{pmatrix} 1 \\ -2-2 \\ 6+2(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$$

$$|\vec{AB}| = \sqrt{1^2 + (-4)^2 + 2^2} = \sqrt{21}$$

Shortest distance $= \sqrt{21}$ units

Shortest Distance Between Two Lines

How do we find the shortest distance between two parallel lines?

- Two **parallel** lines will never intersect
- The shortest distance between two **parallel lines** will be the **perpendicular distance** between them
- Given a line l_1 with equation $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and a line l_2 with equation $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$ then the shortest distance between them can be found using the following steps:
  - STEP 1: Find the vector between $\mathbf{a}_1$ and a general coordinate from l_2 in terms of μ
  - STEP 2: Set the scalar product of the vector found in STEP 1 and the direction vector $\mathbf{d}_1$ equal to zero
    - Remember the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are scalar multiples of each other and so either can be used here
  - STEP 3: Form and solve an equation to find the value of μ
  - STEP 4: Substitute the value of μ back into the equation for l_2 to find the coordinate on l_2 closest to l_1
  - STEP 5: Find the distance between $\mathbf{a}_1$ and the coordinate found in STEP 4
- Alternatively, the formula $\frac{\left|\overrightarrow{AB} \times \mathbf{d}\right|}{|\mathbf{d}|}$ can be used
  - Where $\overrightarrow{AB}$ is the vector connecting the two given coordinates $\mathbf{a}_1$ and $\mathbf{a}_2$
  - **d** is the simplified vector in the direction of $\mathbf{d}_1$ and $\mathbf{d}_2$
  - This is **not** given in the formula booklet

How do we find the shortest distance from a given point on a line to another line?

- The shortest distance from any point on a line to another line will be the **perpendicular** distance from the point to the line
- If the angle between the two lines is known or can be found then right-angled trigonometry can be used to find the perpendicular distance
  - The formula $\frac{\left|\overrightarrow{AB} \times \mathbf{d}\right|}{|\mathbf{d}|}$ given above is derived using this method and can be used
- Alternatively, the equation of the line can be used to find a general coordinate and the steps above can be followed to find the shortest distance

How do we find the shortest distance between two skew lines?

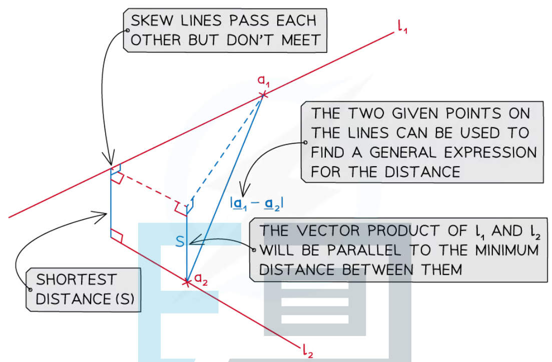
- Two **skew** lines are not parallel but will never intersect
- The shortest distance between two **skew lines** will be perpendicular to **both** of the lines
  - This will be at the point where the two lines pass each other with the perpendicular distance where the point of intersection would be
  - The **vector product** of the two direction vectors can be used to find a vector in the direction of the shortest distance
  - The shortest distance will be a vector **parallel** to the vector product

- To find the shortest distance between two skew lines with equations $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$,
  - STEP 1: Find the vector product of the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$

    $$\mathbf{d} = \mathbf{d}_1 \times \mathbf{d}_2$$
  - STEP 2: Find the vector in the direction of the line between the two general points on l_1 and l_2 in terms of λ and μ

    $$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$
  - STEP 3: Set the two vectors parallel to each other

    $$\mathbf{d} = k\overrightarrow{AB}$$
  - STEP 4: Set up and solve a system of linear equations in the three unknowns, k, λ and μ



Exam Tip

- Exam questions will often ask for the shortest, or minimum, distance within vector questions
- If you're unsure start by sketching a quick diagram
- Sometimes calculus can be used, however usually vector methods are required

Worked Example

Consider the skew lines l_1 and l_2 as defined by:

$$l_1: \boldsymbol{r} = \begin{pmatrix} 6 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$$l_2: \boldsymbol{r} = \begin{pmatrix} -5 \\ 4 \\ -8 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Find the minimum distance between the two lines.

Find the vector product of the direction vectors.

$$\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} (-3)(1)-(4)(2) \\ (4)(-1)-(2)(1) \\ (2)(2)-(-3)(-1) \end{pmatrix} = \begin{pmatrix} -11 \\ -6 \\ 1 \end{pmatrix}$$

Find the vector in the direction of the line between the general coordinates.

$$\overrightarrow{AB} = \begin{pmatrix} -5-\mu \\ 4+2\mu \\ -8+\mu \end{pmatrix} - \begin{pmatrix} 6+2\lambda \\ -4-3\lambda \\ 3+4\lambda \end{pmatrix} = \begin{pmatrix} -11-\mu-2\lambda \\ 8+2\mu+3\lambda \\ -11+\mu-4\lambda \end{pmatrix}$$

A point on l_2 A point on l_1

$$\begin{pmatrix} -11-\mu-2\lambda \\ 8+2\mu+3\lambda \\ -11+\mu-4\lambda \end{pmatrix} = k\begin{pmatrix} -11 \\ -6 \\ 1 \end{pmatrix}$$

$\overrightarrow{AB}$ is parallel to $\begin{pmatrix} -11 \\ -6 \\ 1 \end{pmatrix}$ so $\overrightarrow{AB} = k\begin{pmatrix} -11 \\ -6 \\ 1 \end{pmatrix}$

Set up and solve a system of equations.

$$\left.\begin{aligned} 11k - 2\lambda - \mu &= 11 \\ 6k + 3\lambda + 2\mu &= -8 \\ \mu - 4\lambda - k &= 11 \end{aligned}\right\}$$

Solve using GDC: $k = \frac{31}{79}$ $\lambda = -\frac{238}{79}$ $\mu = -\frac{52}{79}$

Substitute back into the expression for $\overrightarrow{AB}$ and find the magnitude:

$$|\overrightarrow{AB}| = \left|\begin{pmatrix} -11-\left(-\frac{52}{79}\right)-2\left(-\frac{238}{79}\right) \\ 8+2\left(-\frac{52}{79}\right)+3\left(-\frac{238}{79}\right) \\ -11+\left(-\frac{52}{79}\right)-4\left(-\frac{238}{79}\right) \end{pmatrix}\right| = \left|\begin{pmatrix} -\frac{341}{79} \\ -\frac{186}{79} \\ \frac{31}{79} \end{pmatrix}\right| = \sqrt{\left(-\frac{341}{79}\right)^2 + \left(-\frac{186}{79}\right)^2 + \left(\frac{31}{79}\right)^2}$$

Shortest distance = 4.93 units (3 s.f.)

3.11 Vector Planes

3.11.1 Vector Equations of Planes

Equation of a Plane in Vector Form

How do I find the vector equation of a plane?

- A plane is a flat surface which is two-dimensional
  - Imagine a flat piece of paper that continues on forever in both directions
- A plane in often denoted using the capital Greek letter Π
- The vector form of the equation of a plane can be found using **two direction vectors** on the plane
  - The direction vectors must be
    - **parallel** to the plane
    - **not parallel** to each other
    - therefore they will **intersect** at some point on the plane
- The formula for finding the **vector equation** of a plane is
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
    - Where $\boldsymbol{r}$ is the **position vector** of any point on the plane
    - $\boldsymbol{a}$ is the **position vector** of a known point on the plane
    - $\boldsymbol{b}$ and $\boldsymbol{c}$ are two **non-parallel direction** (displacement) **vectors** parallel to the plane
    - λ and μ are scalars
  - The formula is **given in the formula booklet** but you must make sure you know what each part means
- As $\boldsymbol{a}$ could be the position vector of **any** point on the plane and $\boldsymbol{b}$ and $\boldsymbol{c}$ could be **any non-parallel** direction vectors on the plane there are infinite vector equations for a single plane

How do I determine whether a point lies on a plane?

- Given the equation of a plane $\boldsymbol{r} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$ then the point $\boldsymbol{r}$ with position vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is on the plane if there exists a value of λ and μ such that
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix} + \mu \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$
  - This means that there exists a single value of λ and μ that satisfy the three **parametric** equations:
    - $x = a_1 + \lambda b_1 + \mu c_1$
    - $y = a_2 + \lambda b_2 + \mu c_2$

$z = a_3 + \lambda b_3 + \mu c_3$

- Solve two of the equations first to find the values of λ and μ that satisfy the first two equation and then check that this value also satisfies the third equation
- If the values of λ and μ do not satisfy all three equations, then the point ***r*** does not lie on the plane

Exam Tip

- The formula for the vector equation of a plane is given in the formula booklet, make sure you know what each part means
- Be careful to use different letters, e.g. λ and μ as the scalar multiples of the two direction vectors

Worked Example

The points A, B and C have position vectors $\boldsymbol{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\boldsymbol{b} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, and $\boldsymbol{c} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ respectively, relative to the origin O.

(a) Find the vector equation of the plane.

Start by finding the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix}$$

$$\overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

All three points lie on the plane, so choose the position vector of one point, e.g. $\overrightarrow{OA}$, to use as 'a' in the vector equation of a plane formula.

Check that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not parallel.

$$r = \underline{a} + \lambda \overrightarrow{AB} + \mu \overrightarrow{AC}$$

$$\underline{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

(This is one of many correct answers)

(b) Determine whether the point D with coordinates (-2, -3, 5) lies on the plane.

Let D have position vector $\underline{d} = \begin{pmatrix}-2\\-3\\5\end{pmatrix}$, then the point D lies on the plane if there exists a value of λ and μ for which: $\begin{pmatrix}-2\\-3\\5\end{pmatrix} = \begin{pmatrix}3\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}-2\\-4\\5\end{pmatrix} + \mu\begin{pmatrix}1\\-3\\4\end{pmatrix}$

Find the parametric equations:

$-2 = 3 - 2\lambda + \mu \Rightarrow \mu - 2\lambda = -5$ ①

$-3 = 2 - 4\lambda - 3\mu \Rightarrow 3\mu + 4\lambda = 5$ ②

$5 = -1 + 5\lambda + 4\mu \Rightarrow 4\mu + 5\lambda = 6$ ③

(solve two equations ① and ② for λ and μ.)

Find the value of λ and μ from two equations:

2①: $2\mu - 4\lambda = -10$

+②: $3\mu + 4\lambda = 5$

$5\mu = -5$

$\mu = -1$ sub into ①: $(-1) - 2\lambda = -5$

$\lambda = 2$

Check to see if λ and μ satisfy the third equation:

$4(-1) + 5(2) = -4 + 10 = 6$ ✓

The point D lies on the plane.

Equation of a Plane in Cartesian Form

How do I find the vector equation of a plane in cartesian form?

- The **cartesian** equation of a plane is given in the form
  - $ax + by + cz = d$
  - This is **given in the formula booklet**
- A **normal vector** to the plane can be used along with a **known point on the plane** to find the cartesian equation of the plane
  - The normal vector will be a vector that is **perpendicular** to the plane
- The **scalar product** of the normal vector and any **direction vector** on the plane will the **zero**
  - The two vectors will be perpendicular to each other
  - The **direction vector** from a fixed-point *A* to any point on the plane, *R* can be written as ***r* – *a***
  - Then ***n*** • (***r*** – ***a***) = 0 and it follows that (***n*** • ***r***) – (***n*** • ***a***) = 0
- This gives the **equation of a plane using the normal vector:**
  - ***n*** • ***r*** = ***a*** • ***n***

    Where ***r*** is the **position vector** of any point on the plane

    a is the **position vector** of a known point on the plane

    n is a vector that is **normal** to the plane
  - This is **given in the formula booklet**
- If the vector ***r*** is given in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and ***a*** and ***n*** are both known vectors given in the form $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ then the Cartesian equation of the plane can be found using:
  - $\boldsymbol{n} \cdot \boldsymbol{r} = n_1x + n_2y + n_3z$
  - $\boldsymbol{a} \cdot \boldsymbol{n} = a_1n_1 + a_2n_2 + a_3n_3$
  - Therefore $n_1x + n_2y + n_3z = a_1n_1 + a_2n_2 + a_3n_3$
  - This simplifies to the form $ax + by + cz = d$

How do I find the equation of a plane in Cartesian form given the vector form?

- The **Cartesian** equation of a plane can be found if you know
  - the **normal vector** and
  - a **point** on the plane
- The **vector equation of a plane** can be used to find the **normal vector** by finding the **vector product** of the two direction vectors
  - A vector product is always perpendicular to the two vectors from which it was calculated
- The vector ***a*** given in the vector equation of a plane is a **known point** on the plane

- Once you have found the normal vector then the point ***a*** can be used in the formula ***n*** • ***r*** = ***a*** • ***n*** to find the equation in Cartesian form

- To find $ax + by + cz = d$ given $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$:
  - Let $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \boldsymbol{b} \times \boldsymbol{c}$ then $d = \boldsymbol{n} \cdot \boldsymbol{a}$

Exam Tip

- In an exam, using whichever form of the equation of the plane to write down a normal vector to the plane is always a good starting point

Worked Example

A plane Π contains the point A$(2, 6, -3)$ and has a normal vector $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.

a)
Find the equation of the plane in its Cartesian form.

The components of the normal vector are the x-, y- and z- coefficients of the Cartesian form:

$3x - y + 4z = d$

The point $(2, 6, -3)$ is on the plane so

$d = 3(2) - (6) + 4(-3) = 6 - 6 - 12 = -12$

Therefore

$3x - y + 4z = -12$

b)
Determine whether point B with coordinates $(-1, 0, -2)$ lies on the same plane.

Test by putting the coordinates into the equation:

$3(-1) - (0) + 4(-2) = -3 - 8 = -11 \neq -12$

The point with coordinates (-1, 0, 2) does not lie on the plane

3.11.2 Intersections of Lines & Planes

Intersection of Line & Plane

How do I tell if a line is parallel to a plane?

- A line is parallel to a plane if its **direction vector** is **perpendicular** to the plane's **normal vector**
- If you know the Cartesian equation of the plane in the form $ax + by + cz = d$ then the values of *a*, *b*, and *c* are the individual components of a normal vector to the plane
- The **scalar product** can be used to check in the direction vector and the normal vector are perpendicular
  - If two vectors are perpendicular their scalar product will be zero

How do I tell if the line lies inside the plane?

- If the line is parallel to the plane then it will either **never intersect** or it will lie inside the plane
  - Check to see if they have a common point
- If a line is parallel to a plane and they share **any point**, then the line lies inside the plane

How do I find the point of intersection of a line and a plane?

- If a line is **not parallel** to a plane it will **intersect** it at a single point
- If both the **vector equation of the line** and the **Cartesian equation of the plane** is known then this can be found by:
- STEP 1: Set the position vector of the point you are looking for to have the individual components *x*, *y*, and *z* and substitute into the vector equation of the line
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
- STEP 2: Find the parametric equations in terms of *x*, *y*, and *z*
  - $x = x_0 + \lambda l$
  - $y = y_0 + \lambda m$
  - $z = z_0 + \lambda n$
- STEP 3: Substitute these parametric equations into the Cartesian equation of the plane and solve to find λ
  - $a(x_0 + \lambda l) + b(y_0 + \lambda m) + c(z_0 + \lambda n) = d$
- STEP 4: Substitute this value of λ back into the vector equation of the line and use it to find the position vector of the point of intersection
- STEP 5: Check this value in the Cartesian equation of the plane to make sure you have the correct answer

Worked Example

Find the point of intersection of the line $r = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ with the plane $3x - 4y + z = 8$.

Find the parametric form of the equation of the line:

Let $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ then

$x = 1 + 2\lambda$

$y = -3 - \lambda$

$z = 2 - \lambda$

Substitute into the equation of the plane:

$3(1 + 2\lambda) - 4(-3 - \lambda) + (2 - \lambda) = 8$

Solve to find λ:

$3 + 6\lambda + 12 + 4\lambda + 2 - \lambda = 8$

$\lambda = -1$

Substitute $\lambda = -1$ into the vector equation of the line:

$r = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-2 \\ -3+1 \\ 2+1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$

(-1, -2, 3)

Intersection of Planes

How do we find the line of intersection of two planes?

- Two planes will either be **parallel** or they will intersect along a **line**
  - Consider the point where a wall meets a floor or a ceiling
  - You will need to find the **equation of the line** of intersection
- If you have the Cartesian forms of the two planes then the equation of the line of intersection can be found by solving the two equations simultaneously
  - As the solution is a vector equation of a line rather than a unique point you will see below how the equation of the line can be found by part solving the equations
  - For example:

$$2x - y + 3z = 7 \qquad (1)$$
$$x - 3y + 4z = 11 \qquad (2)$$

- STEP 1: Choose one variable and substitute this variable for λ in both equations
  - For example, letting $x = \lambda$ gives:

$$2\lambda - y + 3z = 7 \qquad (1)$$
$$\lambda - 3y + 4z = 11 \qquad (2)$$

- STEP 2: Rearrange the two equations to bring λ to one side
  - Equations (1) and (2) become

$$y - 3z = 2\lambda - 7 \qquad (1)$$
$$3y - 4z = \lambda - 11 \qquad (2)$$

- STEP 3: Solve the equations simultaneously to find the two variables in terms of λ
  - 3(1) – (2) Gives

$$z = 2 - \lambda$$

  - Substituting this into (1) gives

$$y = -1 - \lambda$$

- STEP 4: Write the three parametric equations for *x*, *y*, and *z* in terms of λ and convert into the vector equation of a line in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
  - The parametric equations

$$x = \lambda$$
$$y = -1 - \lambda$$
$$z = 2 - \lambda$$

  - Become

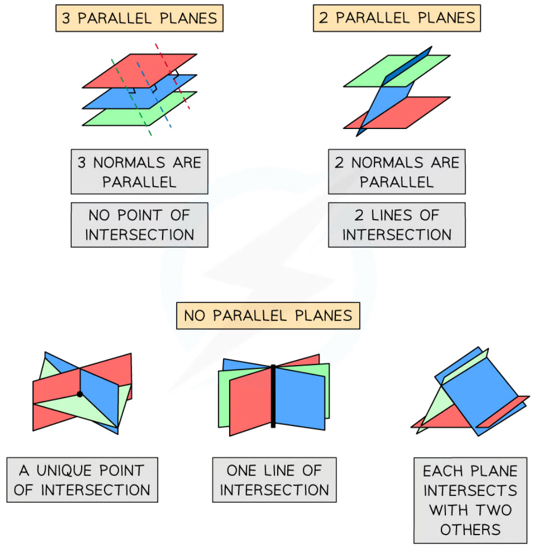
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

- If you have fractions in your direction vector you can change its magnitude by multiplying each one by their common denominator
  - The magnitude of the direction vector can be changed without changing the equation of a line
- An alternative method is to find two points on both planes by setting either *x*, *y*, or *z* to zero and solving the system of equations using your GDC or row reduction
  - Repeat this twice to get two points on both planes

- These two points can then be used to find the vector equation of the line between them
- This will be the line of intersection of the planes
- This method relies on the line of intersection having points where the chosen variables are equal to zero

How do we find the relationship between three planes?

- Three planes could either be **parallel**, intersect at one **point**, or intersect along a **line**
- If the three planes have a **unique point of intersection** this point can be found by using your GDC (or row reduction) to solve the three equations in their Cartesian form
  - Make sure you know how to use your GDC to solve a **system of linear equations**
  - Enter all three equations in for the three variables x, y, and z
  - Your GDC will give you the unique solution which will be the coordinates of the point of intersection
- If the three planes do not intersect at a unique point you will not be able to use your GDC to solve the equations
  - If there are no solutions to the system of Cartesian equations then there is no unique point of intersection
- If the three planes are all **parallel** their **normal vectors** will be parallel to each other
  - Show that the normal vectors all have equivalent **direction vectors**
  - These direction vectors may be **scalar multiples** of each other
- If the three planes have **no point of intersection** and are **not all parallel** they may have a relationship such as:
  - Each plane intersects two other planes such that they form a **prism** (none are parallel)
  - Two planes are parallel with the third plane intersecting each of them
  - Check the normal vectors to see if any two of the planes are parallel to decide which relationship they have
- If the three planes intersect along a line there will not be a unique solution to the three equations but there will be a **vector equation of a line** that will satisfy the three equations
- The system of equations will need to be solved by **elimination** or **row reduction**
  - Choose one variable to substitute for λ
  - Solve two of the equations simultaneously to find the other two variables in terms of λ
  - Write x, y, and z in terms of λ in the parametric form of the equation of the line and convert into the vector form of the equation of a line



Exam Tip

- In an exam you may need to decide the relationship between three planes by using row reduction to determine the number of solutions
  - Make sure you are confident using row reduction to solve systems of linear equations
  - Make sure you remember the different forms three planes can take

Worked Example

Two planes Π_1 and Π_2 are defined by the equations:

$$\Pi_1 : 3x + 4y + 2z = 7$$

$$\Pi_2 : x - 2y + 3z = 5$$

Find the vector equation of the line of intersection of the two planes.

STEP 1: Let $z = \lambda$, then $3x + 4y + 2\lambda = 7$ ①

$x - 2y + 3\lambda = 5$ ②

You can substitute any variable here, look at the equations to see which is easiest.

STEP 2: ① : $3x + 4y = 7 - 2\lambda$

② : $x - 2y = 5 - 3\lambda$

Write the two equations as simultaneous equations for the two remaining constants.

STEP 3: Find x and y in terms of λ:

① − 2②: $(3x + 4y = 7 - 2\lambda)$

$+ (2x - 4y = 10 - 6\lambda)$

$5x = 17 - 8\lambda$

$x = \frac{17}{5} - \frac{8\lambda}{5}$

sub into ②: $\frac{17}{5} - \frac{8\lambda}{5} - 2y + 3\lambda = 5$

$y = \frac{7\lambda}{10} - \frac{8}{10}$

STEP 4: $x = \frac{17}{5} - \frac{8\lambda}{5}$, $y = \frac{7\lambda}{10} - \frac{4}{5}$, $z = \lambda$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{17}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{8}{5} \\ \frac{7}{10} \\ 1 \end{pmatrix}$$

The components of the direction vector can be multiplied by a scalar without changing the direction.

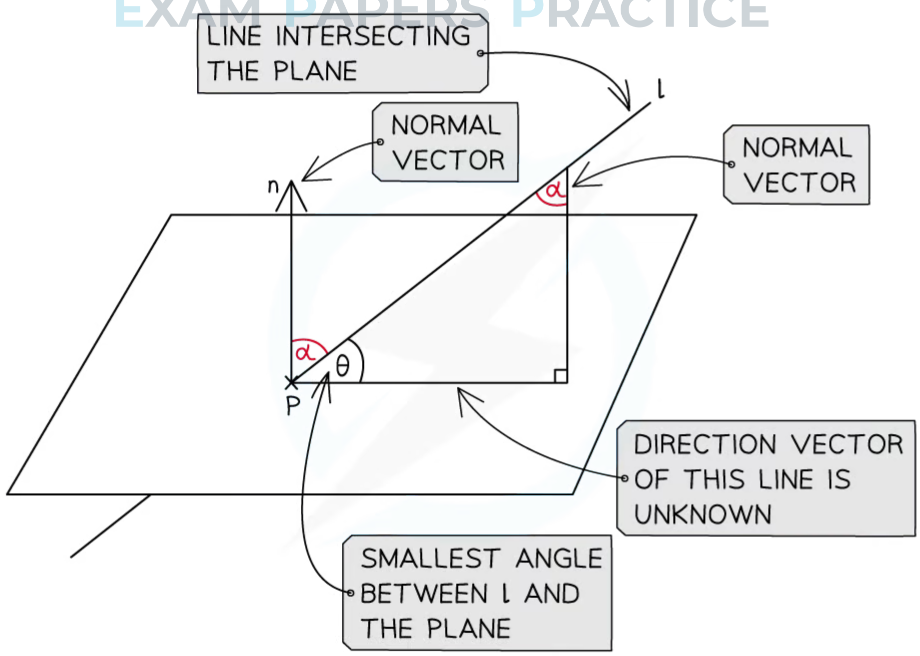
$$\mathbf{r} = \begin{pmatrix} 17/5 \\ -4/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 16 \\ 7 \\ 10 \end{pmatrix}$$

3.11.3 Angles Between Lines & Planes

Angle Between Line & Plane

How do I find the angle between a line and a plane?

- When you find the angle between a line and a plane you will be finding the angle between the line itself and the line on the plane that creates the smallest angle with it
  - This means the line on the plane directly under the line as it joins the plane
- It is easiest to think of these two lines making a right-triangle with the normal vector to the plane
  - The line joining the plane will be the **hypotenuse**
  - The line on the plane will be **adjacent** to the angle
  - The normal will the **opposite** the angle
- As you do not know the angle of the line on the plane you can instead find the angle between the **normal** and the **hypotenuse**
  - This is the angle **opposite** the angle you want to find
  - This angle **can be found** because you will know the direction vector of the line joining the plane and the normal vector to the plane
  - This angle is also equal to the angle made by the line at the point it joins the plane and the normal vector at this point
- The smallest angle between the line and the plane will be 90° minus the angle between the normal vector and the line
  - In radians this will be $\frac{\pi}{2}$ minus the angle between the normal vector and the line



Exam Tip

- Remember that if the scalar product is negative your answer will result in an obtuse angle
  - Taking the absolute value of the scalar product will ensure that you get the acute angle as your answer

Worked Example

Find the angle in radians between the line L with vector equation $\mathbf{r} = (2-\lambda)\mathbf{i} + (\lambda+1)\mathbf{j} + (1-2\lambda)\mathbf{k}$ and the plane Π with Cartesian equation $x - 3y + 2z = 5$.

Rewrite line equation in standard vector form:

$$r = \begin{pmatrix} 2-\lambda \\ 1+\lambda \\ 1-2\lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

↑ direction vector of the line

Find the normal vector of the plane:

$x - 3y + 2z = 5 \Rightarrow$ normal vector $= \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

(components of the normal vector)

Find the angle between the direction vector and the normal vector, α:

Angle between two vectors	$\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert \mathbf{v} \rvert \lvert \mathbf{w} \rvert}$

$$\cos\alpha = \frac{\left| \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right|}{\sqrt{(-1)^2 + (1)^2 + (-2)^2} \times \sqrt{1^2 + (-3)^2 + 2^2}} = \frac{|(-1)(1) + (1)(-3) + (-2)(2)|}{\sqrt{6}\sqrt{14}}$$

$$\theta = \frac{\pi}{2} - \cos^{-1}\alpha$$

$$\theta = \frac{\pi}{2} - \cos^{-1}\left(\frac{|-8|}{\sqrt{6}\sqrt{14}}\right)$$

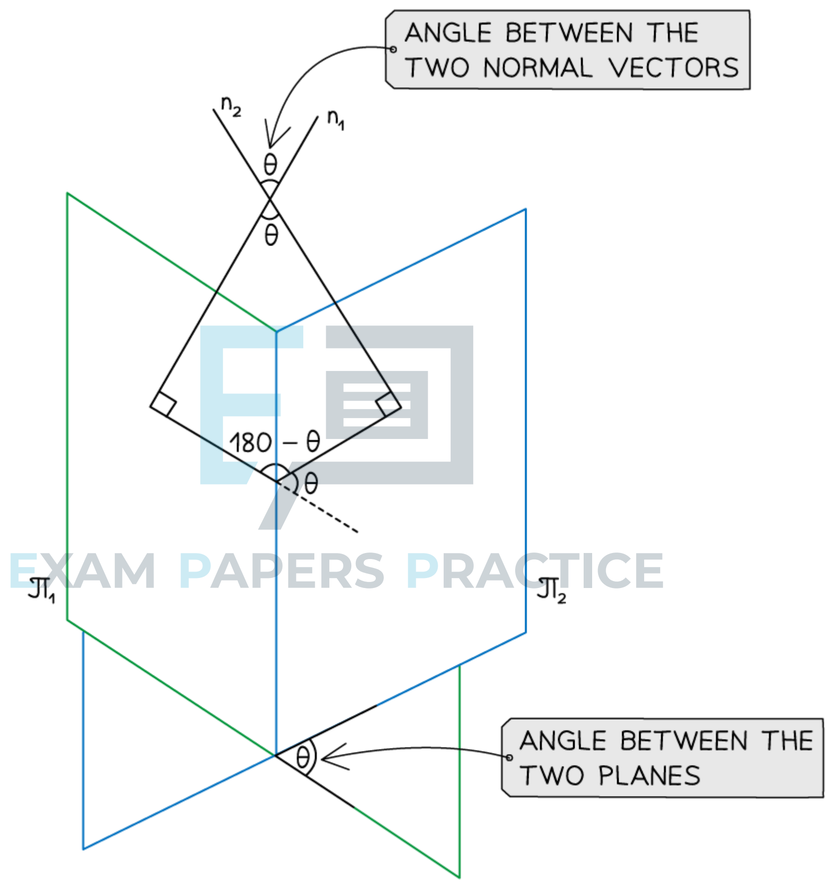
Using the absolute value ensures we find the acute angle.

$\theta = 1.06$ radians (3 s.f.)

Angle Between Two Planes

How do we find the angle between two planes?

- The angle between two planes is equal to the angle between their **normal vectors**
  - It can be found using the **scalar product** of their normal vectors
- If two planes Π_1 and Π_2 with normal vectors n_1 and n_2 meet at an angle then the two planes and the two normal vectors will form a quadrilateral
  - The angles between the planes and the normal will both be 90°
  - The angle between the two planes and the angle opposite it (between the two normal vectors) will add up to 180°



Exam Tip

- In your exam read the question carefully to see if you need to find the acute or obtuse angle
  - When revising, get into the practice of double checking at the end of a question whether your angle is acute or obtuse and whether this fits the question

Worked Example

Find the acute angle between the two planes which can be defined by equations Π_1: $2x-y+3z= 7$ and Π_2: $x+2y-z=20$.

Find the normal vectors of each of the planes:

Π_1: $2x - y + 3z = 7 \Rightarrow$ normal vector, $n_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Π_2: $x + 2y - z = 20 \Rightarrow$ normal vector, $n_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Find the angle between the two normal vectors:

Angle between two vectors	$\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert \mathbf{v} \rvert \lvert \mathbf{w} \rvert}$

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{|(2)(1) + (-1)(2) + (3)(-1)|}{\sqrt{2^2+(-1)^2+3^2} \times \sqrt{1^2+2^2+(-1)^2}} = \frac{|-3|}{\sqrt{14}\times\sqrt{6}}$$

Using the absolute value ensures we find the acute angle.

$$\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)$$

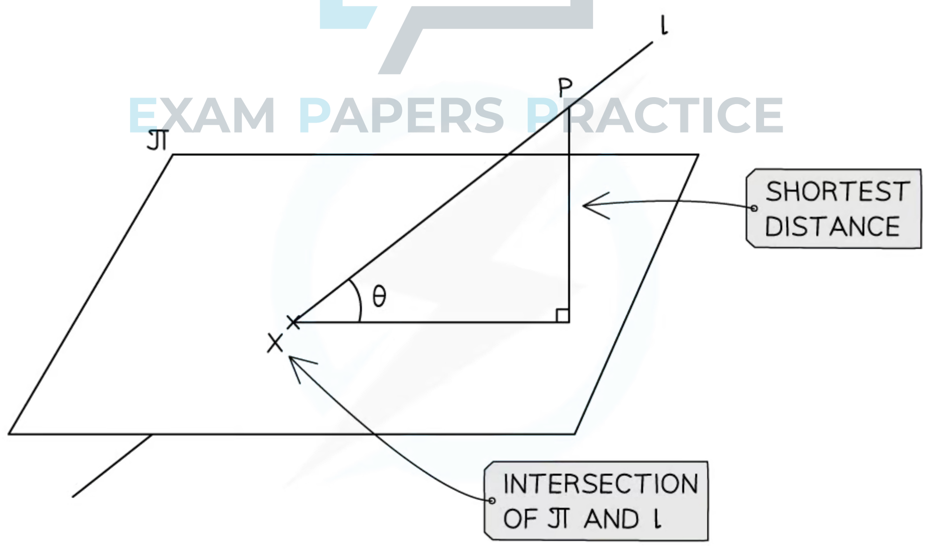
$$\theta = 1.24 \text{ radians (3 s.f.)}$$

3.11.4 Shortest Distances with Planes

Shortest Distance Between a Line and a Plane

How do I find the shortest distance between a given point on a line and a plane?

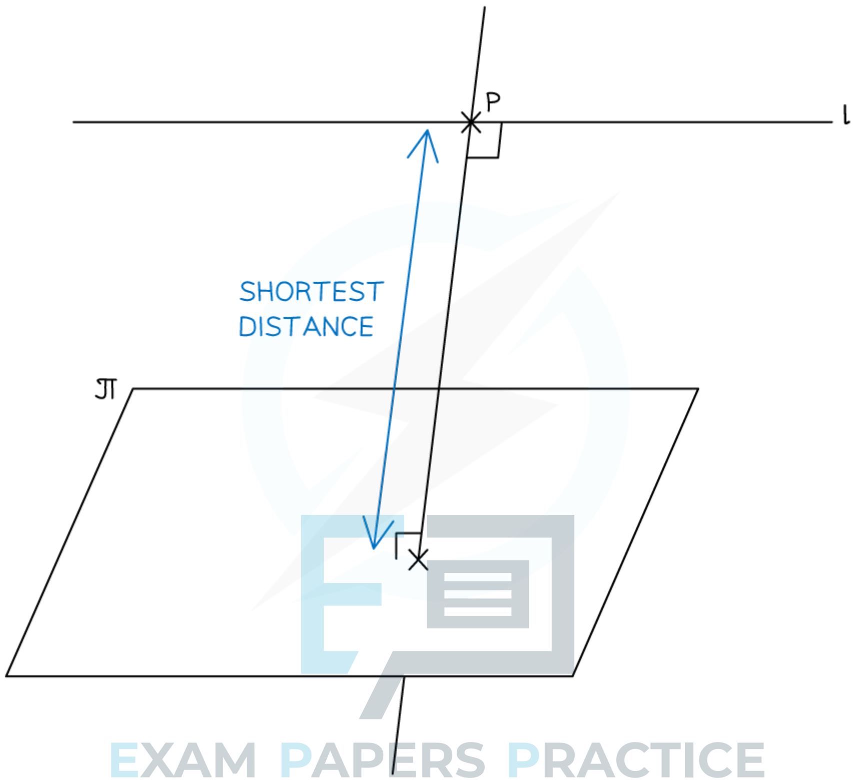
- The shortest distance from any point on a line to a plane will always be the **perpendicular** distance from the point to the plane
- Given a point, *P*, on the line l with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a plane Π with equation $\mathbf{r} \cdot \mathbf{n} = d$
  - STEP 1: Find the vector equation of the line perpendicular to the plane that goes through the point, *P*, on l
    - This will have the position vector of the point, *P*, and the direction vector **n**
  - STEP 2: Find the coordinates of the point of intersection of this new line with Π by substituting the equation of the line into the equation of the plane
  - STEP 3: Find the distance between the given point on the line and the point of intersection
    - This will be the shortest distance from the plane to the point
- A question may provide the acute angle between the line and the plane
  - Use right-angled trigonometry to find the perpendicular distance between the point on the line and the plane
    - Drawing a clear diagram will help



How do I find the shortest distance between a plane and a line parallel to the plane?

- The shortest distance between a line and a plane that are parallel to each other will be the **perpendicular** distance from the line to the plane
- Given a line l_1 with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a plane Π parallel to l_1 with equation $\mathbf{r} \cdot \mathbf{n} = d$
  - Where **n** is the **normal vector** to the plane

- STEP 1: Find the equation of the line l_2 perpendicular to l_1 and Π going through the point **a** in the form $\mathbf{r} = \mathbf{a} + \mu\mathbf{n}$
- STEP 2: Find the point of intersection of the line l_2 and Π
- STEP 3: Find the distance between the point of intersection and the point,



Exam Tip

- Vector planes questions can be tricky to visualise, read the question carefully and sketch a very simple diagram to help you get started

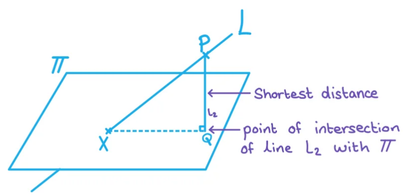
Worked Example

The plane Π has equation $\mathbf{r}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=6$.

The line L has equation $\mathbf{r}=\begin{pmatrix}2\\3\\1\end{pmatrix}+s\begin{pmatrix}1\\-2\\4\end{pmatrix}$.

The point $P\,(-2,\,11,\,-15)$ lies on the line L.

Find the shortest distance between the point P and the plane Π.



STEP 1: Use the given point, P and the known normal to the plane, n to write an equation for the line perpendicular to Π, l_2.

$$\mathbf{r}=\begin{pmatrix}-2\\11\\-15\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

(P, n)

STEP 2: Find the point of intersection, Q, of the new line, l_2, with Π.

$$\left(\begin{pmatrix}-2\\11\\-15\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=6$$

$$2(-2+2\lambda)-(11-\lambda)+(\lambda-15)=6$$

$$-4+4\lambda-11+\lambda+\lambda-15=6$$

$$6\lambda-30=6$$

$$\lambda=6\Rightarrow\overrightarrow{OQ}=\begin{pmatrix}-2\\11\\-15\end{pmatrix}+6\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}10\\5\\-9\end{pmatrix}$$

STEP 3: Find the distance between P and Q.

$$|\overrightarrow{PQ}|=\sqrt{(10--2)^2+(5-11)^2+(-9--15)^2}=6\sqrt{6}\text{ units}$$

Shortest distance $=6\sqrt{6}$ units

Shortest Distance Between Two Planes

How do I find the shortest distance between two parallel planes?

- Two **parallel** planes will never intersect
- The shortest distance between two **parallel planes** will be the **perpendicular distance** between them
- Given a plane Π_1 with equation $\mathbf{r} \cdot \mathbf{n} = d$ and a plane Π_2 with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ then the shortest distance between them can be found
  - STEP 1: The equation of the line perpendicular to both planes and through the point **a** can be written in the form **r** = **a** + s**n**
  - STEP 2: Substitute the equation of the line into $\mathbf{r} \cdot \mathbf{n} = d$ to find the coordinates of the point where the line meets Π_1
  - STEP 3: Find the distance between the two points of intersection of the line with the two planes

How do I find the shortest distance from a given point on a plane to another plane?

- The shortest distance from any point, *P* on a plane, Π_1 , to another plane, Π_2 will be the **perpendicular** distance from the point to Π_2
  - STEP 1: Use the given coordinates of the point *P* on Π_1 and the normal to the plane Π_2 to find the vector equation of the line through *P* that is perpendicular to Π_1
  - STEP 2: Find the point of intersection of this line with the plane Π_2
  - STEP 3: Find the distance between the two points of intersection

Exam Tip

- There are a lot of steps when answering these questions so set your methods out clearly in the exam

Worked Example

Consider the parallel planes defined by the equations:

$$\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 44,$$

$$\Pi_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find the shortest distance between the two planes Π_1 and Π_2.

Find the equation of the line perpendicular to the planes through the point (0,0,3)

$$L : r = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + s \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$$

position vector of Π_2

Normal vector of Π_1

Substitute the equation of L into the equation of Π_1:

$$\begin{pmatrix} 3s \\ -5s \\ 3+2s \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 44$$

$$3(3s) - 5(-5s) + 2(3+2s) = 44$$

$$38s + 6 = 44$$

$$s = 1$$

Substitute $s = 1$ back into the equation of L:

$$r = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 5 \end{pmatrix}$$

Find the distance between (0,0,3) and (3,-5,5)

$$d = \sqrt{3^2 + (-5)^2 + (5-3)^2}$$

$$= \sqrt{38}$$

Shortest distance = $\sqrt{38}$ units