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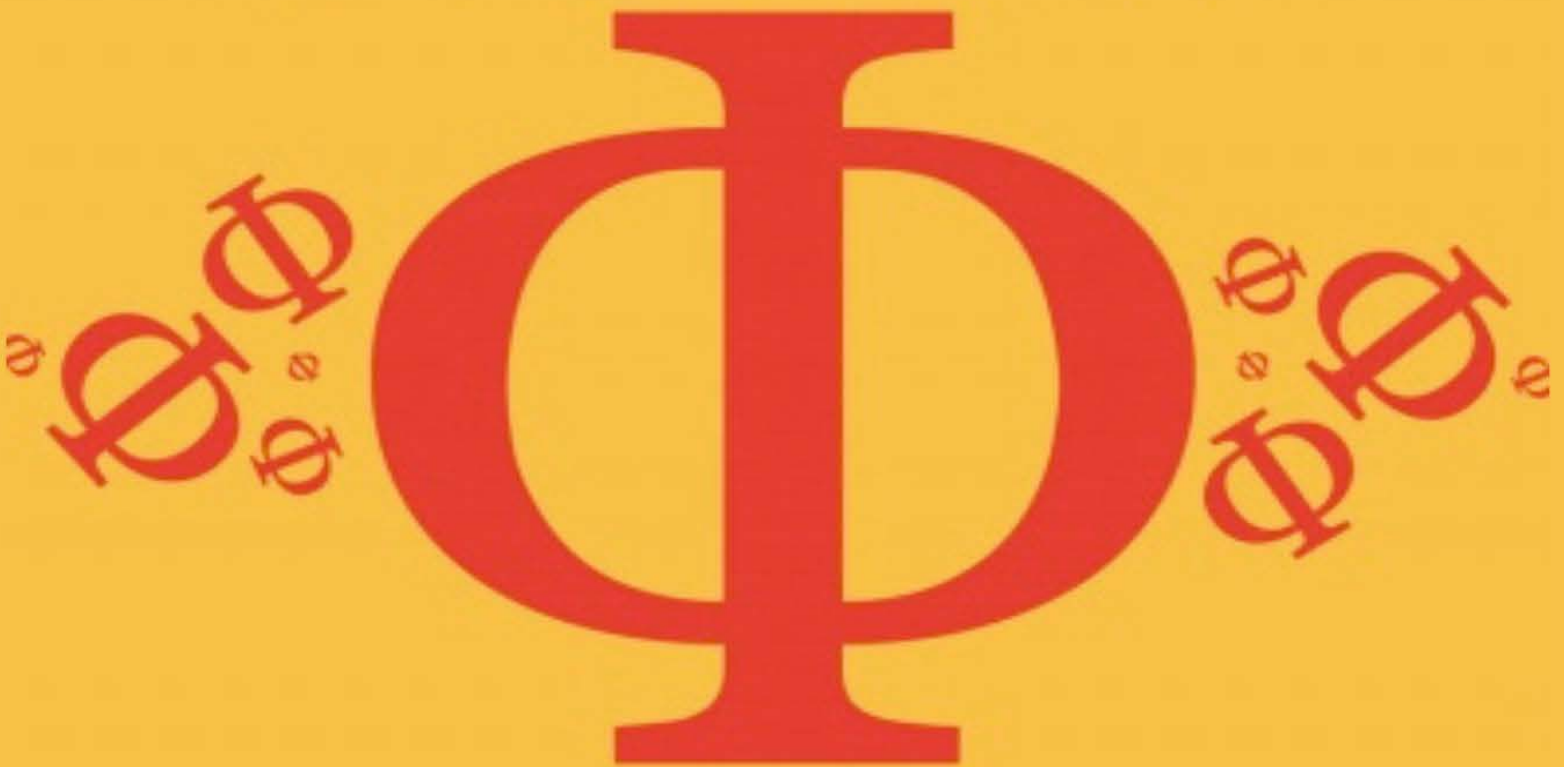
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## **2. Functions**

### **2.3 Functions Toolkit**



# **MATHS**

## **AA HL**

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## 2. Functions

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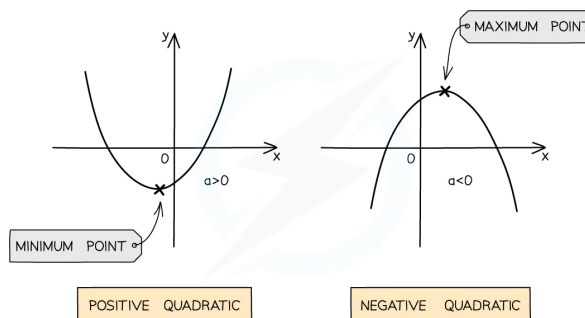
## 2.1 Quadratic Functions & Graphs

### 2.1.1 Quadratic Functions

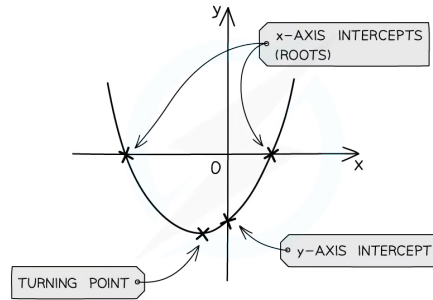
#### Quadratic Functions & Graphs

##### What are the key features of quadratic graphs?

- A **quadratic** graph can be written in the form  $y = ax^2 + bx + c$  where  $a \neq 0$
- The value of  $a$  affects the shape of the curve
  - If  $a$  is **positive** the shape is **concave up**  $\cup$
  - If  $a$  is **negative** the shape is **concave down**  $\cap$
- The **y-intercept** is at the point  $(0, c)$
- The **zeros or roots** are the solutions to  $ax^2 + bx + c = 0$ 
  - These can be found by
    - Factorising
    - Quadratic formula
    - Using your GDC
  - These are also called the x-intercepts
  - There can be 0, 1 or 2 x-intercepts
    - This is determined by the value of the **discriminant**
- There is an **axis of symmetry** at  $x = -\frac{b}{2a}$ 
  - This is given in your **formula booklet**
  - If there are two x-intercepts then the axis of symmetry goes through the midpoint of them
- The **vertex** lies on the axis of symmetry
  - It can be found by **completing the square**
  - The x-coordinate is  $x = -\frac{b}{2a}$
  - The y-coordinate can be found using the GDC or by calculating  $y$  when  $x = -\frac{b}{2a}$
  - If  $a$  is **positive** then the vertex is the **minimum point**
  - If  $a$  is **negative** then the vertex is the **maximum point**







## What are the equations of a quadratic function?

- $f(x) = ax^2 + bx + c$ 
  - This is the **general form**
  - It clearly shows the y-intercept  $(0, c)$
  - You can find the axis of symmetry by  $x = -\frac{b}{2a}$ 
    - This is given in the formula booklet
- $f(x) = a(x - p)(x - q)$ 
  - This is the **factorised form**
  - It clearly shows the roots  $(p, 0)$  &  $(q, 0)$
  - You can find the axis of symmetry by  $x = \frac{p + q}{2}$
- $f(x) = a(x - h)^2 + k$ 
  - This is the **vertex form**
  - It clearly shows the vertex  $(h, k)$
  - The axis of symmetry is therefore  $x = h$
  - It clearly shows how the function can be transformed from the graph  $y = x^2$ 
    - Vertical stretch by scale factor  $a$
    - Translation by vector  $\begin{pmatrix} h \\ k \end{pmatrix}$

## How do I find an equation of a quadratic?

- If you have the **roots**  $x = p$  and  $x = q$ ...
  - Write in **factorised form**  $y = a(x - p)(x - q)$
  - You will need a third point to find the value of  $a$
- If you have the **vertex**  $(h, k)$  then...
  - Write in **vertex form**  $y = a(x - h)^2 + k$
  - You will need a second point to find the value of  $a$
- If you have **three random points**  $(x_1, y_1)$ ,  $(x_2, y_2)$  &  $(x_3, y_3)$  then...
  - Write in the **general form**  $y = ax^2 + bx + c$
  - Substitute the three points into the equation
  - Form and solve a system of three linear equations to find the values of  $a$ ,  $b$  &  $c$



### Exam Tip

- Use your GDC to find the roots and the turning point of a quadratic function
  - You do not need to factorise or complete the square
  - It is good exam technique to sketch the graph from your GDC as part of your working

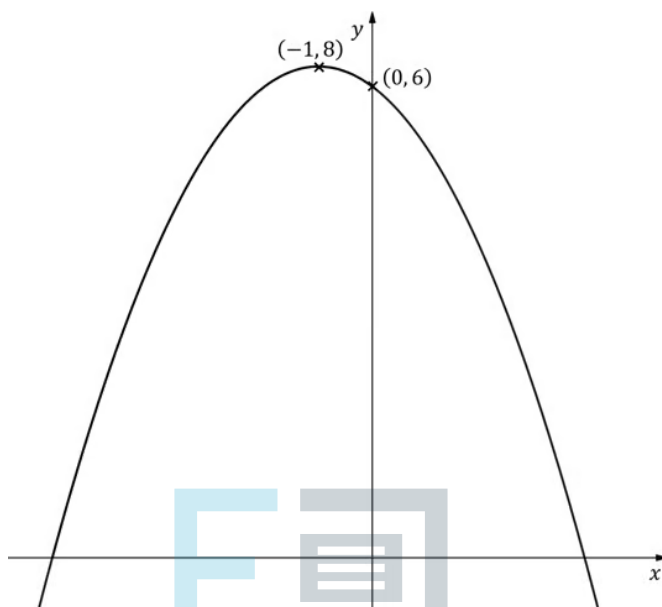




### ? Worked Example

The diagram below shows the graph of  $y = f(x)$ , where  $f(x)$  is a quadratic function.

The intercept with the  $y$ -axis and the vertex have been labelled.



Write down an expression for  $y = f(x)$ .

We have the vertex so use  $y = a(x-h)^2 + k$

$$\text{Vertex } (-1, 8) : y = a(x - (-1))^2 + 8$$

$$y = a(x + 1)^2 + 8$$

Substitute the second point

$$x = 0, y = 6 : 6 = a(0 + 1)^2 + 8$$

$$6 = a + 8$$

$$a = -2$$

$$\boxed{y = -2(x + 1)^2 + 8}$$



## 2.1.2 Factorising & Completing the Square

### Factorising Quadratics

#### Why is factorising quadratics useful?

- Factorising gives **roots (zeroes or solutions)** of a quadratic
- It gives the **x-intercepts** when drawing the graph

#### How do I factorise a monic quadratic of the form $x^2 + bx + c$ ?

- A monic quadratic is a quadratic where the coefficient of the  $x^2$  term is 1
- You might be able to spot the factors by **inspection**
  - Especially if  $c$  is a **prime number**
- Otherwise find two numbers  $m$  and  $n$ ..
  - A sum equal to  $b$ 
    - $p + q = b$
  - A product equal to  $c$ 
    - $pq = c$
- Rewrite  $bx$  as  $mx + nx$
- Use this to factorise  $x^2 + mx + nx + c$
- A shortcut is to write:
  - $(x + p)(x + q)$

#### How do I factorise a non-monic quadratic of the form $ax^2 + bx + c$ ?

- A non-monic quadratic is a quadratic where the coefficient of the  $x^2$  term is not equal to 1
- If  $a$ ,  $b$  &  $c$  have a common factor then first factorise that out to leave a quadratic with coefficients that have **no common factors**
- You might be able to spot the factors by **inspection**
  - Especially if  $a$  and/or  $c$  are **prime numbers**
- Otherwise find two numbers  $m$  and  $n$ ..
  - A sum equal to  $b$ 
    - $m + n = b$
  - A product equal to  $ac$ 
    - $mn = ac$
- Rewrite  $bx$  as  $mx + nx$
- Use this to factorise  $ax^2 + mx + nx + c$
- A shortcut is to write:
  - $\frac{(ax + m)(ax + n)}{a}$
  - Then factorise common factors from numerator to cancel with the  $a$  on the denominator

#### How do I use the difference of two squares to factorise a quadratic of the form $a^2x^2 - c^2$ ?

- The **difference of two squares** can be used when...
  - There is **no x term**
  - The **constant term is a negative**
- Square root the two terms  $a^2x^2$  and  $c^2$
- The two factors are the **sum of square roots** and the **difference of the square roots**



- A shortcut is to write:
  - $(ax + c)(ax - c)$



### Exam Tip

- You can deduce the factors of a quadratic function by using your GDC to find the solutions of a quadratic equation
  - Using your GDC, the quadratic equation  $6x^2 + x - 2 = 0$  has solutions
$$x = -\frac{2}{3} \text{ and } x = \frac{1}{2}$$
  - Therefore the factors would be  $(3x + 2)$  and  $(2x - 1)$
  - i.e.  $6x^2 + x - 2 = (3x + 2)(2x - 1)$





## ? Worked Example

Factorise fully:

a)

$$x^2 - 7x + 12.$$

Find two numbers  $m$  and  $n$  such that

$$m+n=b=-7 \quad mn=c=12$$

$$-4 + -3 = -7 \quad -4 \times -3 = 12$$

Split  $-7x$  up and factorise

$$x^2 - 4x - 3x + 12$$

$$x(x-4) - 3(x-4)$$

$$(x-3)(x-4)$$

Shortcut

$$(x+m)(x+n)$$

$$(x-3)(x-4)$$

b)

$$4x^2 + 4x - 15.$$

Find two numbers  $m$  and  $n$  such that

$$m+n=b=4 \quad mn=ac=4 \times -15=-60$$

$$10 + -6 = 4 \quad 10 \times -6 = -60$$

Split  $4x$  up and factorise

$$4x^2 + 10x - 6x - 15$$

$$2x(2x+5) - 3(2x+5)$$

$$(2x-3)(2x+5)$$

Shortcut

$$\frac{(ax+m)(ax+n)}{a}$$

$$\frac{(4x+10)(4x-6)}{4}$$

$$\frac{2(2x+5) \times 2(2x-3)}{4}$$

$$(2x-3)(2x+5)$$

c)

$$18 - 50x^2.$$



Factorise the common factor

$$2(9 - 25x^2)$$

Use difference of two squares

$$2(3 - 5x)(3 + 5x)$$





## Completing the Square

### Why is completing the square for quadratics useful?

- Completing the square gives the **maximum/minimum** of a quadratic function
  - This can be used to define the **range** of the function
- It gives the **vertex** when drawing the graph
- It can be used to **solve quadratic equations**
- It can be used to derive the **quadratic formula**

### How do I complete the square for a monic quadratic of the form $x^2 + bx + c$ ?

- Half the value of  $b$  and write  $\left(x + \frac{b}{2}\right)^2$ 
  - This is because  $\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$
- Subtract the unwanted  $\frac{b^2}{4}$  term and add on the constant  $c$ 
  - $\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$

### How do I complete the square for a non-monic quadratic of the form $ax^2 + bx + c$ ?

- Factorise out the  $a$  from the terms involving  $x$ 
  - $a\left(x^2 + \frac{b}{a}x\right) + c$
  - Leaving the  $c$  alone will **avoid working with lots of fractions**
- Complete the square on the quadratic term
  - Half  $\frac{b}{a}$  and write  $\left(x + \frac{b}{2a}\right)^2$ 
    - This is because  $\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$
  - Subtract the unwanted  $\frac{b^2}{4a^2}$  term
- Multiply by  $a$  and add the constant  $c$ 
  - $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c$
  - $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$



#### Exam Tip

- Some questions may not use the phrase "completing the square" so ensure you can recognise a quadratic expression or equation written in this form
  - $a(x - h)^2 + k (= 0)$



**Worked Example**

Complete the square:

a)

$$x^2 - 8x + 3.$$

Half  $b$  and subtract its square

$$(x - 4)^2 - 4^2 + 3$$

$$(x - 4)^2 - 13$$

b)

$$3x^2 + 12x - 5.$$

Factorise the 3 from the  $x$  terms

$$3(x^2 + 4x) - 5$$

Complete the square on  $x^2 + 4x$ 

$$3((x+2)^2 - 2^2) - 5$$

Simplify

$$3((x+2)^2 - 4) - 5$$

$$3(x+2)^2 - 12 - 5$$

$$3(x+2)^2 - 17$$

## 2.1.3 Solving Quadratics

### Solving Quadratic Equations

#### How do I decide the best method to solve a quadratic equation?

- A quadratic equation is of the form  $ax^2 + bx + c = 0$
- If it is a calculator paper then use your GDC to solve the quadratic
- If it is a non-calculator paper then...
  - you can always use the **quadratic formula**
  - you can **factorise** if it can be factorised with integers
  - you can always **complete the square**

#### How do I solve a quadratic equation by the quadratic formula?

- If necessary **rewrite** in the form  $ax^2 + bx + c = 0$
- **Clearly identify** the values of  $a$ ,  $b$  &  $c$
- **Substitute** the values into the formula
  - $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ 
    - This is given in the **formula booklet**
- **Simplify** the solutions as much as possible

#### How do I solve a quadratic equation by factorising?

- **Factorise** to rewrite the quadratic equation in the form  $a(x - p)(x - q) = 0$
- Set each factor to zero and **solve**
  - $x - p = 0 \Rightarrow x = p$
  - $x - q = 0 \Rightarrow x = q$

#### How do I solve a quadratic equation by completing the square?

- **Complete the square** to rewrite the quadratic equation in the form  $a(x - h)^2 + k = 0$
- Get the squared term by itself
  - $(x - h)^2 = -\frac{k}{a}$
- If  $\left(-\frac{k}{a}\right)$  is **negative** then there will be **no solutions**
- If  $\left(-\frac{k}{a}\right)$  is **positive** then there will be **two values** for  $x - h$ 
  - $x - h = \pm \sqrt{-\frac{k}{a}}$
- **Solve** for  $x$ 
  - $x = h \pm \sqrt{-\frac{k}{a}}$



### Exam Tip

- When using the quadratic formula with awkward values or fractions you may find it easier to deal with the " $b^2 - 4ac$ " (**discriminant**) first
  - This can help avoid numerical and negative errors, improving accuracy



## ? Worked Example

Solve the equations:

a)

$$4x^2 + 4x - 15 = 0.$$

This can be factorised

$$(2x + 5)(2x - 3) = 0$$

$$2x + 5 = 0 \quad \text{or} \quad 2x - 3 = 0$$

$$x = -\frac{5}{2} \quad \text{or} \quad x = \frac{3}{2}$$

b)

$$3x^2 + 12x - 5 = 0.$$

This can not be factorised but  $3x^2$  and  $12x$  have a common factor so complete the square

$$3(x+2)^2 - 17 = 0$$

$$(x+2)^2 = \frac{17}{3}$$

$$x+2 = \pm \sqrt{\frac{17}{3}}$$

Rearrange

Remember  $\pm$

$$x = -2 \pm \sqrt{\frac{17}{3}}$$

c)

$$7 - 3x - 5x^2 = 0.$$

This can not be factorised so use formula

Formula booklet

|                                   |                                                                                  |
|-----------------------------------|----------------------------------------------------------------------------------|
| Solutions of a quadratic equation | $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$ |
|-----------------------------------|----------------------------------------------------------------------------------|

$$a = -5 \quad b = -3 \quad c = 7$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-5)(7)}}{2(-5)}$$

$$= \frac{3 \pm \sqrt{9 + 140}}{-10}$$

$$x = -\frac{3 \pm \sqrt{149}}{10}$$



## 2.1.4 Quadratic Inequalities

### Quadratic Inequalities

**What affects the inequality sign when rearranging a quadratic inequality?**

- The inequality sign is **unchanged** by...
  - **Adding/subtracting** a term to both sides
  - **Multiplying/dividing** both sides by a **positive term**
- The inequality sign **flips** ( $<$  changes to  $>$ ) when...
  - **Multiplying/dividing** both sides by a **negative term**

**How do I solve a quadratic inequality?**

- **STEP 1: Rearrange** the inequality into quadratic form with a **positive squared term**
  - $ax^2 + bx + c > 0$
  - $ax^2 + bx + c \geq 0$
  - $ax^2 + bx + c < 0$
  - $ax^2 + bx + c \leq 0$
- **STEP 2:** Find the **roots** of the quadratic equation
  - Solve  $ax^2 + bx + c = 0$  to get  $x_1$  and  $x_2$  where  $x_1 < x_2$
- **STEP 3: Sketch** a graph of the quadratic and label the roots
  - As the squared term is positive it will be **concave up** so "U" shaped
- **STEP 4: Identify** the **region** that satisfies the inequality
  - If you want the graph to be **above the x-axis** then choose the region to be the **two intervals outside** of the two roots
  - If you want the graph to be **below the x-axis** then choose the region to be the **interval between** the two roots
  - For  $ax^2 + bx + c > 0$ 
    - The solution is  $x < x_1$  or  $x > x_2$
  - For  $ax^2 + bx + c \geq 0$ 
    - The solution is  $x \leq x_1$  or  $x \geq x_2$
  - For  $ax^2 + bx + c < 0$ 
    - The solution is  $x_1 < x < x_2$
  - For  $ax^2 + bx + c \leq 0$ 
    - The solution is  $x_1 \leq x \leq x_2$

**How do I solve a quadratic inequality of the form  $(x - h)^2 < n$  or  $(x - h)^2 > n$ ?**

- The safest way is by following the steps above
  - Expand and rearrange
- A **common mistake** is writing  $x - h < \pm\sqrt{n}$  or  $x - h > \pm\sqrt{n}$ 
  - This is **NOT correct!**
- The correct solution to  $(x - h)^2 < n$  is
  - $|x - h| < \sqrt{n}$  which can be written as  $-\sqrt{n} < x - h < \sqrt{n}$
  - The **final solution** is  $h - \sqrt{n} < x < h + \sqrt{n}$
- The correct solution to  $(x - h)^2 > n$  is



- $|x - h| > \sqrt{n}$  which can be written as  $x - h < -\sqrt{n}$  or  $x - h > \sqrt{n}$
- The **final solution** is  $x < h - \sqrt{n}$  or  $x > h + \sqrt{n}$



### Exam Tip

- It is easiest to sketch the graph of a quadratic when it has a positive  $x^2$  term, so rearrange first if necessary
- Use your GDC to help select the correct region(s) for the inequality
- Some makes/models of GDC may have the ability to solve inequalities directly
  - However unconventional notation may be used to display the answer (e.g.  $6 > x > 3$  rather than  $3 < x < 6$ )
  - The safest method is to **always** sketch the graph



### Worked Example

Find the set of values which satisfy  $3x^2 + 2x - 6 > x^2 + 4x - 2$ .

STEP 1: Rearrange

$$\begin{aligned}(3x^2 + 2x - 6) - (x^2 + 4x - 2) &> 0 \quad \text{This way gives } a > 0 \\ 2x^2 - 2x - 4 &> 0 \\ x^2 - x - 2 &> 0 \quad \text{Divide by factor of 2}\end{aligned}$$

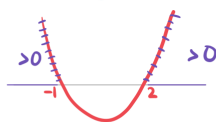
STEP 2: Find the roots

$$\begin{aligned}x^2 - x - 2 &= 0 \\ (x - 2)(x + 1) &= 0 \\ x = 2 \quad \text{or} \quad x = -1\end{aligned}$$

STEP 3: Sketch



STEP 4: Identify region



$$x < -1 \quad \text{or} \quad x > 2$$



## 2.1.5 Discriminants

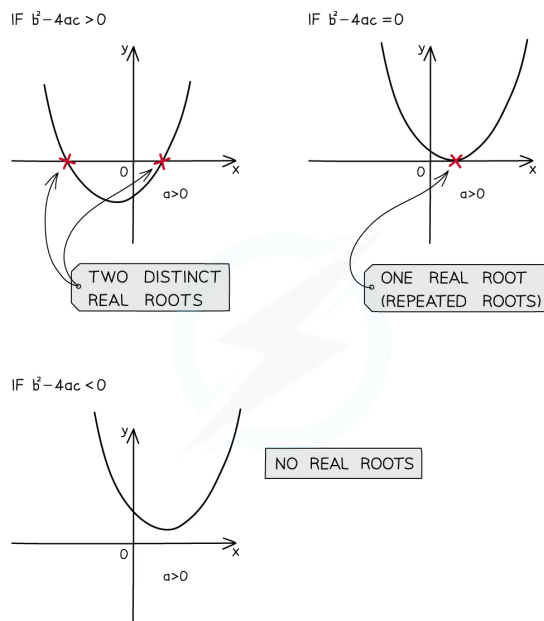
### Discriminants

#### What is the discriminant of a quadratic function?

- The discriminant of a quadratic is denoted by the Greek letter  $\Delta$  (upper case delta)
- For the quadratic function the discriminant is given by
  - $\Delta = b^2 - 4ac$   
This is given in the **formula booklet**
- The discriminant is the expression that is square rooted in the **quadratic formula**

#### How does the discriminant of a quadratic function affect its graph and roots?

- If  $\Delta > 0$  then  $\sqrt{b^2 - 4ac}$  and  $-\sqrt{b^2 - 4ac}$  are **two distinct values**
  - The equation  $ax^2 + bx + c = 0$  has **two distinct real solutions**
  - The graph of  $y = ax^2 + bx + c$  has **two distinct real roots**  
This means the graph **crosses** the x-axis **twice**
- If  $\Delta = 0$  then  $\sqrt{b^2 - 4ac}$  and  $-\sqrt{b^2 - 4ac}$  are **both zero**
  - The equation  $ax^2 + bx + c = 0$  has **one repeated real solution**
  - The graph of  $y = ax^2 + bx + c$  has **one repeated real root**  
This means the graph **touches** the x-axis at **exactly one point**  
This means that the **x-axis** is a **tangent** to the graph
- If  $\Delta < 0$  then  $\sqrt{b^2 - 4ac}$  and  $-\sqrt{b^2 - 4ac}$  are **both undefined**
  - The equation  $ax^2 + bx + c = 0$  has **no real solutions**
  - The graph of  $y = ax^2 + bx + c$  has **no real roots**  
This means the graph **never touches** the **x-axis**  
This means that graph is **wholly above** (or **below**) the **x-axis**



## Forming equations and inequalities using the discriminant

- Often at least one of the coefficients of a quadratic is **unknown**
  - Questions usually use the letter  $k$  for the unknown constant
- You will be given a fact about the quadratic such as:
  - The **number of solutions** of the equation
  - The **number of roots** of the graph
- To find the **value or range of values** of  $k$ 
  - Find an **expression for the discriminant**
    - Use  $\Delta = b^2 - 4ac$
  - Decide whether  $\Delta > 0$ ,  $\Delta = 0$  or  $\Delta < 0$ 
    - If the question says there are **real roots** but does not specify how many then use  $\Delta \geq 0$
  - **Solve** the resulting equation or inequality



### Exam Tip

- Questions will rarely use the word discriminant so it is important to recognise when its use is required
  - Look for
    - a number of roots or solutions being stated
    - whether and/or how often the graph of a quadratic function intercepts the  $x$ -axis
- Be careful setting up inequalities that concern "two real roots" ( $\Delta \geq 0$ ) as opposed to "two real distinct roots" ( $\Delta > 0$ )



### ? Worked Example

A function is given by  $f(x) = 2kx^2 + kx - k + 2$ , where  $k$  is a constant. The graph of  $y = f(x)$  has two distinct real roots.

a)

Show that  $9k^2 - 16k > 0$ .

Two distinct real roots  $\Rightarrow \Delta > 0$

Formula booklet

Discriminant

$\Delta = b^2 - 4ac$

$$a = 2k \quad b = k \quad c = (-k + 2)$$

$$\Delta = k^2 - 4(2k)(-k + 2)$$

$$= k^2 + 8k^2 - 16k$$

$$= 9k^2 - 16k$$

$$\Delta > 0 \Rightarrow 9k^2 - 16k > 0$$

b)

Hence find the set of possible values of  $k$ .

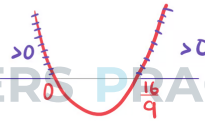
Solve the inequality

$$9k^2 - 16k = 0$$

$$k(9k - 16) = 0$$

$$k = 0 \text{ or } k = \frac{16}{9}$$

$$k < 0 \text{ or } k > \frac{16}{9}$$



## 2.2 Linear Functions & Graphs

### 2.2.1 Equations of a Straight Line

#### Equations of a Straight Line

##### How do I find the gradient of a straight line?

- Find two points that the line passes through with coordinates  $(x_1, y_1)$  and  $(x_2, y_2)$
- The gradient between these two points is calculated by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- This is given in the **formula booklet**
- The gradient of a straight line measures its **slope**
  - A line with gradient 1 will go up 1 unit for every unit it goes to the right
  - A line with gradient -2 will go down two units for every unit it goes to the right

##### What are the equations of a straight line?

- $y = mx + c$ 
  - This is the **gradient-intercept form**
  - It clearly shows the gradient  $m$  and the y-intercept  $(0, c)$
- $y - y_1 = m(x - x_1)$ 
  - This is the **point-gradient form**
  - It clearly shows the gradient  $m$  and a point on the line  $(x_1, y_1)$
- $ax + by + d = 0$ 
  - This is the **general form**
  - You can quickly get the x-intercept  $\left(-\frac{d}{a}, 0\right)$  and y-intercept  $\left(0, -\frac{d}{b}\right)$

##### How do I find an equation of a straight line?

- You will need the gradient
  - If you are given two points then first find the gradient
- It is easiest to start with the **point-gradient form**
  - then rearrange into whatever form is required
    - multiplying both sides by any denominators will get rid of fractions
- You can check your answer by using your GDC
  - Graph your answer and check it goes through the point(s)
  - If you have two points then you can enter these in the **statistics mode** and find the regression line  $y = ax + b$



## Exam Tip

- A sketch of the graph of the straight line(s) can be helpful, even if not demanded by the question
  - Use your GDC to plot them
- Ensure you state equations of straight lines in the format required
  - Usually  $y = mx + c$  or  $ax + by + d = 0$
  - Check whether coefficients need to be integers (they usually are for  $ax + by + d = 0$ )



## Worked Example

The line  $l$  passes through the points  $(-2, 5)$  and  $(6, -7)$ .

Find the equation of  $l$ , giving your answer in the form  $ax + by + d = 0$  where  $a$ ,  $b$  and  $c$  are integers to be found.

Find the gradient between  $(-2, 5)$  and  $(6, -7)$

Formula booklet

$$m = \frac{-7 - 5}{6 - (-2)} = -\frac{3}{2}$$

Gradient formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Use the point-gradient formula

Formula booklet

Equations of a straight line

$$y - y_1 = m(x - x_1)$$

$$(x_1, y_1) = (-2, 5) \quad m = -\frac{3}{2}$$

$$y - 5 = -\frac{3}{2}(x - (-2)) \quad \text{Simplify}$$

$$y - 5 = -\frac{3}{2}(x + 2)$$

$$2(y - 5) = -3(x + 2)$$

$$2y - 10 = -3x - 6$$

$$3x + 2y - 4 = 0$$

Multiply by denominator

Expand

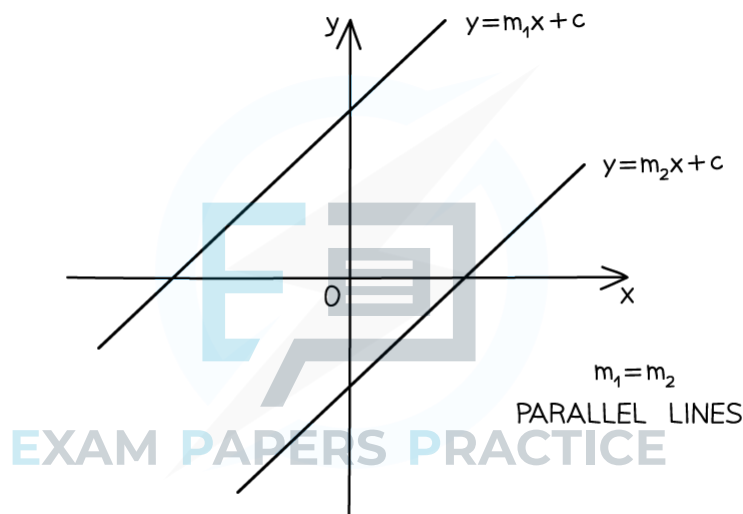
Rearrange



## Parallel Lines

### How are the equations of parallel lines connected?

- **Parallel lines** are always equidistant meaning they never intersect
- Parallel lines have the same gradient
  - If the gradient of line  $l_1$  is  $m_1$  and gradient of line  $l_2$  is  $m_2$  then...
    - $m_1 = m_2 \Rightarrow l_1 \text{ \& } l_2 \text{ are parallel}$
    - $l_1 \text{ \& } l_2 \text{ are parallel} \Rightarrow m_1 = m_2$
- To determine if two lines are parallel:
  - Rearrange into the gradient-intercept form  $y = mx + c$
  - Compare the coefficients of  $x$
  - If they are equal then the lines are parallel



**Worked Example**

The line  $l$  passes through the point  $(4, -1)$  and is parallel to the line with equation  $2x - 5y = 3$ .

Find the equation of  $l$ , giving your answer in the form  $y = mx + c$ .

Rearrange into  $y = mx + c$  to find the gradient

$$5y = 2x - 3 \Rightarrow y = \frac{2}{5}x - \frac{3}{5} \therefore \text{gradient} = \frac{2}{5}$$

Parallel lines  $\Rightarrow m_1 = m_2$

$$m = \frac{2}{5}$$

Use the point-gradient formula

Formula booklet

|                              |
|------------------------------|
| Equations of a straight line |
|------------------------------|

|                        |
|------------------------|
| $y - y_1 = m(x - x_1)$ |
|------------------------|

$$(x_1, y_1) = (4, -1) \quad m = \frac{2}{5}$$

$$y + 1 = \frac{2}{5}(x - 4)$$

$$y + 1 = \frac{2}{5}x - \frac{8}{5}$$

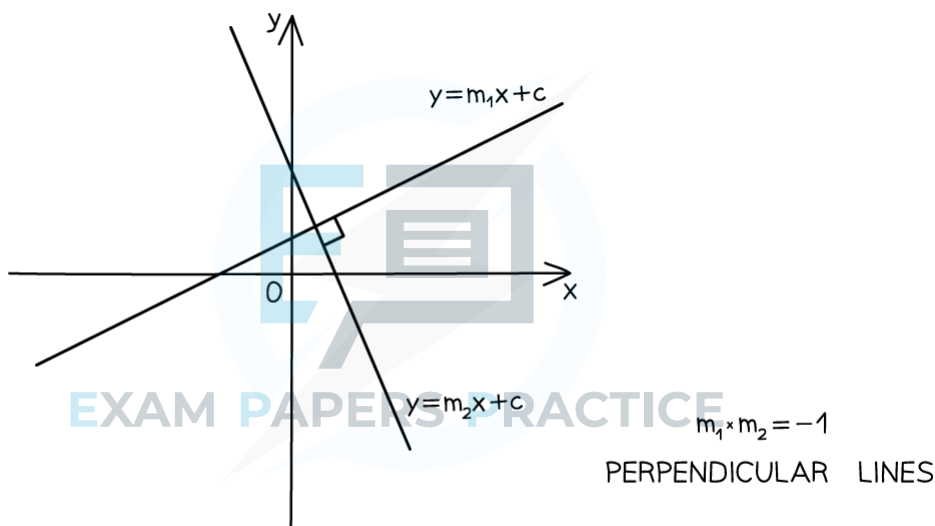
$$y = \frac{2}{5}x - \frac{13}{5}$$



## Perpendicular Lines

### How are the equations of perpendicular lines connected?

- **Perpendicular lines** intersect at right angles
- The gradients of two perpendicular lines are negative reciprocals
  - If the gradient of line  $l_1$  is  $m_1$  and gradient of line  $l_2$  is  $m_2$  then...
    - $m_1 \times m_2 = -1 \Rightarrow l_1 \text{ \& } l_2$  are perpendicular
    - $l_1 \text{ \& } l_2$  are perpendicular  $\Rightarrow m_1 \times m_2 = -1$
- To determine if two lines are perpendicular:
  - Rearrange into the gradient-intercept form  $y = mx + c$
  - Compare the coefficients of  $x$
  - If their product is -1 then they are perpendicular
- Be careful with horizontal and vertical lines
  - $x = p$  and  $y = q$  are perpendicular where  $p$  and  $q$  are constants





### ? Worked Example

The line  $l_1$  is given by the equation  $3x - 5y = 7$ .

The line  $l_2$  is given by the equation  $y = \frac{1}{4} - \frac{5}{3}x$ .

Determine whether  $l_1$  and  $l_2$  are perpendicular. Give a reason for your answer.

Rearrange  $l_1$  into  $y = mx + c$  form

$$5y = 3x - 7 \Rightarrow y = \frac{3}{5}x - \frac{7}{5}$$

Identify gradients

$$m_1 = \frac{3}{5} \quad m_2 = -\frac{5}{3}$$

$m_1 \times m_2 = -1 \Rightarrow$  Perpendicular lines

$$\frac{3}{5} \times -\frac{5}{3} = -1$$

$l_1$  and  $l_2$  are perpendicular as  $m_1 \times m_2 = -1$

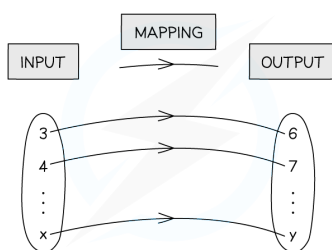
## 2.3 Functions Toolkit

### 2.3.1 Language of Functions

#### Language of Functions

##### What is a mapping?

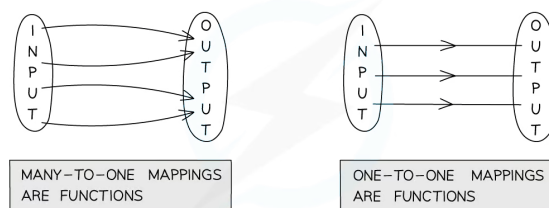
- A **mapping transforms** one set of values (**inputs**) into another set of values (**outputs**)
- Mappings can be:
  - **One-to-one**
    - Each input gets mapped to **exactly one unique** output
    - No two inputs are mapped to the same output
    - For example: A mapping that cubes the input
  - **Many-to-one**
    - Each input gets mapped to **exactly one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that squares the input
  - **One-to-many**
    - An input can be mapped to **more than one** output
    - No two inputs are mapped to the same output
    - For example: A mapping that gives the numbers which when squared equal the input
  - **Many-to-many**
    - An input can be mapped to **more than one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that gives the factors of the input



##### What is a function?

- A **function** is a mapping between two sets of numbers where **each input** gets mapped to **exactly one output**
  - The output does not need to be unique
- **One-to-one** and **many-to-one** mappings are functions
- A mapping is a function if its graph passes the **vertical line test**
  - Any **vertical line** will intersect with the graph **at most once**



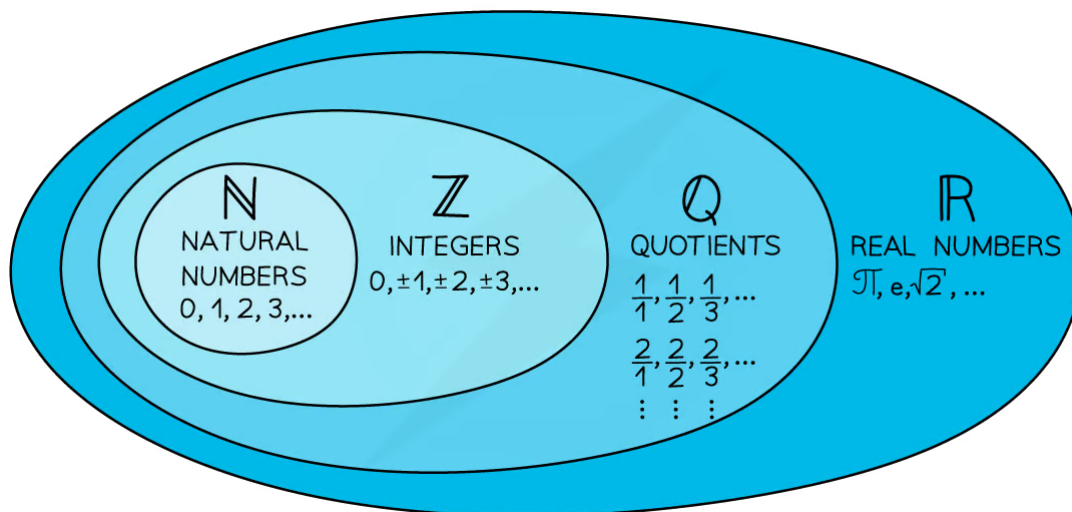


## What notation is used for functions?

- Functions are denoted using letters (such as  $f$ ,  $v$ ,  $g$ , etc)
  - A function is followed by a variable in a bracket
  - This shows the input for the function
  - The letter  $f$  is used most commonly for functions and will be used for the remainder of this revision note
- $f(x)$  represents an expression for the value of the function  $f$  when evaluated for the variable  $x$
- Function notation gets rid of the need for words which makes it **universal**
  - $f = 5$  when  $x = 2$  can simply be written as  $f(2) = 5$

## What are the domain and range of a function?

- The **domain** of a function is the set of values that are used as **inputs**
- A domain should be stated with a function
  - If a domain is not stated then it is assumed the domain is all the real values which would work as inputs for the function
  - Domains are expressed in terms of the input
    - $x \leq 2$
- The **range** of a function is the set of values that are given as **outputs**
  - The range depends on the domain
  - Ranges are expressed in terms of the output
    - $f(x) \geq 0$
- To graph a function we use the **inputs as the x-coordinates** and the **outputs as the y-coordinates**
  - $f(2) = 5$  corresponds to the coordinates (2, 5)
- Graphing the function can help you visualise the range
- Common sets of numbers have special symbols:
  - $\mathbb{R}$  represents all the real numbers that can be placed on a number line
    - $x \in \mathbb{R}$  means  $x$  is a real number
  - $\mathbb{Q}$  represents all the rational numbers  $\frac{a}{b}$  where  $a$  and  $b$  are integers and  $b \neq 0$
  - $\mathbb{Z}$  represents all the integers (positive, negative and zero)
    - $\mathbb{Z}^+$  represents positive integers
  - $\mathbb{N}$  represents the natural numbers (0, 1, 2, 3...)



### What are piecewise functions?

- **Piecewise functions** are defined by different functions depending on which interval the input is in

$$\text{E.g. } f(x) = \begin{cases} x+1 & x \leq 5 \\ 2x-4 & 5 < x < 10 \\ x^2 & 10 \leq x \leq 20 \end{cases}$$

- The region for the individual functions **cannot overlap**
- To evaluate a piecewise function for a particular value  $x = k$ 
  - Find which interval includes  $k$
  - Substitute  $x = k$  into the corresponding function
- The function **may or may not be continuous** at the ends of the intervals
  - In the example above the function is
    - continuous at  $x = 5$  as  $5 + 1 = 2(5) - 4$
    - not continuous at  $x = 10$  as  $2(10) - 4 \neq 10^2$



#### Exam Tip

- Questions may refer to "the largest possible domain"
  - This would usually be  $x \in \mathbb{R}$  unless  $\mathbb{N}$ ,  $\mathbb{Z}$  or  $\mathbb{Q}$  has already been stated
  - There are usually some exceptions
    - e.g. square roots;  $x \geq 0$  for a function involving  $\sqrt{x}$
    - e.g. reciprocal functions;  $x \neq 2$  for a function with denominator  $(x-2)$

**Worked Example**

For the function  $f(x) = x^3 + 1$ ,  $2 \leq x \leq 10$ :

a)

write down the value of  $f(7)$ .

Substitute  $x = 7$

$$f(7) = 7^3 + 1$$

$$f(7) = 344$$

b)

find the range of  $f(x)$ .

Find the values of  $x^3 + 1$  when  $2 \leq x \leq 10$

$$2 \leq x \leq 10$$

$$8 \leq x^3 \leq 1000$$

$$9 \leq x^3 + 1 \leq 1001$$

$$9 \leq f(x) \leq 1001$$



## 2.3.2 Composite & Inverse Functions

### Composite Functions

#### What is a composite function?

- A **composite function** is where a function is applied to another function
- A composite function can be denoted
  - $(f \circ g)(x)$
  - $fg(x)$
  - $f(g(x))$
- The order matters
  - $(f \circ g)(x)$  means:
    - First apply  $g$  to  $x$  to get  $g(x)$
    - Then apply  $f$  to the previous output to get  $f(g(x))$
    - Always start with the function **closest to the variable**
  - $(f \circ g)(x)$  is not usually equal to  $(g \circ f)(x)$

#### How do I find the domain and range of a composite function?

- The domain of  $f \circ g$  is the set of values of  $x$ ...
  - which are a **subset** of the **domain of  $g$**
  - which maps  $g$  to a value that is in the **domain of  $f$**
- The range of  $f \circ g$  is the set of values of  $x$ ...
  - which are a **subset** of the **range of  $f$**
  - found by **applying  $f$**  to the **range of  $g$**
- To find the **domain and range** of  $f \circ g$ 
  - First find the **range of  $g$**
  - **Restrict** these values to the values that are **within the domain of  $f$** 
    - The **domain** is the set of values that **produce the restricted range** of  $g$
    - The **range** is the set of values that are **produced using the restricted range** of  $g$  as the domain for  $f$
- For example: let  $f(x) = 2x + 1$ ,  $-5 \leq x \leq 5$  and  $g(x) = \sqrt{x}$ ,  $1 \leq x \leq 49$ 
  - The **range of  $g$**  is  $1 \leq g(x) \leq 7$ 
    - **Restricting** this to fit the **domain of  $f$**  results in  $1 \leq g(x) \leq 5$
  - The **domain** of  $f \circ g$  is therefore  $1 \leq x \leq 25$ 
    - These are the values of  $x$  which map to  $1 \leq g(x) \leq 5$
  - The **range** of  $f \circ g$  is therefore  $3 \leq (f \circ g)(x) \leq 11$ 
    - These are the values which  $f$  maps  $1 \leq g(x) \leq 5$  to



### Exam Tip

- Make sure you know what your GDC is capable of with regard to functions
  - You may be able to store individual functions and find composite functions and their values for particular inputs
  - You may be able to graph composite functions directly and so deduce their domain and range from the graph
- The link between the domains and ranges of a function and its inverse can act as a check for your solution
- $ff(x)$  is not the same as  $[f(x)]^2$





### ? Worked Example

Given  $f(x) = \sqrt{x+4}$  and  $g(x) = 3 + 2x$ :

a)

Write down the value of  $(g \circ f)(12)$ .

First apply function closest to input

$$(g \circ f)(12) = g(f(12))$$

$$f(12) = \sqrt{12+4} = \sqrt{16} = 4$$

$$g(4) = 3 + 2(4) = 11$$

$$(g \circ f)(12) = 11$$

b)

Write down an expression for  $(f \circ g)(x)$ .

First apply function closest to input

$$(f \circ g)(x) = f(g(x))$$

$$= f(3+2x)$$

$$= \sqrt{3+2x+4}$$

$$(f \circ g)(x) = \sqrt{7+2x}$$

c)

Write down an expression for  $(g \circ g)(x)$ .

$$(g \circ g)(x) = g(g(x))$$

$$= g(3+2x)$$

$$= 3 + 2(3+2x)$$

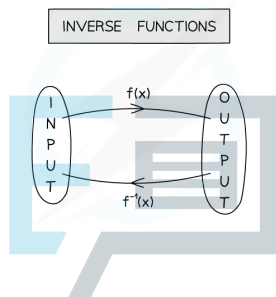
$$= 3 + 6 + 4x$$

$$(g \circ g)(x) = 9 + 4x$$

## Inverse Functions

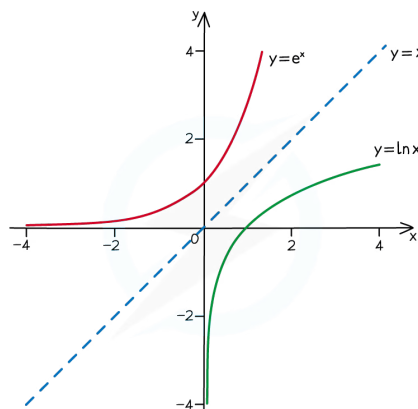
### What is an inverse function?

- Only **one-to-one** functions have inverses
- A function has an inverse if its graph passes the **horizontal line test**
  - Any **horizontal line** will intersect with the graph **at most once**
- The **identity function**  $\text{id}$  maps each value to itself
  - $\text{id}(x) = x$
- If  $f \circ g$  and  $g \circ f$  have the **same effect as the identity function** then  $f$  and  $g$  are **inverses**
- Given a function  $f(x)$  we denote the **inverse function** as  $f^{-1}(x)$
- An inverse function **reverses the effect** of a function
  - $f(2) = 5$  means  $f^{-1}(5) = 2$
- Inverse functions are used to solve equations
  - The solution of  $f(x) = 5$  is  $x = f^{-1}(5)$
- A composite function made of  $f$  and  $f^{-1}$  has the **same effect as the identity function**
  - $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$



### What are the connections between a function and its inverse function?

- The **domain of a function** becomes the **range of its inverse**
- The **range of a function** becomes the **domain of its inverse**
- The graph of  $y = f^{-1}(x)$  is a **reflection** of the graph  $y = f(x)$  in the line  $y = x$ 
  - Therefore solutions to  $f(x) = x$  or  $f^{-1}(x) = x$  will also be solutions to  $f(x) = f^{-1}(x)$ 
    - There could be other solutions to  $f(x) = f^{-1}(x)$  that don't lie on the line  $y = x$



### How do I find the inverse of a function?

- STEP 1: **Swap** the  $x$  and  $y$  in  $y = f(x)$ 
  - If  $y = f^{-1}(x)$  then  $x = f(y)$
- STEP 2: **Rearrange**  $x = f(y)$  to make  $y$  the subject
- Note this can be done in any order
  - Rearrange  $y = f(x)$  to make  $x$  the subject
  - Swap  $x$  and  $y$

### Can many-to-one functions ever have inverses?

- You can **restrict the domain** of a many-to-one function so that it has an inverse
- Choose a subset of the domain where the function is one-to-one
  - The inverse will be determined by the restricted domain
  - Note that a many-to-one function can **only** have an inverse if its domain is restricted first
- For **quadratics** – use the **vertex** as the upper or lower bound for the **restricted domain**
  - For  $f(x) = x^2$  restrict the domain so 0 is either the maximum or minimum value
    - For example:  $x \geq 0$  or  $x \leq 0$
  - For  $f(x) = a(x - h)^2 + k$  restrict the domain so  $h$  is either the maximum or minimum value
    - For example:  $x \geq h$  or  $x \leq h$
- For **trigonometric functions** – use part of a cycle as the **restricted domain**
  - For  $f(x) = \sin x$  restrict the domain to half a cycle between a maximum and a minimum
    - For example:  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
  - For  $f(x) = \cos x$  restrict the domain to half a cycle between maximum and a minimum
    - For example:  $0 \leq x \leq \pi$
  - For  $f(x) = \tan x$  restrict the domain to one cycle between two asymptotes
    - For example:  $-\frac{\pi}{2} < x < \frac{\pi}{2}$

### How do I find the inverse function after restricting the domain?

- The range of the inverse is the same as the restricted domain of the original function
- The inverse function is determined by the restricted domain
  - Restricting the domain differently will change the inverse function
- Use the range of the inverse to help find the inverse function
  - Restricting the domain of  $f(x) = x^2$  to  $x \geq 0$  means the range of the inverse is  $f^{-1}(x) \geq 0$ 
    - Therefore  $f^{-1}(x) = \sqrt{x}$
  - Restricting the domain of  $f(x) = x^2$  to  $x \leq 0$  means the range of the inverse is  $f^{-1}(x) \leq 0$ 
    - Therefore  $f^{-1}(x) = -\sqrt{x}$





### Exam Tip

- Remember that an inverse function is a reflection of the original function in the line  $y = x$ 
  - Use your GDC to plot the function and its inverse on the same graph to visually check this
- $f^{-1}(x)$  is not the same as  $\frac{1}{f(x)}$





### ? Worked Example

The function  $f(x) = (x-2)^2 + 5$ ,  $x \leq m$  has an inverse.

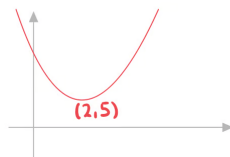
a)

Write down the largest possible value of  $m$ .

Sketch  $y=f(x)$

The graph is one-to-one  
for  $x \leq 2$

$$m = 2$$



b)

Find the inverse of  $f(x)$ .

Let  $y=f^{-1}(x)$  and rearrange  $x=f(y)$

$$x = (y-2)^2 + 5$$

$$x-5 = (y-2)^2$$

$$\pm\sqrt{x-5} = y-2$$

$$2 \pm \sqrt{x-5} = y$$

Range of  $f^{-1}$  is the domain of  $f$

$$f^{-1}(x) \leq 2 \quad \therefore y = 2 - \sqrt{x-5}$$

$$f^{-1}(x) = 2 - \sqrt{x-5}$$

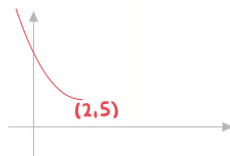
c)

Find the domain of  $f^{-1}(x)$ .

Domain of  $f^{-1}$  is the range of  $f$

Sketch  $y=f(x)$  to  
see range

For  $x \leq 2$ ,  $f(x) \geq 5$



$$\text{Domain of } f^{-1} : x \geq 5$$

d)

Find the value of  $k$  such that  $f(k) = 9$ .



Use inverse  $f(a) = b \Leftrightarrow a = f^{-1}(b)$

$$k = f^{-1}(9) = 2 - \sqrt{9-5}$$

$$k = 0$$





## 2.3.3 Symmetry of Functions

### Odd & Even Functions

#### What are odd functions?

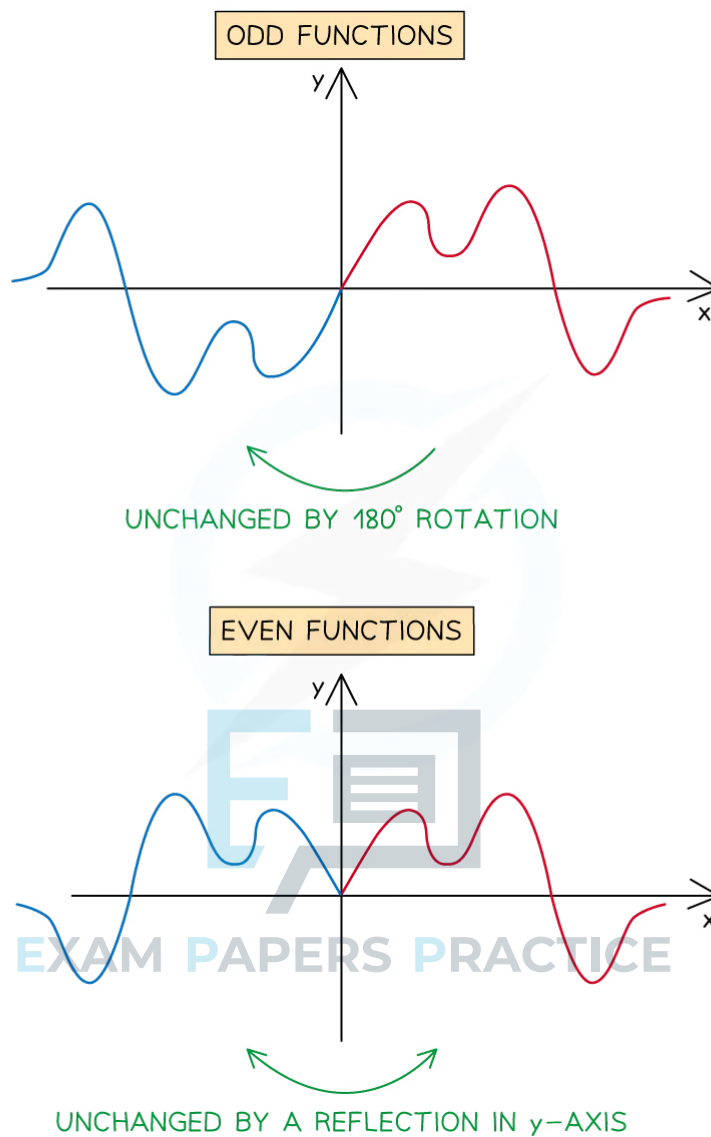
- A function  $f(x)$  is called **odd** if
  - $f(-x) = -f(x)$  for all values of  $x$
- Examples of odd functions include:
  - Power functions with **odd powers**:  $x^{2n+1}$  where  $n \in \mathbb{Z}$   
For example:  $(-x)^3 = -x^3$
  - Some **trig functions**:  $\sin x$ ,  $\operatorname{cosec} x$ ,  $\tan x$ ,  $\cot x$   
For example:  $\sin(-x) = -\sin x$
  - **Linear combinations** of odd functions  
For example:  $f(x) = 3x^5 - 4\sin x + \frac{6}{x}$

#### What are even functions?

- A function  $f(x)$  is called **even** if
  - $f(-x) = f(x)$  for all values of  $x$
- Examples of even functions include:
  - Power functions with **even powers**:  $x^{2n}$  where  $n \in \mathbb{Z}$   
For example:  $(-x)^4 = x^4$
  - Some **trig functions**:  $\cos x$ ,  $\sec x$   
For example:  $\cos(-x) = \cos x$
  - **Modulus function**:  $|x|$
  - **Linear combinations** of even functions  
For example:  $f(x) = 7x^6 + 3|x| - 8\cos x$

#### What are the symmetries of graphs of odd & even functions?

- The graph of an **odd** function has **rotational symmetry**
  - The graph is unchanged by a **180° rotation** about the origin
- The graph of an **even** function has **reflective symmetry**
  - The graph is unchanged by a **reflection** in the **y-axis**



#### Exam Tip

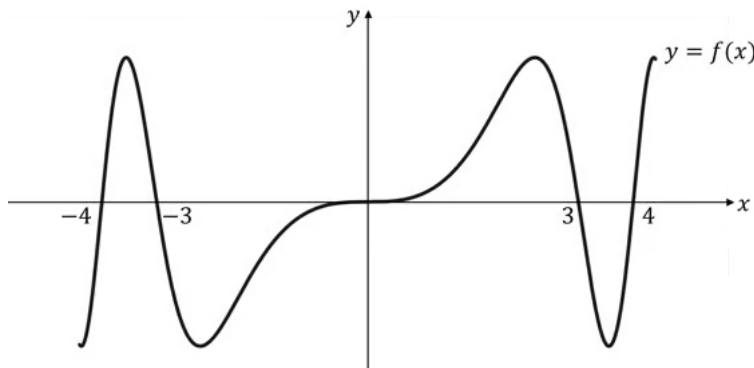
- Turn your GDC upside down for a quick visual check for an odd function!
  - Ignoring axes, etc, if the graph looks exactly the same both ways, it's odd



## ? Worked Example

a)

The graph  $y = f(x)$  is shown below. State, with a reason, whether the function  $f$  is odd, even or neither.



$f$  is an odd function as its graph has rotational symmetry - it is unchanged by a  $180^\circ$  rotation about the origin.

b)

Use algebra to show that  $g(x) = x^3 \sin(x) + 5 \cos(x^5)$  is an even function.

$g$  is even if  $g(-x) = g(x)$  for all  $x$

$$\begin{aligned} g(-x) &= (-x)^3 \sin(-x) + 5 \cos((-x)^5) \\ &= (-x^3)(-\sin(x)) + 5 \cos(-x^5) \quad \leftarrow x^3, x^5, \sin x \text{ are odd} \\ &= x^3 \sin(x) + 5 \cos(x^5) \quad \leftarrow \cos x \text{ is even} \\ &= g(x) \end{aligned}$$

$g$  is even as  $g(-x) = g(x)$  for all  $x$

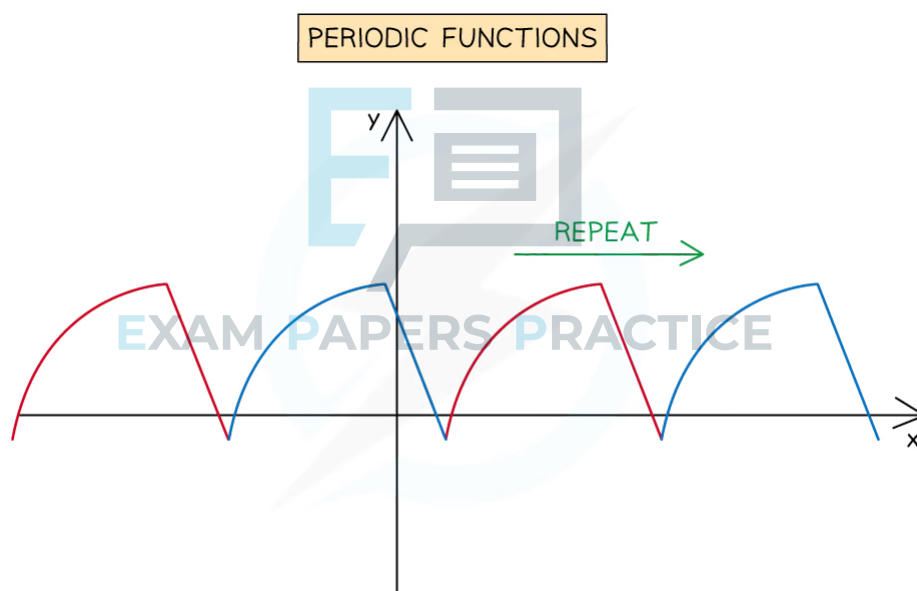
## Periodic Functions

### What are periodic functions?

- A function  $f(x)$  is called **periodic**, with **period  $k$** , if
  - $f(x + k) = f(x)$  for all values of  $x$
- Examples of periodic functions include:
  - $\sin x$  &  $\cos x$ : The period is  $2\pi$  or  $360^\circ$
  - $\tan x$ : The period is  $\pi$  or  $180^\circ$
  - **Linear combinations** of periodic functions with the **same period**
    - For example:  $f(x) = 2\sin(3x) - 5\cos(3x + 2)$

### What are the symmetries of graphs of periodic functions?

- The graph of a **periodic** function has **translational symmetry**
  - The graph is unchanged by **translations** that are **integer multiples of**  $\begin{pmatrix} k \\ 0 \end{pmatrix}$
  - This means that the graph appears to **repeat** the same section (cycle) infinitely



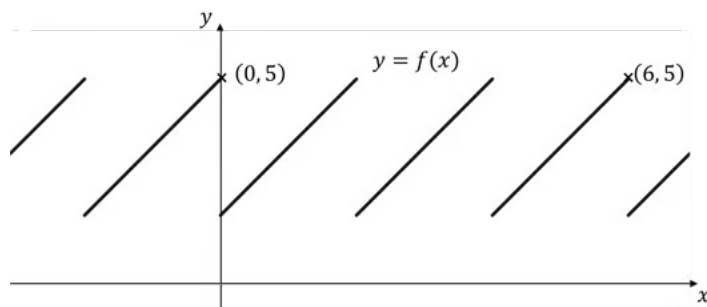
#### Exam Tip

- There may be several intersections between the graph of a periodic function and another function
  - i.e. Equations may have several solutions so only answers within a certain range of  $x$ -values may be required
    - e.g. Solve  $\tan x = \sqrt{3}$  for  $0^\circ \leq x \leq 360^\circ$
    - $x = 60^\circ, 240^\circ$
  - Alternatively you may have to write **all** solutions in a general form
    - e.g.  $x = 60(3k + 1)^\circ$ ,  $k = 0, \pm 1, \pm 2, \dots$



### Worked Example

The graph  $y = f(x)$  is shown below. Given that  $f$  is periodic, write down the period.



Period is the length of the interval of a single cycle

Between  $x=0$  and  $x=6$  there are 3 cycles

$$\text{Period} = \frac{6-0}{3}$$

$$\text{Period} = 2$$





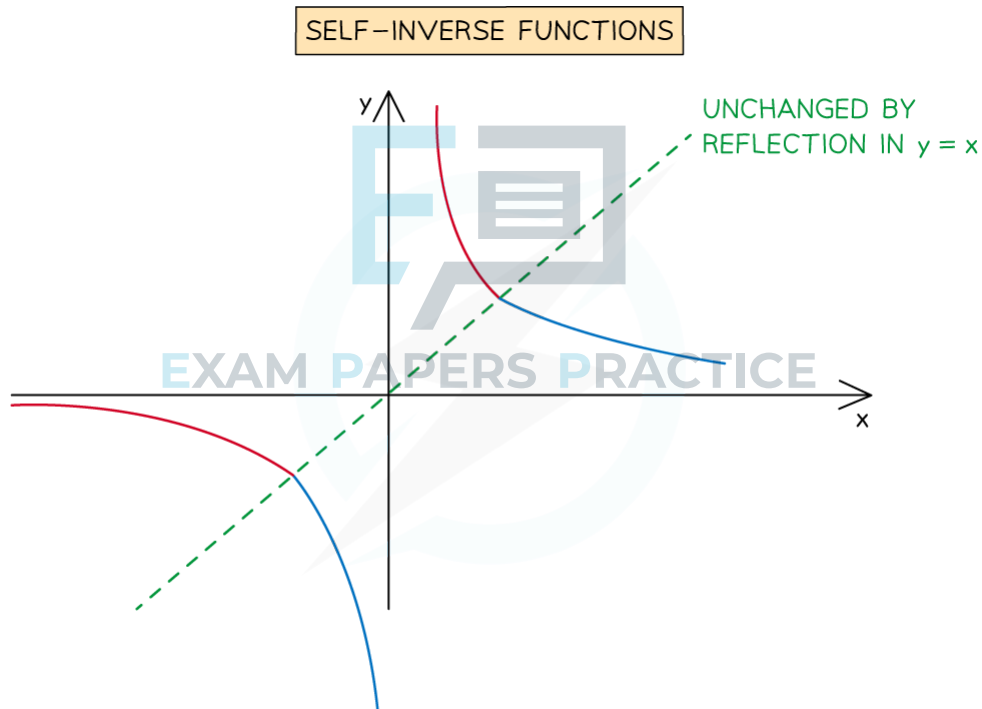
## Self-Inverse Functions

### What are self-inverse functions?

- A function  $f(x)$  is called **self-inverse** if
  - $(f \circ f)(x) = x$  for all values of  $x$
  - $f^{-1}(x) = f(x)$
- Examples of self-inverse functions include:
  - **Identity function:**  $f(x) = x$
  - **Reciprocal function:**  $f(x) = \frac{1}{x}$
  - **Linear functions** with a **gradient of -1**:  $f(x) = -x + c$

### What are the symmetries of graphs of self-inverse functions?

- The graph of a **self-inverse** function has **reflective symmetry**
  - The graph is unchanged by a **reflection** in the line  $y = x$





## Exam Tip

- If your expression for  $f^{-1}(x)$  is not the same as the expression for  $f(x)$  you can check their equivalence by plotting both on your GDC
  - If equivalent the graphs will sit on top of one another and appear as one
  - This will indicate if you have made an error in your algebra, before trying to simplify/rewrite to make the two expressions identical
- It is sometimes easier to consider self inverse functions geometrically rather than algebraically



## Worked Example

Use algebra to show the function defined by  $f(x) = \frac{7x-5}{x-7}$ ,  $x \neq 7$  is self-inverse.

Method 1:  $f^{-1}(x)$

Let  $y = f^{-1}(x)$  so  $x = f(y)$

$$x = \frac{7y-5}{y-7}$$

$$(y-7)x = 7y-5$$

$$xy - 7x = 7y - 5$$

$$xy - 7y = 7x - 5$$

Isolate y on one side

$$(x-7)y = 7x-5$$

$$y = \frac{7x-5}{x-7}$$

$$f^{-1}(x) = \frac{7x-5}{x-7} = f(x)$$

$\therefore f$  is self-inverse

Method 2:  $(f \circ f)(x)$

$$(f \circ f)(x) = f(f(x))$$

$$f(f(x)) = \frac{7f(x)-5}{f(x)-7}$$

$$= \frac{7\left(\frac{7x-5}{x-7}\right)-5}{\frac{7x-5}{x-7}-7}$$

$$= \frac{7(7x-5)-5(x-7)}{7x-5-7(x-7)}$$

$$= \frac{49x-35-5x+35}{7x-5-7x+49}$$

$$= \frac{44x}{44}$$

$$(f \circ f)(x) = x$$

$\therefore f$  is self-inverse



## 2.3.4 Graphing Functions

### Graphing Functions

#### How do I graph the function $y = f(x)$ ?

- A point  $(a, b)$  lies on the graph  $y = f(x)$  if  $f(a) = b$
- The **horizontal axis** is used for the **domain**
- The **vertical axis** is used for the **range**
- You will be able to graph some functions by hand
- For some functions you will need to use your GDC
- You might be asked to graph the **sum** or **difference** of two functions
  - Use your GDC to graph  $y = f(x) + g(x)$  or  $y = f(x) - g(x)$
  - Just type the functions into the graphing mode

#### What is the difference between “draw” and “sketch”?

- If asked to sketch you should:
  - Show the general shape
  - Label any key points such as the intersections with the axes
  - Label the axes
- If asked to draw you should:
  - Use a pencil and ruler
  - Draw to scale
  - Plot any points **accurately**
  - Join points with a straight line or smooth curve
  - Label any key points such as the intersections with the axes
  - Label the axes

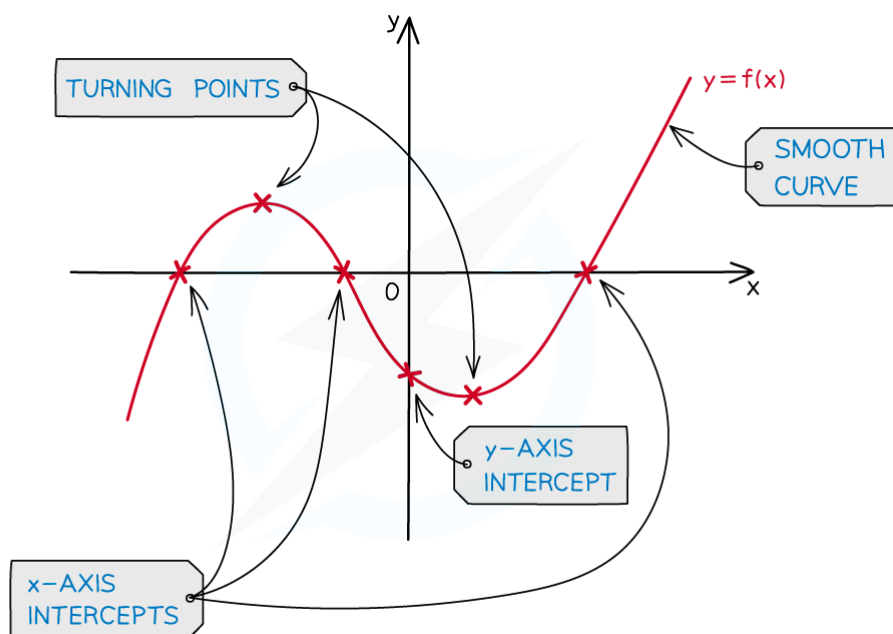
#### How can my GDC help me sketch/draw a graph?

- You use your GDC to plot the graph
  - Check the scales on the graph to make sure you see the full shape
- Use your GDC to find any key points
- Use your GDC to check specific points to help you plot the graph

## Key Features of Graphs

### What are the key features of graphs?

- You should be familiar with the following key features and know how to use your GDC to find them
- Local minimums/maximums
  - These are points where the graph has a minimum/maximum for a small region
  - They are also called **turning points**  
This is where the graph changes its direction between upwards and downwards directions
  - A graph can have multiple local minimums/maximums
  - A local minimum/maximum is not necessarily the minimum/maximum of the whole graph  
This would be called the **global** minimum/maximum
  - For quadratic graphs the minimum/maximum is called the **vertex**
- Intercepts
  - y – intercepts are where the graph crosses the y-axis  
At these points  $x = 0$
  - x – intercepts are where the graph crosses the x-axis  
At these points  $y = 0$   
These points are also called the **zeros of the function** or **roots of the equation**
- Symmetry
  - Some graphs have lines of symmetry  
A quadratic will have a vertical line of symmetry
- Asymptotes
  - These are lines which the graph will get closer to but not cross
  - These can be horizontal or vertical  
Exponential graphs have horizontal asymptotes  
Graphs of variables which vary inversely can have vertical and horizontal asymptotes



### Exam Tip

- Most GDC makes/models will not plot/show asymptotes just from inputting a function
  - Add the asymptotes as additional graphs for your GDC to plot
  - You can then check the equations of your asymptotes visually
  - You may have to zoom in or change the viewing window options to confirm an asymptote
- Even if using your GDC to plot graphs and solve problems sketching them as part of your working is good exam technique
  - Label the key features of the graph and anything else relevant to the question on your sketch



## ? Worked Example

Two functions are defined by

$$f(x) = x^2 - 4x - 5 \text{ and } g(x) = 2 + \frac{1}{x+1}.$$

a)

Draw the graph  $y = f(x)$ .

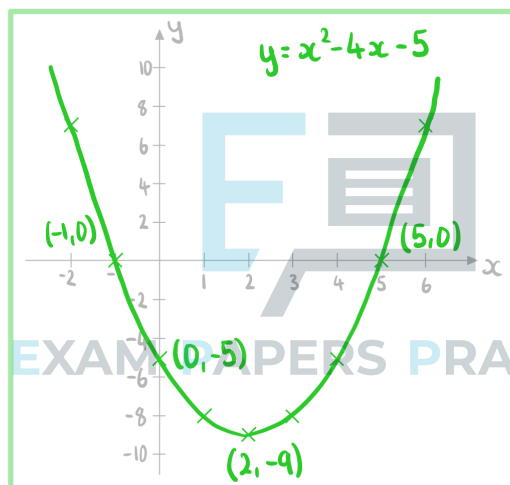
Draw means accurately

Use GDC to find vertex, roots and y-intercepts

Vertex =  $(2, -9)$

Roots =  $(-1, 0)$  and  $(5, 0)$

y-intercept =  $(0, -5)$



b)

Sketch the graph  $y = g(x)$ .



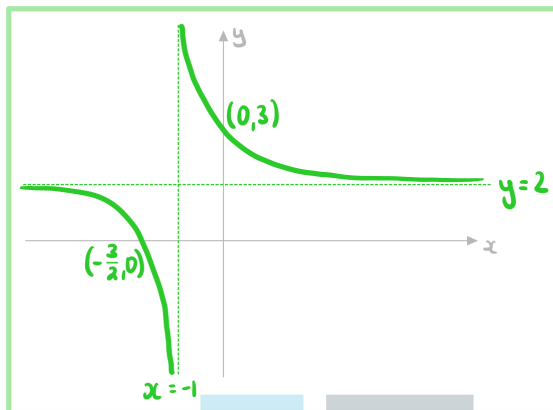
Sketch means rough but showing key points

Use GDC to find  $x$  and  $y$ -intercepts and asymptotes

$$x\text{-intercept} = \left(-\frac{3}{2}, 0\right)$$

$$y\text{-intercept} = (0, 3)$$

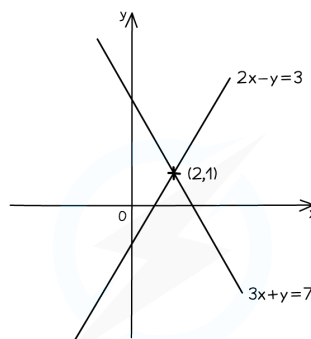
Asymptotes :  $x = -1$  and  $y = 2$



## Intersecting Graphs

### How do I find where two graphs intersect?

- Plot both graphs on your GDC
- Use the intersect function to find the intersections
- Check if there is more than one point of intersection



• LINES INTERSECT AT (2,1)  
 • SOLVING  $2x - y = 3$  AND  $3x + y = 7$   
 SIMULTANEOUSLY IS  $x = 2$ ,  $y = 1$

### How can I use graphs to solve equations?

- One method to solve equations is to use graphs
- To solve  $f(x) = a$ 
  - Plot the two graphs  $y = f(x)$  and  $y = a$  on your GDC
  - Find the points of intersections
  - The **x-coordinates** are the **solutions** of the equation
- To solve  $f(x) = g(x)$ 
  - Plot the two graphs  $y = f(x)$  and  $y = g(x)$  on your GDC
  - Find the points of intersections
  - The **x-coordinates** are the **solutions** of the equation
- Using graphs makes it easier to see **how many solutions** an equation will have



#### Exam Tip

- You can use graphs to solve equations
  - Questions will not necessarily ask for a drawing/sketch or make reference to graphs
  - Use your GDC to plot the equations and find the intersections between the graphs





## ? Worked Example

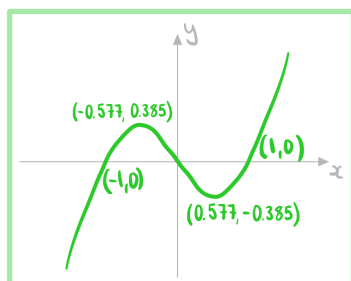
Two functions are defined by

$$f(x) = x^3 - x \text{ and } g(x) = \frac{4}{x}.$$

a)

Sketch the graph  $y = f(x)$ .

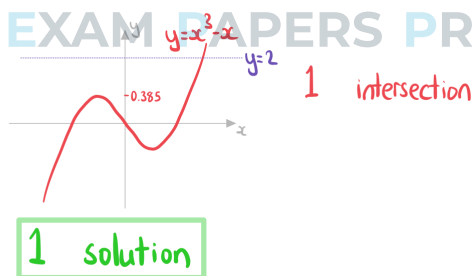
Use GDC to find max, min, intercepts



b)

Write down the number of real solutions to the equation  $x^3 - x = 2$ .

Identify the number of intersections between  
 $y = x^3 - x$  and  $y = 2$

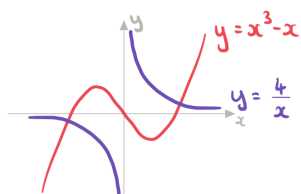


c)

Find the coordinates of the points where  $y = f(x)$  and  $y = g(x)$  intersect.



Use GDC to sketch both graphs



$$(-1.60, -2.50) \text{ and } (1.60, 2.50)$$

d)

Write down the solutions to the equation  $x^3 - x = \frac{4}{x}$ .

Solutions to  $x^3 - x = \frac{4}{x}$  are the  $x$  coordinates of the points of intersection.

$$x = -1.60 \text{ and } x = 1.60$$



## 2.4 Other Functions & Graphs

### 2.4.1 Exponential & Logarithmic Functions

#### Exponential Functions & Graphs

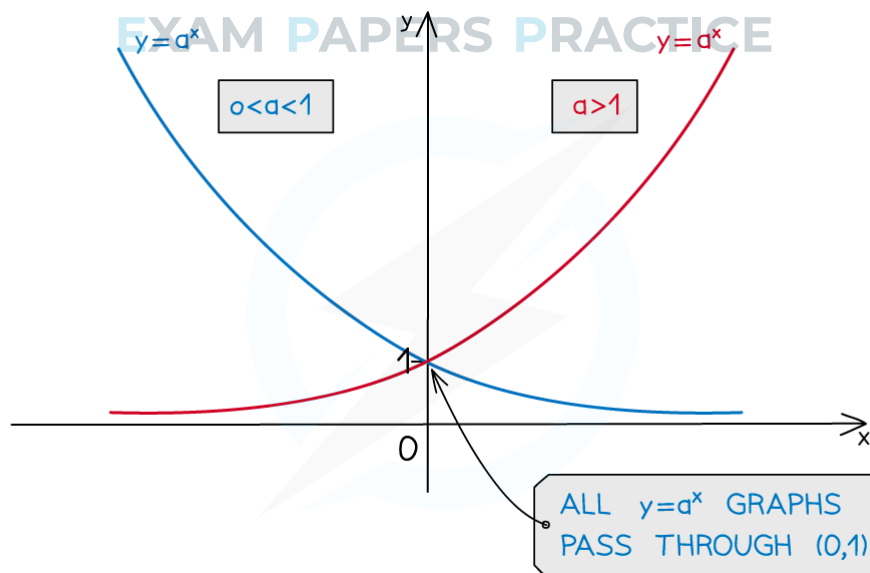
##### What is an exponential function?

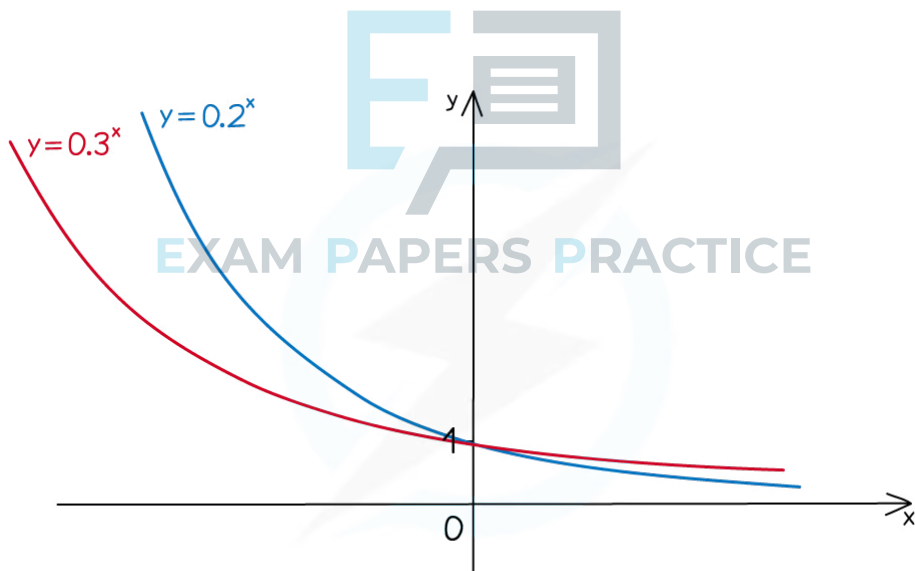
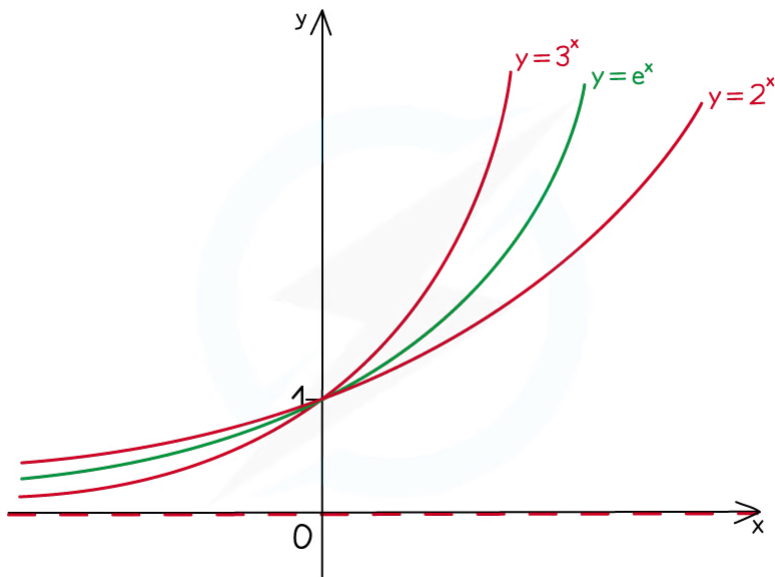
- An **exponential function** is defined by  $f(x) = a^x$ ,  $a > 0$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all positive real values**
- An important exponential function is  $f(x) = e^x$ 
  - Where  $e$  is the mathematical constant 2.718...
- Any exponential function can be written using  $e$ 
  - $a^x = e^{x \ln a}$

This is given in the **formula booklet**

##### What are the key features of exponential graphs?

- The graphs have a **y-intercept** at  $(0, 1)$
- The graphs **do not have any roots**
- The graphs have a **horizontal asymptote** at the x-axis:  $y = 0$ 
  - For  $a > 1$  this is the **limiting value** when  $x$  tends to **negative infinity**
  - For  $0 < a < 1$  this is the **limiting value** when  $x$  tends to **positive infinity**
- The graphs **do not have any minimum or maximum points**





## Logarithmic Functions & Graphs

### What is a logarithmic function?

- A **logarithmic function** is of the form  $f(x) = \log_a x$ ,  $x > 0$
- Its **domain** is the set of all **positive real values**
  - You can't take a log of zero or a negative number
- Its **range** is set of **all real values**
- $\log_a x$  and  $a^x$  are **inverse** functions
- An important logarithmic function is  $f(x) = \ln x$ 
  - This is the natural logarithmic function  $\ln x \equiv \log_e x$
  - This is the inverse of  $e^x$   
 $\ln e^x = x$  and  $e^{\ln x} = x$
- Any logarithmic function can be written using  $\ln$ 
  - $\log_a x = \frac{\ln x}{\ln a}$  using the change of base formula

### What are the key features of logarithmic graphs?

- The graphs **do not have a y-intercept**
- The graphs have **one root** at  $(1, 0)$
- The graphs have a **vertical asymptote** at the y-axis:  $x = 0$
- The graphs **do not have any minimum or maximum points**



### ? Worked Example

The function  $f$  is defined by  $f(x) = \log_5 x$  for  $x > 0$ .

a)

Write down the inverse of  $f$ . Give your answer in the form  $e^{g(x)}$ .

Formula booklet

Exponents & logarithms

$$a^x = b \Leftrightarrow x = \log_a b$$

$$a > 0, b > 0, a \neq 1$$

$$x = \log_5 y \Leftrightarrow y = 5^x$$

Formula booklet

Exponential & logarithmic functions

$$a^x = e^{x \ln a}$$

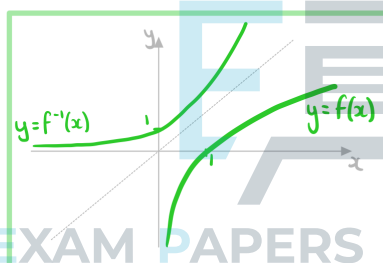
$$5^x = e^{x \ln 5}$$

$$f^{-1}(x) = e^{x \ln 5}$$

b)

Sketch the graphs of  $f$  and its inverse on the same set of axes.

$f$  and  $f^{-1}$  are reflections in line  $y=x$



## 2.4.2 Solving Equations

### Solving Equations Analytically

**How can I solve equations analytically where the unknown appears only once?**

- These equations can be **solved by rearranging**
- For **one-to-one functions** you can just apply the **inverse**
  - Addition and subtraction are inverses
$$y = x + k \Leftrightarrow x = y - k$$
  - Multiplication and division are inverses
$$y = kx \Leftrightarrow x = \frac{y}{k}$$
  - Taking the reciprocal is a self-inverse
$$y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}$$
  - Odd powers and roots are inverses
$$y = x^n \Leftrightarrow x = \sqrt[n]{y}$$

$$y = x^n \Leftrightarrow x = y^{\frac{1}{n}}$$
  - Exponentials and logarithms are inverses
$$y = a^x \Leftrightarrow x = \log_a y$$

$$y = e^x \Leftrightarrow x = \ln y$$
- For **many-to-one functions** you will need to use your knowledge of the functions to find the **other solutions**
  - Even powers lead to positive and negative solutions
$$y = x^n \Leftrightarrow x = \pm \sqrt[n]{y}$$
  - Modulus functions lead to positive and negative solutions
$$y = |x| \Leftrightarrow x = \pm y$$
  - Trigonometric functions lead to infinite solutions using their symmetries
$$y = \sin x \Leftrightarrow x = 2k\pi + \arcsin y \text{ or } x = (1 + 2k)\pi - \arcsin y$$

$$y = \cos x \Leftrightarrow x = 2k\pi \pm \arccos y$$

$$y = \tan x \Leftrightarrow x = k\pi + \arctan y$$
- Take care when you apply **many-to-one functions** to **both sides** of an equation as this can create **additional solutions** which are incorrect
  - For example: squaring both sides
$$x + 1 = 3 \text{ has one solution } x = 2$$

$$(x + 1)^2 = 3^2 \text{ has two solutions } x = 2 \text{ and } x = -4$$
- Always **check your solutions** by substituting back into the **original equation**

**How can I solve equations analytically where the unknown appears more than once?**

- Sometimes it is possible to **simplify expressions** to make the **unknown appear only once**
- **Collect all terms** involving  $x$  on **one side** and try to simplify into one term
  - For **exponents** use



$$a^{f(x)} \times a^{g(x)} = a^{f(x)+g(x)}$$

$$\frac{a^{f(x)}}{a^{g(x)}} = a^{f(x)-g(x)}$$

$$(a^{f(x)})^{g(x)} = a^{f(x) \times g(x)}$$

$$a^{f(x)} = e^{f(x) \ln a}$$

- For **logarithms** use

$$\log_a f(x) + \log_a g(x) = \log_a (f(x) \times g(x))$$

$$\log_a f(x) - \log_a g(x) = \log_a \left( \frac{f(x)}{g(x)} \right)$$

$$n \log_a f(x) = \log_a (f(x))^n$$

### How can I solve equations analytically when the equation can't be simplified?

- Sometimes it is **not possible to simplify** equations
- Most of these equations **cannot be solved analytically**
- A **special case** that can be solved is where the equation can be **transformed into a quadratic** using a substitution
  - These will have **three terms** and involve the same type of function
- **Identify the suitable substitution** by considering which **function is a square of another**
  - For example: the following can be transformed into  $2y^2 + 3y - 4 = 0$ 

$$2x^4 + 3x^2 - 4 = 0 \text{ using } y = x^2$$

$$2x + 3\sqrt{x} - 4 = 0 \text{ using } y = \sqrt{x}$$

$$\frac{2}{x^6} + \frac{3}{x^3} - 4 = 0 \text{ using } y = \frac{1}{x^3}$$

$$2e^{2x} + 3e^x - 4 = 0 \text{ using } y = e^x$$

$$2 \times 25^x + 3 \times 5^x - 4 = 0 \text{ using } y = 5^x$$

$$2^{2x+1} + 3 \times 2^x - 4 = 0 \text{ using } y = 2^x$$

$$2(x^3 - 1)^2 + 3(x^3 - 1) - 4 = 0 \text{ using } y = x^3 - 1$$
- To **solve**:
  - Make the **substitution**  $y = f(x)$
  - **Solve the quadratic equation**  $ay^2 + by + c = 0$  to get  $y_1$  &  $y_2$
  - **Solve**  $f(x) = y_1$  and  $f(x) = y_2$

Note that some equations might have **zero or several solutions**

### Can I divide both sides of an equation by an expression?

- When dividing by an expression you must consider whether the **expression could be zero**
- Dividing by an expression that could be zero could result in you **losing solutions to the original equation**
  - For example:  $(x+1)(2x-1) = 3(x+1)$   
If you divide both sides by  $(x+1)$  you get  $2x-1 = 3$  which gives  $x = 2$   
However  $x = -1$  is also a solution to the original equation
- To ensure you **do not lose solutions** you can:
  - **Split the equation into two equations**  
One where the dividing expression equals zero:  $x+1 = 0$





One where the equation has been divided by the expression:  $2x - 1 = 3$

- **Make the equation equal zero and factorise**

$$(x + 1)(2x - 1) - 3(x + 1) = 0$$

$$(x + 1)(2x - 1 - 3) = 0 \text{ which gives } (x + 1)(2x - 4) = 0$$

Set each factor equal to zero and solve:  $x + 1 = 0$  and  $2x - 4 = 0$



### Exam Tip

- A common mistake that students make in exams is applying functions to each term rather than to each side
  - For example: Starting with the equation  $\ln x + \ln(x - 1) = 5$  it would be incorrect to write  $e^{\ln x} + e^{\ln(x - 1)} = e^5$  or  $x + (x - 1) = e^5$
  - Instead it would be correct to write  $e^{\ln x + \ln(x - 1)} = e^5$  and then simplify from there





### Worked Example

Find the exact solutions for the following equations:

a)

$$5 - 2\log_4 x = 0.$$

Rearrange using inverse functions

$$5 - 2\log_4 x = 0$$

$$2\log_4 x = 5$$

$$\log_4 x = \frac{5}{2}$$

$$x = 4^{\frac{5}{2}}$$

$$x = (\sqrt{4})^5$$

$$x = 32$$

$$y = x - k \Leftrightarrow x = y + k$$

$$y = kx \Leftrightarrow x = \frac{y}{k}$$

$$y = \log_a x \Leftrightarrow x = a^y$$

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

b)

$$x = \sqrt{x+2}.$$

Square both sides (Many-to-one function)

$$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0 \Rightarrow x = 2 \text{ or } x = -1$$

Check whether each solution is valid

$$x = 2: \text{ LHS} = 2 \quad \text{RHS} = \sqrt{2+2} = 2 \quad \checkmark$$

$$x = -1: \text{ LHS} = -1 \quad \text{RHS} = \sqrt{-1+2} = 1 \quad \times$$

$$x = 2$$

c)

$$e^{2x} - 4e^x - 5 = 0.$$



Notice  $e^{2x} = (e^x)^2$ , let  $y = e^x$

$$y^2 - 4y - 5 = 0 \Rightarrow (y+1)(y-5) = 0$$

$$y = -1 \text{ or } y = 5$$

Solve using  $y = e^x$

$e^x = -1$  has no solutions as  $e^x > 0$

$$e^x = 5 \quad \therefore x = \ln 5$$

$$x = \ln 5$$



## Solving Equations Graphically

### How can I solve equations graphically?

- To solve  $f(x) = g(x)$ 
  - One method is to **draw the graphs**  $y = f(x)$  and  $y = g(x)$   
The **solutions** are the **x-coordinates** of the points of **intersection**
  - Another method is to **draw the graph**  $y = f(x) - g(x)$  or  $y = g(x) - f(x)$   
The **solutions** are the **roots (zeros)** of this graph  
This method is sometimes quicker as it involves **drawing only one graph**

### Why do I need to solve equations graphically?

- Some equations **cannot be solved analytically**
  - **Polynomials** of degree higher than 4  
 $x^5 - x + 1 = 0$
  - Equations involving **different types of functions**  
 $e^x = x^2$



#### Exam Tip

- On a calculator paper you are allowed to solve equations using your GDC unless the question asks for an algebraic method
- If your answer needs to be an exact value then you might need to solve analytically to get the exact value

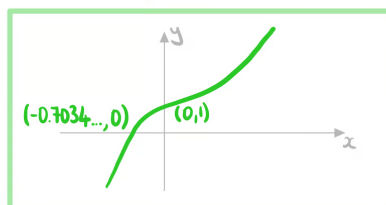


#### Worked Example

a)

Sketch the graph  $y = e^x - x^2$ .

Sketch using GDC



b)

Hence find the solution to  $e^x = x^2$ .

$$e^x = x^2 \quad \text{when} \quad e^x - x^2 = 0$$

Solution is the x-intercept of  $y = e^x - x^2$

$$x = -0.703 \text{ (3sf)}$$

## 2.4.3 Modelling with Functions

### Modelling with Functions

#### What is a mathematical model?

- A **mathematical model** simplifies a real-world situation so it can be described using mathematics
  - The model can then be used to make predictions
- **Assumptions** about the situation are made in order to simplify the mathematics
- Models can be **refined** (improved) if further information is available or if the model is compared to real-world data

#### How do I set up the model?

- The question could:
    - give you the equation of the model
    - tell you about the relationship
      - It might say the relationship is linear, quadratic, etc
    - ask you to suggest a **suitable model**
      - Use your knowledge of **each model**
      - E.g. if it is compound interest then an exponential model is the most appropriate
  - You may have to determine a **reasonable domain**
    - Consider real-life context
      - E.g. if dealing with hours in a day then
      - E.g. if dealing with physical quantities (such as length) then
    - Consider the **possible ranges**
      - If the outcome cannot be negative then you want to choose a domain which corresponds to a range with no negative values
- Sketching the graph** is helpful to determine a suitable domain

#### Which models might I need to use?

- You could be given any model and be expected to use it
- Common models include:
  - **Linear**
    - Arithmetic sequences
    - Linear regression
  - **Quadratic**
    - Projectile motion
    - The height of a cable supporting a bridge
    - Profit
  - **Exponential**
    - Geometric sequences
    - Exponential growth and decay
    - Compound interest
  - **Logarithmic**
    - Richter scale for the magnitude of earthquakes
  - **Rational**

Temperature of a cup of coffee

- **Trigonometric**

The depth of a tide

## How do I use a model?

- You can use a model by substituting in values for the variable to **estimate outputs**
  - For example: Let  $h(t)$  be the height of a football  $t$  seconds after being kicked  
 $h(3)$  will be an estimate for the height of the ball 3 seconds after being kicked
- Given an **output** you can **form an equation** with the model to **estimate the input**
  - For example: Let  $P(n)$  be the profit made by selling  $n$  items  
Solving  $P(n) = 100$  will give you an estimate for the number of items needing to be sold to make a profit of 100
- If your variable is **time** then substituting  $t = 0$  will give you the **initial value** according to the model
- Fully understand the **units for the variables**
  - If the units of  $P$  are measured in **thousand dollars** then  $P = 3$  represents \$3000
- Look out for **key words** such as:
  - Initially
  - Minimum/maximum
  - Limiting value

## What do I do if some of the parameters are unknown?

- A general method is to **form equations** by substituting in given values
  - You can form **multiple equations** and **solve them simultaneously** using your GDC
  - This method **works for all models**
- The **initial value** is the value of the function when the variable is 0
  - This is **normally one of the parameters** in the equation of the model



### ? Worked Example

The temperature,  $T^{\circ}\text{C}$ , of a cup of coffee is monitored. Initially the temperature is  $80^{\circ}\text{C}$  and 5 minutes later it is  $40^{\circ}\text{C}$ . It is suggested that the temperature follows the model:

$$T(t) = Ae^{kt} + 16, \quad t \geq 0.$$

where  $t$  is the time, in minutes, after the coffee has been made.

a)

State the value of  $A$ .

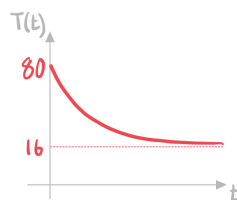
Initially temperature is  $80^{\circ}\text{C}$

$$T(0) = 80$$

$$Ae^{-k(0)} + 16 = 80$$

$$A + 16 = 80$$

$$A = 64$$



b)

Find the exact value of  $k$ .

$$t = 5, \quad T = 40$$

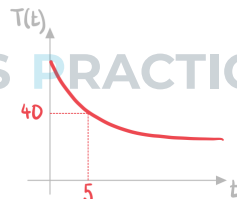
$$40 = 64e^{5k} + 16$$

$$64e^{5k} = 24$$

$$e^{5k} = \frac{3}{8}$$

$$5k = \ln \frac{3}{8}$$

$$k = \frac{1}{5} \ln \frac{3}{8}$$



c)

Find the time taken for the temperature of the coffee to reach  $30^{\circ}\text{C}$ .



Find  $t$  such that  $T(t) = 30$

$$30 = 64e^{kt} + 16$$

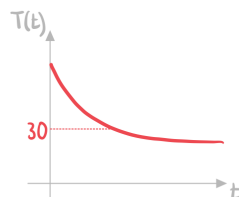
Leave as  $k$  until the end to save writing  $\frac{1}{5} \ln \frac{3}{8}$  each time

$$64e^{kt} = 14$$

$$e^{kt} = \frac{7}{32}$$

$$kt = \ln \frac{7}{32}$$

$$t = \frac{\ln \frac{7}{32}}{k} = \frac{\ln \frac{7}{32}}{\frac{1}{5} \ln \frac{3}{8}} = 7.7476..$$



7.75 minutes (3sf)







## 2.5 Reciprocal & Rational Functions

### 2.5.1 Reciprocal & Rational Functions

#### Reciprocal Functions & Graphs

##### What is the reciprocal function?

- The **reciprocal function** is defined by  $f(x) = \frac{1}{x}$ ,  $x \neq 0$
- Its **domain** is the set of **all real values except 0**
- Its **range** is the set of **all real values except 0**
- The reciprocal function has a **self-inverse** nature
  - $f^{-1}(x) = f(x)$
  - $(f \circ f)(x) = x$

##### What are the key features of the reciprocal graph?

- The graph **does not have a y-intercept**
- The graph **does not have any roots**
- The graph has **two asymptotes**
  - A **horizontal** asymptote at the x-axis:  $y = 0$   
This is the **limiting value** when the absolute value of  $x$  gets very large
  - A **vertical** asymptote at the y-axis:  $x = 0$   
This is the value that causes the **denominator to be zero**
- The graph has **two axes of symmetry**
  - $y = x$
  - $y = -x$
- The graph **does not have any minimum or maximum points**

## Linear Rational Functions & Graphs

### What is a rational function with linear terms?

- A **(linear) rational function** is of the form  $f(x) = \frac{ax + b}{cx + d}, x \neq -\frac{d}{c}$
- Its **domain** is the set of **all real values except**  $-\frac{d}{c}$
- Its **range** is the set of **all real values except**  $\frac{a}{c}$
- The **reciprocal function** is a **special case** of a rational function

### What are the key features of linear rational graphs?

- The graph has a **y-intercept** at  $\left(0, \frac{b}{d}\right)$  provided  $d \neq 0$
- The graph has **one root** at  $\left(-\frac{b}{a}, 0\right)$  provided  $a \neq 0$
- The graph has **two asymptotes**
  - A **horizontal** asymptote:  $y = \frac{a}{c}$   
This is the **limiting value** when the absolute value of  $x$  gets very large
  - A **vertical** asymptote:  $x = -\frac{d}{c}$   
This is the value that causes the **denominator to be zero**
- The graph **does not have any minimum or maximum points**
- If you are asked to **sketch or draw** a rational graph:
  - Give the **coordinates** of any **intercepts** with the axes
  - Give the **equations** of the **asymptotes**



#### Exam Tip

- If you draw a horizontal line anywhere it should only intersect this type of graph once at most
- The only horizontal line that should not intersect the graph is the horizontal asymptote
  - This can be used to check your sketch in an exam



## ? Worked Example

The function  $f$  is defined by  $f(x) = \frac{10-5x}{x+2}$  for  $x \neq -2$ .

a)

Write down the equation of

(i)

the vertical asymptote of the graph of  $f$ ,

(ii)

the horizontal asymptote of the graph of  $f$ .

(i) Vertical asymptote is when denominator equals zero

$$x+2=0 \quad \boxed{x=-2}$$

(ii) Horizontal asymptote is limiting value as  $x$  gets large

$$\lim_{x \rightarrow \infty} \frac{10-5x}{x+2} = \lim_{x \rightarrow \infty} \frac{-5x}{x} \quad \boxed{y=-5}$$

b)

Find the coordinates of the intercepts of the graph of  $f$  with the axes.

$y$ -intercept occurs when  $x=0$

$$y = \frac{10-5(0)}{0+2} = 5 \quad \boxed{(0,5)}$$

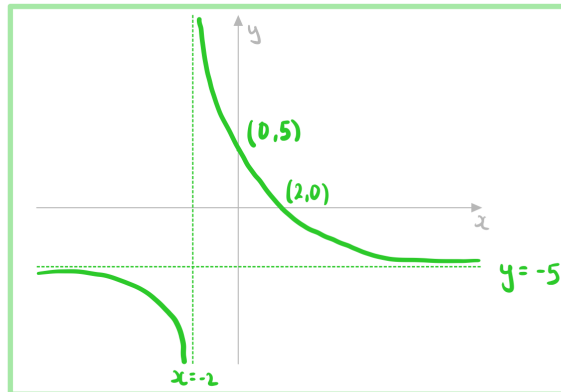
$x$ -intercept occurs when  $y=0$

$$\frac{10-5x}{x+2} = 0 \Rightarrow 10-5x=0 \Rightarrow x=2 \quad \boxed{(2,0)}$$

c)

Sketch the graph of  $f$ .

Include asymptotes and intercepts



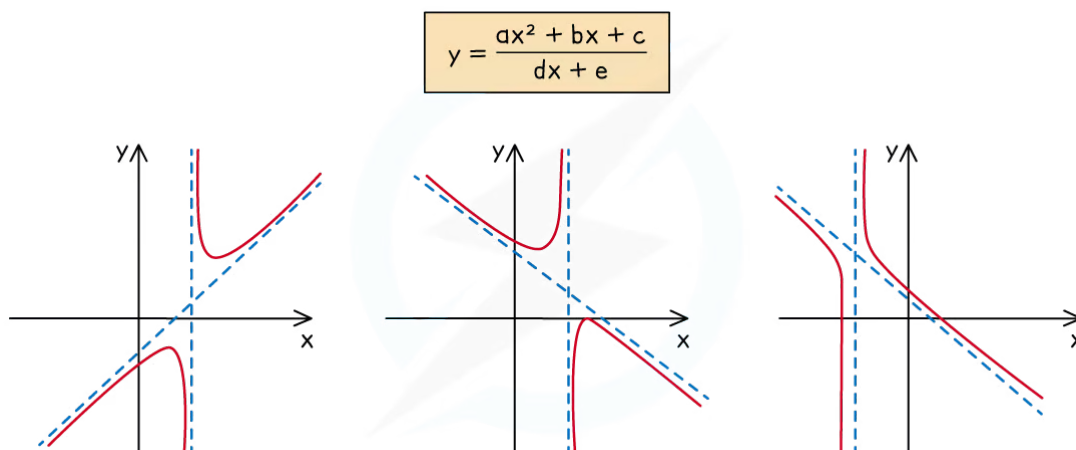
## Quadratic Rational Functions & Graphs

How do I sketch the graph of a rational function where the terms are not linear?

- A rational function can be written  $f(x) = \frac{g(x)}{h(x)}$ 
  - Where  $g$  and  $h$  are polynomials
- To find the **y-intercept** evaluate  $\frac{g(0)}{h(0)}$
- To find the **x-intercept(s)** solve  $g(x) = 0$
- To find the equations of the **vertical asymptote(s)** solve  $h(x) = 0$
- There will also be an **asymptote** determined by what  $f(x)$  tends to as  $x$  approaches infinity
  - In this course it will be either:
    - Horizontal**
    - Oblique (a slanted line)**
      - This can be found by writing  $g(x)$  in the form  $h(x)Q(x) + r(x)$   
You can do this by **polynomial division** or **comparing coefficients**
      - The function then tends to the curve  $y = Q(x)$

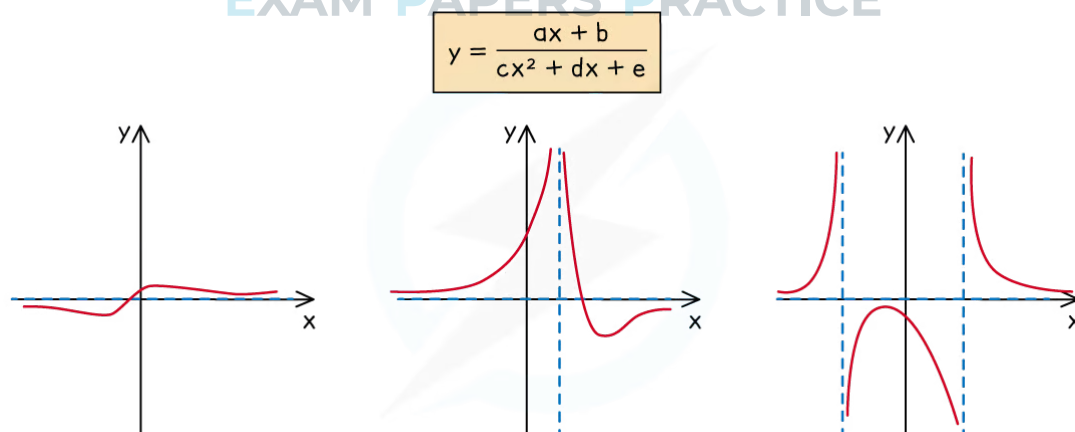
What are the key features of rational graphs: quadratic over linear?

- For the rational function of the form  $f(x) = \frac{ax^2 + bx + c}{dx + e}$
- The graph has a **y-intercept** at  $\left(0, \frac{c}{e}\right)$  provided  $e \neq 0$
- The graph can have **0, 1 or 2 roots**
  - They are the solutions to  $ax^2 + bx + c = 0$
- The graph has **one vertical asymptote**  $x = -\frac{e}{d}$
- The graph has an **oblique asymptote**  $y = px + q$ 
  - Which can be found by writing  $ax^2 + bx + c$  in the form  $(dx + e)(px + q) + r$   
Where  $p, q, r$  are constants  
This can be done by **polynomial division** or **comparing coefficients**



### What are the key features of rational graphs: linear over quadratic?

- For the rational function of the form  $f(x) = \frac{ax + b}{cx^2 + dx + e}$
- The graph has a **y-intercept** at  $\left(0, \frac{b}{e}\right)$  provided  $e \neq 0$
- The graph has **one root** at  $x = -\frac{b}{a}$
- The graph has can have **0, 1 or 2 vertical asymptotes**
  - They are the solutions to  $cx^2 + dx + e = 0$
- The graph has a **horizontal asymptote**



#### Exam Tip

- If you draw a horizontal line anywhere it should only intersect this type of graph twice at most
  - This idea can be used to check your graph or help you sketch it

## ? Worked Example

The function  $f$  is defined by  $f(x) = \frac{2x^2 + 5x - 3}{x + 1}$  for  $x \neq -1$ .

a)

(i)

Show that  $\frac{2x^2 + 5x - 3}{x + 1} = px + q + \frac{r}{x + 1}$  for constants  $p$ ,  $q$  and  $r$  which are to be found.

(ii)

Hence write down the equation of the oblique asymptote of the graph of  $f$ .

(i) Write  $2x^2 + 5x - 3$  as  $(x+1)(px+q) + r$

$$2x^2 + 5x - 3 = px^2 + qx + px + q + r$$

Compare coefficients

$$\begin{array}{ccc} x^2 & x & \text{constant} \\ 2 = p & 5 = q + p & -3 = q + r \end{array}$$

$$\therefore p = 2 \quad \therefore q = 3 \quad \therefore r = -6$$

$$\frac{2x^2 + 5x - 3}{x + 1} = \frac{(x+1)(2x+3) - 6}{x+1} = 2x + 3 - \frac{6}{x+1}$$

(ii)

$$y = 2x + 3$$

b)

Find the coordinates of the intercepts of the graph of  $f$  with the axes.

y-intercept occurs when  $x = 0$

$$y = \frac{2(0)^2 + 5(0) - 3}{(0) + 1} = -3 \quad (0, -3)$$

x-intercept occurs when  $y = 0$

$$\frac{2x^2 + 5x - 3}{x + 1} = 0 \Rightarrow 2x^2 + 5x - 3 = 0 \Rightarrow (2x - 1)(x + 3) \Rightarrow x = 0.5 \text{ or } x = -3$$

$$(0.5, 0) \text{ and } (-3, 0)$$

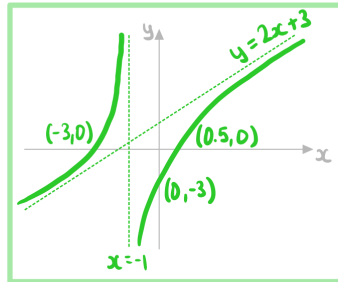
c)

Sketch the graph of  $f$ .



Vertical asymptote when denominator is zero  $x = -1$

Include asymptotes and intercepts





## 2.6 Transformations of Graphs

### 2.6.1 Translations of Graphs

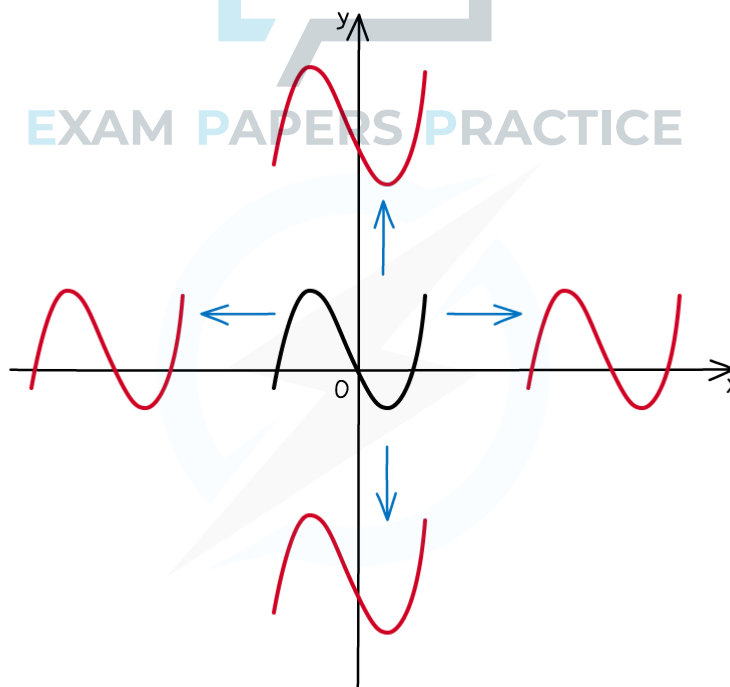
#### Translations of Graphs

##### What are translations of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **translation**:
  - the graph is **moved** (up or down, left or right) in the xy plane  
Its position **changes**
  - the shape, size, and orientation of the graph remain **unchanged**
- A particular translation (how far left/right, how far up/down) is specified by a **translation vector**

**vector**  $\begin{pmatrix} x \\ y \end{pmatrix}$ :

- x is the **horizontal** displacement  
**Positive** moves **right**  
**Negative** moves **left**
- y is the **vertical** displacement  
**Positive** moves **up**  
**Negative** moves **down**

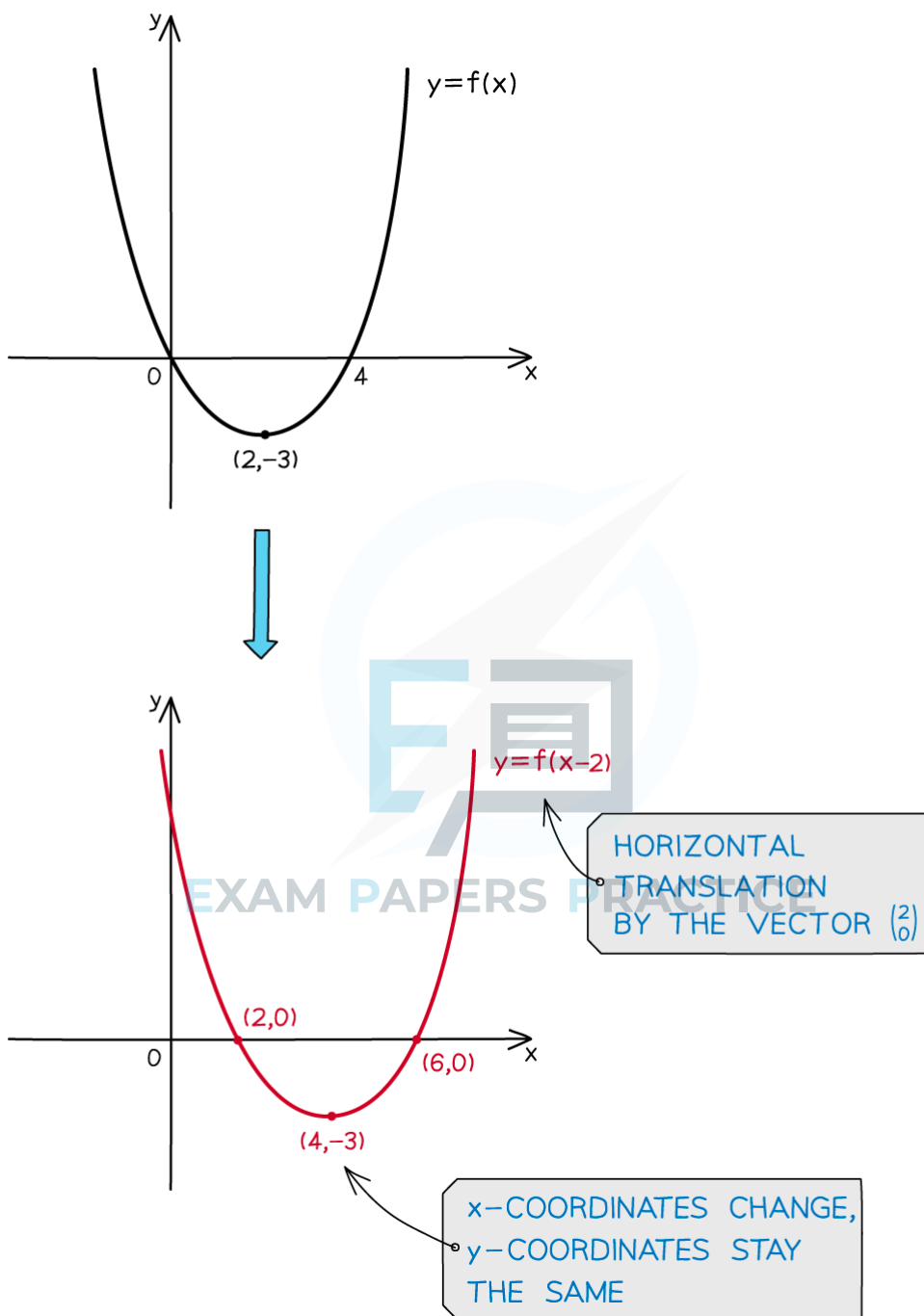


##### What effects do horizontal translations have on the graphs and functions?

- A **horizontal translation** of the graph  $y = f(x)$  by the vector  $\begin{pmatrix} a \\ 0 \end{pmatrix}$  is represented by



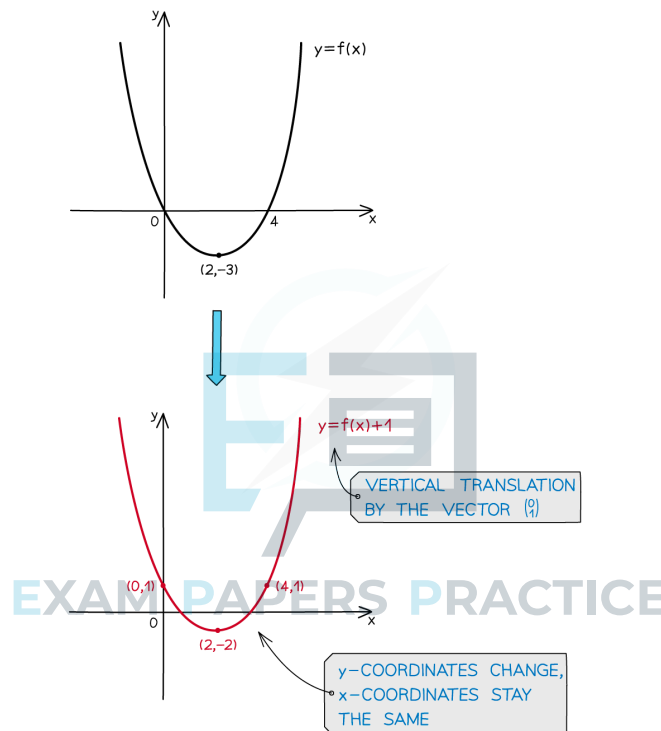
- $y = f(x - a)$
- The **x-coordinates change**
  - The value  $a$  is **subtracted** from them
- The **y-coordinates stay the same**
- The coordinates  $(x, y)$  become  $(x + a, y)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
  - $x = k$  becomes  $x = k + a$



**What effects do vertical translations have on the graphs and functions?**

- A **vertical translation** of the graph  $y = f(x)$  by the vector  $\begin{pmatrix} 0 \\ b \end{pmatrix}$  is represented by
  - $y - b = f(x)$

- This is often rearranged to  $y = f(x) + b$
- The **x-coordinates stay the same**
- The **y-coordinates change**
  - The value  $b$  is **added** to them
- The coordinates  $(x, y)$  become  $(x, y + b)$
- **Horizontal** asymptotes **change**
  - $y = k$  becomes  $y = k + b$
- **Vertical** asymptotes **stay the same**

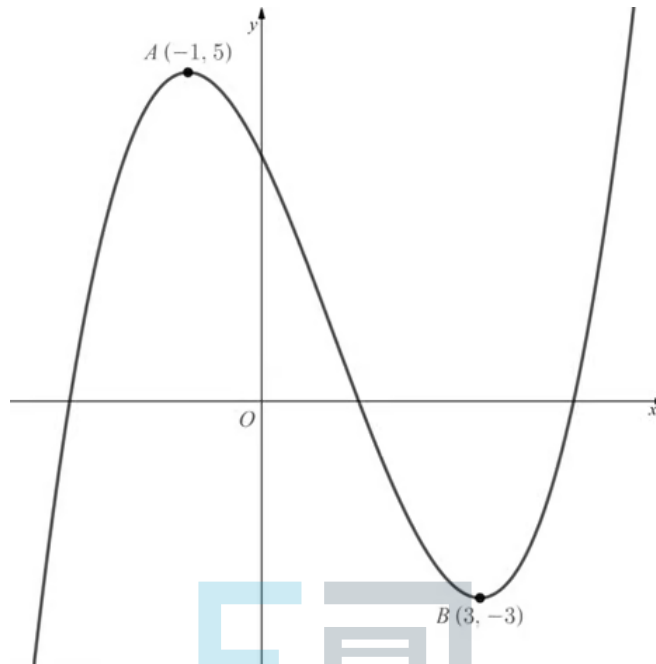


### Exam Tip

- To get full marks in an exam make sure you use correct mathematical terminology
  - For example: Translate by the vector  $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

## ? Worked Example

The diagram below shows the graph of  $y = f(x)$ .



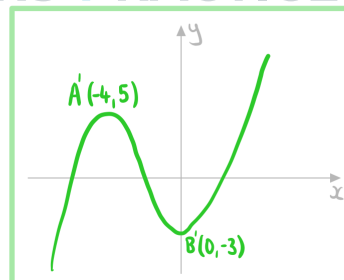
- a)  
Sketch the graph of  $y = f(x + 3)$ .

$y = f(x + k)$  translation by  $\begin{pmatrix} -k \\ 0 \end{pmatrix}$

Translate  $y = f(x)$  by  $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

A becomes  $(-4, 5)$

B becomes  $(0, -3)$



- b)  
Sketch the graph of  $y = f(x) + 3$ .

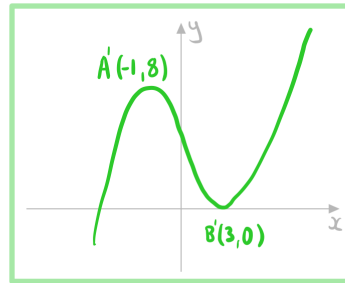


$y = f(x) + k$  translation by  $\begin{pmatrix} 0 \\ k \end{pmatrix}$

Translate  $y = f(x)$  by  $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

A becomes  $(-1, 8)$

B becomes  $(3, 0)$



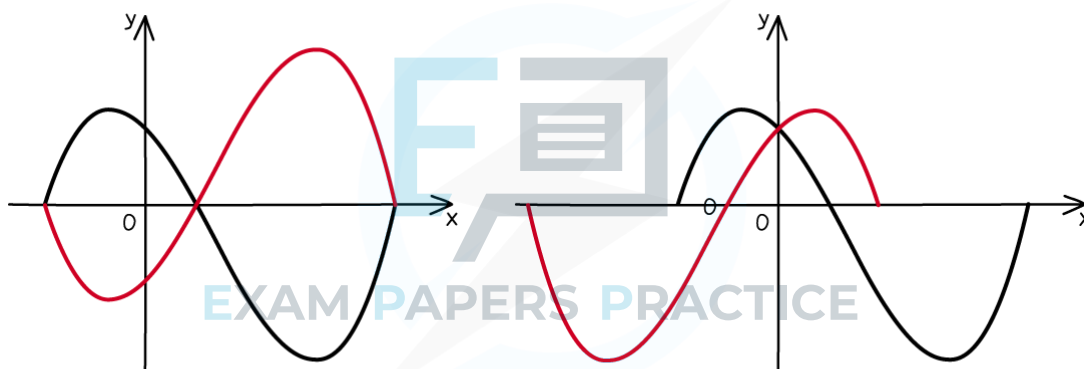


## 2.6.2 Reflections of Graphs

### Reflections of Graphs

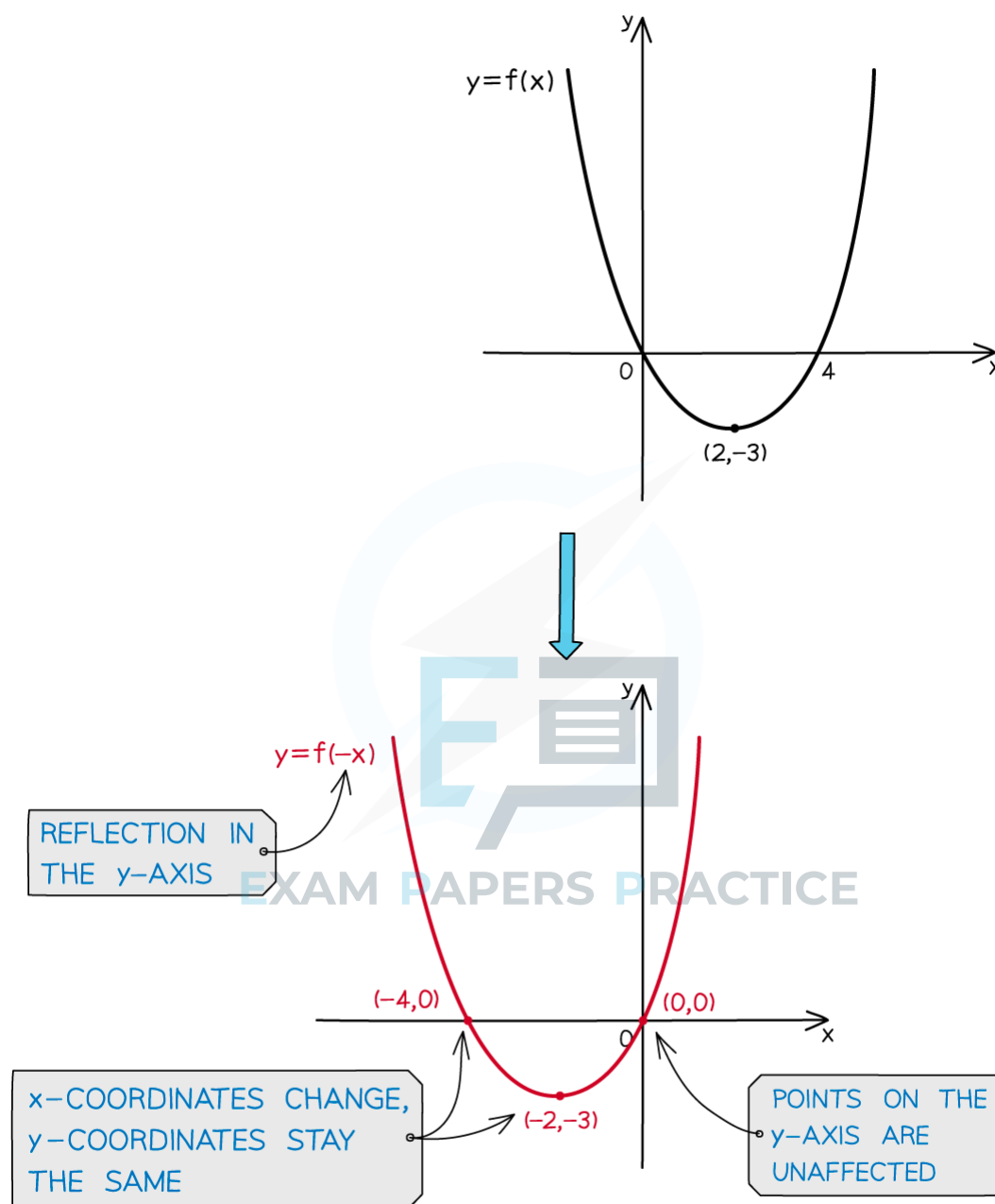
#### What are reflections of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **reflection**:
  - the graph is **flipped** about one of the coordinate axes  
Its orientation **changes**
  - the size of the graph remains **unchanged**
- A particular reflection is specified by an **axis of symmetry**:
  - $y = 0$   
This is the x-axis
  - $x = 0$   
This is the y-axis



#### What effects do horizontal reflections have on the graphs and functions?

- A **horizontal reflection** of the graph  $y = f(x)$  about the y-axis is represented by
  - $y = f(-x)$
- The **x-coordinates change**
  - Their **sign** changes
- The **y-coordinates stay the same**
- The coordinates  $(x, y)$  become  $(-x, y)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
  - $x = k$  becomes  $x = -k$

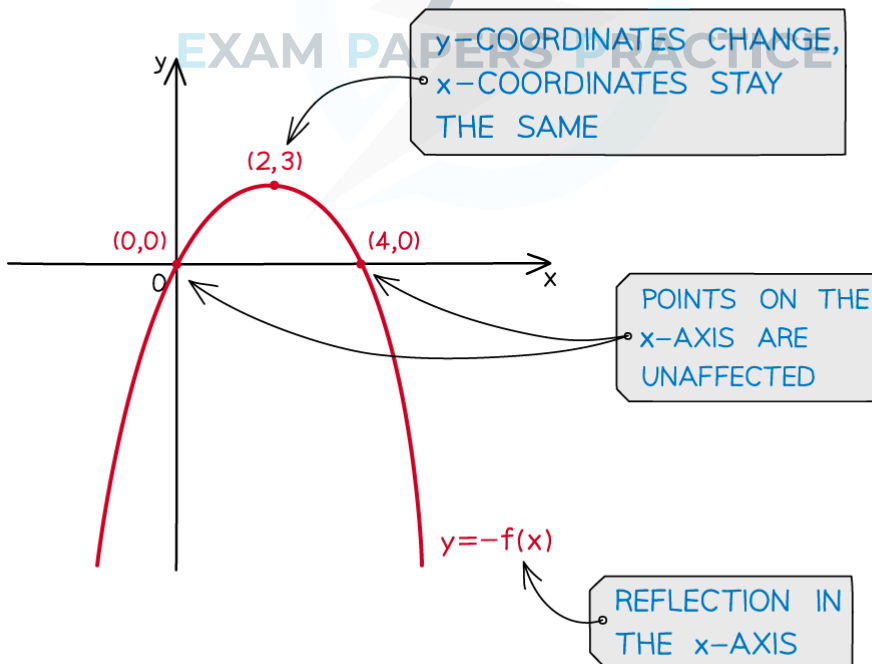
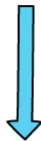
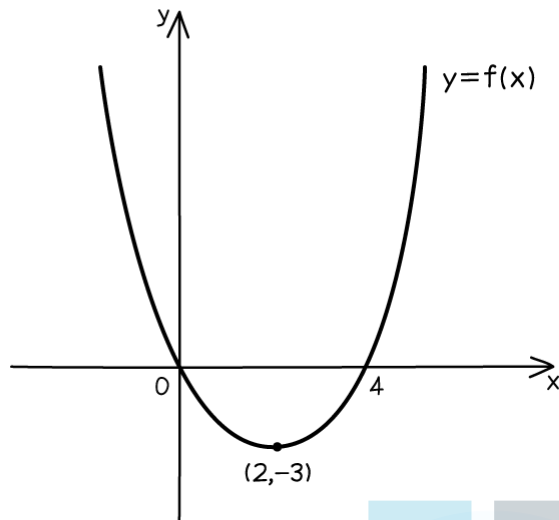


### What effects do vertical reflections have on the graphs and functions?

- A **vertical reflection** of the graph  $y = f(x)$  about the  $x$ -axis is represented by
  - $-y = f(x)$
  - This is often rearranged to  $y = -f(x)$
- The  **$x$ -coordinates stay the same**
- The  **$y$ -coordinates change**
  - Their **sign** changes



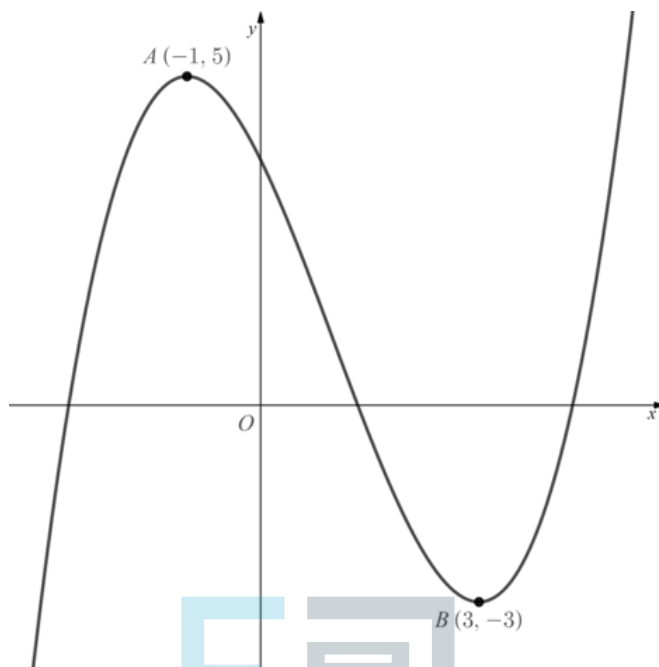
- The coordinates  $(x, y)$  become  $(x, -y)$
- **Horizontal** asymptotes **change**
  - $y = k$  becomes  $y = -k$
- **Vertical** asymptotes **stay the same**





## ? Worked Example

The diagram below shows the graph of  $y = f(x)$ .

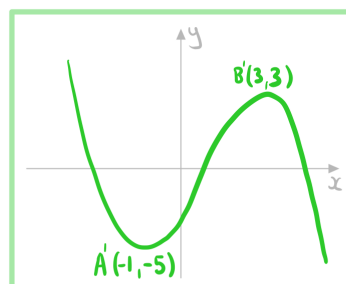


- a)  
Sketch the graph of  $y = -f(x)$ .

$y = -f(x)$  reflection in  $x$ -axis

A becomes  $(-1, -5)$

B becomes  $(3, 3)$



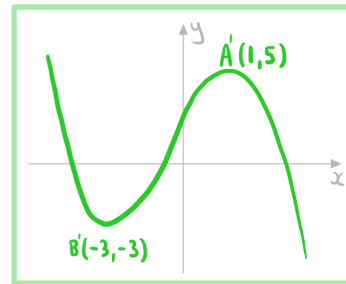
- b)  
Sketch the graph of  $y = f(-x)$ .



$y=f(-x)$  reflection in  $y$ -axis

A becomes  $(1, 5)$

B becomes  $(-3, -3)$





## 2.6.3 Stretches Graphs

### Stretches of Graphs

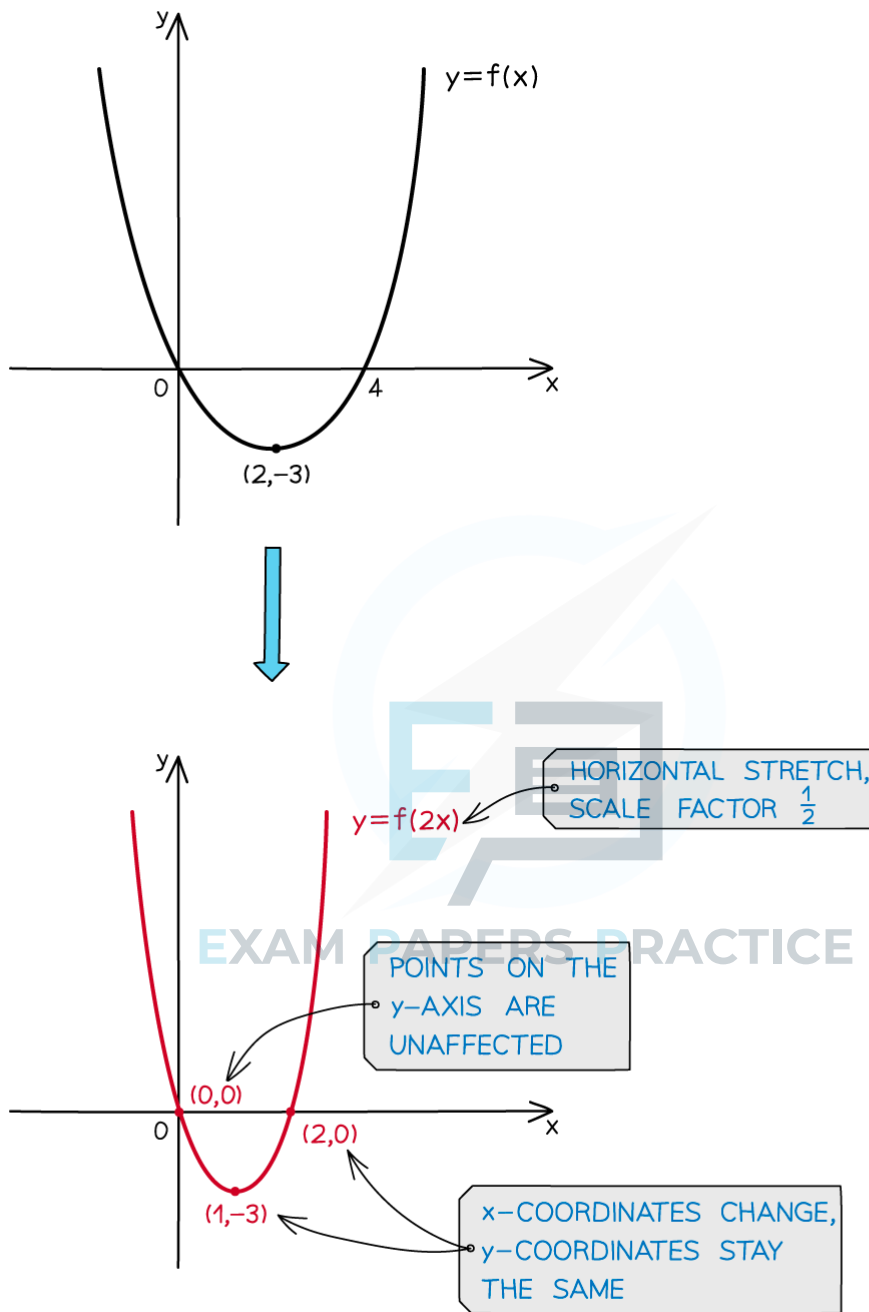
#### What are stretches of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **stretch**:
  - the graph is **stretched** about one of the coordinate axes by a scale factor  
Its size **changes**
  - the orientation of the graph remains **unchanged**
- A particular stretch is specified by a **coordinate axis** and a **scale factor**:
  - The **distance** between a **point** on the graph and the **specified coordinate axis** is **multiplied** by the **constant scale factor**
  - The graph is stretched in the **direction** which is **parallel** to the **other coordinate axis**
  - For scale factors **bigger than 1**  
the points on the graph get **further away** from the **specified coordinate axis**
  - For scale factors **between 0 and 1**  
the points on the graph get **closer** to the **specified coordinate axis**  
This is also sometimes called a **compression** but in your exam you must use the term **stretch** with the appropriate scale factor



#### What effects do horizontal stretches have on the graphs and functions?

- A **horizontal stretch** of the graph  $y = f(x)$  by a scale factor  $q$  centred about the  $y$ -axis is represented by
  - $y = f\left(\frac{x}{q}\right)$
- The  **$x$ -coordinates change**
  - They are **divided** by  $q$
- The  **$y$ -coordinates stay the same**
- The coordinates  $(x, y)$  become  $(qx, y)$
- **Horizontal asymptotes stay the same**
- **Vertical asymptotes change**
  - $x = k$  becomes  $x = qk$

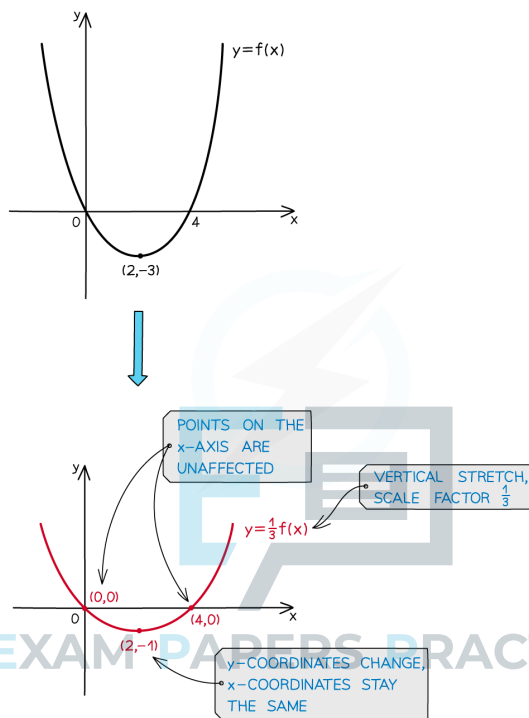


### What effects do vertical stretches have on the graphs and functions?

- A **vertical stretch** of the graph  $y = f(x)$  by a scale factor  $p$  centred about the  $x$ -axis is represented by
  - $\frac{y}{p} = f(x)$
  - This is often rearranged to  $y = pf(x)$



- The **x-coordinates stay the same**
- The **y-coordinates change**
  - They are **multiplied** by  $p$
- The coordinates  $(x, y)$  become  $(x, py)$
- **Horizontal** asymptotes **change**
  - $y = k$  becomes  $y = pk$
- **Vertical** asymptotes **stay the same**

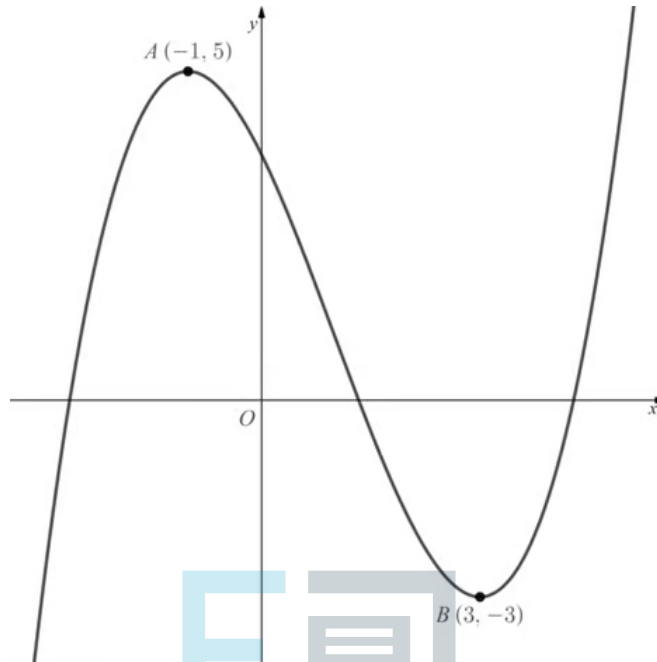


### Exam Tip

- To get full marks in an exam make sure you use correct mathematical terminology
  - For example: Stretch vertically by scale factor  $\frac{1}{2}$
  - Do not use the word "compress" in your exam

## ? Worked Example

The diagram below shows the graph of  $y = f(x)$ .



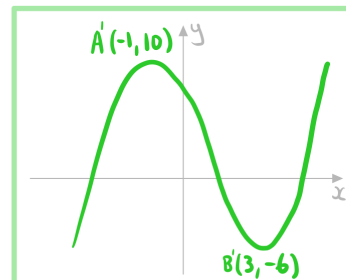
- a)  
Sketch the graph of  $y = 2f(x)$ .

$y = kf(x)$  vertical stretch scale factor  $k$

Stretch  $y = f(x)$  vertically  
scale factor 2

A becomes  $(-1, 10)$

B becomes  $(3, -6)$



- b)  
Sketch the graph of  $y = f(2x)$ .

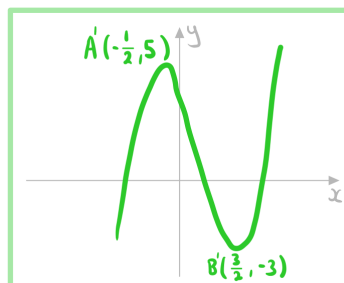


$y=f(kx)$  horizontal stretch scale factor  $\frac{1}{k}$

Stretch  $y=f(x)$  horizontally  
scale factor  $\frac{1}{2}$

A becomes  $(-\frac{1}{2}, 5)$

B becomes  $(\frac{3}{2}, -3)$





## 2.6.4 Composite Transformations of Graphs

### Composite Transformations of Graphs

#### What transformations do I need to know?

- $y = f(x + k)$  is **horizontal translation** by vector  $\begin{pmatrix} -k \\ 0 \end{pmatrix}$ 
  - If  $k$  is **positive** then the graph moves **left**
  - If  $k$  is **negative** then the graph moves **right**
- $y = f(x) + k$  is **vertical translation** by vector  $\begin{pmatrix} 0 \\ k \end{pmatrix}$ 
  - If  $k$  is **positive** then the graph moves **up**
  - If  $k$  is **negative** then the graph moves **down**
- $y = f(kx)$  is a **horizontal stretch** by scale factor  $\frac{1}{k}$  centred about the  $y$ -axis
  - If  $k > 1$  then the graph gets **closer** to the  $y$ -axis
  - If  $0 < k < 1$  then the graph gets **further** from the  $y$ -axis
- $y = kf(x)$  is a **vertical stretch** by scale factor  $k$  centred about the  $x$ -axis
  - If  $k > 1$  then the graph gets **further** from the  $x$ -axis
  - If  $0 < k < 1$  then the graph gets **closer** to the  $x$ -axis
- $y = f(-x)$  is a **horizontal reflection** about the  $y$ -axis
  - A **horizontal reflection** can be viewed as a special case of a **horizontal stretch**
- $y = -f(x)$  is a **vertical reflection** about the  $x$ -axis
  - A **vertical reflection** can be viewed as a special case of a **vertical stretch**

#### How do horizontal and vertical transformations affect each other?

- **Horizontal and vertical transformations** are **independent** of each other
  - The horizontal transformations involved will need to be applied in their correct order
  - The vertical transformations involved will need to be applied in their correct order
- Suppose there are **two horizontal** transformation  $H_1$  then  $H_2$  and **two vertical** transformations  $V_1$  then  $V_2$  then they can be applied in the following orders:
  - Horizontal then vertical:
 
$$H_1 H_2 V_1 V_2$$
  - Vertical then horizontal:
 
$$V_1 V_2 H_1 H_2$$
  - Mixed up (provided that  $H_1$  comes before  $H_2$  and  $V_1$  comes before  $V_2$ ):
 
$$\begin{aligned} &H_1 V_1 H_2 V_2 \\ &H_1 V_1 V_2 H_2 \\ &V_1 H_1 V_2 H_2 \\ &V_1 H_1 H_2 V_2 \end{aligned}$$



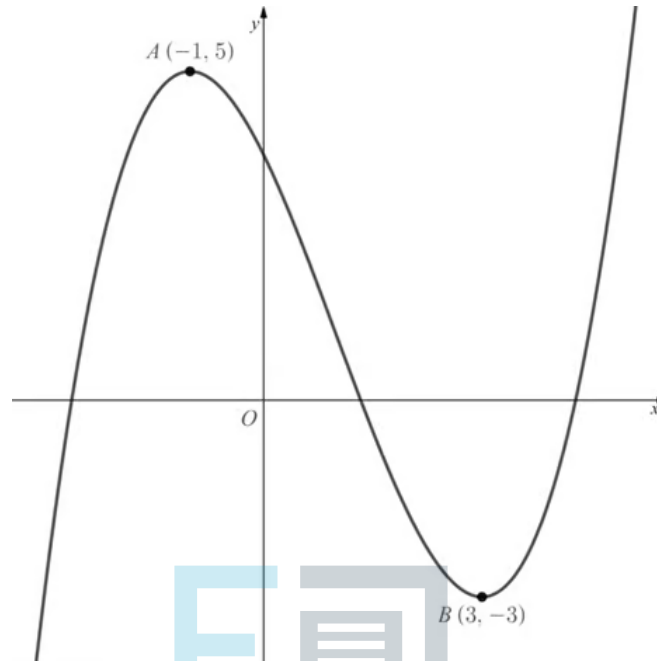
#### Exam Tip

- In an exam you are more likely to get the correct solution if you deal with one transformation at a time and sketch the graph after each transformation



### ? Worked Example

The diagram below shows the graph of  $y = f(x)$ .



Sketch the graph of  $y = \frac{1}{2}f\left(\frac{x}{2}\right)$ .

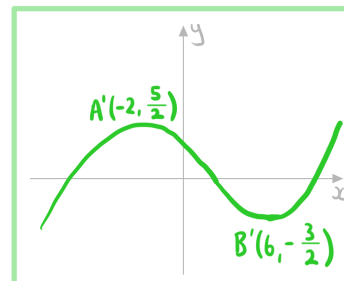
A vertical and horizontal transformation can be done in any order

$y = \frac{1}{2}f(x)$  : vertical stretch scale factor  $\frac{1}{2}$

$y = f\left(\frac{x}{2}\right)$  : horizontal stretch scale factor 2

A becomes  $\left(-2, \frac{5}{2}\right)$

B becomes  $\left(6, -\frac{3}{2}\right)$



## Composite Vertical Transformations $af(x)+b$

### How do I deal with multiple vertical transformations?

- **Order matters** when you have **more than one vertical transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order

- A **vertical stretch** by scale factor  $a$  followed by a **translation** of  $\begin{pmatrix} 0 \\ b \end{pmatrix}$

Stretch:  $y = af(x)$

Then translation:  $y = [af(x)] + b$

Finalequation:  $y = af(x) + b$

- A **translation** of  $\begin{pmatrix} 0 \\ b \end{pmatrix}$  followed by a **vertical stretch** by scale factor  $a$

Translation:  $y = f(x) + b$

Then stretch:  $y = a[f(x) + b]$

Finalequation:  $y = af(x) + ab$

- If you are asked to determine the **order**
  - The order of vertical transformations **follows the order of operations**
  - First write the equation in the form  $y = af(x) + b$ 

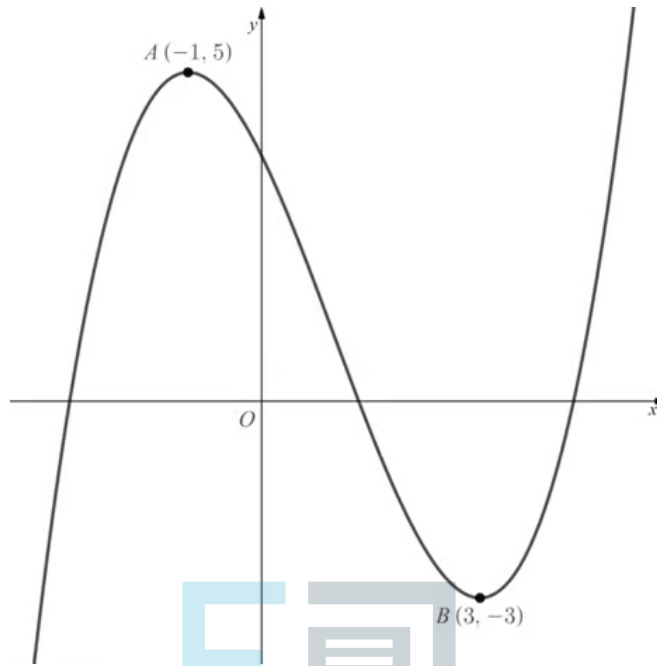
**First stretch vertically** by scale factor  $a$

If  $a$  is negative then the **reflection and stretch** can be **done in any order**

Then translate by  $\begin{pmatrix} 0 \\ b \end{pmatrix}$

### ? Worked Example

The diagram below shows the graph of  $y = f(x)$ .



Sketch the graph of  $y = 3f(x) - 2$ .

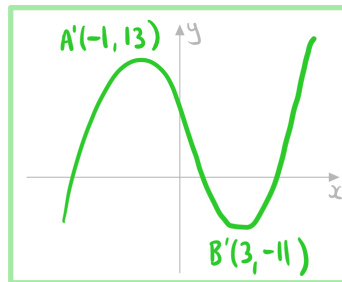
The order vertical transformations follows the order of operations

$y = 3f(x)$ : Vertical stretch scale factor 3

$y = f(x) - 2$ : Translate  $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

A becomes  $(-1, 13)$

B becomes  $(3, -11)$



## Composite Horizontal Transformations $f(ax+b)$

### How do I deal with multiple horizontal transformations?

- **Order matters** when you have **more than one horizontal transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order

- A **horizontal stretch** by scale factor  $\frac{1}{a}$  followed by a **translation** of  $\begin{pmatrix} -b \\ 0 \end{pmatrix}$

Stretch:  $y = f(ax)$

Then translation:  $y = f(a(x + b))$

Final equation:  $y = f(ax + ab)$

- A **translation** of  $\begin{pmatrix} -b \\ 0 \end{pmatrix}$  followed by a **horizontal stretch** by scale factor  $\frac{1}{a}$

Translation:  $y = f(x + b)$

Then stretch:  $y = f((ax) + b)$

Final equation:  $y = f(ax + b)$

- If you are asked to determine the **order**
  - First write the equation in the form  $y = f(ax + b)$
  - The order of horizontal transformations **is the reverse of the order of operations**

**First translate** by  $\begin{pmatrix} -b \\ 0 \end{pmatrix}$

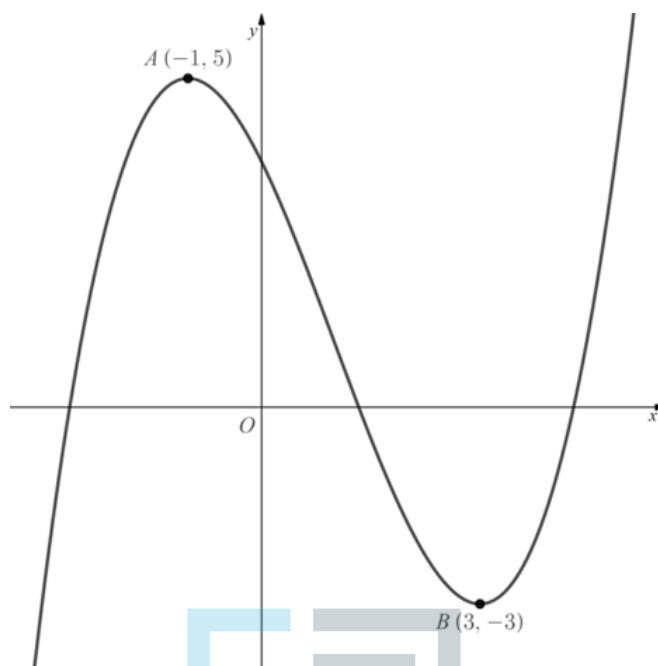
**Then stretch** by scale factor  $\frac{1}{a}$

If  $a$  is negative then the **reflection and stretch** can be **done in any order**



## ? Worked Example

The diagram below shows the graph of  $y = f(x)$ .



Sketch the graph of  $y = f(2x - 1)$ .

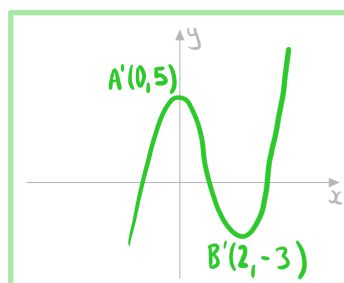
The order of horizontal transformations is the reverse of the order of operations

$y = f(x - 1)$ : Translate  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$y = f(2x)$ : Horizontal stretch scale factor  $\frac{1}{2}$

A becomes  $(0, 5)$

B becomes  $(2, -3)$



## 2.7 Polynomial Functions

### 2.7.1 Factor & Remainder Theorem

#### Factor Theorem

##### What is the factor theorem?

- The **factor theorem** is used to find the linear factors of **polynomial** equations
- This topic is closely tied to finding the **zeros** and **roots** of a **polynomial** function/equation
  - As a rule of thumb a **zero** refers to the polynomial function and a **root** refers to a polynomial equation
- For any **polynomial** function  $P(x)$ 
  - $(x - k)$  is a **factor** of  $P(x)$  if  $P(k) = 0$
  - $P(k) = 0$  if  $(x - k)$  is a **factor** of  $P(x)$

##### How do I use the factor theorem?

- Consider the polynomial function  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  and  $(x - k)$  is a **factor**
  - Then, due to the factor theorem  $P(k) = a_n k^n + a_{n-1} k^{n-1} + \dots + a_1 k + a_0 = 0$
  - $P(x) = (x - k) \times Q(x)$ , where  $Q(x)$  is a **polynomial** that is a factor of  $P(x)$
  - Hence,  $\frac{P(x)}{x - k} = Q(x)$ , where  $Q(x)$  is another factor of  $P(x)$
- If the linear factor has a **coefficient of x** then you must first factorise out the coefficient
  - If the linear factor is  $(ax - b) = a\left(x - \frac{b}{a}\right) \rightarrow P\left(\frac{b}{a}\right) = 0$



#### Exam Tip

- A common mistake in exams is using the incorrect sign for either the root or the factor
- If you are asked to find integer solutions to a polynomial then you only need to consider factors of the constant term



### ? Worked Example

Determine whether  $(x - 2)$  is a factor of the following polynomials:

a)

$$f(x) = x^3 - 2x^2 - x + 2.$$

Step 1: Determine k

Our linear function is  $x - 2$

→ so  $k = 2$

Step 2: Apply factor theorem

For  $x - 2$  to be a factor of  $f(x)$ ,

$f(2)$  has to equal zero

$$\begin{aligned} f(2) &= (2)^3 - 2(2)^2 - (2) + 2 \\ &= 8 - 8 - 2 + 2 \\ &= 0 \end{aligned}$$

$$f(2) = 0,$$

so  $x - 2$  is a factor of  $f(x)$

b)

$$g(x) = 2x^3 + 3x^2 - x + 5.$$



Step 1: Determine k

Our linear function is  $x - 2$

→ so  $k = 2$

Step 2: Apply factor theorem

For  $x - 2$  to be a factor of  $g(x)$ ,  
 $g(2)$  has to equal zero

$$\begin{aligned} g(2) &= 2(2)^3 + 3(2)^2 - (2) + 5 \\ &= 16 - 12 - 2 + 5 \\ &= 7 \end{aligned}$$

$$g(2) = 7,$$

so  $x - 2$  is not a factor of  $g(x)$

It is given that  $(2x - 3)$  is a factor of  $h(x) = 2x^3 - bx^2 + 7x - 6$ .

c)

Find the value of  $b$ .





Step 1: Determine k

Our linear function is  $2x - 3$

$$\rightarrow \text{so } k = \frac{3}{2}$$

Step 2: Apply factor theorem to find b

Since  $2x - 3$  is a factor of  $h(x)$ ,

$$h\left(\frac{3}{2}\right) = 0$$

$$\begin{aligned} 0 &= 2\left(\frac{3}{2}\right)^3 - b\left(\frac{3}{2}\right)^2 + 7\left(\frac{3}{2}\right) - 6 \\ &= \frac{54}{8} - \frac{9}{4}b + \frac{21}{2} - 6 \end{aligned}$$

$$b = 5$$



## Remainder Theorem

### What is the remainder theorem?

- The **remainder theorem** is used to find the remainder when we divide a **polynomial** function by a linear function
- When any polynomial  $P(x)$  is divided by any linear function  $(x - k)$  the value of the remainder  $R$  is given by  $P(k) = R$ 
  - Note, when  $P(k) = 0$  then  $(x - k)$  is a factor of  $P(x)$

### How do I use the remainder theorem?

- Consider the polynomial function  $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$  and the linear function  $(x - k)$ 
  - Then, due to the remainder theorem  $P(k) = a_nk^n + a_{n-1}k^{n-1} + \dots + a_1k + a_0 = R$
  - $P(x) = (x - k) \times Q(x) + R$ , where  $Q(x)$  is a **polynomial**
  - Hence,  $\frac{P(x)}{x - k} = Q(x) + \frac{R}{x - k}$ , where  $R$  is the remainder
- If the linear factor has a **coefficient of x** then you must first factorise out the coefficient
  - If the linear factor is  $(ax - b) = a\left(x - \frac{b}{a}\right) \rightarrow P\left(\frac{b}{a}\right) = R$

? **Worked Example**

Let  $f(x) = 2x^4 - 2x^3 - x^2 - 3x + 1$ , find the remainder  $R$  when  $f(x)$  is divided by:

a)  
 $x - 3$ .

Step 1: Determine  $k$

Our linear function is  $x - 3$

→ so  $k = 3$

Step 2: Apply remainder theorem

$$f(3) = R$$

$$f(3) = 2(3)^4 - 2(3)^3 - (3)^2 - 3(3) + 1$$

$$f(3) = 162 - 54 - 9 - 9 + 1$$

$$f(3) = 91$$

$$R = 91$$



b)  
 $x + 2$ .

EXAM PAPERS PRACTICE

Step 1: Determine  $k$

Our linear function is  $x + 2$

→ so  $k = -2$

Step 2: Apply remainder theorem

$$f(-2) = R$$

$$f(-2) = 2(-2)^4 - 2(-2)^3 - (-2)^2 - 3(-2) + 1$$

$$f(-2) = 32 + 16 - 4 + 6 + 1$$

$$f(-2) = 51$$

$$R = 51$$



The remainder when  $f(x)$  is divided by  $(2x + k)$  is  $\frac{893}{8}$ .

c)

Given that  $k > 0$ , find the value of  $k$ .

Step 1: Apply remainder theorem

$$2x + k = 2\left(x + \frac{k}{2}\right) \quad f\left(-\frac{k}{2}\right) = \frac{893}{8}$$

$$\frac{893}{8} = 2\left(-\frac{k}{2}\right)^4 - 2\left(-\frac{k}{2}\right)^3 - \left(-\frac{k}{2}\right)^2 - 3\left(-\frac{k}{2}\right) + 1$$

Step 2: Solve for  $k$  using your GDC

$$k = 5$$





## 2.7.2 Polynomial Division

### Polynomial Division

#### What is polynomial division?

- Polynomial division is the process of **dividing two polynomials**
  - This is usually only useful when the **degree of the denominator** is **less than or equal to** the **degree of the numerator**
- To do this we use an algorithm similar to that used for **division of integers**
- To divide the polynomial  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  by the polynomial  $D(x) = b_k x^k + b_{k-1} x^{k-1} + \dots + b_1 x + b_0$  where  $k \leq n$

#### STEP 1

**Divide** the **leading term of the polynomial**  $P(x)$  by the **leading term of the divisor**  $D(x)$

$$\frac{a_n x^n}{b_k x^k} = q_m x^m$$

#### STEP 2

**Multiply the divisor** by this term:  $D(x) \times q_m x^m$

#### STEP 3

**Subtract this** from the **original polynomial**  $P(x)$  to cancel out the leading term:

$$R(x) = P(x) - D(x) \times q_m x^m$$

- Repeat steps 1 – 3 using the new polynomial  $R(x)$  in place of  $P(x)$  until the subtraction results in an expression for  $R(x)$  with degree less than the divisor

The quotient  $Q(x)$  is the **sum of the terms** you multiplied the divisor by:

$$Q(x) = q_m x^m + q_{m-1} x^{m-1} + \dots + q_1 x + q_0$$

The remainder  $R(x)$  is the polynomial after the final subtraction

#### Division by linear functions

- If  $P(x)$  has degree  $n$  and is divided by a linear function  $(ax + b)$  then
  - $\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$  where
    - $ax + b$  is the **divisor** (degree 1)
    - $Q(x)$  is the **quotient** (degree  $n - 1$ )
    - $R$  is the **remainder** (degree 0)
  - Note that  $P(x) = Q(x) \times (ax + b) + R$

#### Division by quadratic functions

- If  $P(x)$  has degree  $n$  and is divided by a quadratic function  $(ax^2 + bx + c)$  then
  - $\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{ex + f}{ax^2 + bx + c}$  where
    - $ax^2 + bx + c$  is the **divisor** (degree 2)
    - $Q(x)$  is the **quotient** (degree  $n - 2$ )

- $ex + f$  is the **remainder** (degree less than 2)
- The remainder will be **linear** (degree 1) if  $e \neq 0$ , and **constant** (degree 0) if  $e = 0$
- Note that  $P(x) = Q(x) \times (ax^2 + bx + c) + ex + f$

### Division by polynomials of degree $k \leq n$

- If  $P(x)$  has degree  $n$  and is divided by a polynomial  $D(x)$  with degree  $k \leq n$ 
  - $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$  where
    - $D(x)$  is the **divisor** (degree  $k$ )
    - $Q(x)$  is the **quotient** (degree  $n - k$ )
    - $R(x)$  is the **remainder** (degree less than  $k$ )
  - Note that  $P(x) = Q(x) \times D(x) + R(x)$

### Are there other methods for dividing polynomials?

- **Synthetic division** is a faster and shorter way of setting out a division when dividing by a linear term of the form
  - To divide  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  by  $(x - c)$ :
    - Set  $b_n = a_n$
    - Calculate  $b_{n-1} = a_{n-1} + c \times b_n$
    - Continue this iterative process  $b_{i-1} = a_{i-1} + c \times a_i$
    - The quotient is  $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \dots + b_2 x + b_1$  and the remainder is  $r = b_0$
- You can also find quotients and remainders by **comparing coefficients**
  - Given a polynomial  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
  - And a divisor  $D(x) = d_k x^k + d_{k-1} x^{k-1} + \dots + d_1 x + d_0$
  - Write  $Q(x) = q_{n-k} x^{n-k} + \dots + q_1 x + q_0$  and  $R(x) = r_{k-1} x^{k-1} + \dots + r_1 x + r_0$
  - Write  $P(x) = Q(x)D(x) + R(x)$ 
    - Expand the right-hand side
    - Equate the coefficients
    - Solve to find the unknowns  $q$ 's &  $r$ 's



#### Exam Tip

- In an exam you can use whichever method to divide polynomials – just make sure your method is written clearly so that if you make a mistake you can still get a mark for your method!



### ? Worked Example

a)

Perform the division  $\frac{x^4 + 11x^2 - 1}{x + 3}$ . Hence write  $x^4 + 11x^2 - 1$  in the form

$$Q(x) \times (x + 3) + R.$$

Step 1: what do we multiply  $x$  by to get  $x^4$ ?

$$\begin{array}{r} x^3 \\ x+3 \overline{) x^4 + 0x^3 + 11x^2 + 0x - 1} \end{array}$$

Note:  $0x^3$  and  $0x$  are used to keep like terms together.

Step 2: subtract  $x^3(x+3) = x^4 + 3x^3$  from  $x^4 + 0x^3$

$$\begin{array}{r} x^3 \\ x+3 \overline{) x^4 + 0x^3 + 11x^2 + 0x - 1} \\ \underline{-(x^4 + 3x^3)} \\ -3x^3 \end{array}$$



Step 3: bring the  $11x^2$  down and return to step 1.

$$\begin{array}{r}
 x^3 - 3x^2 + 20x - 60 \\
 x + 3 \overline{) x^4 + 0x^3 + 11x^2 + 0x - 1} \\
 \underline{-(x^4 + 3x^3)} \phantom{+ 0x^2 + 0x - 1} \\
 -3x^3 + 11x^2 \phantom{+ 0x - 1} \\
 \underline{-(-3x^3 - 9x^2)} \phantom{+ 0x - 1} \\
 20x^2 + 0x \phantom{- 1} \\
 \underline{-(20x^2 + 60x)} \phantom{- 1} \\
 -60x - 1 \\
 \underline{-(-60x - 180)} \\
 179
 \end{array}$$

$$\begin{aligned}
 &x^4 + 11x^2 - 1 \\
 &= (x^3 - 3x^2 + 20x - 60)(x + 3) + 179
 \end{aligned}$$

b)

Find the quotient and remainder for  $\frac{x^4 + 4x^3 - x + 1}{x^2 - 2x}$ . Hence write  $x^4 + 4x^3 - x + 1$  in the form  $Q(x) \times (x^2 - 2x) + R(x)$ .

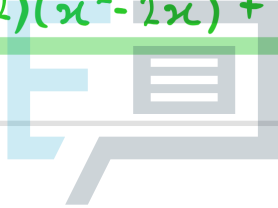




When dividing by quadratics use the same steps as above.

$$\begin{array}{r} x^2 + 6x + 12 \\ x^2 - 2x \overline{) x^4 + 4x^3 + 0x^2 - x + 1} \\ \underline{-(x^4 - 2x^3)} \phantom{+ 1} \\ 6x^3 + 0x^2 \phantom{- x + 1} \\ \underline{-(6x^3 - 12x^2)} \phantom{+ 1} \\ 12x^2 - x \phantom{+ 1} \\ \underline{-(12x^2 - 24x)} \phantom{+ 1} \\ 23x + 1 \end{array}$$

$$\begin{aligned} & x^4 + 4x^3 - x + 1 \\ & = (x^2 + 6x + 12)(x^2 - 2x) + 23x + 1 \end{aligned}$$





## 2.7.3 Polynomial Functions

### Sketching Polynomial Graphs

In exams you'll commonly be asked to sketch the graphs of different polynomial functions with and without the use of your GDC.

#### What's the relationship between a polynomial's degree and its zeros?

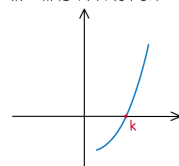
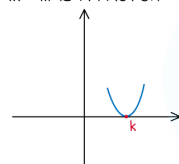
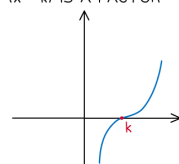
- If a **real polynomial**  $P(x)$  has **degree  $n$** , it will have  **$n$  zeros** which can be written in the form  $a + bi$ , where  $a, b \in \mathbb{R}$ 
  - For example:
    - A quadratic will have 2 zeros
    - A cubic function will have 3 zeros
    - A quartic will have 4 zeros
  - Some of the zeros may be **repeated**
- Every **real polynomial** of **odd degree** has **at least one real zero**

#### How do I sketch the graph of a polynomial function without a GDC?

- Suppose  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  is a **real polynomial** with **degree  $n$**
- To sketch the graph of a polynomial you need to know three things:
  - The **y-intercept**  
Find this by **substituting  $x = 0$**  to get  **$y = a_0$**
  - The **roots**  
You can find these by **factorising** or solving  **$y = 0$**
  - The **shape**  
This is determined by the **degree ( $n$ )** and the sign of the **leading coefficient ( $a_n$ )**

#### How does the multiplicity of a real root affect the graph of the polynomial?

- The **multiplicity** of a root is the number of times it is **repeated** when the polynomial is factorised
  - If  $x = k$  is a root with **multiplicity  $m$**  then  $(x - k)^m$  is a **factor** of the polynomial
- The graph either **crosses** the x-axis or **touches** the x-axis at a **root  $x = k$**  where  $k$  is a real number
  - If  $x = k$  has **multiplicity 1** then the graph **crosses** the x-axis at  $(k, 0)$
  - If  $x = k$  has **multiplicity 2** then the graph has a **turning point** at  $(k, 0)$  so **touches** the x-axis
    - If  $x = k$  has **odd multiplicity  $m \geq 3$**  then the graph has a **stationary point of inflection** at  $(k, 0)$  so **crosses** the x-axis
    - If  $x = k$  has **even multiplicity  $m \geq 4$**  then the graph has a **turning point** at  $(k, 0)$  so **touches** the x-axis

 $(x - k)$  IS A FACTORCURVE CROSSES THE  $x$  - AXIS $(x - k)^2$  IS A FACTORCURVE TOUCHES THE  $x$  - AXIS  
AT THE TURNING POINT $(x - k)^3$  IS A FACTORCURVE CROSSES THE  $x$  - AXIS  
AT THE STATIONARY POINT OF INFLECTION

## How do I determine the shape of the graph of the polynomial?

- Consider what happens as  $x$  tends to  $\pm \infty$ 
  - If  $a_n$  is **positive** and  $n$  is **even** then the graph **approaches from the top left and tends to the top right**

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow +\infty} f(x) = +\infty$$
  - If  $a_n$  is **negative** and  $n$  is **even** then the graph **approaches from the bottom left and tends to the bottom right**

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow +\infty} f(x) = -\infty$$
  - If  $a_n$  is **positive** and  $n$  is **odd** then the graph **approaches from the bottom left and tends to the top right**

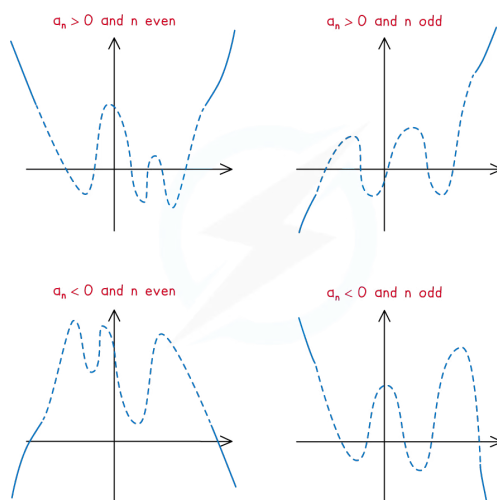
$$\lim_{x \rightarrow -\infty} f(x) = -\infty \text{ and } \lim_{x \rightarrow +\infty} f(x) = +\infty$$
  - If  $a_n$  is **negative** and  $n$  is **odd** then the graph **approaches from the top left and tends to the bottom right**

$$\lim_{x \rightarrow -\infty} f(x) = +\infty \text{ and } \lim_{x \rightarrow +\infty} f(x) = -\infty$$
- Once you know the **shape**, the **real roots** and the **y-intercept** then you simply connect the points using a **smooth curve**
- There will be **at least one turning point** in-between each pair of roots
  - If the degree is  $n$  then there is **at most  $n - 1$  stationary points** (some will be **turning points**)
    - Every real polynomial of **even degree** has **at least one turning point**
    - Every real polynomial of **odd degree bigger than 1** has **at least one point of inflection**
  - If it is a calculator paper then you can use your GDC to find the coordinates of the turning points



- You won't need to find their location without a GDC unless the question asks you to

$$y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$



### Exam Tip

- If it is a calculator paper then you can use your GDC to find the coordinates of any turning points
- If it is the non-calculator paper then you will not be required to find the turning points when sketching unless specifically asked to



## ? Worked Example

a)

The function  $f$  is defined by  $f(x) = (x+1)(2x-1)(x-2)^2$ . Sketch the graph of  $y = f(x)$ .

Find the y-intercept

$$x = 0: y = (1)(-1)(-2)^2 = -4$$

Find the roots and determine if graphs crosses or touches the x-axis

$$(x+1)(2x-1)(x-2)^2$$

$$(-1, 0) \quad \left(\frac{1}{2}, 0\right) \quad (2, 0)$$

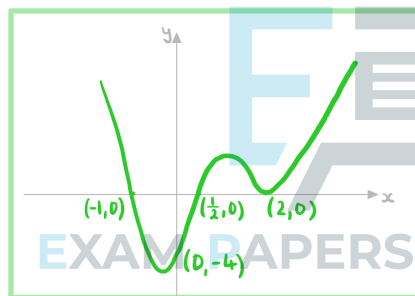
cross cross touch

Determine the shape by looking at the leading term

$$\text{Leading term is } (x)(2x)(x)^2 = 2x^4$$

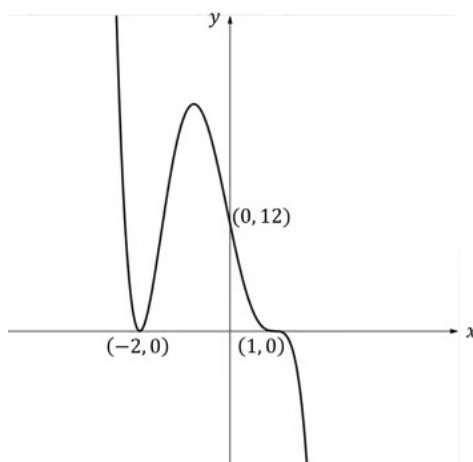
$$\text{As } x \rightarrow -\infty \quad y \rightarrow +\infty$$

$$\text{As } x \rightarrow +\infty \quad y \rightarrow +\infty$$



b)

The graph below shows a polynomial function. Find a possible equation of the polynomial.





Touches at  $(-2,0)$   $(x+2)^2$  is a factor

Point of inflection at  $(1,0)$   $(x-1)^3$  is a factor

Write in the form of:  $y = a(x+2)^2(x-1)^3$

Use the y-intercept to find a

$$12 = a(2)^2(-1)^3 \Rightarrow -4a = 12 \quad \therefore a = -3$$

$$y = -3(x+2)^2(x-1)^3$$



## Solving Polynomial Equations

### What is “The Fundamental Theorem of Algebra”?

- Every **real polynomial** with degree  $n$  can be factorised into  $n$  **complex linear factors**
  - Some of which may be **repeated**
  - This means the polynomial will have  $n$  zeros (some may be repeats)
- Every **real polynomial** can be expressed as a product of **real linear factors** and **real irreducible quadratic factors**
  - An irreducible quadratic is where it **does not have real roots**  
The **discriminant** will be negative:  $b^2 - 4ac < 0$
- If  $a + bi$  ( $b \neq 0$ ) is a **zero** of a **real polynomial** then its **complex conjugate**  $a - bi$  is also a **zero**
- Every **real polynomial** of **odd degree** will have **at least one real zero**

### How do I solve polynomial equations?

- Suppose you have an equation  $P(x) = 0$  where  $P(x)$  is a **real polynomial of degree  $n$** 
  - $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
- You may be given one zero or you might have to find a zero  $x = k$  by substituting values into  $P(x)$  until it equals 0
- If you know a **root** then you know a **factor**
  - If you know  $x = k$  is a **root** then  **$(x - k)$  is a factor**
  - If you know  $x = a + bi$  is a **root** then you know a **quadratic factor**  $(x - (a + bi))(x - (a - bi))$   
Which can be written as  $((x - a) - bi)((x - a) + bi)$  and **expanded quickly using difference of two squares**
- You can then **divide**  $P(x)$  by this factor to get **another factor**
  - For example: dividing a cubic by a linear factor will give you a quadratic factor
- You then may be able to factorise this new factor

### Exam Tip

- If a polynomial has three or less terms check whether a substitution can turn it into a quadratic
  - For example:  $x^6 + 3x^3 + 2$  can be written as  $(x^3)^2 + 3(x^3) + 2$

**Worked Example**

Given that  $x = \frac{1}{2}$  is a zero of the polynomial defined by  $f(x) = 2x^3 - 3x^2 + 5x - 2$ ,  
find all three zeros of  $f$ .

$x = \frac{1}{2}$  is a root  $\therefore (2x-1)$  is a factor

Find the quadratic factor  $(2x^3 - 3x^2 + 5x - 2) = (2x-1)(ax^2 + bx + c)$

Compare coefficients :  $2x^3 = 2ax^3 \quad \therefore a=1$

$-2 = -c \quad \therefore c=2$

$5x = 2cx - bx \Rightarrow 5 = 4 - b \quad \therefore b=-1$

Solve the quadratic :  $x^2 - x + 2 = 0$

Formula booklet

|                                   |                                                                        |
|-----------------------------------|------------------------------------------------------------------------|
| Solutions of a quadratic equation | $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
|-----------------------------------|------------------------------------------------------------------------|

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{-7}}{2} = \frac{1}{2} \pm \frac{\sqrt{7}}{2} i$$

$$\text{Roots : } \frac{1}{2}, \frac{1}{2} + \frac{\sqrt{7}}{2} i, \frac{1}{2} - \frac{\sqrt{7}}{2} i$$





## 2.7.4 Roots of Polynomials

### Sum & Product of Roots

#### How do I find the sum & product of roots of polynomials?

- Suppose  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  is a **polynomial** of **degree**  $n$  with  $n$  roots  $\alpha_1, \alpha_2, \dots, \alpha_n$

- The polynomial is written as  $\sum_{r=0}^n a_r x^r = 0$ ,  $a_n \neq 0$  in the **formula booklet**

- $a_n$  is the coefficient of the **leading term**

- $a_{n-1}$  is the coefficient of the  **$x^{n-1}$  term**

Be careful: this could be equal to zero

- $a_0$  is the **constant term**

Be careful: this could be equal to zero

- In factorised form:  $P(x) = a_n (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$

- Comparing coefficients of the  **$x^{n-1}$  term** and the **constant term** gives

$$a_{n-1} = a_n (-\alpha_1 - \alpha_2 - \dots - \alpha_n)$$

$$a_0 = a_n (-\alpha_1) \times (-\alpha_2) \times \dots \times (-\alpha_n)$$

- The **sum** of the roots is given by:

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = -\frac{a_{n-1}}{a_n}$$

- The **product** of the roots is given by:

$$\alpha_1 \times \alpha_2 \times \dots \times \alpha_n = \frac{(-1)^n a_0}{a_n}$$

both of these formulae are in your **formula booklet**

#### How can I find unknowns if I am given the sum and/or product of the roots of a polynomial?

- If you know a complex root of a real polynomial then its **complex conjugate** is **another root**
- Form **two equations** using the roots
  - One using the **sum of the roots formula**
  - One using the **product of the roots formula**
- Solve** for any unknowns



#### Exam Tip

- Examiners might trick you by not having an  $x^{n-1}$  term or a constant term
- To make sure you do not get tricked you can write out the full polynomial using 0 as a coefficient where needed
  - For example: Write  $x^4 + 2x^2 - 5x$  as  $x^4 + 0x^3 + 2x^2 - 5x + 0$



## ? Worked Example

$2 - 3i$ ,  $\frac{5}{3}i$  and  $\alpha$  are three roots of the equation

$$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k = 0.$$

a)

Use the sum of all the roots to find the value of  $\alpha$ .

It is a real polynomial so if  $a+bi$  is a root then  $a-bi$  is also a root

$$\text{Roots: } 2-3i, 2+3i, \frac{5}{3}i, -\frac{5}{3}i, \alpha$$

Formula booklet

|                                                                                           |                               |
|-------------------------------------------------------------------------------------------|-------------------------------|
| Sum & product of the roots of polynomial equations of the form $\sum_{r=0}^n a_r x^r = 0$ | Sum is $-\frac{a_{n-1}}{a_n}$ |
|-------------------------------------------------------------------------------------------|-------------------------------|

$$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k$$

$$a_n = 18 \quad a_{n-1} = -9$$

$$(2-3i) + (2+3i) + \left(\frac{5}{3}i\right) + \left(-\frac{5}{3}i\right) + \alpha = \frac{-(-9)}{18}$$

$$4 + \alpha = \frac{1}{2}$$

$$\alpha = -\frac{7}{2}$$

b)

Use the product of all the roots to find the value of  $k$ .

Formula booklet

|                                                                                           |                                     |
|-------------------------------------------------------------------------------------------|-------------------------------------|
| Sum & product of the roots of polynomial equations of the form $\sum_{r=0}^n a_r x^r = 0$ | product is $\frac{(-1)^n a_0}{a_n}$ |
|-------------------------------------------------------------------------------------------|-------------------------------------|

$$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k$$

$$a_n = 18 \quad n = 5 \quad a_0 = k$$

$$(2-3i)(2+3i)\left(\frac{5}{3}i\right)\left(-\frac{5}{3}i\right)\left(-\frac{7}{2}\right) = \frac{(-1)^5 k}{18}$$

$$(13)\left(\frac{25}{9}\right)\left(-\frac{7}{2}\right) = \frac{-k}{18}$$

$$-\frac{2275}{18} = -\frac{k}{18}$$

$$k = 2275$$

## 2.8 Inequalities

### 2.8.1 Solving Inequalities Graphically

#### Solving Inequalities Graphically

##### How can I solve inequalities graphically?

- Consider the inequality  $f(x) \leq g(x)$ , where  $f(x)$  and  $g(x)$  are functions of  $x$ 
  - if we **move  $g(x)$  to the LHS** we get  

$$f(x) - g(x) \leq 0$$
- Solve  $f(x) - g(x) = 0$  to find the **zeros** of  $f(x) - g(x)$ 
  - These correspond to the  $x$ -coordinates of the points of intersection of the graphs  $y = f(x)$  and  $y = g(x)$
- To solve the inequality we can use a **graph**
  - Graph  $y = f(x) - g(x)$**  and label its zeros
  - Hence find the intervals of  $x$  that satisfy the inequality  $f(x) - g(x) \leq 0$   
 These are the **intervals which satisfies the original inequality**  $f(x) \leq g(x)$
  - This method is particularly useful when finding the intersections between the functions is difficult due to needing **large  $x$  and  $y$  windows** on your GDC

##### Be careful when rearranging inequalities!

- Remember to **flip the sign** of the inequality when you **multiply or divide** both sides by a **negative** number
  - e.  $1 < 2 \rightarrow [\text{times both sides by } (-1)] \rightarrow -1 > -2$  (sign flips)
- Never multiply or divide by a variable** as this could be **positive or negative**
  - You can only multiply by a term if you are certain it is always positive (or always negative)  
 Such as  $x^2$ ,  $|x|$ ,  $e^x$
- Some **functions reverse the inequality**
  - Taking reciprocals of positive values  

$$0 < x < y \Rightarrow \frac{1}{x} > \frac{1}{y}$$
  - Taking logarithms when the base is  $0 < a < 1$   

$$0 < x < y \Rightarrow \log_a(x) > \log_a(y)$$
- The **safest way** to rearrange is simply to add & subtract to move all the terms onto one side



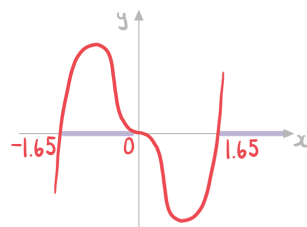
### ? Worked Example

Use a GDC to solve the inequality  $2x^3 < x^5 - 2x$ .

Rearrange to get one side as zero

$$x^5 - 2x^3 - 2x > 0$$

On GDC sketch  $y = x^5 - 2x^3 - 2x$  and find zeros



Identify the sections where the graph is above the x-axis

$$-1.65 < x < 0 \text{ or } x > 1.65$$



## 2.8.2 Polynomial Inequalities

### Polynomial Inequalities

#### How do I solve polynomial inequalities?

- **STEP 1: Rearrange the inequality** so that **one of the sides is equal to zero**
  - For example:  $P(x) \leq 0$
- **STEP 2:** Find the **roots** of the polynomial
  - You can do this by factorising or using GDC to solve  $P(x) = 0$
- **STEP 3:** Choose one of the following methods:
  - **Graph method**
    - Sketch a graph of the polynomial (with or without a GDC)
    - Choose the intervals for  $x$  corresponding to the sections of the graph that satisfy the inequality
 

For example: for  $P(x) \leq 0$  you would want the sections below the  $x$ -axis
  - **Sign table method**
    - If you are unsure how to sketch a polynomial graph then this method is best
    - **Split the real numbers** into the possible **intervals** using the roots
 

If the roots are  $a$  and  $b$  then the intervals would be  $x < a$ ,  $a < x < b$ ,  $x > b$
    - **Test a value** from each interval using the inequality
 

Choose a value within an interval and substitute into  $P(x)$  to determine if it is positive or negative
    - Alternatively if the polynomial is factorised you can **determine the sign of each factor** in each interval
 

An odd number of negative factors in an interval will mean the polynomial is negative on that interval
    - If the value satisfies the inequality then that interval is part of the solution



#### Exam Tip

- In exams most solutions will be intervals but some could be a single point
  - For example: Solution to  $(x - 3)^2 \leq 0$  is  $x = 3$



## ? Worked Example

Solve the inequality  $x^3 + 2x^2 > x + 2$  using an algebraic method.

Rearrange  $x^3 + 2x^2 - x - 2 > 0$

Let  $P(x) = x^3 + 2x^2 - x - 2$

Find a factor  $P(1) = 0 \therefore (x-1)$  is a factor

Factorise  $(x-1)(x^2 + 3x + 2) > 0$  Compare coefficients or use division

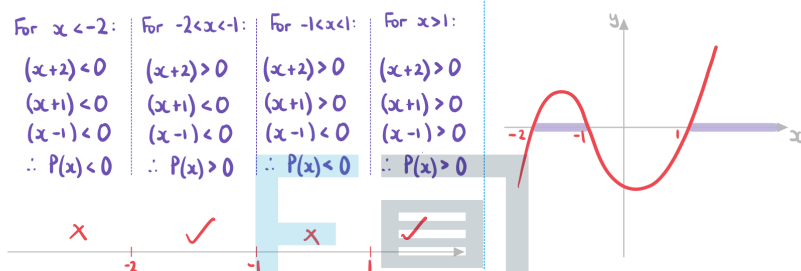
$(x-1)(x+1)(x+2) > 0$

Find the roots  $1, -1, -2$

Construct a sign table

| For $x < -2$ :        | For $-2 < x < -1$ :   | For $-1 < x < 1$ :    | For $x > 1$ :         |
|-----------------------|-----------------------|-----------------------|-----------------------|
| $(x+2) < 0$           | $(x+2) > 0$           | $(x+2) > 0$           | $(x+2) > 0$           |
| $(x+1) < 0$           | $(x+1) < 0$           | $(x+1) > 0$           | $(x+1) > 0$           |
| $(x-1) < 0$           | $(x-1) < 0$           | $(x-1) < 0$           | $(x-1) > 0$           |
| $\therefore P(x) < 0$ | $\therefore P(x) > 0$ | $\therefore P(x) < 0$ | $\therefore P(x) > 0$ |

Or sketch



Choose the regions that satisfy the inequality

$-2 < x < -1$  or  $x > 1$

## 2.9 Further Functions & Graphs

### 2.9.1 Modulus Functions

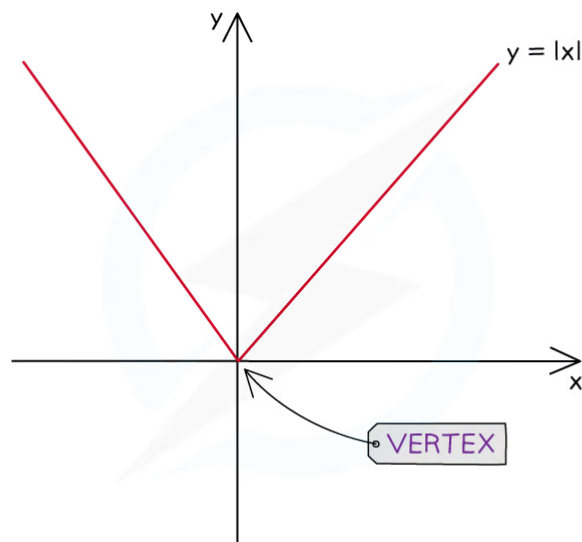
#### Modulus Functions & Graphs

##### What is the modulus function?

- The **modulus function** is defined by  $f(x) = |x|$ 
  - $|x| = \sqrt{x^2}$
  - Equivalently it can be defined  $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all real non-negative values**
- The modulus function gives the **distance** between 0 and  $x$ 
  - This is also called the **absolute value** of  $x$

##### What are the key features of the modulus graph: $y = |x|$ ?

- The graph has a **y-intercept** at  $(0, 0)$
- The graph has **one root** at  $(0, 0)$
- The graph has a **vertex** at  $(0, 0)$
- The graph is **symmetrical** about the **y-axis**
- At the **origin**
  - The function is **continuous**
  - The function is **not differentiable**



##### What are the key features of the modulus graph: $y = a|x + p| + q$ ?

- Every **modulus graph** which is formed by **linear transformations** can be written in this form using key features of the modulus function

- $|ax| = |a||x|$   
For example:  $|2x + 1| = 2\left|x + \frac{1}{2}\right|$
- $|p - x| = |x - p|$   
For example:  $|4 - x| = |x - 4|$
- The graph has a **y-intercept** when  $x = 0$
- The graph can have 0, 1 or 2 **roots**
  - If  $a$  and  $q$  have the **same sign** then there will be **0 roots**
  - If  $q = 0$  then there will be **1 root** at  $(-p, 0)$
  - If  $a$  and  $q$  have **different signs** then there will be **2 roots** at  $\left(-p \pm \frac{q}{a}, 0\right)$
- The graph has a **vertex** at  $(-p, q)$
- The graph is **symmetrical** about the line  $x = -p$
- The value of  $a$  determines the **shape** and the **steepness** of the graph
  - If  $a$  is **positive** the graph looks like  $\vee$
  - If  $a$  is **negative** the graph looks like  $\wedge$
  - The **larger** the value of  $|a|$  the **steeper** the lines
- At the **vertex**
  - The function is **continuous**
  - The function is **not differentiable**



## 2.9.2 Modulus Transformations

### Modulus Transformations

**How do I sketch the graph of the modulus of a function:  $y = |f(x)|$ ?**

- **STEP 1:** Keep the parts of the graph of  $y = f(x)$  that are **on or above the x-axis**
- **STEP 2:** Any parts of the **graph below the x-axis** get **reflected** in the x-axis anything

**How do I sketch the graph of a function of a modulus:  $y = f(|x|)$ ?**

- **STEP 1:** Keep the graph of  $y = f(x)$  **only for  $x \geq 0$**
- **STEP 2:** **Reflect** this in the **y-axis**

**What is the difference between  $y = |f(x)|$  and  $y = f(|x|)$ ?**

- The graph of  $y = |f(x)|$  **never goes below the x-axis**
  - It does not have to have any lines of symmetry
- The graph of  $y = f(|x|)$  is **always symmetrical about the y-axis**
  - It can go below the y-axis

**When multiple transformations are involved how do I determine the order?**

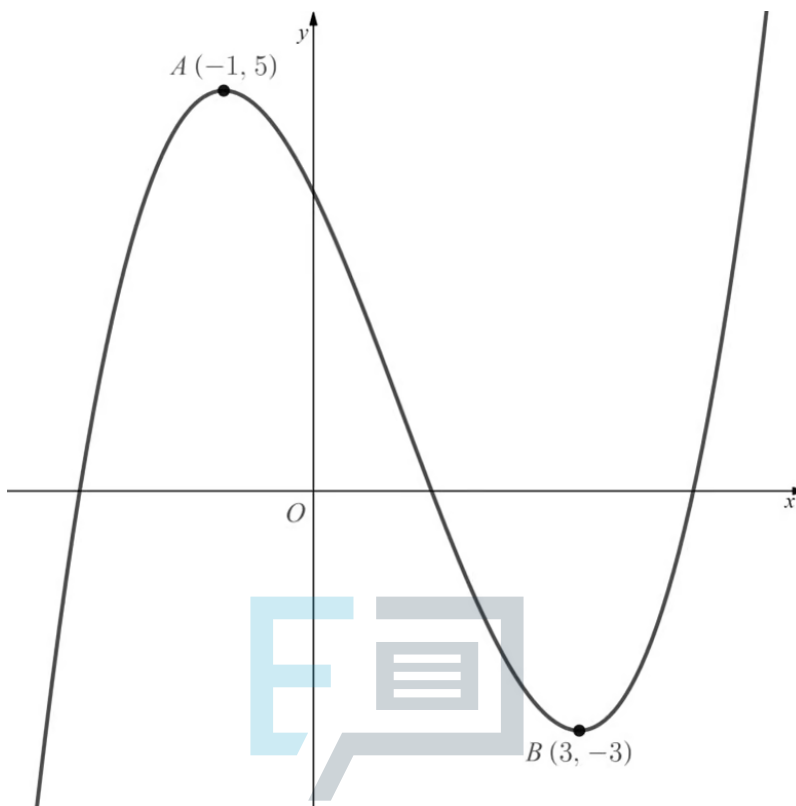
- The transformations **outside the function** follow the **same order** as the **order of operations**
  - $y = |af(x) + b|$   
Deal with the  $a$  then the  $b$  then the modulus
  - $y = a|f(x)| + b$   
Deal with the modulus then the  $a$  then the  $b$
- The transformations **inside the function** are in the **reverse order** to the **order of operations**
  - $y = f(|ax + b|)$   
Deal with the modulus then the  $b$  then the  $a$
  - $y = f(a|x| + b)$   
Deal with the  $b$  then the  $a$  then the modulus

#### Exam Tip

- When sketching one of these transformations in an exam question make sure that the graphs do not look smooth at the points where the original graph have been reflected
  - For  $y = |f(x)|$  the graph should look "sharp" at the points where it has been reflected on the x-axis
  - For  $y = f(|x|)$  the graph should look "sharp" at the point where it has been reflected on the y-axis

## ? Worked Example

The diagram below shows the graph of  $y = f(x)$ .



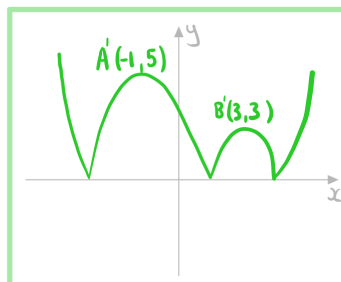
(a) Sketch the graph of  $y = |f(x)|$ .

If the graph is on or above the  $x$ -axis then it stays the same

If the graph is below the  $x$ -axis then it is reflected in the  $x$ -axis

A stays the same  $(-1, 5)$

B becomes  $(3, 3)$



(b) Sketch the graph of  $y = f(|x|)$ .

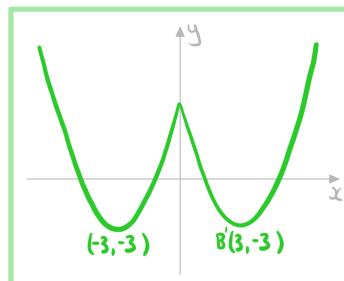


keep the graph for  $x \geq 0$

Reflect this in the y-axis

A disappears

B stays the same  $(3, -3)$



## 2.9.3 Modulus Equations & Inequalities

### Modulus Equations

#### How do I find the modulus of a function?

- The modulus of a function  $f(x)$  is
  - $|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$  or
  - $|f(x)| = \sqrt{[f(x)]^2}$

#### How do I solve modulus equations graphically?

- To solve  $|f(x)| = g(x)$  graphically
  - Draw  $y = |f(x)|$  and  $y = g(x)$  into your GDC
  - Find the  $x$ -coordinates of the **points of intersection**

#### How do I solve modulus equations analytically?

- To solve  $|f(x)| = g(x)$  analytically
  - Form **two equations**

$$f(x) = g(x)$$

$$f(x) = -g(x)$$
  - Solve both equations
  - **Check solutions** work in the original equation
 

For example:  $x - 2 = 2x - 3$  has solution  $x = 1$   
 But  $|(1) - 2| = 1$  and  $2(1) - 3 = -1$   
 So  $x = 1$  is not a solution to  $|x - 2| = 2x - 3$



## Worked Example

Solve for  $x$ :

a)

$$\left| \frac{2x+3}{2-x} \right| = 5$$

Analytically  
Split into two equations

$$\frac{2x+3}{2-x} = \pm 5$$

Solve individually

$$\frac{2x+3}{2-x} = 5$$

$$2x+3 = 10-5x$$

$$7x = 7$$

$$x = 1$$

$$\frac{2x+3}{2-x} = -5$$

$$2x+3 = 5x-10$$

$$13 = 3x$$

$$x = \frac{13}{3}$$

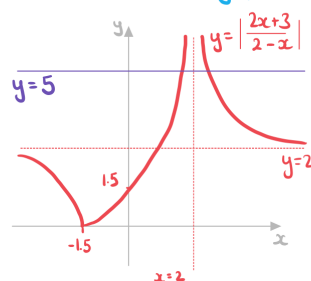
Check:

$$\left| \frac{2(1)+3}{2-(1)} \right| = 5 \checkmark$$

$$\left| \frac{2(\frac{13}{3})+3}{2-(\frac{13}{3})} \right| = 5 \checkmark$$

$$x = 1 \text{ or } x = \frac{13}{3}$$

Graphically  
Sketch the two graphs



Find the points of intersection

$$(1, 5) \quad (4.33, 5)$$

Choose the  $x$ -coordinates

$$x = 1 \text{ or } x = 4.33 \text{ (3sf)}$$

b)

$$|3x-1| = 5x-11$$

Analytically  
Split into two equations

$$3x-1 = \pm(5x-11)$$

Solve individually

$$3x-1 = 5x-11$$

$$10 = 2x$$

$$x = 5$$

$$3x-1 = 11-5x$$

$$8x = 12$$

$$x = 1.5$$

Check:

$$|3(5)-1| = 14 \checkmark$$

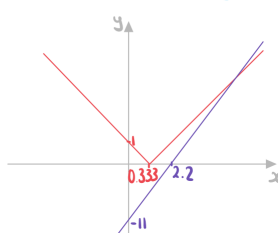
$$5(5)-11 = 14 \checkmark$$

$$|3(1.5)-1| = 3.5$$

$$5(1.5)-11 = -3.5 \times$$

$$x = 5$$

Graphically  
Sketch the two graphs



Find the points of intersection

$$(5, 14)$$

Choose the  $x$ -coordinates

$$x = 5$$

## Modulus Inequalities

### How do I solve modulus inequalities analytically?

- To solve **any** modulus inequality
  - First solve the corresponding modulus equation  
Remembering to **check whether solutions are valid**
  - Then use a graphical method or a sign table to find the intervals that satisfy the inequality
- Another method is to solve **two pairs of inequalities**
  - For  $|f(x)| < g(x)$  solve:  
 $f(x) < g(x)$  when  $f(x) \geq 0$   
 $f(x) > -g(x)$  when  $f(x) \leq 0$
  - For  $|f(x)| > g(x)$  solve:  
 $f(x) > g(x)$  when  $f(x) \geq 0$   
 $f(x) < -g(x)$  when  $f(x) \leq 0$



#### Exam Tip

- If a question on this appears on a calculator paper then use the same ideas as solving other inequalities
  - Sketch the graphs and find the intersections

**Worked Example**Solve the following inequalities for  $x$ .

a)

$$|2x - 1| < 4$$

Solve for  $2x - 1 \geq 0$ 

$$\text{For } x \geq \frac{1}{2}: 2x - 1 < 4 \Rightarrow x < \frac{5}{2} \quad \therefore \frac{1}{2} \leq x < \frac{5}{2}$$

Solve for  $2x - 1 \leq 0$ 

$$\text{For } x \leq \frac{1}{2}: 2x - 1 > -4 \Rightarrow x > -\frac{3}{2} \quad \therefore -\frac{3}{2} < x \leq \frac{1}{2}$$

Combine inequalities

$$-\frac{3}{2} < x < \frac{5}{2}$$

b)

$$|x + 1| < |2x + 3|$$

Solve the corresponding equation

$$|x + 1| = |2x + 3| \Rightarrow x + 1 = \pm(2x + 3)$$

$$\text{Solve } x + 1 = 2x + 3$$

$$x = -2$$

$$x + 1 = -2x - 3$$

$$x = -\frac{4}{3}$$

$$\text{Check } |(-2) + 1| = 1$$

$$|2(-2) + 3| = 1$$

$$|(-\frac{4}{3}) + 1| = \frac{1}{3}$$

$$|2(-\frac{4}{3}) + 3| = \frac{1}{3}$$

Use a sign table

Check  $x = -3$ 

$$|(-3) + 1| < |2(-3) + 3|$$
$$2 < 3$$

True

✓

Check  $x = -1.5$ 

$$|(-1.5) + 1| < |2(-1.5) + 3|$$
$$0.5 < 0$$

False

✗

Check  $x = 0$ 

$$|0 + 1| < |2(0) + 3|$$
$$1 < 3$$

True

✓

Write solution

$$x < -2 \text{ or } x > -\frac{4}{3}$$



## 2.9.4 Reciprocal & Square Transformations

### Reciprocal Transformations

What effects do reciprocal transformations have on the graphs?

- The **x-coordinates stay the same**
- The **y-coordinates change**
  - Their values become their **reciprocals**
- The coordinates  $(x, y)$  become  $\left(x, \frac{1}{y}\right)$  where  $y \neq 0$ 
  - If  $y = 0$  then a vertical asymptote goes through the original coordinate
  - Points that lie on the line  **$y = 1$**  or the line  **$y = -1$**  stay the same

How do I sketch the graph of the reciprocal of a function:  $y = 1/f(x)$ ?

- Sketch the **reciprocal transformation** by considering the **different features** of the original graph
- Consider key points on the original graph

- If  $(x_1, y_1)$  is a point on  $y = f(x)$  where  $y_1 \neq 0$

$$\left(x_1, \frac{1}{y_1}\right) \text{ is a point on } y = \frac{1}{f(x)}$$

If  $|y_1| < 1$  then the point gets **further away from the x-axis**

If  $|y_1| > 1$  then the point gets **closer to the x-axis**

- If  $y = f(x)$  has a **y-intercept** at  $(0, c)$  where  $c \neq 0$

The reciprocal graph  $y = \frac{1}{f(x)}$  has a **y-intercept** at  $\left(0, \frac{1}{c}\right)$

- If  $y = f(x)$  has a **root** at  $(a, 0)$

The reciprocal graph  $y = \frac{1}{f(x)}$  has a **vertical asymptote** at  $x = a$

- If  $y = f(x)$  has a **vertical asymptote** at  $x = a$

The reciprocal graph  $y = \frac{1}{f(x)}$  has a **discontinuity** at  $(a, 0)$

The **discontinuity** will look like a **root**

- If  $y = f(x)$  has a **local maximum** at  $(x_1, y_1)$  where  $y_1 \neq 0$

The reciprocal graph  $y = \frac{1}{f(x)}$  has a **local minimum** at  $\left(x_1, \frac{1}{y_1}\right)$

- If  $y = f(x)$  has a **local minimum** at  $(x_1, y_1)$  where  $y_1 \neq 0$



The reciprocal graph  $y = \frac{1}{f(x)}$  has a **local maximum** at  $\left(x_1, \frac{1}{y_1}\right)$

- Consider key regions on the original graph

- If  $y = f(x)$  is **positive** then  $y = \frac{1}{f(x)}$  is **positive**

If  $y = f(x)$  is **negative** then  $y = \frac{1}{f(x)}$  is **negative**

- If  $y = f(x)$  is **increasing** then  $y = \frac{1}{f(x)}$  is **decreasing**

If  $y = f(x)$  is **decreasing** then  $y = \frac{1}{f(x)}$  is **increasing**

- If  $y = f(x)$  has a **horizontal asymptote** at  $y = k$

$y = \frac{1}{f(x)}$  has a **horizontal asymptote** at  $y = \frac{1}{k}$  if  $k \neq 0$

$y = \frac{1}{f(x)}$  **tends to  $\pm \infty$**  if  $k = 0$

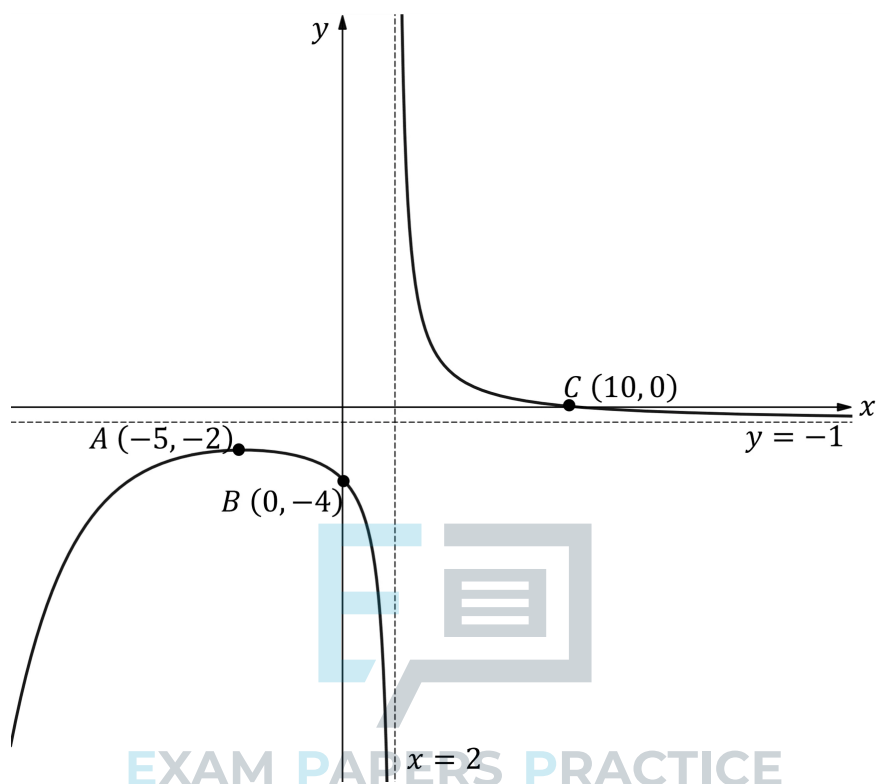
- If  $y = f(x)$  **tends to  $\pm \infty$**  as  $x$  tends to  $+\infty$  or  $-\infty$

$y = \frac{1}{f(x)}$  has a **horizontal asymptote** at  $y = 0$



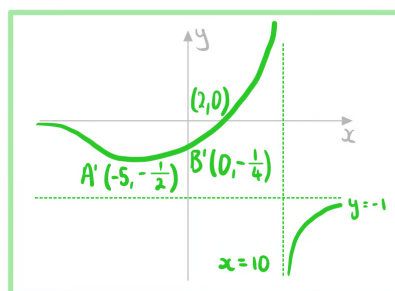
### Worked Example

The diagram below shows the graph of  $y = f(x)$  which has a local maximum at the point A.



Sketch the graph of  $y = \frac{1}{f(x)}$ .

A becomes local minimum  $(-5, -\frac{1}{2})$   
Vertical asymptote becomes root  $(2, 0)$   
B becomes  $(0, -\frac{1}{4})$   
C becomes vertical asymptote  $x = 10$   
Horizontal asymptote  $y = -1$  remains



## Square Transformations

### What effects do square transformations have on the graphs?

- The effects are **similar to** the transformation  $y = |f(x)|$ 
  - The parts **below the x-axis are reflected**
  - The **vertical distance** between a point and the x-axis is **squared**  
This has the effect of **smoothing the curve** at the x-axis
- $y = [f(x)]^2$  is **never below the x-axis**
- The **x-coordinates stay the same**
- The **y-coordinates change**
  - Their values are **squared**
- The coordinates  $(x, y)$  become  $(x, y^2)$ 
  - Points that lie on the **x-axis** or the line  **$y = 1$**  stay the same

### How do I sketch the graph of the square of a function: $y = [f(x)]^2$ ?

- Sketch the **square transformation** by considering the **different features** of the original graph
- Consider key points on the original graph
  - If  $(x_1, y_1)$  is a point on  $y = f(x)$   
 $(x_1, y_1^2)$  is a point on  $y = [f(x)]^2$   
 If  $|y_1| < 1$  then the point gets **closer to the x-axis**  
 If  $|y_1| > 1$  then the point gets **further away from the x-axis**
  - If  $y = f(x)$  has a **y-intercept** at  $(0, c)$   
 The square graph  $y = [f(x)]^2$  has a **y-intercept** at  $(0, c^2)$
  - If  $y = f(x)$  has a **root** at  $(a, 0)$   
 The square graph  $y = [f(x)]^2$  has a **root and turning point** at  $(a, 0)$
  - If  $y = f(x)$  has a **vertical asymptote** at  $x = a$   
 The square graph  $y = [f(x)]^2$  has a **vertical asymptote** at  $x = a$
  - If  $y = f(x)$  has a **local maximum** at  $(x_1, y_1)$   
 The square graph  $y = [f(x)]^2$  has a **local maximum** at  $(x_1, y_1^2)$  if  $y_1 > 0$   
 The square graph  $y = [f(x)]^2$  has a **local minimum** at  $(x_1, y_1^2)$  if  $y_1 \leq 0$
  - If  $y = f(x)$  has a **local minimum** at  $(x_1, y_1)$   
 The square graph  $y = [f(x)]^2$  has a **local minimum** at  $(x_1, y_1^2)$  if  $y_1 \geq 0$   
 The square graph  $y = [f(x)]^2$  has a **local maximum** at  $(x_1, y_1^2)$  if  $y_1 < 0$



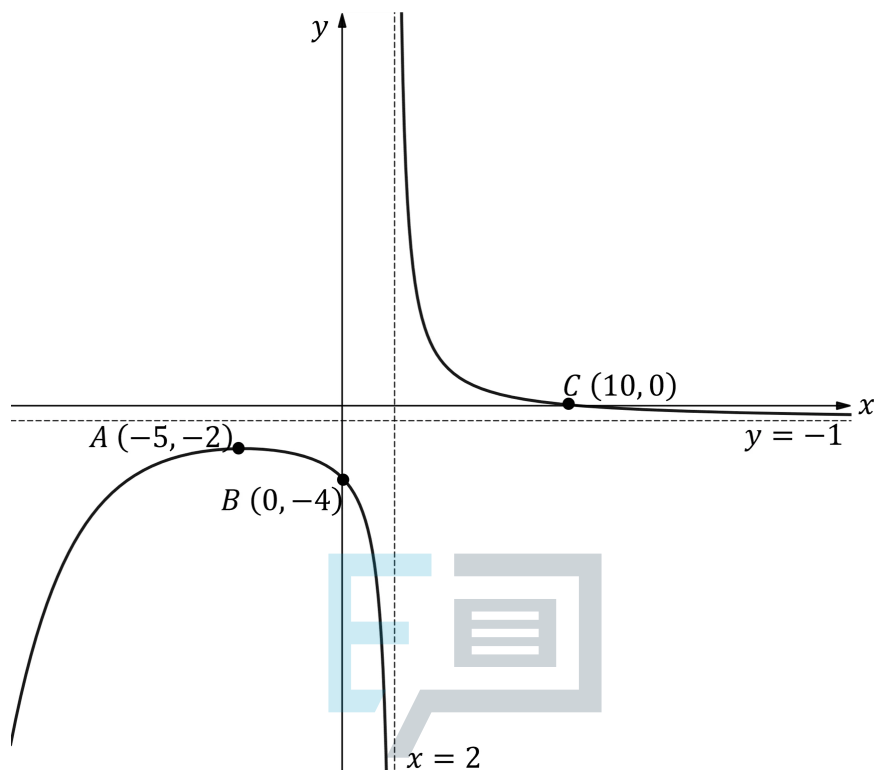
#### Exam Tip

- In an exam question when sketching  $y = [f(x)]^2$  make it clear that the points where the new graph touches the x-axis are smooth
  - This will make it clear to the examiner that you understand the difference between the roots of the graphs  $y = |f(x)|$  and  $y = [f(x)]^2$



## ? Worked Example

The diagram below shows the graph of  $y = f(x)$  which has a local maximum at the point A.



Sketch the graph of  $y = [f(x)]^2$ .

A becomes local minimum  $(-5, 4)$

Vertical asymptote  $x = 2$  remains

B becomes  $(0, 16)$

C becomes local minimum

Horizontal asymptote becomes  $y = 1$

