EXAM PAPERS PRACTICE

Boost your performance and confidence with these topic-based exam questions

Practice questions created by actual examiners and assessment experts

Detailed mark scheme

Suitable for all boards

Designed to test your ability and thoroughly prepare you

2. **Functions**

2.1 Quadratic Functions & Graphs

MATHS

AA HL

EXAM PAPERS PRACTICE

IB Maths DP

2. Functions

CONTENTS

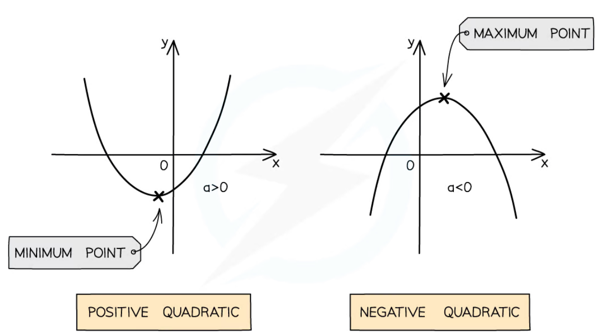
2.1 Quadratic Functions & Graphs

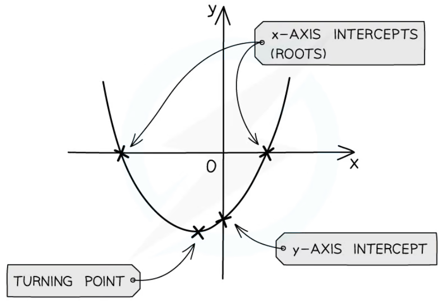
2.1.1 Quadratic Fuctions

Quadratic Functions & Graphs

What are the key features of quadratic graphs?

- A **quadratic** graph can be written in the form $y = ax^2 + bx + c$ where $a \neq 0$
- The value of *a* affects the shape of the curve
  - If *a* is **positive** the shape is **concave up** ∪
  - If *a* is **negative** the shape is **concave down** ∩
- The **y-intercept** is at the point (0, c)
- The **zeros or roots** are the solutions to $ax^2 + bx + c = 0$
  - These can be found by
    - Factorising
    - Quadratic formula
    - Using your GDC
  - These are also called the *x*-intercepts
  - There can be 0, 1 or 2 *x*-intercepts
    - This is determined by the value of the **discriminant**
- There is an **axis of symmetry** at $x = -\frac{b}{2a}$
  - This is given in your **formula booklet**
  - If there are two *x*-intercepts then the axis of symmetry goes through the midpoint of them
- The **vertex** lies on the axis of symmetry
  - It can be found by **completing the square**
  - The *x*-coordinate is $x = -\frac{b}{2a}$
  - The *y*-coordinate can be found using the GDC or by calculating *y* when $x = -\frac{b}{2a}$
  - If *a* is **positive** then the vertex is the **minimum point**
  - If *a* is **negative** then the vertex is the **maximum point**





What are the equations of a quadratic function?

- $f(x) = ax^2 + bx + c$
  - This is the **general form**
  - It clearly shows the *y*-intercept (0, *c*)
  - You can find the axis of symmetry by $x = -\frac{b}{2a}$
    - This is given in the formula booklet
- $f(x) = a(x-p)(x-q)$
  - This is the **factorised form**
  - It clearly shows the roots (*p*, 0) & (*q*, 0)
  - You can find the axis of symmetry by $x = \frac{p+q}{2}$
- $f(x) = a(x-h)^2 + k$
  - This is the **vertex form**
  - It clearly shows the vertex (*h*, *k*)
  - The axis of symmetry is therefore $x = h$
  - It clearly shows how the function can be transformed from the graph $y = x^2$
    - Vertical stretch by scale factor *a*
    - Translation by vector $\begin{pmatrix} h \\ k \end{pmatrix}$

How do I find an equation of a quadratic?

- If you have the **roots** *x* = *p* and *x* = *q*...
  - Write in **factorised form** $y = a(x-p)(x-q)$
  - You will need a third point to find the value of *a*
- If you have the **vertex** (*h*, *k*) then...
  - Write in **vertex form** $y = a(x-h)^2 + k$
  - You will need a second point to find the value of *a*
- If you have **three random points** (x_1, y_1), (x_2, y_2) & (x_3, y_3) then...
  - Write in the **general form** $y = ax^2 + bx + c$
  - Substitute the three points into the equation
  - Form and solve a system of three linear equations to find the values of *a*, *b* & *c*

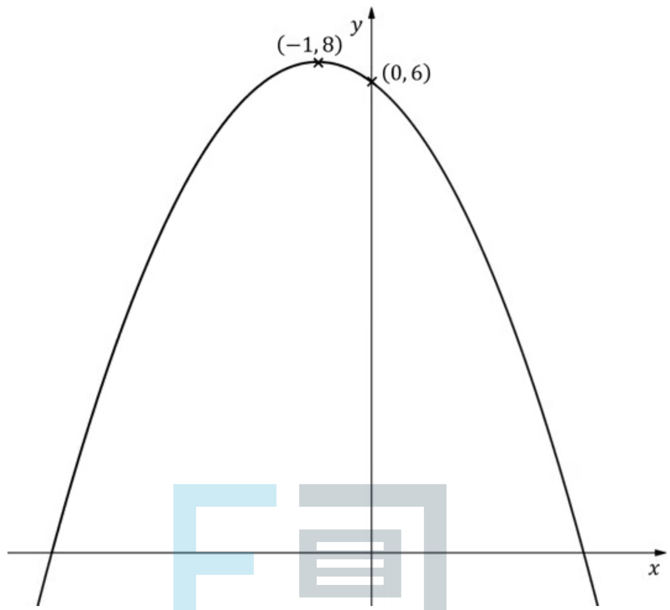
Exam Tip

- Use your GDC to find the roots and the turning point of a quadratic function
  - You do not need to factorise or complete the square
  - It is good exam technique to sketch the graph from your GDC as part of your working

Worked Example

The diagram below shows the graph of $y = f(x)$, where $f(x)$ is a quadratic function.

The intercept with the y-axis and the vertex have been labelled.



Write down an expression for $y = f(x)$.

We have the vertex so use $y = a(x-h)^2 + k$

Vertex $(-1, 8)$: $y = a(x-(-1))^2 + 8$

$y = a(x+1)^2 + 8$

Substitute the second point

$x = 0, y = 6$: $6 = a(0+1)^2 + 8$

$6 = a + 8$

$a = -2$

$y = -2(x+1)^2 + 8$

2.1.2 Factorising & Completing the Square

Factorising Quadratics

Why is factorising quadratics useful?

- Factorising gives **roots (zeroes or solutions)** of a quadratic
- It gives the ***x*-intercepts** when drawing the graph

How do I factorise a monic quadratic of the form $x^2 + bx + c$?

- A monic quadratic is a quadratic where the coefficient of the x^2 term is 1
- You might be able to spot the factors by **inspection**
  - Especially if c is a **prime number**
- Otherwise find two numbers *m* and *n* ..
  - A sum equal to *b*
    - $p + q = b$
  - A product equal to c
    - $pq = c$
- Rewrite *bx* as $mx + nx$
- Use this to factorise $x^2 + mx + nx + c$
- A shortcut is to write:
  - $(x + p)(x + q)$

How do I factorise a non-monic quadratic of the form $ax^2 + bx + c$?

- A non-monic quadratic is a quadratic where the coefficient of the x^2 term is not equal to 1
- If *a*, *b* & c have a common factor then first factorise that out to leave a quadratic with coefficients that have **no common factors**
- You might be able to spot the factors by **inspection**
  - Especially if *a* and/or c are **prime numbers**
- Otherwise find two numbers *m* and *n* ..
  - A sum equal to *b*
    - $m + n = b$
  - A product equal to *ac*
    - $mn = ac$
- Rewrite *bx* as $mx + nx$
- Use this to factorise $ax^2 + mx + nx + c$
- A shortcut is to write:
  - $\frac{(ax + m)(ax + n)}{a}$
  - Then factorise common factors from numerator to cancel with the *a* on the denominator

How do I use the difference of two squares to factorise a quadratic of the form $a^2x^2 - c^2$?

- The **difference of two squares** can be used when...
  - There is **no *x* term**
  - The **constant term is a negative**
- Square root the two terms a^2x^2 and c^2
- The two factors are the **sum of square roots** and the **difference of the square roots**

- A shortcut is to write:
  - $(ax + c)(ax - c)$

Exam Tip

- You can deduce the factors of a quadratic function by using your GDC to find the solutions of a quadratic equation
  - Using your GDC, the quadratic equation $6x^2 + x - 2 = 0$ has solutions $x = -\frac{2}{3}$ and $x = \frac{1}{2}$
  - Therefore the factors would be $(3x + 2)$ and $(2x - 1)$
  - i.e. $6x^2 + x - 2 = (3x + 2)(2x - 1)$

Worked Example

Factorise fully:

a)
$x^2 - 7x + 12$.

Find two numbers m and n such that

$m + n = b = -7$ $mn = c = 12$

$-4 + -3 = -7$ $-4 \times -3 = 12$

Split $-7x$ up and factorise

$x^2 - 4x - 3x + 12$

$x(x-4) - 3(x-4)$

$(x-3)(x-4)$

Shortcut

$(x+m)(x+n)$

$(x-3)(x-4)$

b)
$4x^2 + 4x - 15$.

Find two numbers m and n such that

$m + n = b = 4$ $mn = ac = 4 \times -15 = -60$

$10 + -6 = 4$ $10 \times -6 = -60$

Split $4x$ up and factorise

$4x^2 + 10x - 6x - 15$

$2x(2x+5) - 3(2x+5)$

$(2x-3)(2x+5)$

Shortcut

$\frac{(ax+m)(ax+n)}{a}$

$\frac{(4x+10)(4x-6)}{4}$

$\frac{2(2x+5)2(2x-3)}{4}$

$(2x-3)(2x+5)$

c)
$18 - 50x^2$.

Factorise the common factor

$2(9-25x^2)$

Use difference of two squares

$2(3-5x)(3+5x)$

Completing the Square

Why is completing the square for quadratics useful?

- Completing the square gives the **maximum/minimum** of a quadratic function
  - This can be used to define the **range** of the function
- It gives the **vertex** when drawing the graph
- It can be used to **solve quadratic equations**
- It can be used to derive the **quadratic formula**

How do I complete the square for a monic quadratic of the form $x^2 + bx + c$?

- **Half the value of *b*** and write $\left(x + \frac{b}{2}\right)^2$
  - This is because $\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$
- **Subtract the unwanted** $\frac{b^2}{4}$ **term** and **add on the constant c**
  - $\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$

How do I complete the square for a non-monic quadratic of the form $ax^2 + bx + c$?

- **Factorise out the *a*** from the **terms involving *x***
  - $a\left(x^2 + \frac{b}{a}x\right) + x$
  - Leaving the c alone will **avoid working with lots of fractions**
- **Complete the square** on the quadratic term
  - **Half** $\frac{b}{a}$ and write $\left(x + \frac{b}{2a}\right)^2$
    - This is because $\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$
  - **Subtract the unwanted** $\frac{b^2}{4a^2}$ **term**
- **Multiply by *a*** and **add the constant c**
  - $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c$
  - $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$

> **Exam Tip**
>
> - Some questions may not use the phrase "completing the square" so ensure you can recognise a quadratic expression or equation written in this form
>   - $a(x - h)^2 + k\ (= 0)$

Worked Example

Complete the square:

a)
$x^2 - 8x + 3$.

Half b and subtract its square

$(x-4)^2 - 4^2 + 3$

$(x-4)^2 - 13$

b)
$3x^2 + 12x - 5$.

Factorise the 3 from the x terms

$3(x^2 + 4x) - 5$

Complete the square on $x^2 + 4x$

$3\left((x+2)^2 - 2^2\right) - 5$

Simplify

$3\left((x+2)^2 - 4\right) - 5$

$3(x+2)^2 - 12 - 5$

$3(x+2)^2 - 17$

2.1.3 Solving Quadratics

Solving Quadratic Equations

How do I decide the best method to solve a quadratic equation?

- A quadratic equation is of the form $ax^2 + bx + c = 0$
- If it is a calculator paper then use your GDC to solve the quadratic
- If it is a non-calculator paper then...
  - you can always use the **quadratic formula**
  - you can **factorise** if it can be factorised with integers
  - you can always **complete the square**

How do I solve a quadratic equation by the quadratic formula?

- If necessary **rewrite** in the form $ax^2 + bx + c = 0$
- **Clearly identify** the values of *a*, *b* & *c*
- **Substitute** the values into the formula
  - $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
    - This is given in the **formula booklet**
- **Simplify** the solutions as much as possible

How do I solve a quadratic equation by factorising?

- **Factorise** to rewrite the quadratic equation in the form $a(x-p)(x-q) = 0$
- Set each factor to zero and **solve**
  - $x - p = 0 \Rightarrow x = p$
  - $x - q = 0 \Rightarrow x = q$

How do I solve a quadratic equation by completing the square?

- **Complete the square** to rewrite the quadratic equation in the form $a(x-h)^2 + k = 0$
- Get the squared term by itself
  - $(x-h)^2 = -\dfrac{k}{a}$
- If $\left(-\dfrac{k}{a}\right)$ is **negative** then there will be **no solutions**
- If $\left(-\dfrac{k}{a}\right)$ is **positive** then there will be **two values** for $x - h$
  - $x - h = \pm\sqrt{-\dfrac{k}{a}}$
- **Solve** for *x*
  - $x = h \pm \sqrt{-\dfrac{k}{a}}$

Exam Tip

- When using the quadratic formula with awkward values or fractions you may find it easier to deal with the " $b^2 - 4ac$ " (**discriminant**) first
    - This can help avoid numerical and negative errors, improving accuracy

Worked Example

Solve the equations:

a)
$4x^2 + 4x - 15 = 0$.

This can be factorised

$(2x + 5)(2x - 3) = 0$

$2x + 5 = 0$ or $2x - 3 = 0$

$x = -\frac{5}{2}$ or $x = \frac{3}{2}$

b)
$3x^2 + 12x - 5 = 0$.

This can not be factorised but $3x^2$ and $12x$ have a common factor so complete the square

$3(x+2)^2 - 17 = 0$

Rearrange

$(x+2)^2 = \frac{17}{3}$

Remember ±

$x + 2 = \pm\sqrt{\frac{17}{3}}$

$x = -2 \pm \sqrt{\frac{17}{3}}$

c)
$7 - 3x - 5x^2 = 0$.

This can not be factorised so use formula

Formula booklet

Solutions of a quadratic equation	$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$

$a = -5$ $b = -3$ $c = 7$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-5)(7)}}{2(-5)}$$

$$= \frac{3 \pm \sqrt{9 + 140}}{-10}$$

$$x = -\frac{3 \pm \sqrt{149}}{10}$$

2.1.4 Quadratic Inequalities

Quadratic Inequalities

What affects the inequality sign when rearranging a quadratic inequality?

- The inequality sign is **unchanged** by...
  - **Adding/subtracting** a term to both sides
  - **Multiplying/dividing** both sides by a **positive term**
- The inequality sign **flips** (< changes to >) when...
  - **Multiplying/dividing** both sides by a **negative term**

How do I solve a quadratic inequality?

- **STEP 1**: **Rearrange** the inequality into quadratic form with a **positive squared term**
  - $ax^2 + bx + c > 0$
  - $ax^2 + bx + c \geq 0$
  - $ax^2 + bx + c < 0$
  - $ax^2 + bx + c \leq 0$
- **STEP 2**: Find the **roots** of the quadratic equation
  - Solve $ax^2 + bx + c = 0$ to get x_1 and x_2 where $x_1 < x_2$
- **STEP 3**: **Sketch** a graph of the quadratic and label the roots
  - As the squared term is positive it will be **concave up** so "U" shaped
- **STEP 4**: **Identify** the **region** that satisfies the inequality
  - If you want the graph to be **above the x-axis** then choose the region to be the **two intervals outside** of the two roots
  - If you want the graph to be **below the x-axis** then choose the region to be the **interval between** the two roots
  - For $ax^2 + bx + c > 0$
    - The solution is $x < x_1$ **or** $x > x_2$
  - For $ax^2 + bx + c \geq 0$
    - The solution is $x \leq x_1$ **or** $x \geq x_2$
  - For $ax^2 + bx + c < 0$
    - The solution is $x_1 < x < x_2$
  - For $ax^2 + bx + c \leq 0$
    - The solution is $x_1 \leq x \leq x_2$

How do I solve a quadratic inequality of the form $(x - h)^2 < n$ or $(x - h)^2 > n$?

- The safest way is by following the steps above
  - Expand and rearrange
- A **common mistake** is writing $x - h < \pm\sqrt{n}$ or $x - h > \pm\sqrt{n}$
  - This is **NOT correct**!
- The correct solution to $(x - h)^2 < n$ is
  - $|x - h| < \sqrt{n}$ which can be written as $-\sqrt{n} < x - h < \sqrt{n}$
  - The **final solution** is $h - \sqrt{n} < x < h + \sqrt{n}$
- The correct solution to $(x - h)^2 > n$ is

- $|x-h| > \sqrt{n}$ which can be written as $x-h < -\sqrt{n}$ or $x-h > \sqrt{n}$
- The **final solution** is $x < h-\sqrt{n}$ **or** $x > h+\sqrt{n}$

Exam Tip

- It is easiest to sketch the graph of a quadratic when it has a positive x^2 term, so rearrange first if necessary
- Use your GDC to help select the correct region(s) for the inequality
- Some makes/models of GDC may have the ability to solve inequalities directly
  - However unconventional notation may be used to display the answer (e.g. $6 > x > 3$ rather than $3 < x < 6$)
  - The safest method is to **always** sketch the graph

Worked Example

Find the set of values which satisfy $3x^2+2x-6 > x^2+4x-2$.

Step 1 : Rearrange

$(3x^2+2x-6)-(x^2+4x-2) > 0$ — This way gives $a>0$

$2x^2-2x-4 > 0$

$x^2-x-2 > 0$ — Divide by factor of 2

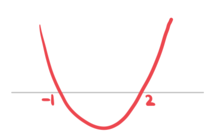
Step 2 : Find the roots

$x^2-x-2=0$

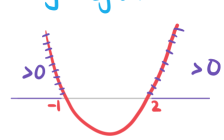
$(x-2)(x+1)=0$

$x=2$ or $x=-1$

Step 3 : Sketch



Step 4 : Identify region



$x < -1$ or $x > 2$

2.1.5 Discriminants

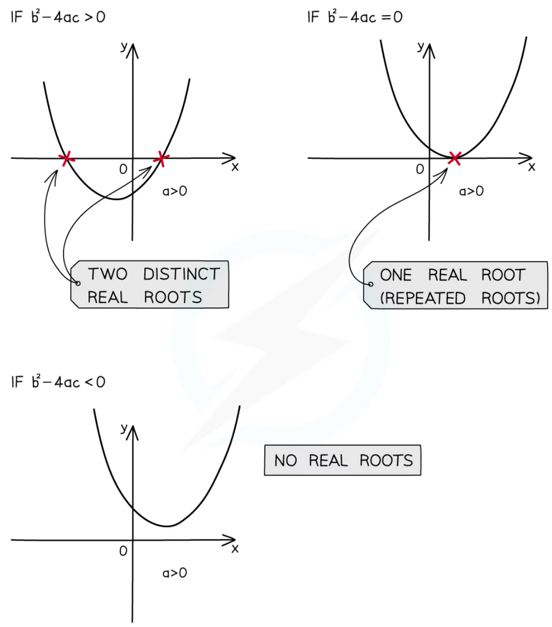
Discriminants

What is the discriminant of a quadratic function?

- The discriminant of a quadratic is denoted by the Greek letter Δ (upper case delta)
- For the quadratic function the discriminant is given by
  - $\Delta = b^2 - 4ac$
    This is given in the **formula booklet**
- The discriminant is the expression that is square rooted in the **quadratic formula**

How does the discriminant of a quadratic function affect its graph and roots?

- If Δ > 0 then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **two distinct values**
  - The equation $ax^2 + bx + c = 0$ has **two distinct real solutions**
  - The graph of $y = ax^2 + bx + c$ has **two distinct real roots**
    This means the graph **crosses** the *x*-axis **twice**
- If Δ = 0 then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **both zero**
  - The equation $ax^2 + bx + c = 0$ has **one repeated real solution**
  - The graph of $y = ax^2 + bx + c$ has **one repeated real root**
    This means the graph **touches** the *x*-axis at **exactly one point**
    This means that the ***x*-axis** is a **tangent** to the graph
- If Δ < 0 then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **both undefined**
  - The equation $ax^2 + bx + c = 0$ has **no real solutions**
  - The graph of $y = ax^2 + bx + c$ has **no real roots**
    This means the graph **never touches** the ***x*-axis**
    This means that graph is **wholly above** (or **below**) the ***x*-axis**



Forming equations and inequalities using the discriminant

- Often at least one of the coefficients of a quadratic is **unknown**
  - Questions usually use the letter *k* for the unknown constant
- You will be given a fact about the quadratic such as:
  - The **number of solutions** of the equation
  - The **number of roots** of the graph
- To find the **value or range of values** of *k*
  - Find an **expression for the discriminant**
    - Use $\Delta = b^2 - 4ac$
  - Decide whether $\Delta > 0$, $\Delta = 0$ or $\Delta < 0$
    - If the question says there are **real roots** but does not specify how many then use $\Delta \geq 0$
  - **Solve** the resulting equation or inequality

Exam Tip

- Questions will rarely use the word discriminant so it is important to recognise when its use is required
  - Look for
    - a number of roots or solutions being stated
    - whether and/or how often the graph of a quadratic function intercepts the x-axis
- Be careful setting up inequalities that concern "two real roots" ($\Delta \geq 0$) as opposed to "two real distinct roots" ($\Delta > 0$)

Worked Example

A function is given by $f(x) = 2kx^2 + kx - k + 2$, where k is a constant. The graph of $y = f(x)$ has two distinct real roots.

a)

Show that $9k^2 - 16k > 0$.

Two distinct real roots $\Rightarrow$ $\Delta > 0$

Formula booklet

Discriminant	$\Delta = b^2 - 4ac$

$a = 2k \quad b = k \quad c = (-k+2)$

$\Delta = k^2 - 4(2k)(-k+2)$

$= k^2 + 8k^2 - 16k$

$= 9k^2 - 16k$

$\Delta > 0 \Rightarrow 9k^2 - 16k > 0$

b)

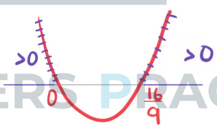
Hence find the set of possible values of k.

Solve the inequality

$9k^2 - 16k = 0$

$k(9k - 16) = 0$

$k = 0$ or $k = \frac{16}{9}$



$k < 0$ or $k > \frac{16}{9}$

2.2 Linear Functions & Graphs

2.2.1 Equations of a Straight Line

Equations of a Straight Line

How do I find the gradient of a straight line?

- Find two points that the line passes through with coordinates (x_1, y_1) and (x_2, y_2)
- The gradient between these two points is calculated by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  - This is given in the **formula booklet**
- The gradient of a straight line measures its **slope**
  - A line with gradient 1 will go up 1 unit for every unit it goes to the right
  - A line with gradient -2 will go down two units for every unit it goes to the right

What are the equations of a straight line?

- $y = mx + c$
  - This is the **gradient-intercept form**
  - It clearly shows the gradient m and the y-intercept (0, c)
- $y - y_1 = m(x - x_1)$
  - This is the **point-gradient form**
  - It clearly shows the gradient m and a point on the line (x_1, y_1)
- $ax + by + d = 0$
  - This is the **general form**
  - You can quickly get the x-intercept $\left(-\frac{d}{a}, 0\right)$ and y-intercept $\left(0, -\frac{d}{b}\right)$

How do I find an equation of a straight line?

- You will need the gradient
  - If you are given two points then first find the gradient
- It is easiest to start with the **point-gradient form**
  - then rearrange into whatever form is required
    - multiplying both sides by any denominators will get rid of fractions
- You can check your answer by using your GDC
  - Graph your answer and check it goes through the point(s)
  - If you have two points then you can enter these in the **statistics mode** and find the regression line $y = ax + b$

Exam Tip

- A sketch of the graph of the straight line(s) can be helpful, even if not demanded by the question
  - Use your GDC to plot them
- Ensure you state equations of straight lines in the format required
  - Usually $y = mx + c$ or $ax + by + d = 0$
  - Check whether coefficients need to be integers (they usually are for $ax + by + d = 0$)

Worked Example

The line l passes through the points $(-2, 5)$ and $(6, -7)$.

Find the equation of l, giving your answer in the form $ax + by + d = 0$ where a, b and c are integers to be found.

Find the gradient between (-2,5) and (6,-7)

Formula booklet

$$m = \frac{-7 - 5}{6 - -2} = -\frac{3}{2}$$

Gradient formula	$m = \frac{y_2 - y_1}{x_2 - x_1}$

Use the point-gradient formula

Formula booklet

Equations of a straight line	$y - y_1 = m(x - x_1)$

$(x_1, y_1) = (-2, 5) \quad m = -\frac{3}{2}$

$y - 5 = -\frac{3}{2}(x - -2)$ Simplify

$y - 5 = -\frac{3}{2}(x + 2)$ Multiply by denominator

$2(y - 5) = -3(x + 2)$ Expand

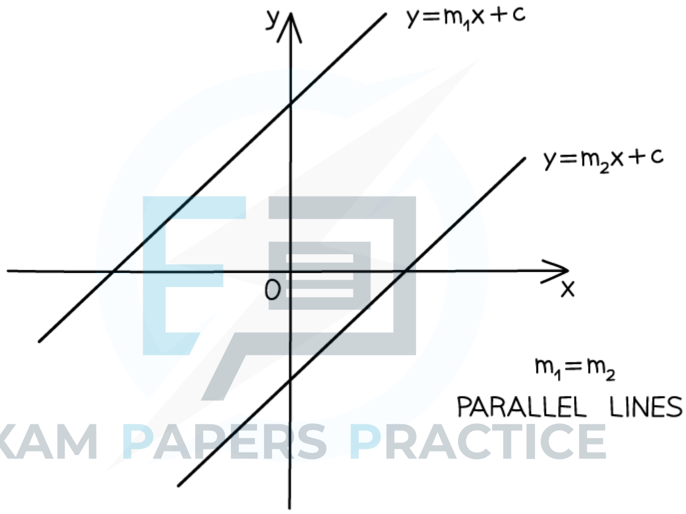
$2y - 10 = -3x - 6$ Rearrange

$$\boxed{3x + 2y - 4 = 0}$$

Parallel Lines

How are the equations of parallel lines connected?

- **Parallel lines** are always equidistant meaning they never intersect
- Parallel lines have the same gradient
  - If the gradient of line l_1 is m_1 and gradient of line l_2 is m_2 then...
    - $m_1 = m_2 \Rightarrow l_1\ \&\ l_2$ are parallel
    - $l_1\ \&\ l_2$ are parallel $\Rightarrow m_1 = m_2$
- To determine if two lines are parallel:
  - Rearrange into the gradient-intercept form $y = mx + c$
  - Compare the coefficients of x
  - If they are equal then the lines are parallel



Worked Example

The line l passes through the point $(4, -1)$ and is parallel to the line with equation $2x - 5y = 3$.

Find the equation of l, giving your answer in the form $y = mx + c$.

Rearrange into $y = mx + c$ to find the gradient

$5y = 2x - 3 \Rightarrow y = \frac{2}{5}x - \frac{3}{5}$ $\therefore$ gradient $= \frac{2}{5}$

Parallel lines $\Rightarrow m_1 = m_2$

$m = \frac{2}{5}$

Use the point-gradient formula

Formula booklet

Equations of a straight line	$y - y_1 = m(x - x_1)$

$(x_1, y_1) = (4, -1)$ $m = \frac{2}{5}$

$y + 1 = \frac{2}{5}(x - 4)$

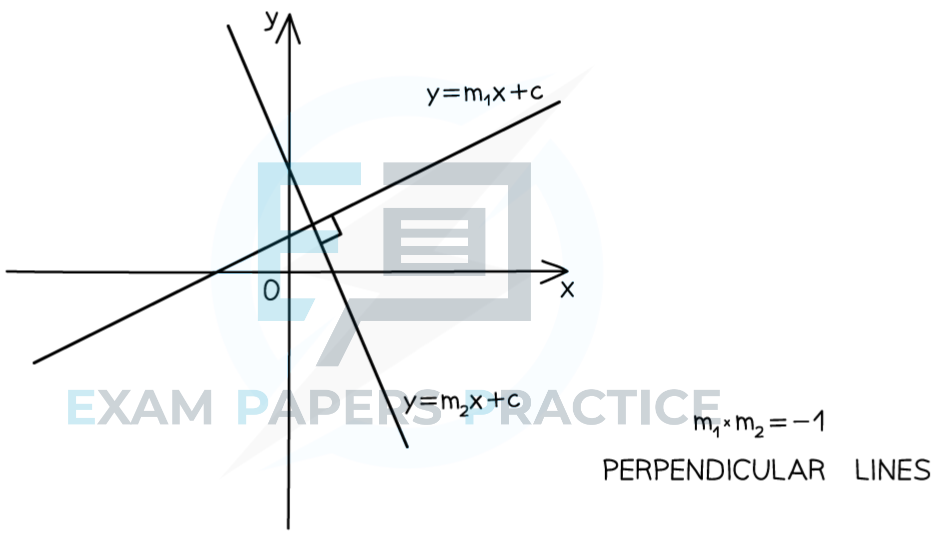
$y + 1 = \frac{2}{5}x - \frac{8}{5}$

$y = \frac{2}{5}x - \frac{13}{5}$

Perpendicular Lines

How are the equations of perpendicular lines connected?

- **Perpendicular lines** intersect at right angles
- The gradients of two perpendicular lines are negative reciprocals
  - If the gradient of line l_1 is m_1 and gradient of line l_2 is m_2 then...
    - $m_1 \times m_2 = -1 \Rightarrow l_1$ & l_2 are perpendicular
    - l_1 & l_2 are perpendicular $\Rightarrow m_1 \times m_2 = -1$
- To determine if two lines are perpendicular:
  - Rearrange into the gradient-intercept form $y = mx + c$
  - Compare the coefficients of x
  - If their product is -1 then they are perpendicular
- Be careful with horizontal and vertical lines
  - $x = p$ and $y = q$ are perpendicular where p and q are constants



Worked Example

The line l_1 is given by the equation $3x - 5y = 7$.

The line l_2 is given by the equation $y = \frac{1}{4} - \frac{5}{3}x$.

Determine whether l_1 and l_2 are perpendicular. Give a reason for your answer.

Rearrange l_1 into $y = mx + c$ form

$5y = 3x - 7 \Rightarrow y = \frac{3}{5}x - \frac{7}{5}$

Identity gradients

$m_1 = \frac{3}{5} \quad m_2 = -\frac{5}{3}$

$m_1 \times m_2 = -1 \Rightarrow$ Perpendicular lines

$\frac{3}{5} \times -\frac{5}{3} = -1$

l_1 and l_2 are perpendicular as $m_1 \times m_2 = -1$

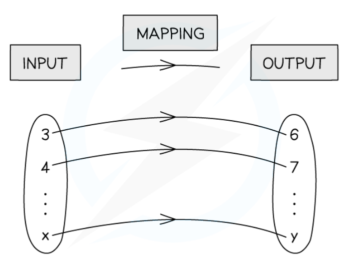
2.3 Functions Toolkit

2.3.1 Language of Functions

Language of Functions

What is a mapping?

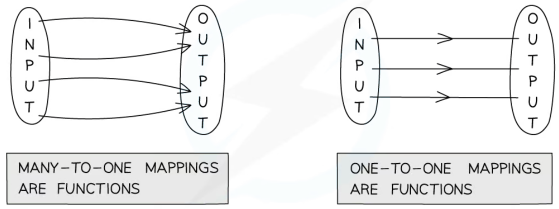
- A **mapping transforms** one set of values (**inputs**) into another set of values (**outputs**)
- Mappings can be:
  - **One-to-one**
    - Each input gets mapped to **exactly one unique** output
    - No two inputs are mapped to the same output
    - For example: A mapping that cubes the input
  - **Many-to-one**
    - Each input gets mapped to **exactly one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that squares the input
  - **One-to-many**
    - An input can be mapped to **more than one** output
    - No two inputs are mapped to the same output
    - For example: A mapping that gives the numbers which when squared equal the input
  - **Many-to-many**
    - An input can be mapped to **more than one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that gives the factors of the input



What is a function?

- A **function** is a mapping between two sets of numbers where **each input** gets mapped to **exactly one output**
  - The output does not need to be unique

- **One-to-one** and **many-to-one** mappings are functions
- A mapping is a function if its graph passes the **vertical line test**
  - Any **vertical line** will intersect with the graph **at most once**

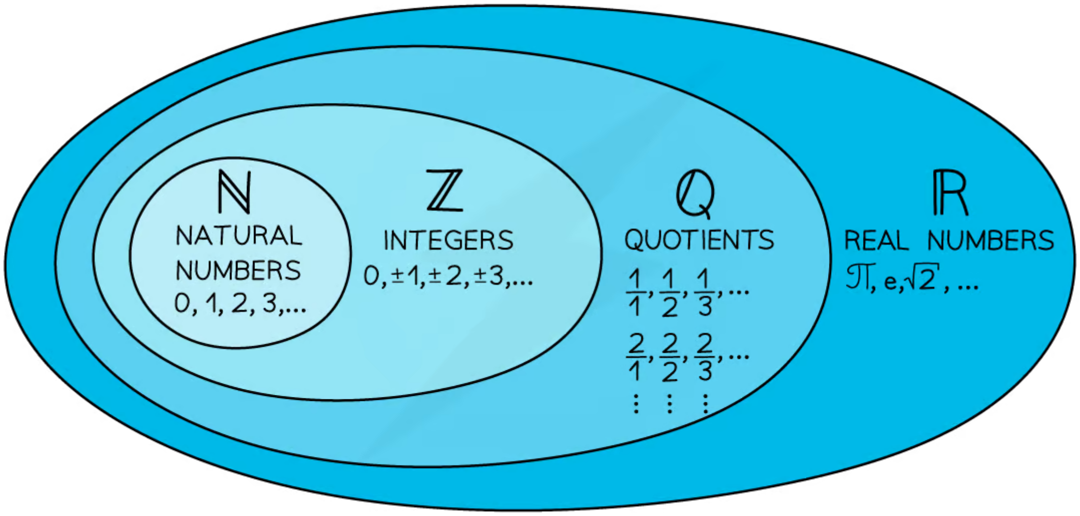


What notation is used for functions?

- Functions are denoted using letters (such as f, v, g, etc)
  - A function is followed by a variable in a bracket
  - This shows the input for the function
  - The letter f is used most commonly for functions and will be used for the remainder of this revision note
- $f(x)$ represents an expression for the value of the function f when evaluated for the variable x
- Function notation gets rid of the need for words which makes it **universal**
  - $f = 5$ when $x = 2$ can simply be written as $f(2) = 5$

What are the domain and range of a function?

- The **domain** of a function is the set of values that are used as **inputs**
- A domain should be stated with a function
  - If a domain is not stated then it is assumed the domain is all the real values which would work as inputs for the function
  - Domains are expressed in terms of the input
    - $x \le 2$
- The **range** of a function is the set of values that are given as **outputs**
  - The range depends on the domain
  - Ranges are expressed in terms of the output
    - $f(x) \ge 0$
- To graph a function we use the **inputs as the *x*-coordinates** and the **outputs as the *y*-coordinates**
  - $f(2) = 5$ corresponds to the coordinates (2, 5)
- Graphing the function can help you visualise the range
- Common sets of numbers have special symbols:
  - $\mathbb{R}$ represents all the real numbers that can be placed on a number line
    - $x \in \mathbb{R}$ means x is a real number
  - $\mathbb{Q}$ represents all the rational numbers $\frac{a}{b}$ where *a* and *b* are integers and *b* ≠ 0
  - $\mathbb{Z}$ represents all the integers (positive, negative and zero)
    - $\mathbb{Z}^+$ represents positive integers
  - $\mathbb{N}$ represents the natural numbers (0,1,2,3...)



What are piecewise functions?

- **Piecewise functions** are defined by different functions depending on which interval the input is in
  - E.g. $f(x) = \begin{cases} x+1 & x \le 5 \\ 2x-4 & 5 < x < 10 \\ x^2 & 10 \le x \le 20 \end{cases}$
- The region for the individual functions **cannot overlap**
- To evaluate a piecewise function for a particular value $x = k$
  - Find which interval includes k
  - Substitute $x = k$ into the corresponding function
- The function **may or may not be continuous** at the ends of the intervals
  - In the example above the function is
    - continuous at $x = 5$ as $5 + 1 = 2(5) - 4$
    - not continuous at $x = 10$ as $2(10) - 4 \ne 10^2$

Exam Tip

- Questions may refer to "the largest possible domain"
  - This would usually be $x \in \mathbb{R}$ unless $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$ has already been stated
  - There are usualy some exceptions
    - e.g. square roots; $x \ge 0$ for a function involving $\sqrt{x}$
    - e.g. reciprocal functions; $x \ne 2$ for a function with denominator $(x-2)$

Worked Example

For the function $f(x) = x^3 + 1,\ 2 \le x \le 10$:

a)
write down the value of $f(7)$.

Substitute $x = 7$

$f(7) = 7^3 + 1$

$f(7) = 344$

b)
find the range of $f(x)$.

Find the values of x^3+1 when $2 \le x \le 10$

$2 \le x \le 10$

$8 \le x^3 \le 1000$

$9 \le x^3+1 \le 1001$

$9 \le f(x) \le 1001$

2.3.2 Composite & Inverse Functions

Composite Functions

What is a composite function?

- A **composite function** is where a function is applied to another function
- A composite function can be denoted
  - $(f \circ g)(x)$
  - $fg(x)$
  - $f(g(x))$
- The order matters
  - $(f \circ g)(x)$ means:
    - First apply *g* to *x* to get $g(x)$
    - Then apply *f* to the previous output to get $f(g(x))$
    - Always start with the function **closest to the variable**
  - $(f \circ g)(x)$ is not usually equal to $(g \circ f)(x)$

How do I find the domain and range of a composite function?

- The domain of $f \circ g$ is the set of values of x...
  - which are a **subset** of the **domain of g**
  - which maps g to a value that is in the **domain of f**
- The range of $f \circ g$ is the set of values of x...
  - which are a **subset** of the **range of f**
  - found by **applying f** to the **range of g**
- To find the **domain** and **range** of $f \circ g$
  - First find the **range of g**
  - **Restrict** these values to the values that are **within the domain** of **f**
    - The **domain** is the set of values that **produce the restricted range** of *g*
    - The **range** is the set of values that are **produced using the restricted range** of *g* as the domain for *f*
- For example: let $f(x) = 2x + 1,\ -5 \le x \le 5$ and $g(x) = \sqrt{x},\ 1 \le x \le 49$
  - The **range of g** is $1 \le g(x) \le 7$
    - **Restricting** this to fit the **domain of f** results in $1 \le g(x) \le 5$
  - The **domain** of $f \circ g$ is therefore $1 \le x \le 25$
    - These are the values of *x* which map to $1 \le g(x) \le 5$
  - The **range** of $f \circ g$ is therefore $3 \le (f \circ g)(x) \le 11$
    - These are the values which *f* maps $1 \le g(x) \le 5$ to

Exam Tip

- Make sure you know what your GDC is capable of with regard to functions
  - You may be able to store individual functions and find composite functions and their values for particular inputs
  - You may be able to graph composite functions directly and so deduce their domain and range from the graph
- The link between the domains and ranges of a function and its inverse can act as a check for your solution
- $ff(x)$ is not the same as $[f(x)]^2$

Worked Example

Given $f(x) = \sqrt{x+4}$ and $g(x) = 3 + 2x$:

a)
Write down the value of $(g \circ f)(12)$.

First apply function closest to input

$(g \circ f)(12) = g(f(12))$

$f(12) = \sqrt{12+4} = \sqrt{16} = 4$

$g(4) = 3 + 2(4) = 11$

$(g \circ f)(12) = 11$

b)
Write down an expression for $(f \circ g)(x)$.

First apply function closest to input

$(f \circ g)(x) = f(g(x))$

$= f(3+2x)$

$= \sqrt{3+2x+4}$

$(f \circ g)(x) = \sqrt{7+2x}$

c)
Write down an expression for $(g \circ g)(x)$.

$(g \circ g)(x) = g(g(x))$

$= g(3+2x)$

$= 3 + 2(3+2x)$

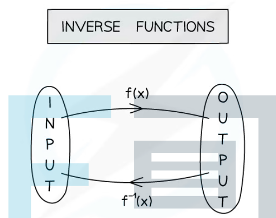
$= 3 + 6 + 4x$

$(g \circ g)(x) = 9 + 4x$

Inverse Functions

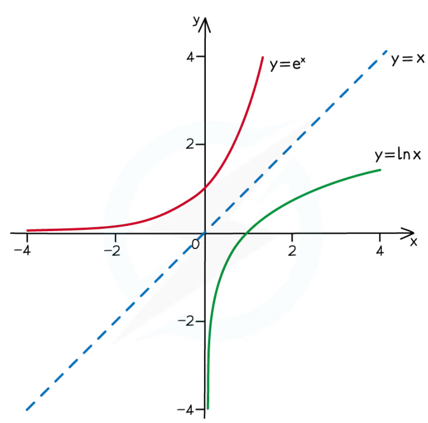
What is an inverse function?

- **Only one-to-one** functions have inverses
- A function has an inverse if its graph passes the **horizontal line test**
  - Any **horizontal line** will intersect with the graph **at most once**
- The **identity function** id maps each value to itself
  - $\mathrm{id}(x) = x$
- If $f \circ g$ and $g \circ f$ have the **same effect as the identity function** then f and g are **inverses**
- Given a function $f(x)$ we denote the **inverse function** as $f^{-1}(x)$
- An inverse function **reverses the effect** of a function
  - $f(2) = 5$ means $f^{-1}(5) = 2$
- Inverse functions are used to solve equations
  - The solution of $f(x) = 5$ is $x = f^{-1}(5)$
- A composite function made of f and f^{-1} has the **same effect as the identity function**
  - $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$



What are the connections between a function and its inverse function?

- The **domain of a function** becomes the **range of its inverse**
- The **range of a function** becomes the **domain of its inverse**
- The graph of $y = f^{-1}(x)$ is a **reflection** of the graph $y = f(x)$ in the line $y = x$
  - Therefore solutions to $f(x) = x$ or $f^{-1}(x) = x$ will also be solutions to $f(x) = f^{-1}(x)$
    - There could be other solutions to $f(x) = f^{-1}(x)$ that don't lie on the line $y = x$



How do I find the inverse of a function?

- STEP 1: **Swap** the *x* and *y* in $y = f(x)$
  - If $y = f^{-1}(x)$ then $x = f(y)$
- STEP 2: **Rearrange** $x = f(y)$ to make y the subject
- Note this can be done in any order
  - Rearrange $y = f(x)$ to make x the subject
  - Swap x and y

Can many-to-one functions ever have inverses?

- You can **restrict the domain** of a many-to-one function so that it has an inverse
- Choose a subset of the domain where the function is one-to-one
  - The inverse will be determined by the restricted domain
  - Note that a many-to-one function can **only** have an inverse if its domain is restricted first
- For **quadratics** – use the **vertex** as the upper or lower bound for the **restricted domain**
  - For $f(x) = x^2$ restrict the domain so 0 is either the maximum or minimum value
    - For example: $x \geq 0$ or $x \leq 0$
  - For $f(x) = a(x-h)^2 + k$ restrict the domain so *h* is either the maximum or minimum value
    - For example: $x \geq h$ or $x \leq h$
- For **trigonometric functions** – use part of a cycle as the **restricted domain**
  - For $f(x) = \sin x$ restrict the domain to half a cycle between a maximum and a minimum
    - For example: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
  - For $f(x) = \cos x$ restrict the domain to half a cycle between maximum and a minimum
    - For example: $0 \leq x \leq \pi$
  - For $f(x) = \tan x$ restrict the domain to one cycle between two asymptotes
    - For example: $-\frac{\pi}{2} < x < \frac{\pi}{2}$

How do I find the inverse function after restricting the domain?

- The range of the inverse is the same as the restricted domain of the original function
- The inverse function is determined by the restricted domain
  - Restricting the domain differently will change the inverse function
- Use the range of the inverse to help find the inverse function
  - Restricting the domain of $f(x) = x^2$ to $x \geq 0$ means the range of the inverse is $f^{-1}(x) \geq 0$
    - Therefore $f^{-1}(x) = \sqrt{x}$
  - Restricting the domain of $f(x) = x^2$ to $x \leq 0$ means the range of the inverse is $f^{-1}(x) \leq 0$
    - Therefore $f^{-1}(x) = -\sqrt{x}$

Exam Tip

- Remember that an inverse function is a reflection of the original function in the line $y = x$
  - Use your GDC to plot the function and its inverse on the same graph to visually check this
- $f^{-1}(x)$ is not the same as $\frac{1}{f(x)}$

Worked Example

The function $f(x) = (x-2)^2 + 5, \ x \le m$ has an inverse.

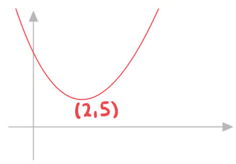
a)

Write down the largest possible value of m.

Sketch $y = f(x)$

The graph is one-to-one for $x \le 2$

$m = 2$



b)

Find the inverse of $f(x)$.

Let $y = f^{-1}(x)$ and rearrange $x = f(y)$

$x = (y-2)^2 + 5$

$x - 5 = (y-2)^2$

$\pm\sqrt{x-5} = y - 2$

$2 \pm \sqrt{x-5} = y$

Range of f^{-1} is the domain of f

$f^{-1}(x) \le 2 \quad \therefore \ y = 2 - \sqrt{x-5}$

$f^{-1}(x) = 2 - \sqrt{x-5}$

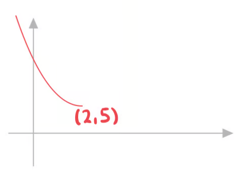
c)

Find the domain of $f^{-1}(x)$.

Domain of f^{-1} is the range of f

Sketch $y = f(x)$ to see range

For $x \le 2$, $f(x) \ge 5$



Domain of f^{-1} : $x \ge 5$

d)

Find the value of k such that $f(k) = 9$.

Use inverse $f(a) = b \iff a = f^{-1}(b)$

$k = f^{-1}(9) = 2 - \sqrt{9-5}$

$k = 0$

2.3.3 Symmetry of Functions

Odd & Even Functions

What are odd functions?

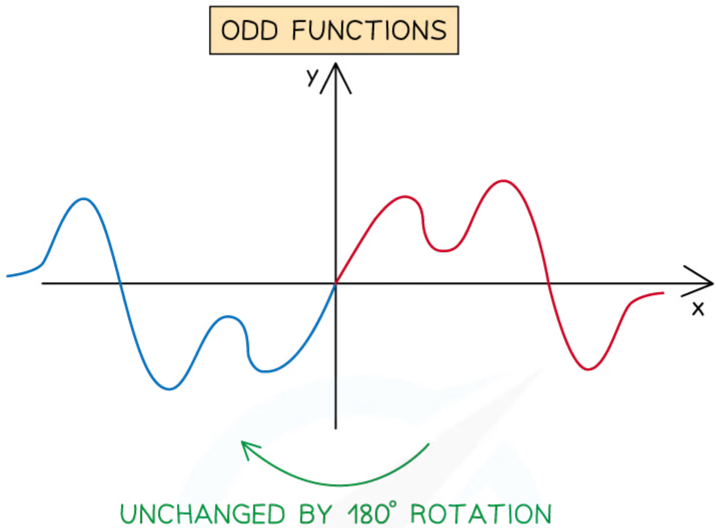
- A function $f(x)$ is called **odd** if
  - $f(-x) = -f(x)$ for all values of x
- Examples of odd functions include:
  - Power functions with **odd powers**: x^{2n+1} where $n \in \mathbb{Z}$
    For example: $(-x)^3 = -x^3$
  - Some **trig functions**: $\sin x$, $\operatorname{cosec} x$, $\tan x$, $\cot x$
    For example: $\sin(-x) = -\sin x$
  - **Linear combinations** of odd functions
    For example: $f(x) = 3x^5 - 4\sin x + \dfrac{6}{x}$

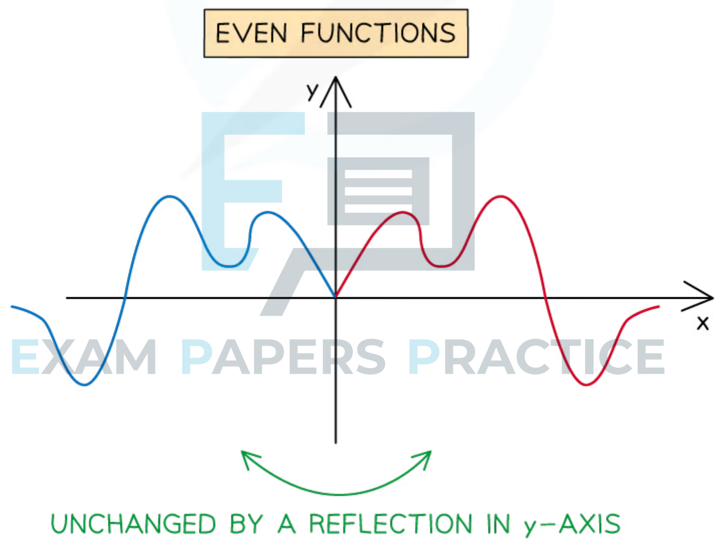
What are even functions?

- A function $f(x)$ is called **even** if
  - $f(-x) = f(x)$ for all values of x
- Examples of even functions include:
  - Power functions with **even powers**: x^{2n} where $n \in \mathbb{Z}$
    For example: $(-x)^4 = x^4$
  - Some **trig functions**: $\cos x$, $\sec x$
    For example: $\cos(-x) = \cos x$
  - **Modulus function**: $|x|$
  - **Linear combinations** of even functions
    For example: $f(x) = 7x^6 + 3|x| - 8\cos x$

What are the symmetries of graphs of odd & even functions?

- The graph of an **odd** function has **rotational symmetry**
  - The graph is unchanged by a **180° rotation** about the origin
- The graph of an **even** function has **reflective symmetry**
  - The graph is unchanged by a **reflection** in the ***y*-axis**





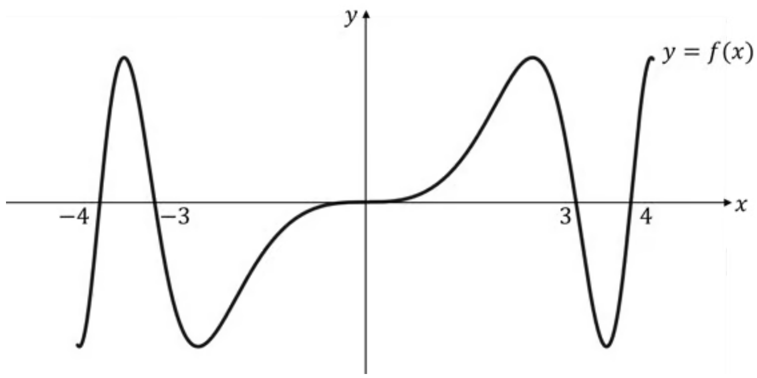
Exam Tip

- Turn your GDC upside down for a quick visual check for an odd function!
  - Ignoring axes, etc, if the graph looks exactly the same both ways, it's odd

Worked Example

a)

The graph $y = f(x)$ is shown below. State, with a reason, whether the function f is odd, even or neither.



f is an odd function as its graph has rotational symmetry – it is unchanged by a 180° rotation about the origin.

b)

Use algebra to show that $g(x) = x^3\sin(x) + 5\cos(x^5)$ is an even function.

g is even if $g(-x) = g(x)$ for all x

$g(-x) = (-x)^3\sin(-x) + 5\cos((-x)^5)$

$= (-x^3)(-\sin(x)) + 5\cos(-x^5)$ ← x^3, x^5, $\sin x$ are odd

$= x^3\sin(x) + 5\cos(x^5)$ ← $\cos x$ is even

$= g(x)$

g is even as $g(-x) = g(x)$ for all x

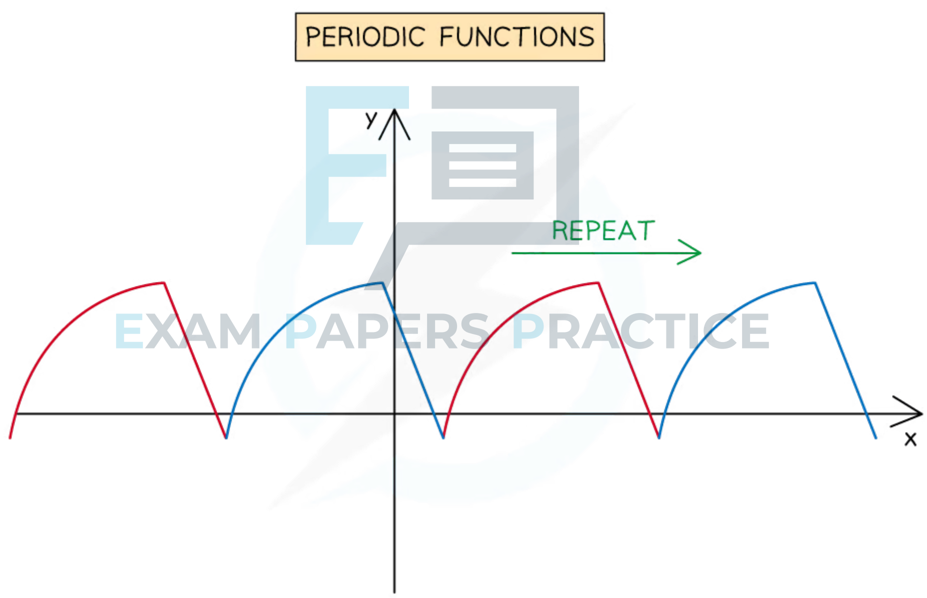
Periodic Functions

What are periodic functions?

- A function $f(x)$ is called **periodic**, with **period *k***, if
  - $f(x+k) = f(x)$ for all values of x
- Examples of periodic functions include:
  - sin *x* & cos *x*: The period is 2π or 360°
  - tan *x*: The period is π or 180°
  - **Linear combinations** of periodic functions with the **same period**
    - For example: $f(x) = 2\sin(3x) - 5\cos(3x+2)$

What are the symmetries of graphs of periodic functions?

- The graph of a **periodic** function has **translational symmetry**
  - The graph is unchanged by **translations** that are **integer multiples of** $\begin{pmatrix} k \\ 0 \end{pmatrix}$
  - The means that the graph appears to **repeat** the same section (cycle) infinitely

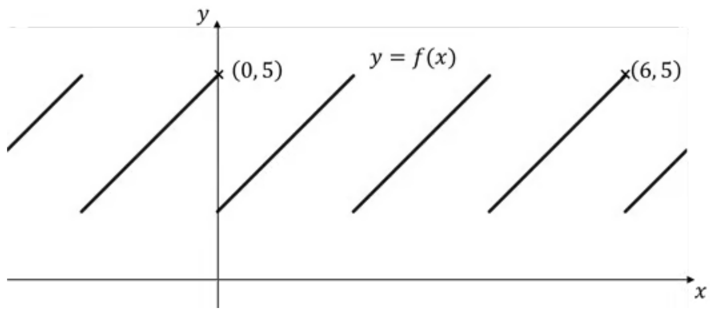


Exam Tip

- There may be several intersections between the graph of a periodic function and another function
  - i.e. Equations may have several solutions so only answers within a certain range of x-values may be required
    - e.g. Solve $\tan x = \sqrt{3}$ for $0° \le x \le 360°$
    - $x = 60°,\ 240°$
  - Alternatively you may have to write **all** solutions in a general form
    - e.g. $x = 60(3k+1)°,\quad k = 0,\ \pm1,\ \pm2,\ \ldots$

Worked Example

The graph $y = f(x)$ is shown below. Given that f is periodic, write down the period.



Period is the length of the interval of a single cycle

Between $x = 0$ and $x = 6$ there are 3 cycles

$$\text{Period} = \frac{6-0}{3}$$

Period = 2

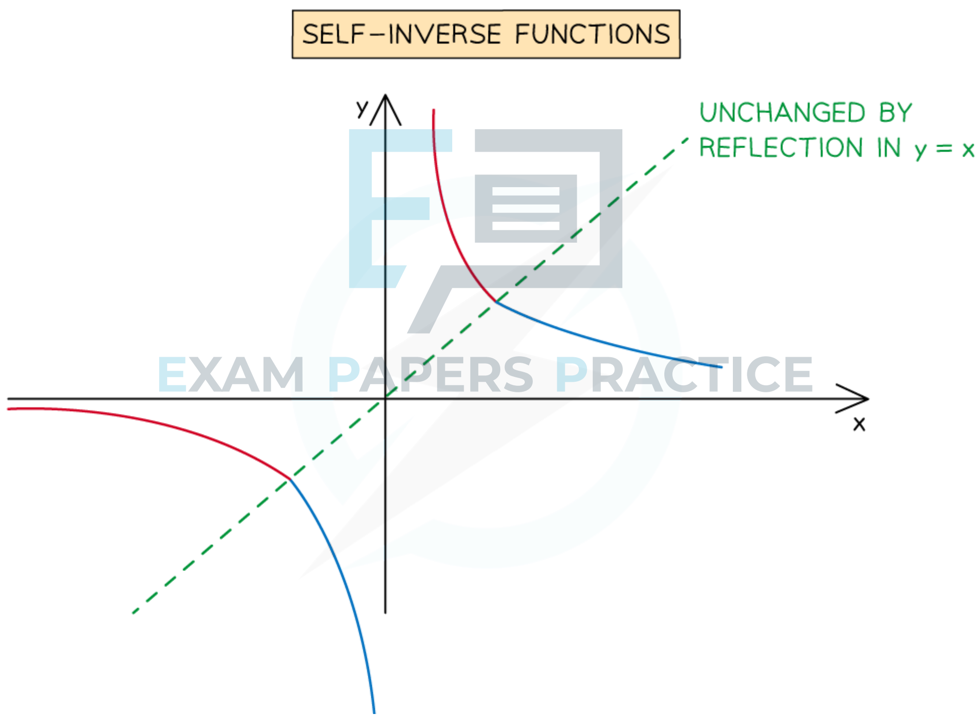
Self-Inverse Functions

What are self-inverse functions?

- A function $f(x)$ is called **self-inverse** if
  - $(f \circ f)(x) = x$ for all values of x
  - $f^{-1}(x) = f(x)$
- Examples of self-inverse functions include:
  - **Identity function**: $f(x) = x$
  - **Reciprocal function**: $f(x) = \frac{1}{x}$
  - **Linear functions** with a **gradient of -1**: $f(x) = -x + c$

What are the symmetries of graphs of self-inverse functions?

- The graph of a **self-inverse** function has **reflective symmetry**
  - The graph is unchanged by a **reflection** in the line ***y* = *x***



Exam Tip

- If your expression for $f^{-1}(x)$ is not the same as the expression for $f(x)$ you can check their equivalence by plotting both on your GDC
  - If equivalent the graphs will sit on top of one another and appear as one
  - This will indicate if you have made an error in your algebra, before trying to simplify/rewrite to make the two expressions identical
- It is sometimes easier to consider self inverse functions geometrically rather than algebraically

Worked Example

Use algebra to show the function defined by $f(x) = \dfrac{7x-5}{x-7}$, $x \neq 7$ is self-inverse.

Method 1: $f^{-1}(x)$

Let $y = f^{-1}(x)$ so $x = f(y)$

$$x = \frac{7y-5}{y-7}$$

$$(y-7)x = 7y-5$$

$$xy - 7x = 7y - 5$$

$$xy - 7y = 7x - 5$$

$$(x-7)y = 7x-5$$

$$y = \frac{7x-5}{x-7}$$

Isolate y on one side

$$f^{-1}(x) = \frac{7x-5}{x-7} = f(x)$$

$\therefore f$ is self-inverse

Method 2: $(f \circ f)(x)$

$$(f \circ f)(x) = f(f(x))$$

$$f(f(x)) = \frac{7f(x)-5}{f(x)-7}$$

$$= \frac{7\left(\frac{7x-5}{x-7}\right)-5}{\frac{7x-5}{x-7}-7}$$

$\times \dfrac{x-7}{x-7}$

$$= \frac{7(7x-5)-5(x-7)}{7x-5-7(x-7)}$$

$$= \frac{49x-35-5x+35}{7x-5-7x+49}$$

$$= \frac{44x}{44}$$

$$(f \circ f)(x) = x$$

$\therefore f$ is self-inverse

EXAM PAPERS PRACTICE

2.3.4 Graphing Functions

Graphing Functions

How do I graph the function $y = f(x)$?

- A point (a, b) lies on the graph $y = f(x)$ if $f(a) = b$
- The **horizontal axis** is used for the **domain**
- The **vertical axis** is used for the **range**
- You will be able to graph some functions by hand
- For some functions you will need to use your GDC
- You might be asked to graph the **sum** or **difference** of two functions
    - Use your GDC to graph $y = f(x) + g(x)$ or $y = f(x) - g(x)$
    - Just type the functions into the graphing mode

What is the difference between "draw" and "sketch"?

- If asked to sketch you should:
    - Show the general shape
    - Label any key points such as the intersections with the axes
    - Label the axes
- If asked to draw you should:
    - Use a pencil and ruler
    - Draw to scale
    - Plot any points **accurately**
    - Join points with a straight line or smooth curve
    - Label any key points such as the intersections with the axes
    - Label the axes

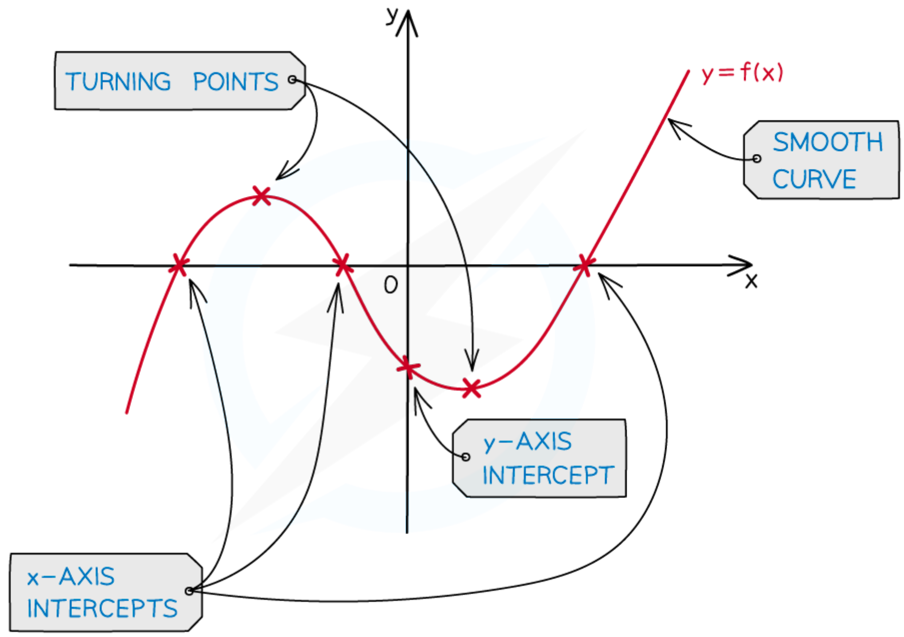
How can my GDC help me sketch/draw a graph?

- You use your GDC to plot the graph
    - Check the scales on the graph to make sure you see the full shape
- Use your GDC to find any key points
- Use your GDC to check specific points to help you plot the graph

Key Features of Graphs

What are the key features of graphs?

- You should be familiar with the following key features and know how to use your GDC to find them
- Local minimums/maximums
  - These are points where the graph has a minimum/maximum for a small region
  - They are also called **turning points**
    - This is where the graph changes its direction between upwards and downwards directions
  - A graph can have multiple local minimums/maximums
  - A local minimum/maximum is not necessarily the minimum/maximum of the whole graph
    - This would be called the **global** minimum/maximum
  - For quadratic graphs the minimum/maximum is called the **vertex**
- Intercepts
  - y – intercepts are where the graph crosses the y-axis
    - At these points $x = 0$
  - x – intercepts are where the graph crosses the x-axis
    - At these points $y = 0$
    - These points are also called the **zeros of the function** or **roots of the equation**
- Symmetry
  - Some graphs have lines of symmetry
    - A quadratic will have a vertical line of symmetry
- Asymptotes
  - These are lines which the graph will get closer to but not cross
  - These can be horizontal or vertical
    - Exponential graphs have horizontal asymptotes
    - Graphs of variables which vary inversely can have vertical and horizontal asymptotes



Exam Tip

- Most GDC makes/models will not plot/show asymptotes just from inputting a function
  - Add the asymptotes as additional graphs for your GDC to plot
  - You can then check the equations of your asymptotes visually
  - You may have to zoom in or change the viewing window options to confirm an asymptote
- Even if using your GDC to plot graphs and solve problems sketching them as part of your working is good exam technique
  - Label the key features of the graph and anything else relevant to the question on your sketch

Worked Example

Two functions are defined by

$$f(x) = x^2 - 4x - 5 \text{ and } g(x) = 2 + \frac{1}{x+1}.$$

a)
Draw the graph $y = f(x)$.

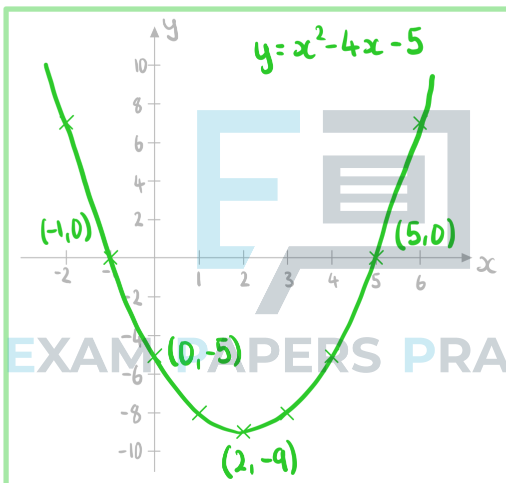
Draw means accurately

Use GDC to find vertex, roots and y-intercepts

Vertex = (2, -9)

Roots = (-1, 0) and (5, 0)

y-intercept = (0, -5)



b)
Sketch the graph $y = g(x)$.

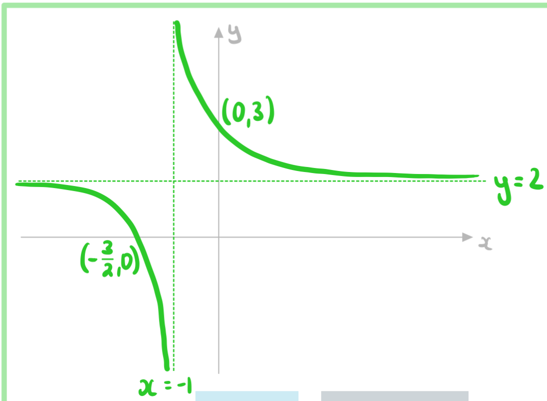
Sketch means rough but showing key points

Use GDC to find x and y-intercepts and asymptotes

x-intercept $= \left(-\frac{3}{2}, 0\right)$

y-intercept $= (0, 3)$

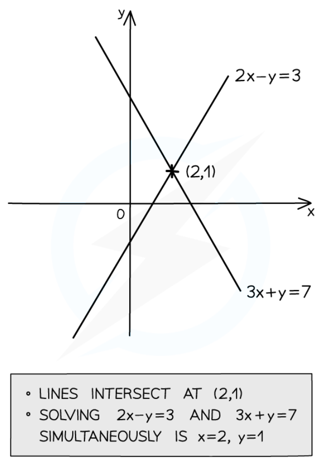
Asymptotes : $x = -1$ and $y = 2$



Intersecting Graphs

How do I find where two graphs intersect?

- Plot both graphs on your GDC
- Use the intersect function to find the intersections
- Check if there is more than one point of intersection



How can I use graphs to solve equations?

- One method to solve equations is to use graphs
- To solve $f(x) = a$
  - Plot the two graphs $y = f(x)$ and $y = a$ on your GDC
  - Find the points of intersections
  - The ***x*-coordinates** are the **solutions** of the equation
- To solve $f(x) = g(x)$
  - Plot the two graphs $y = f(x)$ and $y = g(x)$ on your GDC
  - Find the points of intersections
  - The ***x*-coordinates** are the **solutions** of the equation
- Using graphs makes it easier to see **how many solutions** an equation will have

Exam Tip

- You can use graphs to solve equations
  - Questions will not necessarily ask for a drawing/sketch or make reference to graphs
  - Use your GDC to plot the equations and find the intersections between the graphs

Worked Example

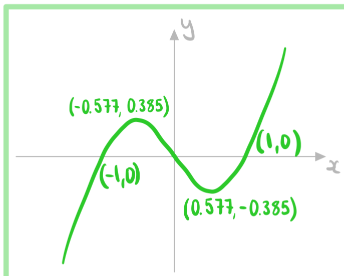
Two functions are defined by

$$f(x) = x^3 - x \text{ and } g(x) = \frac{4}{x}.$$

a)

Sketch the graph $y = f(x)$.

Use GDC to find max, min, intercepts

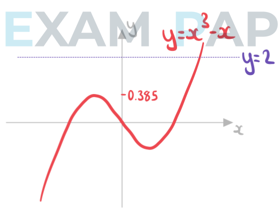


b)

Write down the number of real solutions to the equation $x^3 - x = 2$.

Identify the number of intersections between

$y = x^3 - x$ and $y = 2$



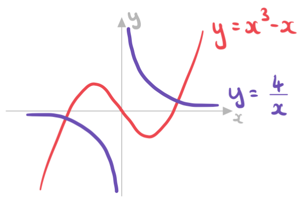
1 intersection

1 solution

c)

Find the coordinates of the points where $y = f(x)$ and $y = g(x)$ intersect.

Use GDC to sketch both graphs



(−1.60, −2.50) and (1.60, 2.50)

d)

Write down the solutions to the equation $x^3 - x = \dfrac{4}{x}$.

Solutions to $x^3 - x = \frac{4}{x}$ are the x coordinates of the points of intersection.

$x = -1.60$ and $x = 1.60$

2.4 Other Functions & Graphs

2.4.1 Exponential & Logarithmic Functions

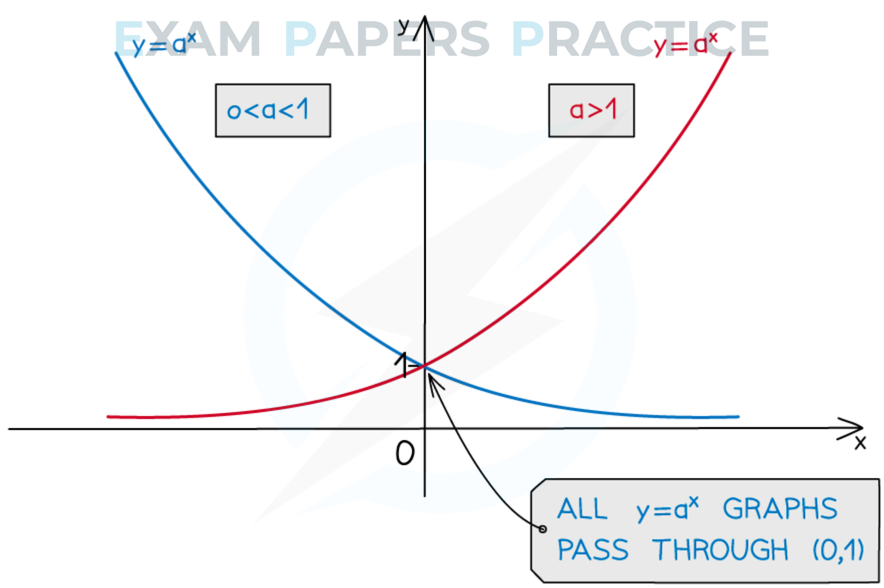
Exponential Functions & Graphs

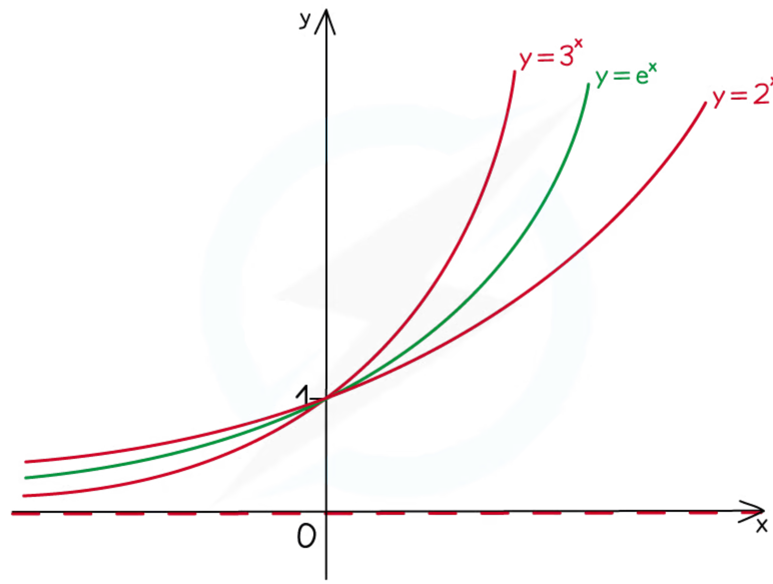
What is an exponential function?

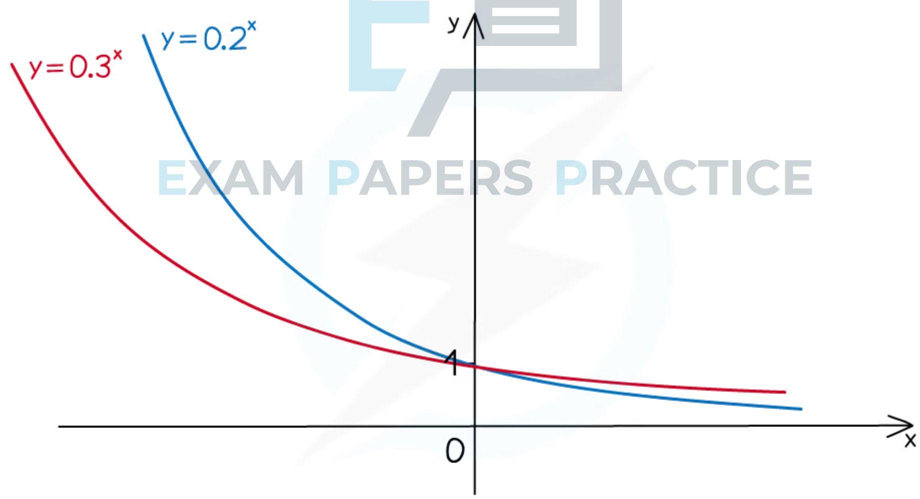
- An **exponential function** is defined by $f(x) = a^x,\ a > 0$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all positive real values**
- An important exponential function is $f(x) = \mathrm{e}^x$
  - Where e is the mathematical constant 2.718…
- Any exponential function can be written using e
  - $a^x = \mathrm{e}^{x \ln a}$
    - This is given in the **formula booklet**

What are the key features of exponential graphs?

- The graphs have a ***y*-intercept** at (0, 1)
- The graphs **do not have any roots**
- The graphs have **a horizontal asymptote** at the *x*-axis: $y = 0$
  - For ***a* > 1** this is the **limiting value** when *x* tends to **negative infinity**
  - For **0 < *a* < 1** this is the **limiting value** when *x* tends to **positive infinity**
- The graphs **do not have any minimum or maximum points**




y
y = 3^x
y = e^x
y = 2^x
1
0
x


y = 0.3^x
y = 0.2^x
y
1
0
x

Logarithmic Functions & Graphs

What is a logarithmic function?

- A **logarithmic function** is of the form $f(x) = \log_a x,\ x > 0$
- Its **domain** is the set of all **positive real values**
  - You can't take a log of zero or a negative number
- Its **range** is set of **all real values**
- $\log_a x$ and a^x are **inverse** functions
- An important logarithmic function is $f(x) = \ln x$
  - This is the natural logarithmic function $\ln x \equiv \log_e x$
  - This is the inverse of e^x

    $\ln e^x = x$ and $e^{\ln x} = x$
- Any logarithmic function can be written using ln
  - $\log_a x = \dfrac{\ln x}{\ln a}$ using the change of base formula

What are the key features of logarithmic graphs?

- The graphs **do not have a *y*-intercept**
- The graphs have **one root** at (1, 0)
- The graphs have a **vertical asymptote** at the *y*-axis: $x = 0$
- The graphs **do not have any minimum or maximum points**

Worked Example

The function f is defined by $f(x) = \log_5 x$ for $x > 0$.

a)

Write down the inverse of f. Give your answer in the form $e^{g(x)}$.

Formula booklet

Exponents & logarithms	$a^x = b \Leftrightarrow x = \log_a b$	$a > 0, b > 0, a \neq 1$

$x = \log_5 y \Leftrightarrow y = 5^x$

Formula booklet

Exponential & logarithmic functions	$a^x = e^{x \ln a}$

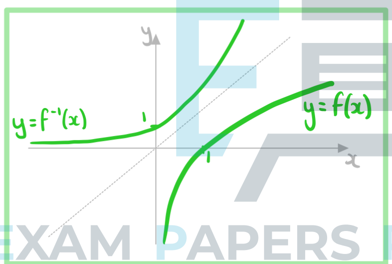
$5^x = e^{x\ln 5}$

$f^{-1}(x) = e^{x\ln 5}$

b)

Sketch the graphs of f and its inverse on the same set of axes.

f and f^{-1} are reflections in line $y = x$



EXAM PAPERS PRACTICE

2.4.2 Solving Equations

Solving Equations Analytically

How can I solve equations analytically where the unknown appears only once?

- These equations can be **solved by rearranging**
- For **one-to-one functions** you can just apply the **inverse**
  - Addition and subtraction are inverses
    $$y = x + k \Leftrightarrow x = y - k$$
  - Multiplication and division are inverses
    $$y = kx \Leftrightarrow x = \frac{y}{k}$$
  - Taking the reciprocal is a self-inverse
    $$y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}$$
  - Odd powers and roots are inverses
    $$y = x^n \Leftrightarrow x = \sqrt[n]{y}$$
    $$y = x^n \Leftrightarrow x = y^{\frac{1}{n}}$$
  - Exponentials and logarithms are inverses
    $$y = a^x \Leftrightarrow x = \log_a y$$
    $$y = e^x \Leftrightarrow x = \ln y$$
- For **many-to-one functions** you will need to use your knowledge of the functions to find the **other solutions**
  - Even powers lead to positive and negative solutions
    $$y = x^n \Leftrightarrow x = \pm\sqrt[n]{y}$$
  - Modulus functions lead to positive and negative solutions
    $$y = |x| \Leftrightarrow x = \pm y$$
  - Trigonometric functions lead to infinite solutions using their symmetries
    $$y = \sin x \Leftrightarrow x = 2k\pi + \arcsin y \text{ or } x = (1 + 2k)\pi - \arcsin y$$
    $$y = \cos x \Leftrightarrow x = 2k\pi \pm \arccos y$$
    $$y = \tan x \Leftrightarrow x = k\pi + \arctan y$$
- Take care when you apply **many-to-one functions** to **both sides** of an equation as this can create **additional solutions** which are incorrect
  - For example: squaring both sides
    $x + 1 = 3$ has one solution $x = 2$
    $(x + 1)^2 = 3^2$ has two solutions $x = 2$ and $x = -4$
- Always **check your solutions** by substituting back into the **original equation**

How can I solve equations analytically where the unknown appears more than once?

- Sometimes it is possible to **simplify expressions** to make the **unknown appear only once**
- **Collect all terms** involving *x* on **one side** and try to simplify into one term
  - For **exponents** use

$$a^{f(x)} \times a^{g(x)} = a^{f(x)+g(x)}$$
$$\frac{a^{f(x)}}{a^{g(x)}} = a^{f(x)-g(x)}$$
$$\left(a^{f(x)}\right)^{g(x)} = a^{f(x)\times g(x)}$$
$$a^{f(x)} = e^{f(x)\ln a}$$

- For **logarithms** use

$$\log_a f(x) + \log_a g(x) = \log_a\left(f(x) \times g(x)\right)$$
$$\log_a f(x) - \log_a g(x) = \log_a\left(\frac{f(x)}{g(x)}\right)$$
$$n\log_a f(x) = \log_a\left(f(x)\right)^n$$

How can I solve equations analytically when the equation can't be simplified?

- Sometimes it is **not possible to simplify** equations
- Most of these equations **cannot be solved analytically**
- A **special case** that can be solved is where the equation can be **transformed into a quadratic** using a substitution
  - These will have **three terms** and involve the same type of function
- **Identify** the **suitable substitution** by considering which **function is a square of another**
  - For example: the following can be transformed into $2y^2 + 3y - 4 = 0$

    $2x^4 + 3x^2 - 4 = 0$ using $y = x^2$

    $2x + 3\sqrt{x} - 4 = 0$ using $y = \sqrt{x}$

    $\frac{2}{x^6} + \frac{3}{x^3} - 4 = 0$ using $y = \frac{1}{x^3}$

    $2e^{2x} + 3e^x - 4 = 0$ using $y = e^x$

    $2 \times 25^x + 3 \times 5^x - 4 = 0$ using $y = 5^x$

    $2^{2x+1} + 3 \times 2^x - 4 = 0$ using $y = 2^x$

    $2(x^3 - 1)^2 + 3(x^3 - 1) - 4 = 0$ using $y = x^3 - 1$
- To **solve**:
  - Make the **substitution** $y = f(x)$
  - **Solve** the **quadratic equation** $ay^2 + by + c = 0$ to get y_1 & y_2
  - **Solve** $f(x) = y_1$ and $f(x) = y_2$

    Note that some equations might have **zero or several solutions**

Can I divide both sides of an equation by an expression?

- When dividing by an expression you must consider whether the **expression could be zero**
- Dividing by an expression that could be zero could result in you **losing solutions to the original equation**
  - For example: $(x + 1)(2x - 1) = 3(x + 1)$

    If you divide both sides by $(x + 1)$ you get $2x - 1 = 3$ which gives $x = 2$

    However $x = -1$ is also a solution to the original equation
- To ensure you **do not lose solutions** you can:
  - **Split the equation into two equations**

    One where the dividing expression equals zero: $x + 1 = 0$

One where the equation has been divided by the expression: $2x - 1 = 3$

- **Make the equation equal zero and factorise**

  $(x+1)(2x-1) - 3(x+1) = 0$

  $(x+1)(2x-1-3) = 0$ which gives $(x+1)(2x-4) = 0$

  Set each factor equal to zero and solve: $x + 1 = 0$ and $2x - 4 = 0$

Exam Tip

- A common mistake that students make in exams is applying functions to each term rather than to each side
  - For example: Starting with the equation $\ln x + \ln(x-1) = 5$ it would be incorrect to write $e^{\ln x} + e^{\ln(x-1)} = e^5$ or $x + (x-1) = e^5$
  - Instead it would be correct to write $e^{\ln x + \ln(x-1)} = e^5$ and then simplify from there

Worked Example

Find the exact solutions for the following equations:

a)

$5 - 2\log_4 x = 0$.

Rearrange using inverse functions

$5 - 2\log_4 x = 0$

$2\log_4 x = 5$ — $y = x - k \Leftrightarrow x = y + k$

$\log_4 x = \frac{5}{2}$ — $y = kx \Leftrightarrow x = \frac{y}{k}$

$x = 4^{5/2}$ — $y = \log_a x \Leftrightarrow x = a^y$

$x = (\sqrt{4})^5$ — $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$

$\boxed{x = 32}$

b)

$x = \sqrt{x + 2}$.

Square both sides (Many-to-one function)

$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0 \Rightarrow x = 2$ or $x = -1$

Check whether each solution is valid

$x = 2$: LHS $= 2$ RHS $= \sqrt{2+2} = 2$ ✓

$x = -1$: LHS $= -1$ RHS $= \sqrt{-1+2} = 1$ ✗

$\boxed{x = 2}$

c)

$e^{2x} - 4e^x - 5 = 0$.

EXAM PAPERS PRACTICE

Notice $e^{2x} = (e^x)^2$, let $y = e^x$

$y^2 - 4y - 5 = 0 \Rightarrow (y+1)(y-5) = 0$

$y = -1$ or $y = 5$

Solve using $y = e^x$

$e^x = -1$ has no solutions as $e^x > 0$

$e^x = 5 \quad \therefore \quad x = \ln 5$

$\boxed{x = \ln 5}$

Solving Equations Graphically

How can I solve equations graphically?

- To solve $f(x) = g(x)$
  - One method is to **draw the graphs** $y = f(x)$ and $y = g(x)$
    - The **solutions** are the **x-coordinates** of the points of **intersection**
  - Another method is to **draw the graph** $y = f(x) - g(x)$ or $y = g(x) - f(x)$
    - The **solutions** are the **roots (zeros)** of this graph
      - This method is sometimes quicker as it involves **drawing only one graph**

Why do I need to solve equations graphically?

- Some equations **cannot be solved analytically**
  - **Polynomials** of degree higher than 4
    - $x^5 - x + 1 = 0$
  - Equations involving **different types of functions**
    - $e^x = x^2$

Exam Tip

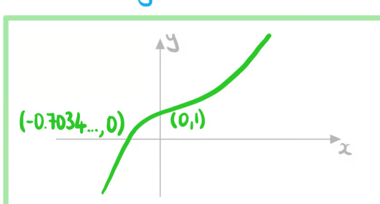
- On a calculator paper you are allowed to solve equations using your GDC unless the question asks for an algebraic method
- If your answer needs to be an exact value then you might need to solve analytically to get the exact value

Worked Example

a)

Sketch the graph $y = e^x - x^2$.

Sketch using GDC



b)

Hence find the solution to $e^x = x^2$.

$e^x = x^2$ when $e^x - x^2 = 0$

Solution is the x-intercept of $y = e^x - x^2$

$x = -0.703$ (3sf)

2.4.3 Modelling with Functions

Modelling with Functions

What is a mathematical model?

- A **mathematical model** simplifies a real-world situation so it can be described using mathematics
  - The model can then be used to make predictions
- **Assumptions** about the situation are made in order to simplify the mathematics
- Models can be **refined** (improved) if further information is available or if the model is compared to real-world data

How do I set up the model?

- The question could:
  - give you the equation of the model
  - tell you about the relationship
    - It might say the relationship is linear, quadratic, etc
  - ask you to suggest a **suitable model**
    - Use your knowledge of each model
    - E.g. if it is compound interest then an exponential model is the most appropriate
- You may have to determine a **reasonable domain**
  - Consider real-life context
    - E.g. if dealing with hours in a day then
    - E.g. if dealing with physical quantities (such as length) then
  - Consider the **possible ranges**
    - If the outcome cannot be negative then you want to choose a domain which corresponds to a range with no negative values
    - **Sketching the graph** is helpful to determine a suitable domain

Which models might I need to use?

- You could be given any model and be expected to use it
- Common models include:
  - **Linear**
    - Arithmetic sequences
    - Linear regression
  - **Quadratic**
    - Projectile motion
    - The height of a cable supporting a bridge
    - Profit
  - **Exponential**
    - Geometric sequences
    - Exponential growth and decay
    - Compound interest
  - **Logarithmic**
    - Richter scale for the magnitude of earthquakes
  - **Rational**

Temperature of a cup of coffee

- **Trigonometric**
  - The depth of a tide

How do I use a model?

- You can use a model by substituting in values for the variable to **estimate outputs**
  - For example: Let $h(t)$ be the height of a football t seconds after being kicked
    $h(3)$ will be an estimate for the height of the ball 3 seconds after being kicked
- Given an **output** you can **form an equation** with the model to **estimate the input**
  - For example: Let $P(n)$ be the profit made by selling n items
    Solving $P(n) = 100$ will give you an estimate for the number of items needing to be sold to make a profit of 100
- If your variable is **time** then substituting $t = 0$ will give you the **initial value** according to the model
- Fully understand the **units for the variables**
  - If the units of P are measured in **thousand dollars** then $P = 3$ represents \$3000
- Look out for **key words** such as:
  - Initially
  - Minimum/maximum
  - Limiting value

What do I do if some of the parameters are unknown?

- A general method is to **form equations** by substituting in given values
  - You can form **multiple equations** and **solve them simultaneously** using your GDC
  - This method **works for all models**
- The **initial value** is the value of the function when the variable is 0
  - This is **normally one of the parameters** in the equation of the model

Worked Example

The temperature, T°C, of a cup of coffee is monitored. Initially the temperature is 80°C and 5 minutes later it is 40°C. It is suggested that the temperature follows the model:

$$T(t) = Ae^{kt} + 16,\ t \geq 0.$$

where t is the time, in minutes, after the coffee has been made.

a)
State the value of A.

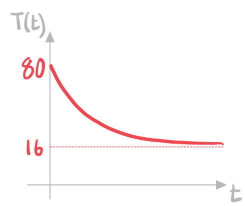
Initially temperature is 80°C

$T(0) = 80$

$Ae^{-k(0)} + 16 = 80$

$A + 16 = 80$

$A = 64$



b)
Find the exact value of k.

$t = 5$, $T = 40$

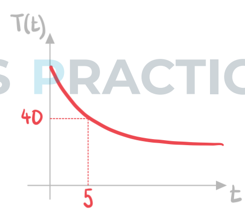
$40 = 64e^{5k} + 16$

$64e^{5k} = 24$

$e^{5k} = \frac{3}{8}$

$5k = \ln\frac{3}{8}$

$k = \frac{1}{5}\ln\frac{3}{8}$



c)
Find the time taken for the temperature of the coffee to reach 30°C.

Find t such that $T(t) = 30$

$30 = 64e^{kt} + 16$

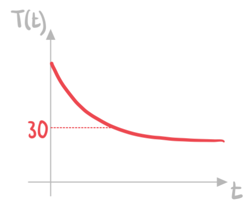
Leave as k until the end to save writing $\frac{1}{5}\ln\frac{3}{8}$ each time

$64e^{kt} = 14$

$e^{kt} = \frac{7}{32}$

$kt = \ln\frac{7}{32}$

$t = \frac{\ln\frac{7}{32}}{k} = \frac{\ln\frac{7}{32}}{\frac{1}{5}\ln\frac{3}{8}} = 7.7476...$



7.75 minutes (3sf)

2.5 Reciprocal & Rational Functions

2.5.1 Reciprocal & Rational Functions

Reciprocal Functions & Graphs

What is the reciprocal function?

- The **reciprocal function** is defined by $f(x) = \frac{1}{x}$, $x \neq 0$
- Its **domain** is the set of **all real values except 0**
- Its **range** is the set of **all real values except 0**
- The reciprocal function has a **self-inverse** nature
  - $f^{-1}(x) = f(x)$
  - $(f \circ f)(x) = x$

What are the key features of the reciprocal graph?

- The graph **does not have a *y*-intercept**
- The graph **does not have any roots**
- The graph has **two asymptotes**
  - A **horizontal** asymptote at the *x*-axis: $y = 0$
    - This is the **limiting value** when the absolute value of *x* gets very large
  - A **vertical** asymptote at the *y*-axis: $x = 0$
    - This is the value that causes the **denominator to be zero**
- The graph has **two axes of symmetry**
  - $y = x$
  - $y = -x$
- The graph **does not have any minimum or maximum points**

Linear Rational Functions & Graphs

What is a rational function with linear terms?

- A **(linear) rational function** is of the form $f(x) = \frac{ax+b}{cx+d},\ x \neq -\frac{d}{c}$
- Its **domain** is the set of **all real values except** $-\frac{d}{c}$
- Its **range** is the set of **all real values except** $\frac{a}{c}$
- The **reciprocal function** is a **special case** of a rational function

What are the key features of linear rational graphs?

- The graph has a **y-intercept** at $\left(0,\ \frac{b}{d}\right)$ provided $d \neq 0$
- The graph has **one root** at $\left(-\frac{b}{a},\ 0\right)$ provided $a \neq 0$
- The graph has **two asymptotes**
  - A **horizontal** asymptote: $y = \frac{a}{c}$

    This is the **limiting value** when the absolute value of *x* gets very large
  - A **vertical** asymptote: $x = -\frac{d}{c}$

    This is the value that causes the **denominator to be zero**
- The graph **does not have any minimum or maximum points**
- If you are asked to **sketch or draw** a rational graph:
  - Give the **coordinates** of any **intercepts** with the axes
  - Give the **equations** of the **asymptotes**

Exam Tip

- If you draw a horizontal line anywhere it should only intersect this type of graph once at most
- The only horizontal line that should not intersect the graph is the horizontal asymptote
  - This can be used to check your sketch in an exam

Worked Example

The function f is defined by $f(x) = \dfrac{10-5x}{x+2}$ for $x \neq -2$.

a)
Write down the equation of
(i)
the vertical asymptote of the graph of f,
(ii)
the horizontal asymptote of the graph of f.

(i) Vertical asymptote is when denominator equals zero

$x+2=0$ $\quad$ $x=-2$

(ii) Horizontal asymptote is limiting value as x gets large

$\lim_{x\to\infty} \dfrac{10-5x}{x+2} = \lim_{x\to\infty} \dfrac{-5x}{x}$ $\quad$ $y=-5$

b)
Find the coordinates of the intercepts of the graph of f with the axes.

y-intercept occurs when $x=0$

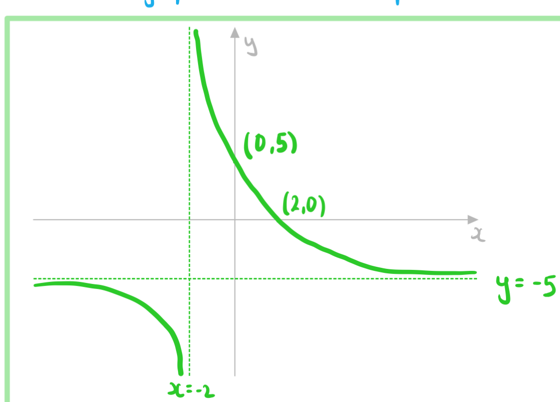
$y = \dfrac{10-5(0)}{0+2} = 5$ $\quad$ $(0,5)$

x-intercept occurs when $y=0$

$\dfrac{10-5x}{x+2} = 0 \Rightarrow 10-5x=0 \Rightarrow x=2$ $\quad$ $(2,0)$

c)
Sketch the graph of f.

Include asymptotes and intercepts



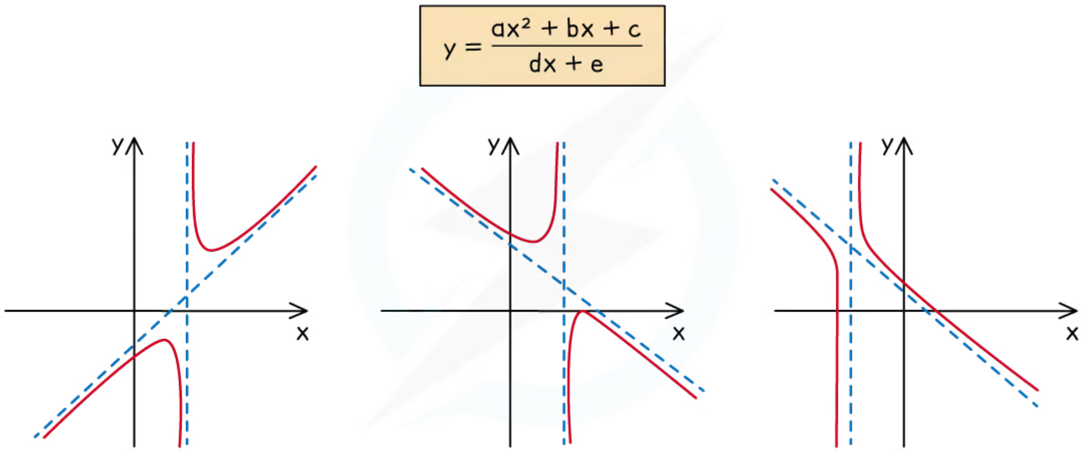
Quadratic Rational Functions & Graphs

How do I sketch the graph of a rational function where the terms are not linear?

- A rational function can be written $f(x) = \frac{g(x)}{h(x)}$
  - Where *g* and *h* are polynomials
- To find the **y-intercept** evaluate $\frac{g(0)}{h(0)}$
- To find the **x-intercept(s)** solve $g(x) = 0$
- To find the equations of the **vertical asymptote(s)** solve $h(x) = 0$
- There will also be an **asymptote** determined by what *f*(*x*) tends to as *x* approaches infinity
  - In this course it will be either:
    - **Horizontal**
    - **Oblique (a slanted line)**
  - This can be found by writing $g(x)$ in the form $h(x)Q(x) + r(x)$
    - You can do this by **polynomial division** or **comparing coefficients**
  - The function then tends to the curve $y = Q(x)$

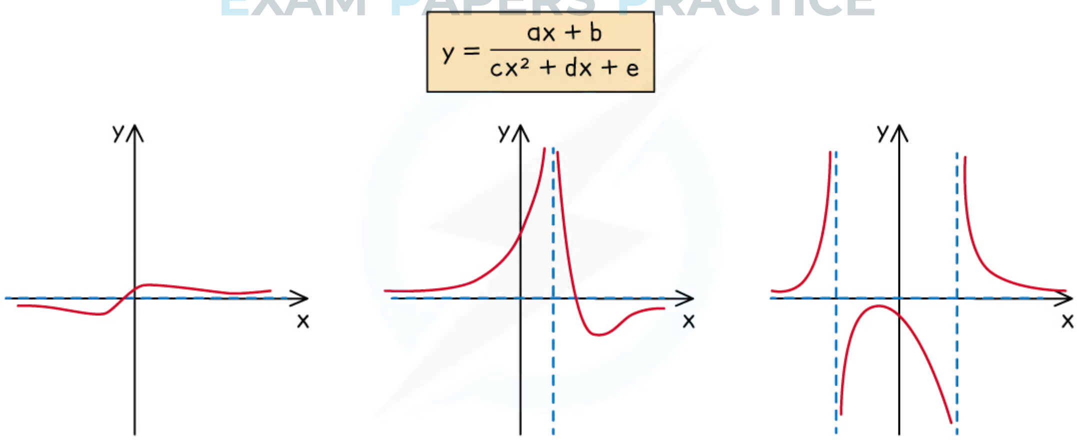
What are the key features of rational graphs: quadratic over linear?

- For the rational function of the form $f(x) = \frac{ax^2 + bx + c}{dx + e}$
- The graph has a **y-intercept** at $\left(0, \frac{c}{e}\right)$ provided $e \neq 0$
- The graph can have **0, 1 or 2 roots**
  - They are the solutions to $ax^2 + bx + c = 0$
- The graph has **one vertical asymptote** $x = -\frac{e}{d}$
- The graph has an **oblique asymptote** $y = px + q$
  - Which can be found by writing $ax^2 + bx + c$ in the form $(dx + e)(px + q) + r$
    - Where *p*, *q*, *r* are constants
    - This can be done by **polynomial division** or **comparing coefficients**



What are the key features of rational graphs: linear over quadratic?

- For the rational function of the form $f(x) = \dfrac{ax + b}{cx^2 + dx + e}$
- The graph has a **y-intercept** at $\left(0, \dfrac{b}{e}\right)$ provided $e \neq 0$
- The graph has **one root** at $x = -\dfrac{b}{a}$
- The graph has can have **0, 1 or 2 vertical asymptotes**
    - They are the solutions to $cx^2 + dx + e = 0$
- The graph has a **horizontal asymptote**



Exam Tip

- If you draw a horizontal line anywhere it should only intersect this type of graph twice at most
    - This idea can be used to check your graph or help you sketch it

Worked Example

The function f is defined by $f(x) = \frac{2x^2+5x-3}{x+1}$ for $x \neq -1$.

a)
(i)
Show that $\frac{2x^2+5x-3}{x+1} = px + q + \frac{r}{x+1}$ for constants p, q and r which are to be found.
(ii)
Hence write down the equation of the oblique asymptote of the graph of f.

(i) Write $2x^2+5x-3$ as $(x+1)(px+q)+r$

$2x^2+5x-3 = px^2+qx+px+q+r$

Compare coefficients

x^2: $2 = p$ $\quad$ x: $5 = q + p$ $\quad$ constant: $-3 = q + r$

$\therefore p = 2$ $\quad$ $\therefore q = 3$ $\quad$ $\therefore r = -6$

$$\frac{2x^2+5x-3}{x+1} = \frac{(x+1)(2x+3)-6}{x+1} = 2x+3-\frac{6}{x+1}$$

(ii) $y = 2x+3$

b)
Find the coordinates of the intercepts of the graph of f with the axes.

y-intercept occurs when $x = 0$

$y = \frac{2(0)^2+5(0)-3}{(0)+1} = -3$ $\quad$ $(0, -3)$

x-intercept occurs when $y = 0$

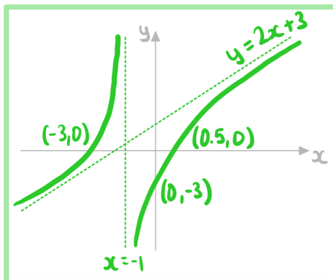
$\frac{2x^2+5x-3}{x+1} = 0 \Rightarrow 2x^2+5x-3=0 \Rightarrow (2x-1)(x+3) \Rightarrow x = 0.5$ or $x = -3$

$(0.5, 0)$ and $(-3, 0)$

c)
Sketch the graph of f.

Vertical asymptote when denominator is zero $x = -1$

Include asymptotes and intercepts



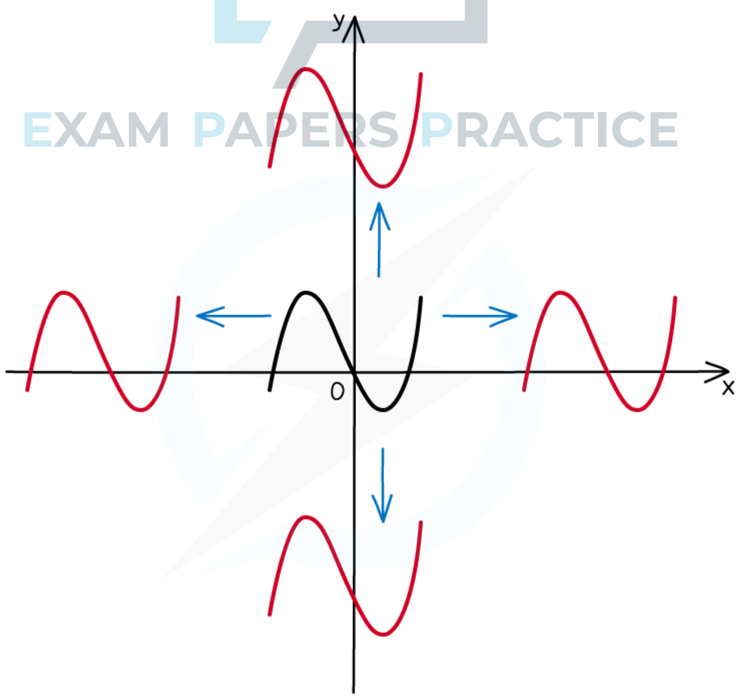
2.6 Transformations of Graphs

2.6.1 Translations of Graphs

Translations of Graphs

What are translations of graphs?

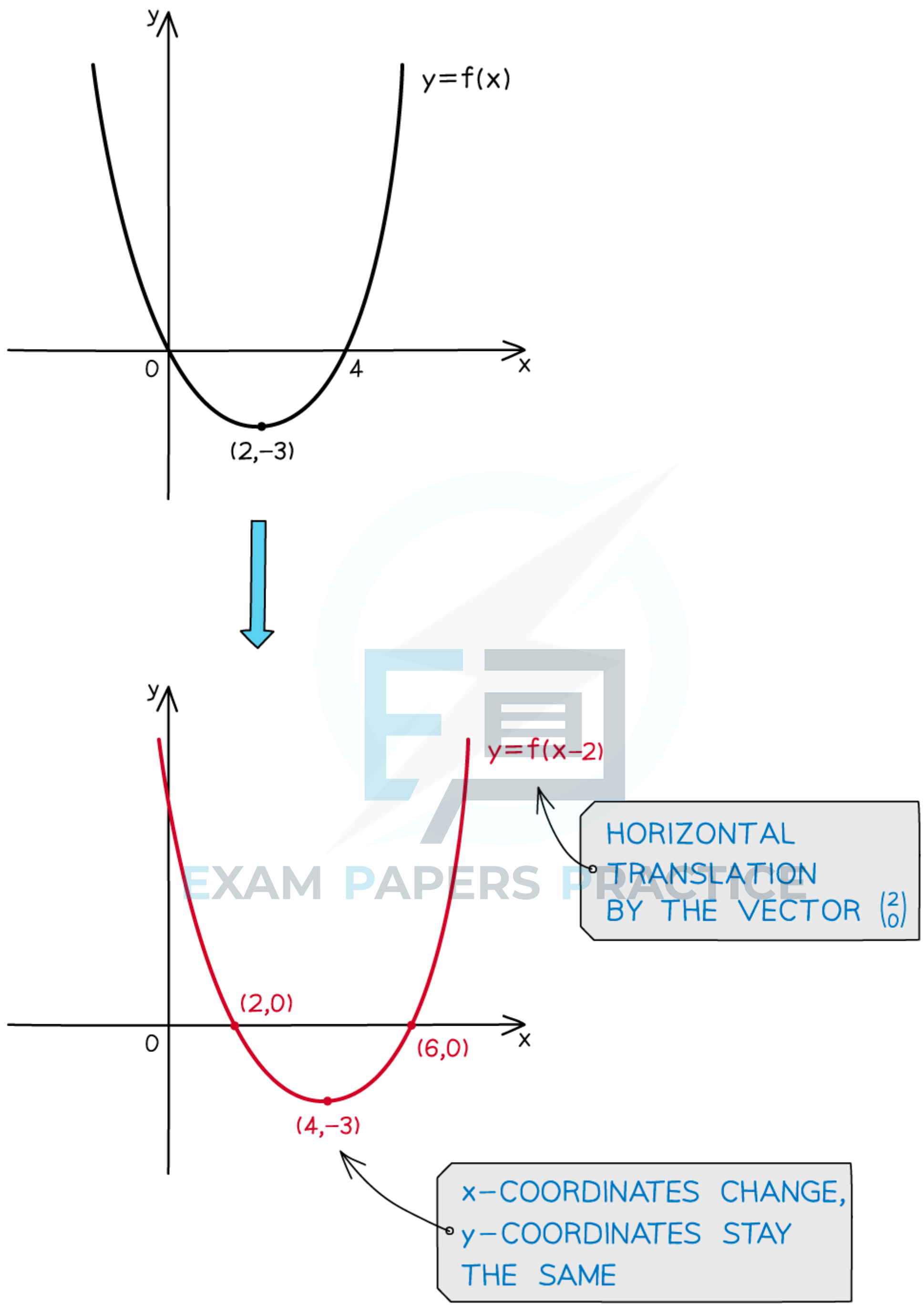
- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **translation**:
  - the graph is **moved** (up or down, left or right) in the xy plane
    - Its position **changes**
  - the shape, size, and orientation of the graph remain **unchanged**
- A particular translation (how far left/right, how far up/down) is specified by a **translation vector** $\begin{pmatrix} x \\ y \end{pmatrix}$:
  - x is the **horizontal** displacement
    - **Positive** moves **right**
    - **Negative** moves **left**
  - y is the **vertical** displacement
    - **Positive** moves **up**
    - **Negative** moves **down**



What effects do horizontal translations have on the graphs and functions?

- A **horizontal translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is represented by

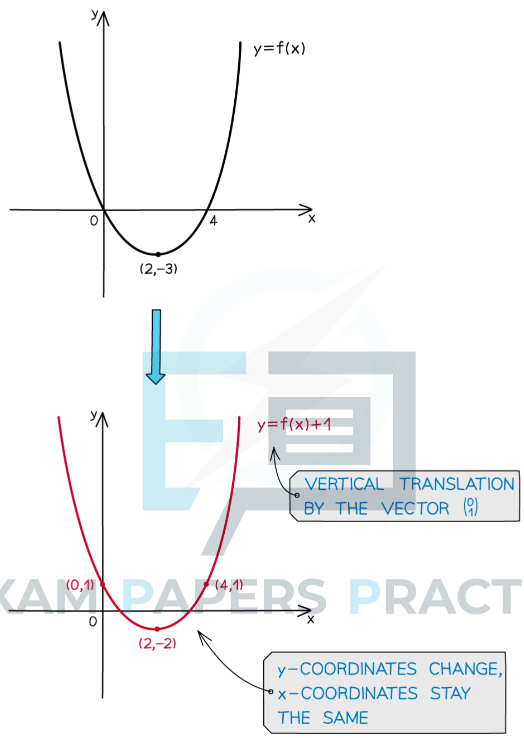
- $y = f(x - a)$
- The ***x*-coordinates change**
  - The value *a* is **subtracted** from them
- The ***y*-coordinates stay the same**
- The coordinates (x, y) become $(x + a, y)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
  - $x = k$ becomes $x = k + a$



What effects do vertical translations have on the graphs and functions?

- A **vertical translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ b \end{pmatrix}$ is represented by
  - $y - b = f(x)$

- This is often rearranged to $y = f(x) + b$
- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
  - The value *b* is **added** to them
- The coordinates $(x,\ y)$ become $(x,\ y + b)$
- **Horizontal** asymptotes **change**
  - $y = k$ becomes $y = k + b$
- **Vertical** asymptotes **stay the same**

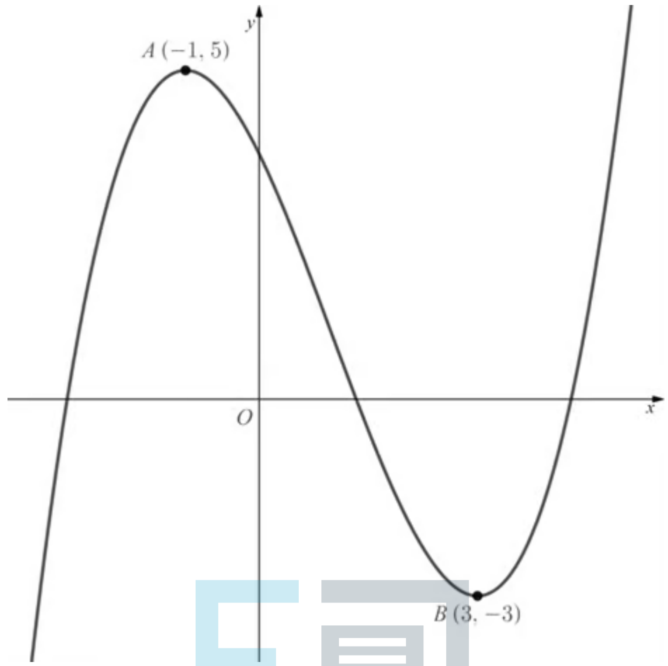


Exam Tip

- To get full marks in an exam make sure you use correct mathematical terminology
  - For example: Translate by the vector $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

Worked Example

The diagram below shows the graph of $y = f(x)$.



a)

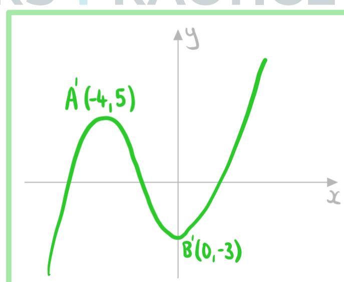
Sketch the graph of $y = f(x+3)$.

$y = f(x+k)$ translation by $\begin{pmatrix} -k \\ 0 \end{pmatrix}$

Translate $y = f(x)$ by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

A becomes $(-4, 5)$

B becomes $(0, -3)$



b)

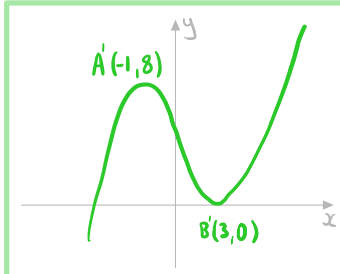
Sketch the graph of $y = f(x) + 3$.

$y = f(x) + k$ translation by $\begin{pmatrix} 0 \\ k \end{pmatrix}$

Translate $y = f(x)$ by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

A becomes $(-1, 8)$

B becomes $(3, 0)$

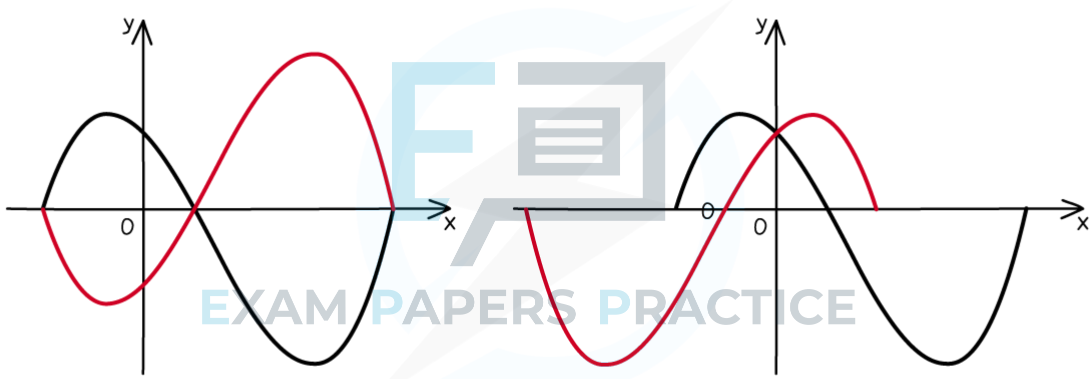


2.6.2 Reflections of Graphs

Reflections of Graphs

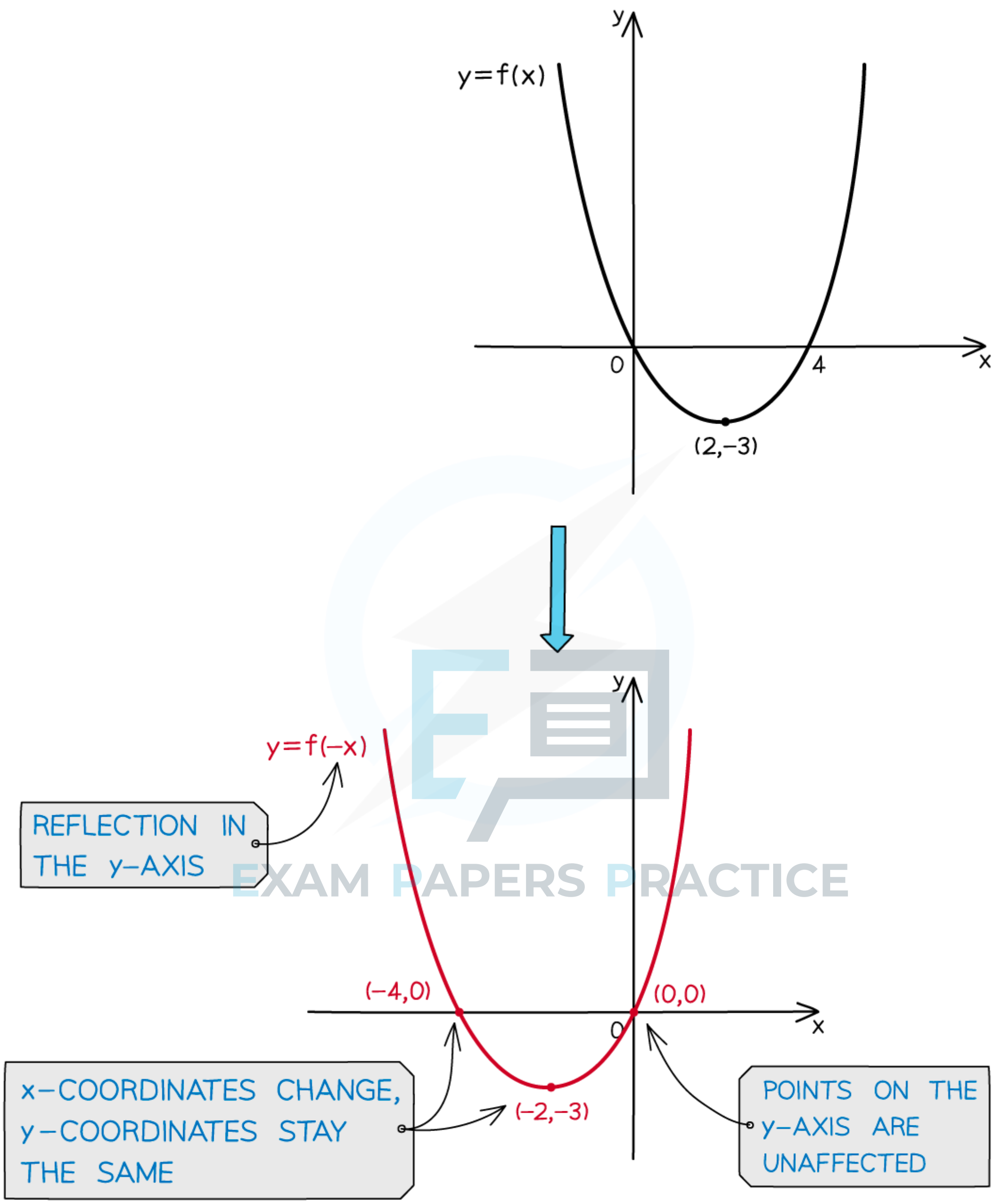
What are reflections of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **reflection**:
  - the graph is **flipped** about one of the coordinate axes
    - Its orientation **changes**
  - the size of the graph remains **unchanged**
- A particular reflection is specified by an **axis of symmetry**:
  - $y = 0$
    - This is the *x*-axis
  - $x = 0$
    - This is the *y*-axis



What effects do horizontal reflections have on the graphs and functions?

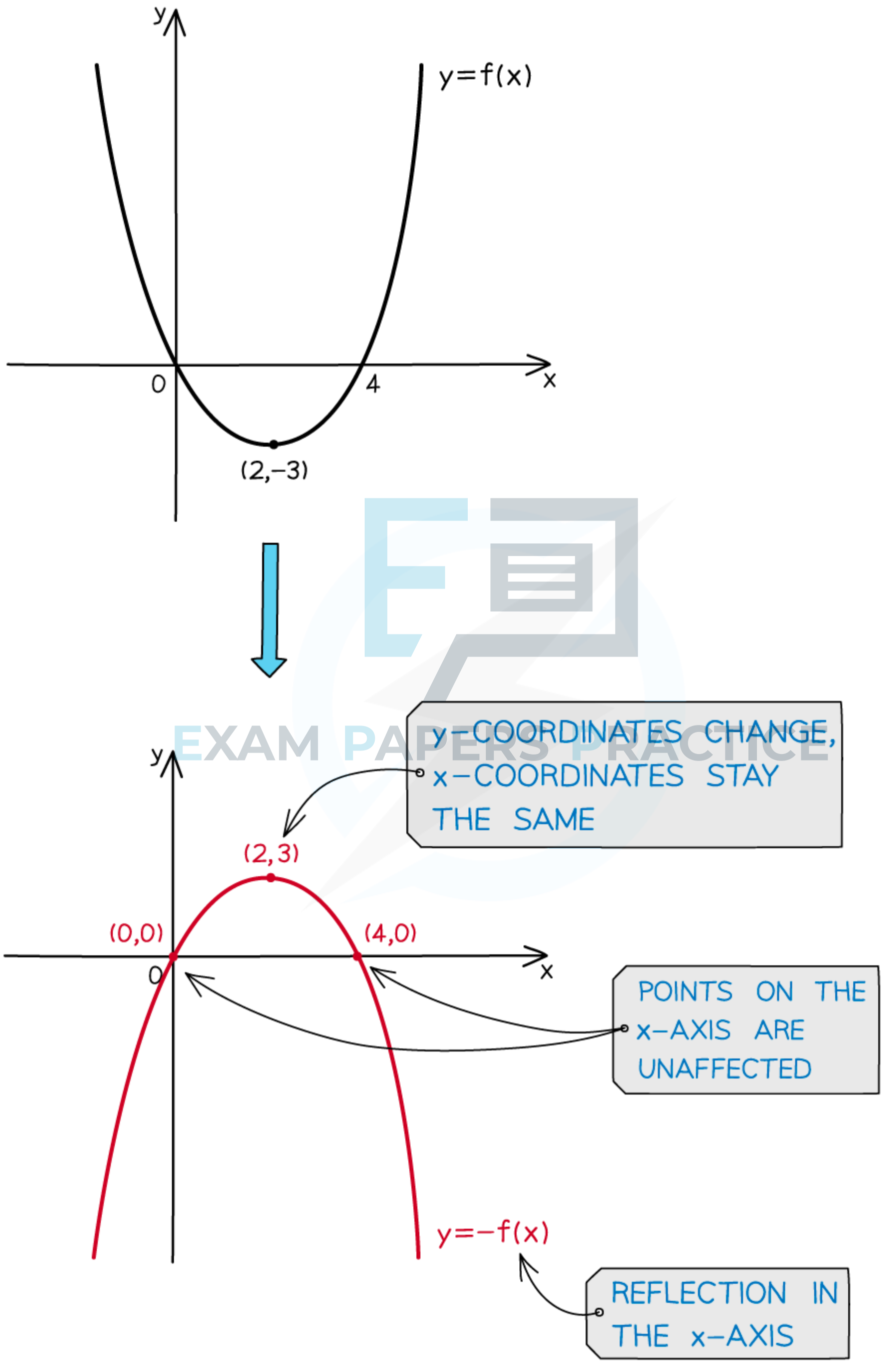
- A **horizontal reflection** of the graph $y = f(x)$ about the *y*-axis is represented by
  - $y = f(-x)$
- The **x-coordinates change**
  - Their **sign** changes
- The ***y*-coordinates stay the same**
- The coordinates (x, y) become $(-x, y)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
  - $x = k$ becomes $x = -k$



What effects do vertical reflections have on the graphs and functions?

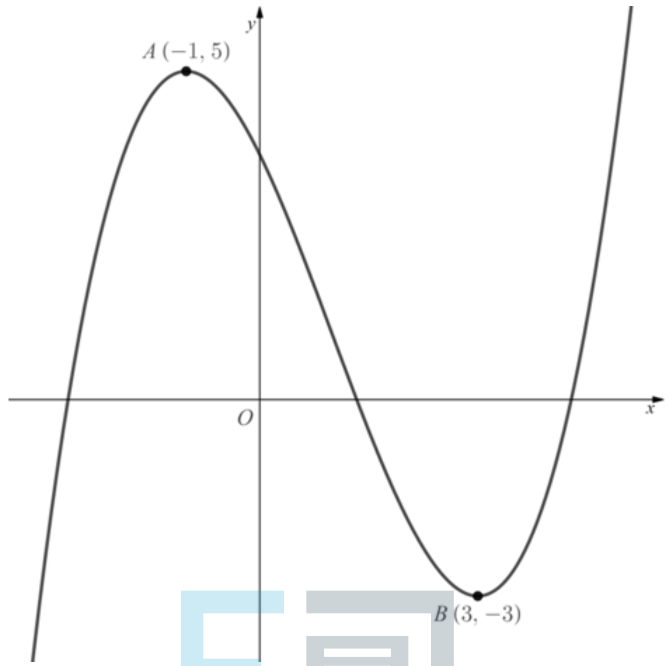
- A **vertical reflection** of the graph $y = f(x)$ about the *x*-axis is represented by
  - $-y = f(x)$
  - This is often rearranged to $y = -f(x)$
- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
  - Their **sign** changes

- The coordinates (x, y) become $(x, -y)$
- **Horizontal** asymptotes **change**
  - $y = k$ becomes $y = -k$
- **Vertical** asymptotes **stay the same**



Worked Example

The diagram below shows the graph of $y = f(x)$.

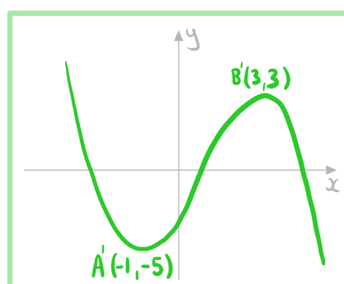


a)
Sketch the graph of $y = -f(x)$.

$y = -f(x)$ reflection in x-axis

A becomes $(-1, -5)$

B becomes $(3, 3)$

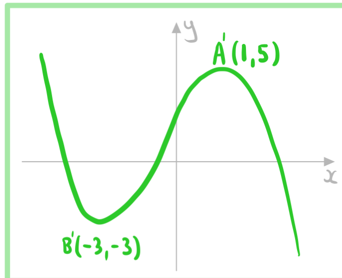


b)
Sketch the graph of $y = f(-x)$.

$y = f(-x)$ reflection in y-axis

A becomes $(1, 5)$

B becomes $(-3, -3)$



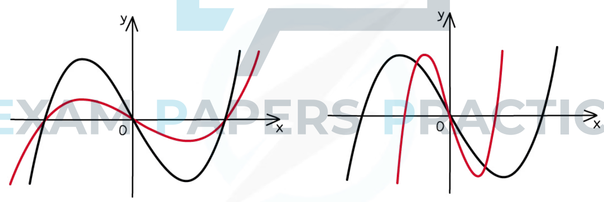
EXAM PAPERS PRACTICE

2.6.3 Stretches Graphs

Stretches of Graphs

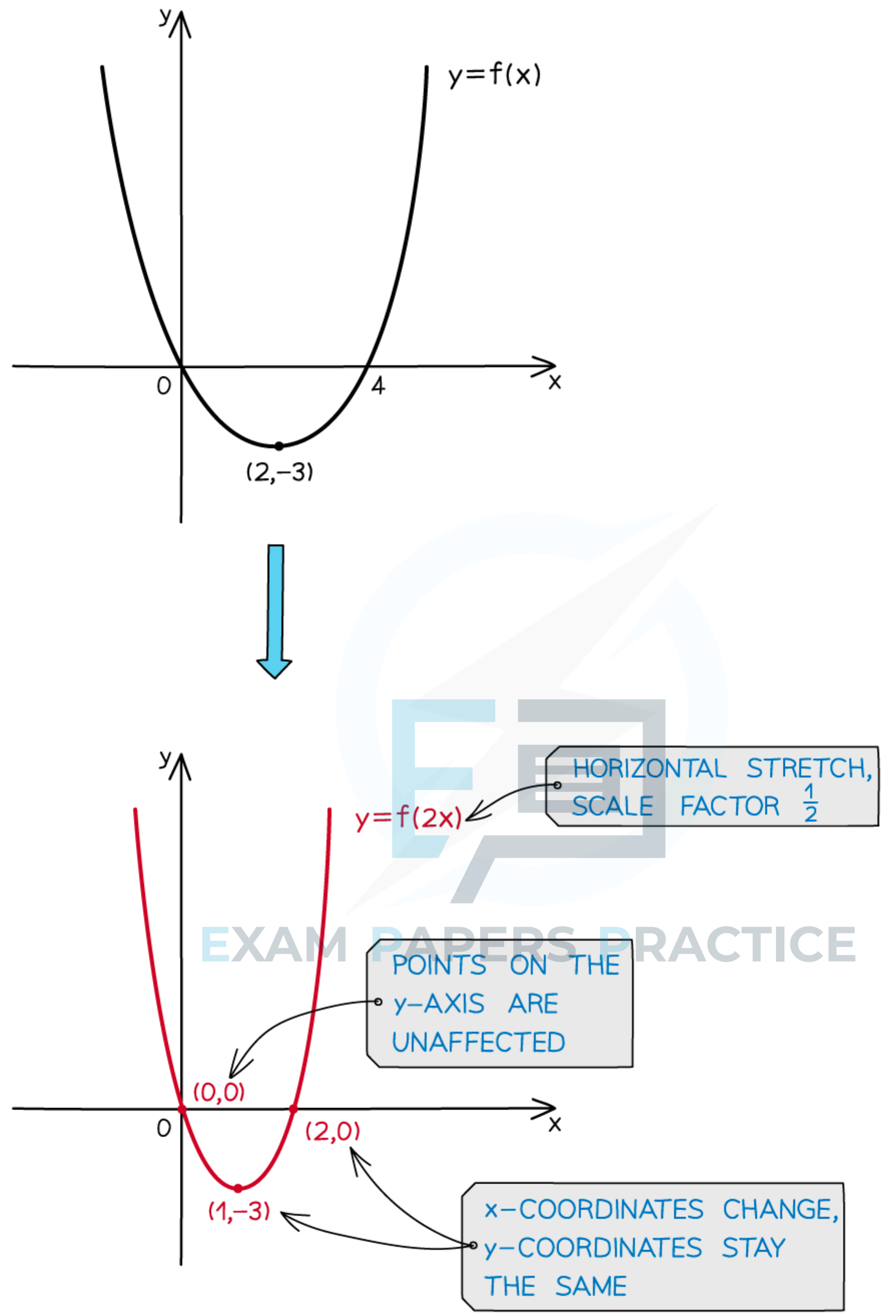
What are stretches of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **stretch**:
  - the graph is **stretched** about one of the coordinate axes by a scale factor
    - Its size **changes**
  - the orientation of the graph remains **unchanged**
- A particular stretch is specified by a **coordinate axis** and a **scale factor**:
  - The **distance** between a **point** on the graph and the **specified coordinate axis** is **multiplied** by the **constant scale factor**
  - The graph is stretched in the **direction** which is **parallel** to the **other coordinate axis**
  - For scale factors **bigger than 1**
    - the points on the graph get **further away** from the **specified coordinate axis**
  - For scale factors **between 0 and 1**
    - the points on the graph get **closer** to the **specified coordinate axis**
    - This is also sometimes called a **compression** but in your exam you must use the term **stretch** with the appropriate scale factor



What effects do horizontal stretches have on the graphs and functions?

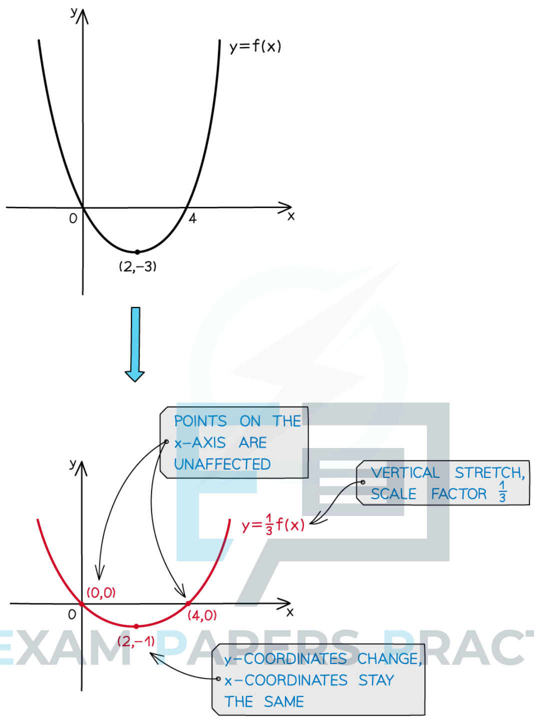
- A **horizontal stretch** of the graph $y = f(x)$ by a scale factor q centred about the y-axis is represented by
  - $y = f\left(\frac{x}{q}\right)$
- The **x-coordinates change**
  - They are **divided** by q
- The **y-coordinates stay the same**
- The coordinates (x, y) become (qx, y)
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
  - $x = k$ becomes $x = qk$



What effects do vertical stretches have on the graphs and functions?

- A **vertical stretch** of the graph $y = f(x)$ by a scale factor *p* centred about the *x*-axis is represented by
  - $\frac{y}{p} = f(x)$
  - This is often rearranged to $y = pf(x)$

- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
    - They are **multiplied** by *p*
- The coordinates (x, y) become (x, py)
- **Horizontal** asymptotes **change**
    - $y = k$ becomes $y = pk$
- **Vertical** asymptotes **stay the same**

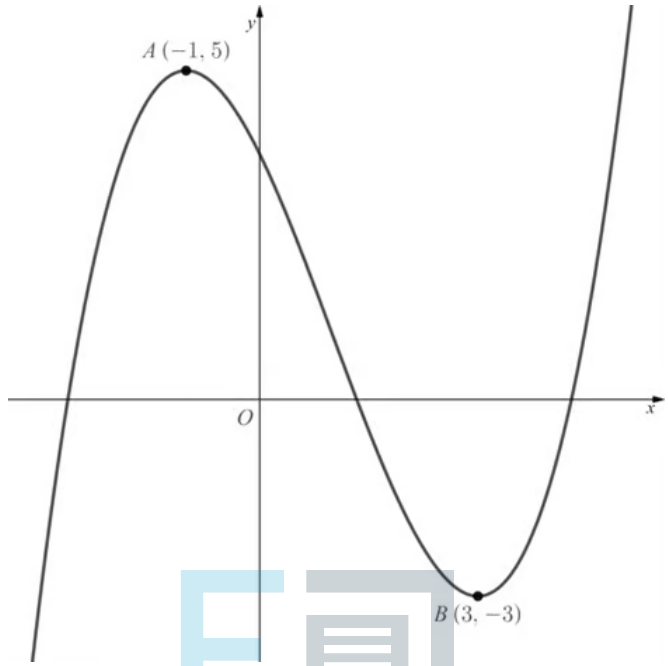


Exam Tip

- To get full marks in an exam make sure you use correct mathematical terminology
    - For example: Stretch vertically by scale factor ½
    - Do not use the word "compress" in your exam

Worked Example

The diagram below shows the graph of $y = f(x)$.

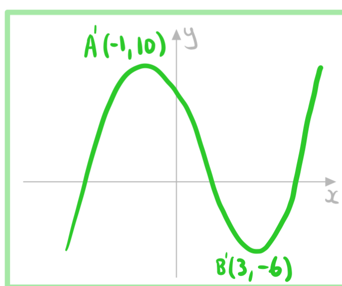


a)
Sketch the graph of $y = 2f(x)$.

$y = kf(x)$ vertical stretch scale factor k

Stretch $y = f(x)$ vertically
scale factor 2

A becomes $(-1, 10)$
B becomes $(3, -6)$



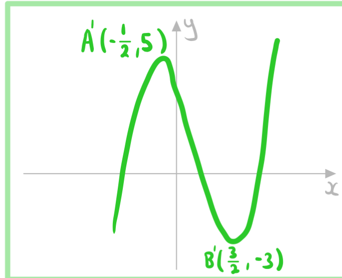
b)
Sketch the graph of $y = f(2x)$.

$y = f(kx)$ horizontal stretch scale factor $\frac{1}{k}$

Stretch $y = f(x)$ horizontally scale factor $\frac{1}{2}$

A becomes $(-\frac{1}{2}, 5)$

B becomes $(\frac{3}{2}, -3)$



2.6.4 Composite Transformations of Graphs

Composite Transformations of Graphs

What transformations do I need to know?

- $y = f(x + k)$ is **horizontal translation** by vector $\begin{pmatrix} -k \\ 0 \end{pmatrix}$
  - If k is **positive** then the graph moves **left**
  - If k is **negative** then the graph moves **right**
- $y = f(x) + k$ is **vertical translation** by vector $\begin{pmatrix} 0 \\ k \end{pmatrix}$
  - If k is **positive** then the graph moves **up**
  - If k is **negative** then the graph moves **down**
- $y = f(kx)$ is a **horizontal stretch** by scale factor $\frac{1}{k}$ centred about the y-axis
  - If $k > 1$ then the graph gets **closer** to the y-axis
  - If $0 < k < 1$ then the graph gets **further** from the y-axis
- $y = kf(x)$ is a **vertical stretch** by scale factor k centred about the x-axis
  - If $k > 1$ then the graph gets **further** from the x-axis
  - If $0 < k < 1$ then the graph gets **closer** to the x-axis
- $y = f(-x)$ is a **horizontal reflection** about the y-axis
  - A **horizontal reflection** can be viewed as a special case of a **horizontal stretch**
- $y = -f(x)$ is a **vertical reflection** about the x-axis
  - A **vertical reflection** can be viewed as a special case of a **vertical stretch**

How do horizontal and vertical transformations affect each other?

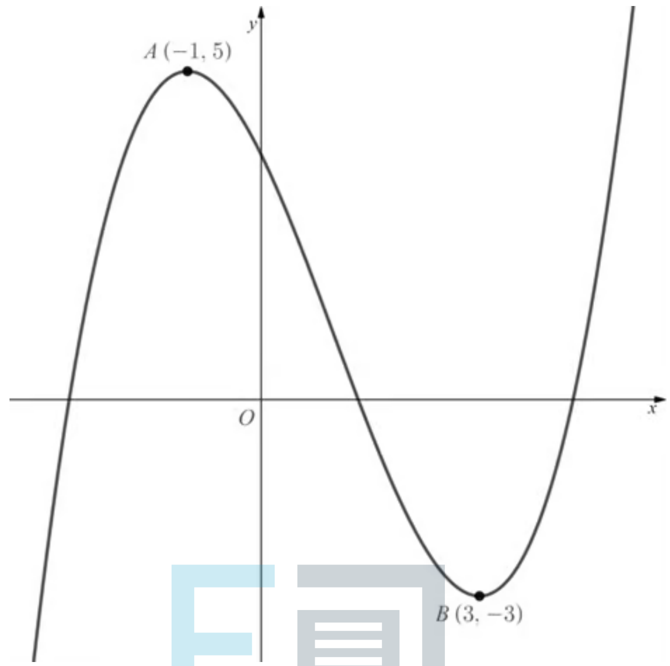
- **Horizontal and vertical transformations** are **independent** of each other
  - The horizontal transformations involved will need to be applied in their correct order
  - The vertical transformations involved will need to be applied in their correct order
- Suppose there are **two horizontal** transformation **H_1 then H_2** and **two vertical** transformations **V_1 then V_2** then they can be applied in the following orders:
  - Horizontal then vertical:
    - $H_1H_2V_1V_2$
  - Vertical then horizontal:
    - $V_1V_2H_1H_2$
  - Mixed up (provided that **H_1 comes before H_2** and **V_1 comes before V2**):
    - $H_1V_1H_2V_2$
    - $H_1V_1V_2H_2$
    - $V_1H_1V_2H_2$
    - $V_1H_1H_2V_2$

Exam Tip

- In an exam you are more likely to get the correct solution if you deal with one transformation at a time and sketch the graph after each transformation

Worked Example

The diagram below shows the graph of $y = f(x)$.



Sketch the graph of $y = \frac{1}{2} f\left(\frac{x}{2}\right)$.

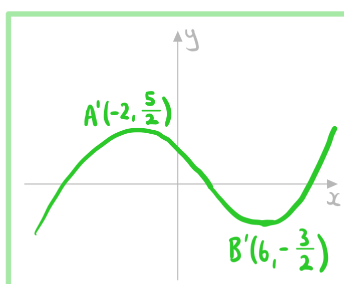
A vertical and horizontal transformation can be done in any order

$y = \frac{1}{2} f(x)$: vertical stretch scale factor $\frac{1}{2}$

$y = f\left(\frac{x}{2}\right)$: horizontal stretch scale factor 2

A becomes $\left(-2, \frac{5}{2}\right)$

B becomes $\left(6, -\frac{3}{2}\right)$



Composite Vertical Transformations af(x)+b

How do I deal with multiple vertical transformations?

- **Order matters** when you have **more than one vertical transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order
  - A **vertical stretch** by scale factor *a* followed by a **translation** of $\begin{pmatrix} 0 \\ b \end{pmatrix}$

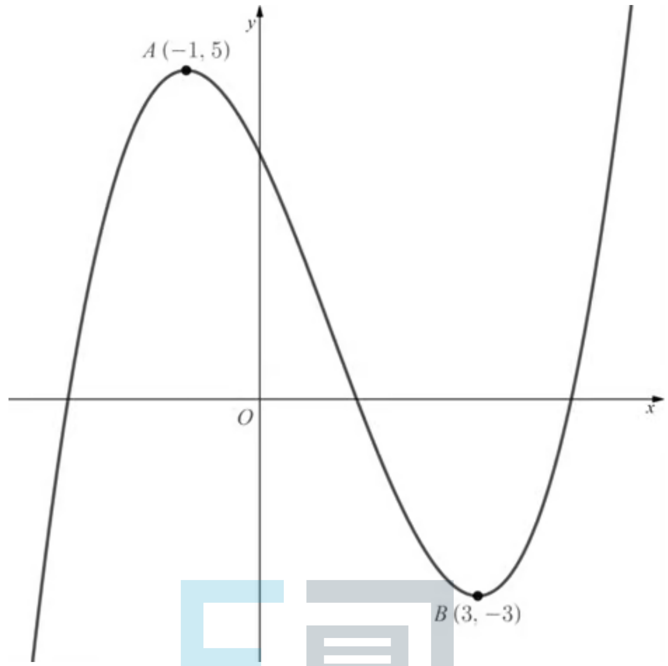
    Stretch: $y = af(x)$
    Then translation: $y = [af(x)] + b$
    Final equation: $y = af(x) + b$
  - A **translation** of $\begin{pmatrix} 0 \\ b \end{pmatrix}$ followed by a **vertical stretch** by scale factor *a*

    Translation: $y = f(x) + b$
    Then stretch: $y = a[f(x) + b]$
    Final equation: $y = af(x) + ab$
- If you are asked to determine the **order**
  - The order of vertical transformations **follows the order of operations**
  - First write the equation in the form $y = af(x) + b$

    First stretch vertically by scale factor *a*
    If *a* is negative then the **reflection and stretch** can be **done in any order**
    Then translate by $\begin{pmatrix} 0 \\ b \end{pmatrix}$

Worked Example

The diagram below shows the graph of $y = f(x)$.



Sketch the graph of $y = 3f(x) - 2$.

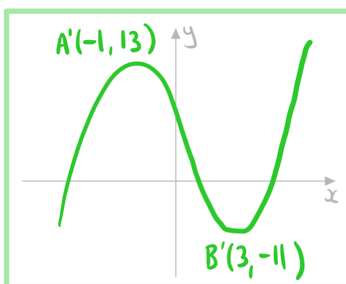
The order vertical transformations follows the order of operations

$y = 3f(x)$: Vertical stretch scale factor 3

$y = f(x) - 2$: Translate $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

A becomes $(-1, 13)$

B becomes $(3, -11)$



Composite Horizontal Transformations f(ax+b)

How do I deal with multiple horizontal transformations?

- **Order matters** when you have **more than one horizontal transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order
  - A **horizontal stretch** by scale factor $\frac{1}{a}$ followed by a **translation** of $\begin{pmatrix} -b \\ 0 \end{pmatrix}$

    Stretch: $y = f(ax)$
    Then translation: $y = f(a(x+b))$
    Final equation: $y = f(ax + ab)$
  - A **translation** of $\begin{pmatrix} -b \\ 0 \end{pmatrix}$ followed by a **horizontal stretch** by scale factor $\frac{1}{a}$

    Translation: $y = f(x+b)$
    Then stretch: $y = f((ax) + b)$
    Final equation: $y = f(ax + b)$
- If you are asked to determine the **order**
  - First write the equation in the form $y = f(ax + b)$
  - The order of horizontal transformations **is the reverse of the order of operations**

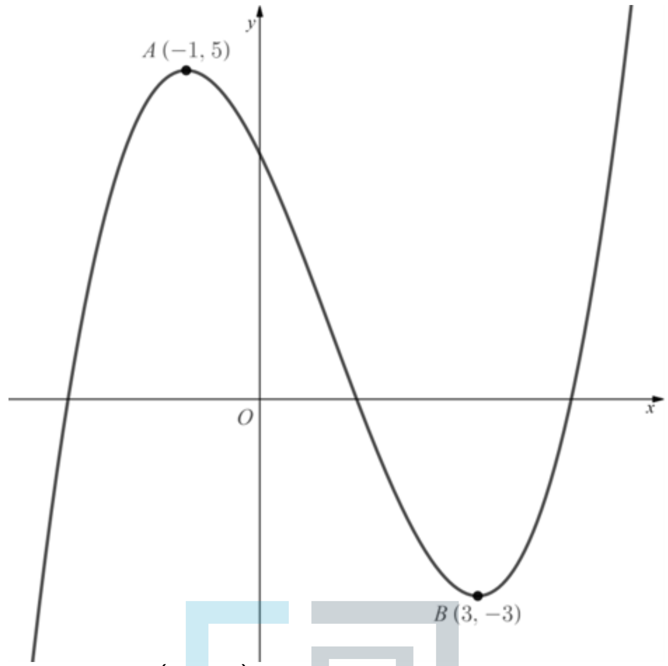
    First translate by $\begin{pmatrix} -b \\ 0 \end{pmatrix}$

    Then stretch by scale factor $\frac{1}{a}$

    If *a* is negative then the **reflection and stretch** can be **done in any order**

Worked Example

The diagram below shows the graph of $y = f(x)$.



Sketch the graph of $y = f(2x - 1)$.

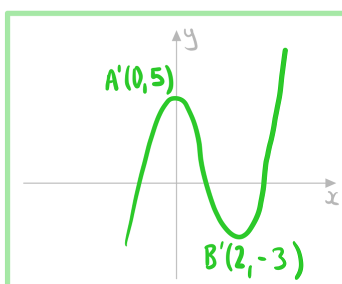
The order of horizontal transformations is the reverse of the order of operations

$y = f(x-1)$: Translate $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$y = f(2x)$: Horizontal stretch scale factor $\frac{1}{2}$

A becomes $(0, 5)$

B becomes $(2, -3)$



2.7 Polynomial Functions

2.7.1 Factor & Remainder Theorem

Factor Theorem

What is the factor theorem?

- The **factor theorem** is used to find the linear factors of **polynomial** equations
- This topic is closely tied to finding the **zeros** and **roots** of a **polynomial** function/equation
  - As a rule of thumb a **zero** refers to the polynomial function and a **root** refers to a polynomial equation
- For any **polynomial** function $P(x)$
  - $(x - k)$ is a **factor** of $P(x)$ if $\mathbf{P(k) = 0}$
  - $\mathbf{P(k) = 0}$ if $(x - k)$ is a **factor** of $P(x)$

How do I use the factor theorem?

- Consider the polynomial function $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ and $(x - k)$ is a **factor**
  - Then, due to the factor theorem $P(k) = a_n k^n + a_{n-1} k^{n-1} + \ldots + a_1 k + a_0 = 0$
  - $P(x) = (x - k) \times Q(x)$, where $Q(x)$ is a **polynomial** that is a factor of $P(x)$
  - Hence, $\frac{P(x)}{x - k} = Q(x)$, where $Q(x)$ is another factor of $P(x)$
- If the linear factor has a **coefficient of *x*** then you must first factorise out the coefficient
  - If the linear factor is $(ax - b) = a\left(x - \frac{b}{a}\right) \rightarrow P\left(\frac{b}{a}\right) = 0$

> **Exam Tip**
>
> - A common mistake in exams is using the incorrect sign for either the root or the factor
> - If you are asked to find integer solutions to a polynomial then you only need to consider factors of the constant term

Worked Example

Determine whether $(x-2)$ is a factor of the following polynomials:

a)
$f(x)=x^3-2x^2-x+2$.

Step 1: Determine k

Our linear function is $x-2$

→ so $k=2$

Step 2: Apply factor theorem

For $x-2$ to be a factor of $f(x)$,

$f(2)$ has to equal zero

$$f(2)=(2)^3-2(2)^2-(2)+2$$
$$=8-8-2+2$$
$$=0$$

$f(2)=0$,
so $x-2$ is a factor of $f(x)$

b)
$g(x)=2x^3+3x^2-x+5$.

Step 1: Determine k

Our linear function is $x - 2$

→ so $k = 2$

Step 2: Apply factor theorem

For $x-2$ to be a factor of $g(x)$,
$g(2)$ has to equal zero

$$g(2) = 2(2)^3 + 3(2)^2 - (2) + 5$$
$$= 16 - 12 - 2 + 5$$
$$= 7$$

$g(2) = 7$,
so $x - 2$ is not a factor of $g(x)$

It is given that $(2x-3)$ is a factor of $h(x) = 2x^3 - bx^2 + 7x - 6$.

c)
Find the value of b.

Step 1: Determine k

Our linear function is $2x-3$

$\rightarrow$ so $k = \frac{3}{2}$

Step 2: Apply factor theorem to find b

Since $2x-3$ is a factor of $h(x)$,

$h\left(\frac{3}{2}\right) = 0$

$$0 = 2\left(\frac{3}{2}\right)^3 - b\left(\frac{3}{2}\right)^2 + 7\left(\frac{3}{2}\right) - 6$$

$$= \frac{54}{8} - \frac{9}{4}b + \frac{21}{2} - 6$$

$b = 5$

Remainder Theorem

What is the remainder theorem?

- The **remainder theorem** is used to find the remainder when we divide a **polynomial** function by a linear function
- When any polynomial $P(x)$ is divided by any linear function $(x - k)$ the value of the remainder R is given by $P(k) = R$
  - Note, when $P(k) = 0$ then $(x - k)$ is a factor of $P(x)$

How do I use the remainder theorem?

- Consider the polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ and the linear function $(x - k)$
  - Then, due to the remainder theorem $P(k) = a_n k^n + a_{n-1}k^{n-1} + \ldots + a_1 k + a_0 = R$
  - $P(x) = (x - k) \times Q(x) + R$, where $Q(x)$ is a **polynomial**
  - Hence, $\frac{P(x)}{x-k} = Q(x) + \frac{R}{x-k}$, where R is the remainder
- If the linear factor has a **coefficient of *x*** then you must first factorise out the coefficient
  - If the linear factor is $(ax - b) = a\left(x - \frac{b}{a}\right) \rightarrow P\left(\frac{b}{a}\right) = R$

Worked Example

Let $f(x) = 2x^4 - 2x^3 - x^2 - 3x + 1$, find the remainder R when $f(x)$ is divided by:

a)
$x - 3$.

Step 1: Determine k

Our linear function is $x - 3$

→ so $k = 3$

Step 2: Apply remainder theorem

$f(3) = R$

$f(3) = 2(3)^4 - 2(3)^3 - (3)^2 - 3(3) + 1$

$f(3) = 162 - 54 - 9 - 9 + 1$

$f(3) = 91$

$R = 91$

b)
$x + 2$.

Step 1: Determine k

Our linear function is $x + 2$

→ so $k = -2$

Step 2: Apply remainder theorem

$f(-2) = R$

$f(-2) = 2(-2)^4 - 2(-2)^3 - (-2)^2 - 3(-2) + 1$

$f(-2) = 32 + 16 - 4 + 6 + 1$

$f(-2) = 51$

$R = 51$

The remainder when $f(x)$ is divided by $(2x+k)$ is $\frac{893}{8}$.

c)
Given that $k>0$, find the value of k.

Step 1: Apply remainder theorem

$2x + k = 2\left(x + \frac{k}{2}\right)$ $\quad f\left(-\frac{k}{2}\right) = \frac{893}{8}$

$\frac{893}{8} = 2\left(-\frac{k}{2}\right)^4 - 2\left(-\frac{k}{2}\right)^3 - \left(-\frac{k}{2}\right)^2 - 3\left(-\frac{k}{2}\right) + 1$

Step 2: Solve for k using your GDC

$k = 5$

2.7.2 Polynomial Division

Polynomial Division

What is polynomial division?

- Polynomial division is the process of **dividing two polynomials**
  - This is usually only useful when the **degree of the denominator** is **less than or equal** to the **degree of the numerator**
- To do this we use an algorithm similar to that used for **division of integers**
- To divide the polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ by the polynomial $D(x) = b_k x^k + b_{k-1} x^{k-1} + \ldots + b_1 x + b_0$ where $k \le n$
  - STEP 1
    Divide the **leading term of the polynomial** $P(x)$ by **the leading term of the divisor** $D(x)$: $\frac{a_n x^n}{b_b x^k} = q_m x^m$
  - STEP 2
    Multiply the divisor by this term: $D(x) \times q_m x^m$
  - STEP 3
    Subtract this from the **original polynomial** $P(x)$ to cancel out the leading term:
    $R(x) = P(x) - D(x) \times q_m x^m$
  - Repeat steps 1 – 3 using the new polynomial $R(x)$ in place of $P(x)$ until the subtraction results in an expression for $R(x)$ with degree less than the divisor
    The quotient $Q(x)$ is the **sum of the terms** you multiplied the divisor by:
    $Q(x) = q_m x^m + q_{m-1} x^{m-1} + \ldots + q_1 x + q_0$
    The remainder $R(x)$ is the polynomial after the final subtraction

Division by linear functions

- If $P(x)$ has degree n and is divided by a linear function $(ax + b)$ then
  - $\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$ where
    $ax + b$ is the **divisor** (degree 1)
    $Q(x)$ is the **quotient** (degree n – 1)
    R is the **remainder** (degree 0)
  - Note that $P(x) = Q(x) \times (ax + b) + R$

Division by quadratic functions

- If $P(x)$ has degree n and is divided by a quadratic function $(ax^2 + bx + c)$ then
  - $\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{ex + f}{ax^2 + bx + c}$ where
    $ax^2 + bx + c$ is the **divisor** (degree 2)
    $Q(x)$ is the **quotient** (degree n – 2)

$ex + f$ is the **remainder** (degree less than 2)

- The remainder will be **linear** (degree 1) if e ≠ 0, and **constant** (degree 0) if e = 0
- Note that $P(x) = Q(x) \times (ax^2 + bx + c) + ex + f$

Division by polynomials of degree *k* ≤ *n*

- If *P*(*x*) has degree *n* and is divided by a polynomial *D*(*x*) with degree *k* ≤ *n*
  - $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$ where

    D(*x*) is the **divisor** (degree *k*)

    Q(*x*) is the **quotient** (degree *n* – *k*)

    R(*x*) is the **remainder** (degree less than *k*)
  - Note that $P(x) = Q(x) \times D(x) + R(x)$

Are there other methods for dividing polynomials?

- **Synthetic division** is a faster and shorter way of setting out a division when dividing by a linear term of the form
  - To divide $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ by $(x - c)$:

    Set $b_n = a_n$

    Calculate $b_{n-1} = a_{n-1} + c \times b_n$

    Continue this iterative process $b_{i-1} = a_{i-1} + c \times a_i$

    The quotient is $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \ldots + b_2 x + b_1$ and the remainder is $r = b_0$
- You can also find quotients and remainders by **comparing coefficients**
  - Given a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$
  - And a divisor $D(x) = d_k x^k + d_{k-1} x^{k-1} + \ldots + d_1 x + d_0$
  - Write $Q(x) = q_{n-k} x^{n-k} + \ldots + q_1 x + q_0$ and $R(x) = r_{k-1} x^{k-1} + \ldots + r_1 x + r_0$
  - Write $P(x) = Q(x)D(x) + R(x)$

    Expand the right-hand side

    Equate the coefficients

    Solve to find the unknowns *q*'s & *r*'s

Exam Tip

- In an exam you can use whichever method to divide polynomials - just make sure your method is written clearly so that if you make a mistake you can still get a mark for your method!

Worked Example

a)

Perform the division $\frac{x^4 + 11x^2 - 1}{x + 3}$. Hence write $x^4 + 11x^2 - 1$ in the form $Q(x) \times (x+3) + R$.

Step 1: what do we multiply x by to get x^4?

$$\begin{array}{r|l} & x^3 \\ x+3 & x^4 + 0x^3 + 11x^2 + 0x - 1 \end{array}$$

Note: $0x^3$ and $0x$ are used to keep like terms together.

Step 2: subtract $x^3(x+3) = x^4 + 3x^3$ from $x^4 + 0x^3$

$$\begin{array}{r|l} & x^3 \\ x+3 & x^4 + 0x^3 + 11x^2 + 0x - 1 \\ & -(x^4 + 3x^3) \\ \hline & \quad -3x^3 \end{array}$$

Step 3: bring the $11x^2$ down and return to step 1.

$$
\begin{array}{r|l}
 & x^3 - 3x^2 + 20x - 60 \\
\hline
x + 3 & x^4 + 0x^3 + 11x^2 + 0x - 1 \\
 & -(x^4 + 3x^3) \\
 & -3x^3 + 11x^2 \\
 & -(-3x^3 - 9x^2) \\
 & 20x^2 + 0x \\
 & -(20x^2 + 60x) \\
 & -60x - 1 \\
 & -(-60x - 180) \\
 & 179
\end{array}
$$

$x^4 + 11x^2 - 1$
$= (x^3 - 3x^2 + 20x - 60)(x + 3) + 179$

b)

Find the quotient and remainder for $\frac{x^4 + 4x^3 - x + 1}{x^2 - 2x}$. Hence write $x^4 + 4x^3 - x + 1$ in the form $Q(x) \times (x^2 - 2x) + R(x)$.

When dividing by quadratics use the same steps as above.

$$
\begin{array}{r}
x^2 + 6x + 12 \\
x^2 - 2x \,\overline{\big)\, x^4 + 4x^3 + 0x^2 - x + 1} \\
-\ \underline{(x^4 - 2x^3)} \qquad\qquad\qquad \\
6x^3 + 0x^2 \qquad\quad \\
-\ \underline{(6x^3 - 12x^2)} \qquad\quad \\
12x^2 - x \quad \\
-\ \underline{(12x^2 - 24x)} \\
23x + 1
\end{array}
$$

$x^4 + 4x^3 - x + 1$

$= (x^2 + 6x + 12)(x^2 - 2x) + 23x + 1$

2.7.3 Polynomial Functions

Sketching Polynomial Graphs

In exams you'll commonly be asked to sketch the graphs of different polynomial functions with and without the use of your GDC.

What's the relationship between a polynomial's degree and its zeros?

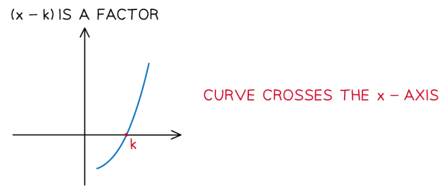
- If a **real polynomial** $P(x)$ has **degree *n***, it will have ***n* zeros** which can be written in the form $a + bi$, where $a, b \in \mathbb{R}$
  - For example:
    - A quadratic will have 2 zeros
    - A cubic function will have 3 zeros
    - A quartic will have 4 zeros
  - Some of the zeros may be **repeated**
- Every **real polynomial** of **odd degree** has **at least one real zero**

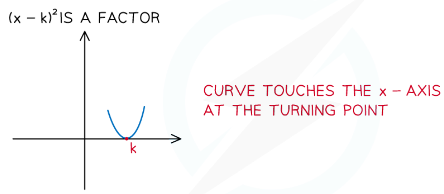
How do I sketch the graph of a polynomial function without a GDC?

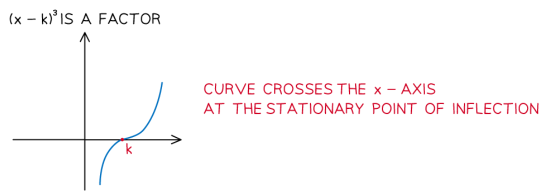
- Suppose $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is a **real polynomial** with **degree *n***
- To sketch the graph of a polynomial you need to know three things:
  - The ***y*-intercept**
    - Find this by **substituting $x = 0$** to get $y = a_0$
  - The **roots**
    - You can find these by **factorising** or solving $y = 0$
  - The **shape**
    - This is determined by the **degree** (n) and the sign of the **leading coefficient** (a_n)

How does the multiplicity of a real root affect the graph of the polynomial?

- The **multiplicity** of a root is the number of times it is **repeated** when the polynomial is factorised
  - If $x = k$ is a root with **multiplicity *m*** then $(x - k)^m$ is a **factor** of the polynomial
- The graph either **crosses** the x-axis or **touches** the x-axis at a **root $x = k$** where k is a real number
  - If $x = k$ has **multiplicity 1** then the graph **crosses** the x-axis at $(k, 0)$
  - If $x = k$ has **multiplicity 2** then the graph has a **turning point** at $(k, 0)$ so **touches** the x-axis
    - If $x = k$ has **odd multiplicity** $m \geq 3$ then the graph has a **stationary point of inflection** at $(k, 0)$ so **crosses** the x-axis
    - If $x = k$ has **even multiplicity** $m \geq 4$ then the graph has a **turning point** at $(k, 0)$ so **touches** the x-axis







How do I determine the shape of the graph of the polynomial?

- Consider what happens as ***x* tends to ± ∞**
  - If a_n is **positive** and *n* is **even** then the graph **approaches from the top left** and **tends to the top right**

    $$\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = +\infty$$

  - If a_n is **negative** and *n* is **even** then the graph **approaches from the bottom left** and **tends to the bottom right**

    $$\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = +\infty$$

  - If a_n is **positive** and *n* is **odd** then the graph **approaches from the bottom left** and **tends to the top right**

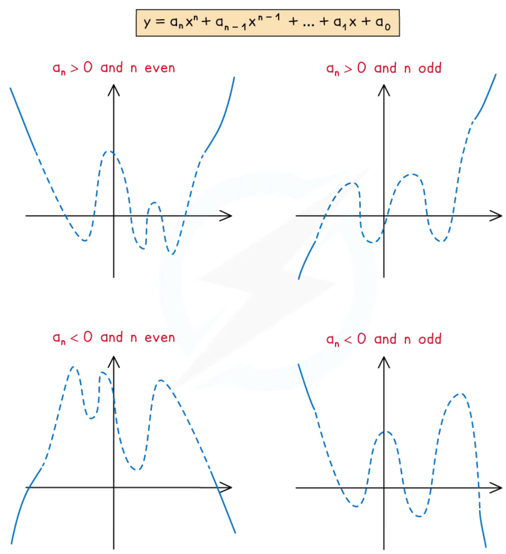
    $$\lim_{x \to -\infty} f(x) = -\infty \text{ and } \lim_{x \to +\infty} f(x) = +\infty$$

  - If a_n is **negative** and *n* is **odd** then the graph **approaches from the top left** and **tends to the bottom right**

    $$\lim_{x \to -\infty} f(x) = +\infty \text{ and } \lim_{x \to +\infty} f(x) = -\infty$$

- Once you know the **shape**, the **real roots** and the ***y*-intercept** then you simply connect the points using a **smooth curve**
- There will be **at least one turning point** in-between each pair of roots
  - If the degree is *n* then there is **at most *n* – 1 stationary points (**some will be **turning points**)
    - Every real polynomial of **even degree** has **at least one turning point**
    - Every real polynomial of **odd degree bigger than 1** has **at least one point of inflection**
  - If it is a calculator paper then you can use your GDC to find the coordinates of the turning points

* You won't need to find their location without a GDC unless the question asks you to



Exam Tip

- If it is a calculator paper then you can use your GDC to find the coordinates of any turning points
- If it is the non-calculator paper then you will not be required to find the turning points when sketching unless specifically asked to

Worked Example

a)

The function f is defined by $f(x) = (x + 1)(2x - 1)(x - 2)^2$. Sketch the graph of $y = f(x)$.

Find the y-intercept

$x = 0: \quad y = (1)(-1)(-2)^2 = -4$

Find the roots and determine if graphs crosses or touches the x-axis

$(x + 1)(2x - 1)(x - 2)^2$

$(-1, 0)$ $\left(\frac{1}{2}, 0\right)$ $(2, 0)$

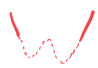
cross cross touch

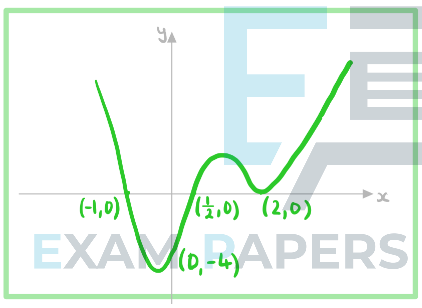
Determine the shape by looking at the leading term

Leading term is $(x)(2x)(x)^2 = 2x^4$

As $x \to -\infty$ $y \to +\infty$

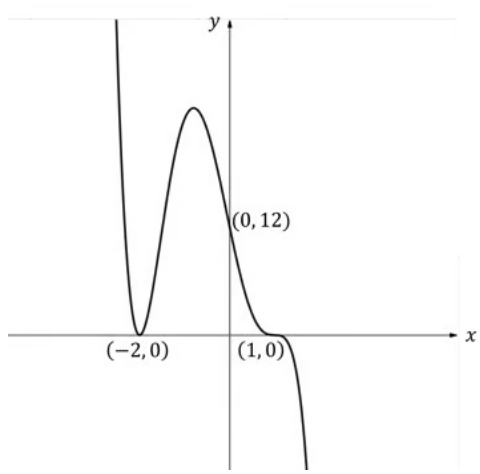
As $x \to +\infty$ $y \to +\infty$





b)

The graph below shows a polynomial function. Find a possible equation of the polynomial.



Touches at $(-2,0)$ $\quad (x+2)^2$ is a factor

Point of inflection at $(1,0)$ $\quad (x-1)^3$ is a factor

Write in the form of: $y = a(x+2)^2(x-1)^3$

Use the y-intercept to find a

$12 = a(2)^2(-1)^3 \Rightarrow -4a = 12 \quad \therefore a = -3$

$y = -3(x+2)^2(x-1)^3$

Solving Polynomial Equations

What is "The Fundamental Theorem of Algebra"?

- Every **real polynomial** with degree *n* can be factorised into ***n* complex linear factors**
  - Some of which may be **repeated**
  - This means the polynomial will have *n* zeros (some may be repeats)
- Every **real polynomial** can be expressed as a product of **real linear factors** and **real irreducible quadratic factors**
  - An irreducible quadratic is where it **does not have real roots**
    - The **discriminant** will be negative: $b^2 - 4ac < 0$
- If $a + bi$ ($b \neq 0$) is a **zero** of a **real polynomial** then its **complex conjugate** $a - bi$ is also a **zero**
- Every **real polynomial** of **odd degree** will have **at least one real zero**

How do I solve polynomial equations?

- Suppose you have an equation $P(x) = 0$ where $P(x)$ is a **real polynomial of degree *n***
  - $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ ... + a_1 x + a_0$
- You may be given one zero or you might have to find a zero $x = k$ by substituting values into $P(x)$ until it equals 0
- If you know a **root** then you know a **factor**
  - If you know $x = k$ **is a root** then $(x - k)$ **is a factor**
  - If you know $x = a + bi$ **is a root** then you know a **quadratic factor** $(x - (a + bi))(x - (a - bi))$
    - Which can be written as $((x - a) - bi)((x - a) + bi)$ and **expanded quickly using difference of two squares**
- You can then **divide** $P(x)$ by this factor to get **another factor**
  - For example: dividing a cubic by a linear factor will give you a quadratic factor
- You then may be able to factorise this new factor

Exam Tip

- If a polynomial has three or less terms check whether a substitution can turn it into a quadratic
  - For example: $x^6 + 3x^3 + 2$ can be written as $(x^3)^2 + 3(x^3) + 2$

Worked Example

Given that $x = \frac{1}{2}$ is a zero of the polynomial defined by $f(x) = 2x^3 - 3x^2 + 5x - 2$, find all three zeros of f.

$x = \frac{1}{2}$ is a root $\therefore (2x-1)$ is a factor

Find the quadratic factor $(2x^3 - 3x^2 + 5x - 2) = (2x-1)(ax^2 + bx + c)$

Compare coefficients : $2x^3 = 2ax^3 \quad \therefore a = 1$

$-2 = -c \quad \therefore c = 2$

$5x = 2cx - bx \Rightarrow 5 = 4 - b \quad \therefore b = -1$

Solve the quadratic : $x^2 - x + 2 = 0$

Formula booklet

Solutions of a quadratic equation	$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{-7}}{2} = \frac{1}{2} \pm \frac{\sqrt{7}}{2}i$$

Roots : $\frac{1}{2}, \ \frac{1}{2} + \frac{\sqrt{7}}{2}i, \ \frac{1}{2} - \frac{\sqrt{7}}{2}i$

EXAM PAPERS PRACTICE

2.7.4 Roots of Polynomials

Sum & Product of Roots

How do I find the sum & product of roots of polynomials?

- Suppose $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ ... + a_1 x + a_0$ is a **polynomial** of **degree** *n* with *n* roots $\alpha_1, \alpha_2, ..., \alpha_n$
  - The polynomial is written as $\sum_{r=0}^{n} a_r x^r = 0,\ a_n \neq 0$ in the **formula booklet**
  - a_n is the coefficient of the **leading term**
  - a_{n-1} is the coefficient of the x^{n-1} **term**
    - Be careful: this could be equal to zero
  - a_0 is the **constant term**
    - Be careful: this could be equal to zero
- In factorised form: $P(x) = a_n(x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$
  - Comparing coefficients of the x^{n-1} **term** and the **constant term** gives
    - $a_{n-1} = a_n(-\alpha_1 - \alpha_2 - ... - \alpha_n)$
    - $a_0 = a_n(-\alpha_1) \times (-\alpha_2) \times ... \times (-\alpha_n)$
- The **sum** of the roots is given by:
  - $\alpha_1 + \alpha_2 + \ ... + \alpha_n = -\frac{a_{n-1}}{a_n}$
- The **product** of the roots is given by:
  - $\alpha_1 \times \alpha_2 \times \ ... \times \alpha_n = \frac{(-1)^n a_0}{a_n}$
    - both of these formulae are in your **formula booklet**

How can I find unknowns if I am given the sum and/or product of the roots of a polynomial?

- If you know a complex root of a real polynomial then its **complex conjugate** is **another root**
- Form **two equations** using the roots
  - One using the **sum of the roots formula**
  - One using the **product of the roots formula**
- **Solve** for any unknowns

Exam Tip

- Examiners might trick you by not having an x^{n-1} term or a constant term
- To make sure you do not get tricked you can write out the full polynomial using 0 as a coefficient where needed
  - For example: Write $x^4 + 2x^2 - 5x$ as $x^4 + 0x^3 + 2x^2 - 5x + 0$

Worked Example

$2-3\text{i}$, $\frac{5}{3}\text{i}$ and α are three roots of the equation

$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k = 0$.

a)
Use the sum of all the roots to find the value of α.

It is a real polynomial so if $a+bi$ is a root then $a-bi$ is also a root

Roots: $2-3i$, $2+3i$, $\frac{5}{3}i$, $-\frac{5}{3}i$, α

Formula booklet

Sum & product of the roots of polynomial equations of the form $\sum_{r=0}^{n} a_r x^r = 0$	Sum is $\frac{-a_{n-1}}{a_n}$

$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k$

$a_n = 18 \quad a_{n-1} = -9$

$(2-3i) + (2+3i) + \left(\frac{5}{3}i\right) + \left(-\frac{5}{3}i\right) + \alpha = \frac{-(-9)}{18}$

$4 + \alpha = \frac{1}{2}$

$\alpha = -\frac{7}{2}$

b)
Use the product of all the roots to find the value of k.

Formula booklet

Sum & product of the roots of polynomial equations of the form $\sum_{r=0}^{n} a_r x^r = 0$	product is $\frac{(-1)^n a_0}{a_n}$

$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k$

$a_n = 18 \quad n = 5 \quad a_0 = k$

$(2-3i)(2+3i)\left(\frac{5}{3}i\right)\left(-\frac{5}{3}i\right)\left(-\frac{7}{2}\right) = \frac{(-1)^5 k}{18}$

$(13)\left(\frac{25}{9}\right)\left(-\frac{7}{2}\right) = \frac{-k}{18}$

$-\frac{2275}{18} = -\frac{k}{18}$

$k = 2275$

2.8 Inequalities

2.8.1 Solving Inequalities Graphically

Solving Inequalities Graphically

How can I solve inequalities graphically?

- Consider the inequality ***f*(*x*) ≤ *g*(*x*)**, where *f*(*x*) and *g*(*x*) are functions of *x*
  - if we **move *g*(*x*) to the LHS** we get
    - ***f*(*x*) – *g*(*x*) ≤ 0**
- Solve ***f*(*x*) – *g*(*x*) = 0 to** find the **zeros** of *f*(*x*) – *g*(*x*)
  - These correspond to the *x*-coordinates of the points of intersection of the graphs ***y* = *f*(*x*)** and ***y* = *g*(*x*)**
- To solve the inequality we can use a **graph**
  - **Graph *y* = *f*(*x*) – *g*(*x*)** and label its zeros
  - Hence find the intervals of *x* that satisfy the inequality *f*(*x*) – *g*(*x*) ≤ 0
    - These are the **intervals which satisfies the original inequality** *f*(*x*) ≤ *g*(*x*)
  - This method is particularly useful when finding the intersections between the functions is difficult due to needing large *x* and *y* windows on your GDC

Be careful when rearranging inequalities!

- Remember to **flip the sign** of the inequality when you **multiply or divide** both sides by a **negative** number
  - e. **1 < 2** → [times both sides by (–1)] → **–1 > –2** (sign flips)
- **Never multiply or divide** by a **variable** as this could be **positive or negative**
  - You can only multiply by a term if you are certain it is always positive (or always negative)
    - Such as x^2, $|x|$, e^x
- Some **functions reverse the inequality**
  - Taking reciprocals of positive values
    - $0 < x < y \Rightarrow \frac{1}{x} > \frac{1}{y}$
  - Taking logarithms when the base is 0 < *a* < 1
    - $0 < x < y \Rightarrow \log_a(x) > \log_a(y)$
- The **safest way** to rearrange is simply to add & subtract to move all the terms onto one side

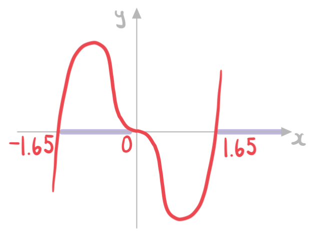
Worked Example

Use a GDC to solve the inequality $2x^3 < x^5 - 2x$.

Rearrange to get one side as zero

$x^5 - 2x^3 - 2x > 0$

On GDC sketch $y = x^5 - 2x^3 - 2x$ and find zeros



Identify the sections where the graph is above the x-axis

$-1.65 < x < 0$ or $x > 1.65$

2.8.2 Polynomial Inequalities

Polynomial Inequalities

How do I solve polynomial inequalities?

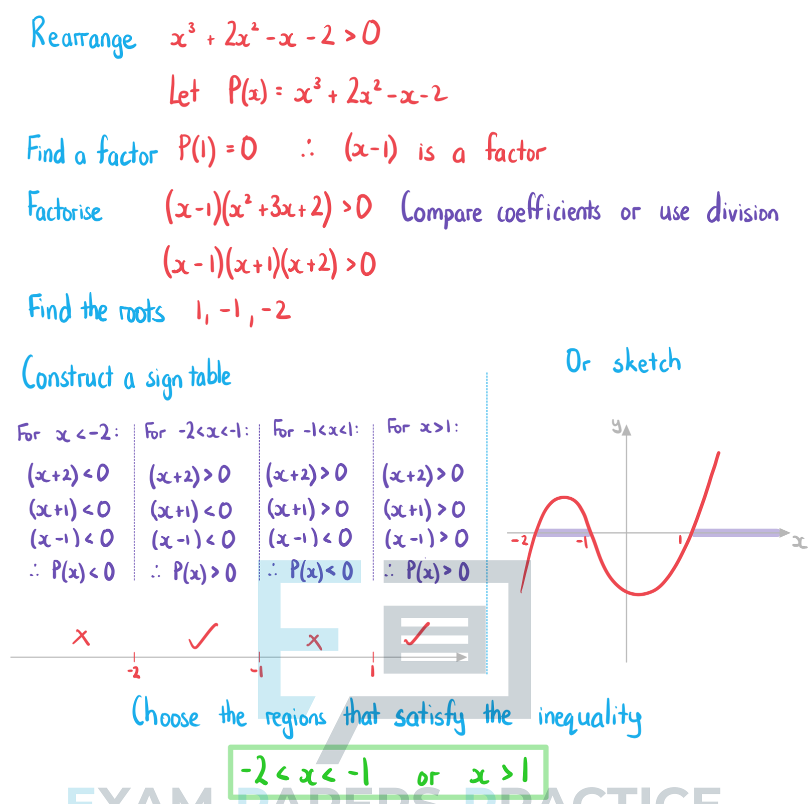
- **STEP 1**: **Rearrange the inequality** so that **one of the sides is equal to zero**
  - For example: $P(x) \le 0$
- **STEP 2**: Find the **roots** of the polynomial
  - You can do this by factorising or using GDC to solve $P(x) = 0$
- **STEP 3**: Choose one of the following methods:
- **Graph method**
  - Sketch a graph of the polynomial (with or without a GDC)
  - Choose the intervals for x corresponding to the sections of the graph that satisfy the inequality
    - For example: for $P(x) \le 0$ you would want the sections below the x-axis
- **Sign table method**
  - If you are unsure how to sketch a polynomial graph then this method is best
  - **Split the real numbers** into the possible **intervals** using the roots
    - If the roots are a and b then the intervals would be $x<a$, $a<x<b$, $x>b$
  - **Test a value** from each interval using the inequality
    - Choose a value within an interval and substitute into $P(x)$ to determine if it is positive or negative
  - Alternatively if the polynomial is factorised you can **determine the sign of each factor** in each interval
    - An odd number of negative factors in an interval will mean the polynomial is negative on that interval
  - If the value satisfies the inequality then that interval is part of the solution

Exam Tip

- In exams most solutions will be intervals but some could be a single point
  - For example: Solution to $(x-3)^2 \le 0$ is $x = 3$

Worked Example

Solve the inequality $x^3 + 2x^2 > x + 2$ using an algebraic method.



2.9 Further Functions & Graphs

2.9.1 Modulus Functions

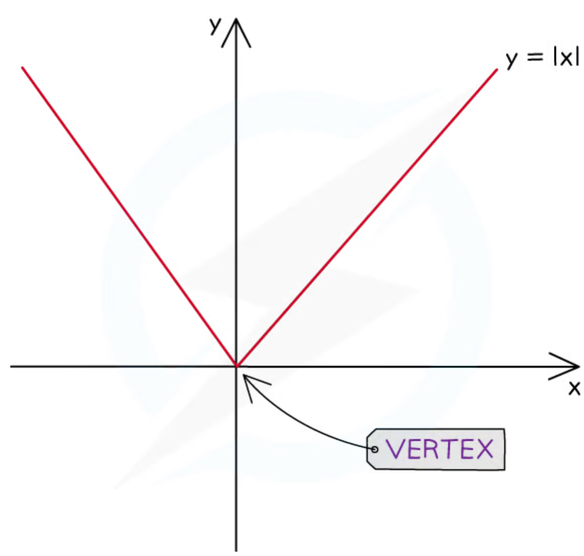
Modulus Functions & Graphs

What is the modulus function?

- The **modulus function** is defined by $f(x) = |x|$
  - $|x| = \sqrt{x^2}$
  - Equivalently it can be defined $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all real non-negative values**
- The modulus function gives the **distance** between 0 and *x*
  - This is also called the **absolute value** of *x*

What are the key features of the modulus graph: $y = |x|$?

- The graph has a ***y*-intercept** at (0, 0)
- The graph has **one root** at (0, 0)
- The graph has a **vertex** at (0, 0)
- The graph is **symmetrical** about the ***y*-axis**
- At the **origin**
  - The function is **continuous**
  - The function is **not differentiable**



What are the key features of the modulus graph: $y = a|x + p| + q$?

- Every **modulus graph** which is formed by **linear transformations** can be written in this form using key features of the modulus function

- $|ax| = |a||x|$

  For example: $|2x+1| = 2\left|x+\frac{1}{2}\right|$
- $|p-x| = |x-p|$

  For example: $|4-x| = |x-4|$

- The graph has a ***y*-intercept** when *x* = 0
- The graph can have 0, 1 or 2 **roots**
  - If *a* and *q* have the **same sign** then there will be **0 roots**
  - If *q* = 0 then there will be **1 root** at (-*p*, 0)
  - If *a* and *q* have **different signs** then there will be **2 roots** at $\left(-p \pm \frac{q}{a}, 0\right)$
- The graph has a **vertex** at (-*p*, *q*)
- The graph is **symmetrical** about the line ***x* = -*p***
- The value of *a* determines the **shape** and the **steepness** of the graph
  - If *a* is **positive** the graph looks like ∨
  - If *a* is **negative** the graph looks like ∧
  - The **larger** the value of |*a*| the **steeper** the lines
- At the **vertex**
  - The function is **continuous**
  - The function is **not differentiable**

2.9.2 Modulus Transformations

Modulus Transformations

How do I sketch the graph of the modulus of a function: $y = |f(x)|$?

- **STEP 1**: Keep the parts of the graph of $y = f(x)$ that are **on or above the x-axis**
- **STEP 2**: Any parts of the **graph below the x-axis** get **reflected** in the x-axis anything

How do I sketch the graph of a function of a modulus: $y = f(|x|)$?

- **STEP 1**: Keep the graph of $y = f(x)$ **only for $x \geq 0$**
- **STEP 2**: **Reflect** this in the **y-axis**

What is the difference between $y = |f(x)|$ and $y = f(|x|)$?

- The graph of $y = |f(x)|$ **never goes below the x-axis**
  - It does not have to have any lines of symmetry
- The graph of $y = f(|x|)$ is **always symmetrical about the y-axis**
  - It can go below the y-axis

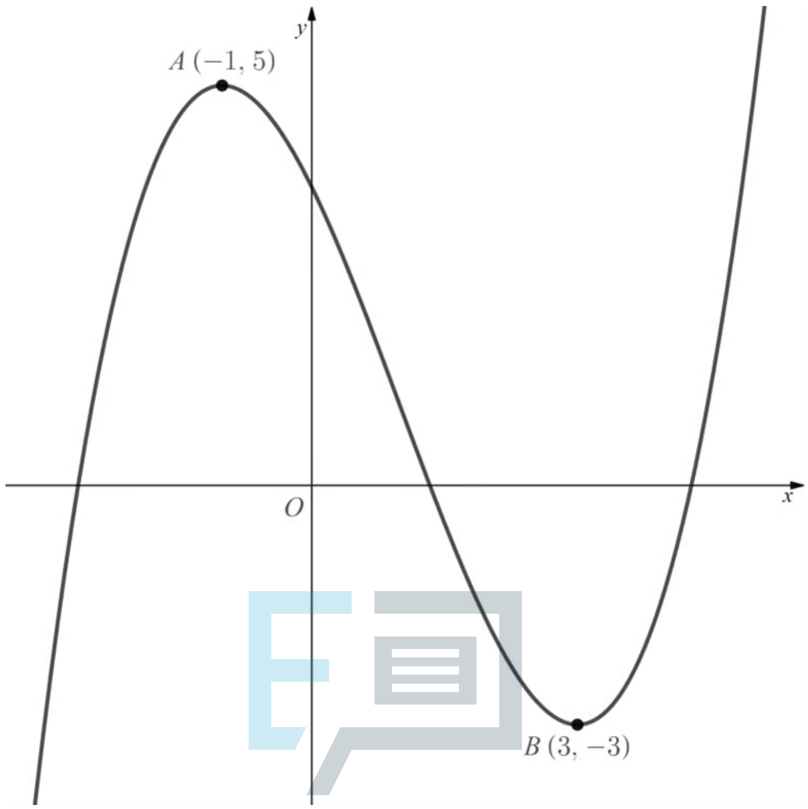
When multiple transformations are involved how do I determine the order?

- The transformations **outside the function** follow the **same order** as the **order of operations**
  - $y = |af(x) + b|$
    Deal with the a then the b then the modulus
  - $y = a|f(x)| + b$
    Deal with the modulus then the a then the b
- The transformations **inside the function** are in the **reverse order** to the **order of operations**
  - $y = f(|ax + b|)$
    Deal with the modulus then the b then the a
  - $y = f(a|x| + b)$
    Deal with the b then the a then the modulus

> **Exam Tip**
>
> - When sketching one of these transformations in an exam question make sure that the graphs do not look smooth at the points where the original graph have been reflected
>   - For $y = |f(x)|$ the graph should look "sharp" at the points where it has been reflected on the x-axis
>   - For $y = f(|x|)$ the graph should look "sharp" at the point where it has been reflected on the y-axis

Worked Example

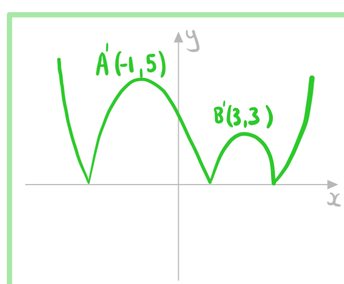
The diagram below shows the graph of $y = f(x)$.



(a) Sketch the graph of $y = |f(x)|$.

If the graph is on or above the x-axis then it stays the same
If the graph is below the x-axis the it is reflected in the x-axis

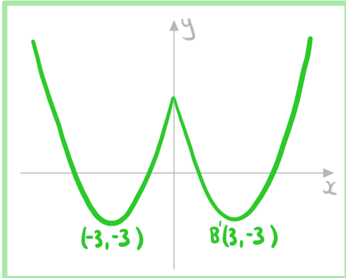
A stays the same (-1,5)
B becomes (3,3)



(b) Sketch the graph of $y = f(|x|)$.

keep the graph for $x \geq 0$
Reflect this in the y-axis

A disappears
B stays the same $(3, -3)$



2.9.3 Modulus Equations & Inequalities

Modulus Equations

How do I find the modulus of a function?

- The **modulus of a function *f*(*x*)** is
  - $|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$ or
  - $|f(x)| = \sqrt{[f(x)]^2}$

How do I solve modulus equations graphically?

- To solve $|f(x)| = g(x)$ graphically
  - Draw $y = |f(x)|$ and $y = g(x)$ into your GDC
  - Find the x-coordinates of the **points of intersection**

How do I solve modulus equations analytically?

- To solve $|f(x)| = g(x)$ analytically
  - Form **two equations**
    - $f(x) = g(x)$
    - $f(x) = -g(x)$
  - Solve both equations
  - **Check solutions** work in the original equation
    - For example: $x - 2 = 2x - 3$ has solution $x = 1$
    - But $|(1) - 2| = 1$ and $2(1) - 3 = -1$
    - So $x = 1$ is not a solution to $|x - 2| = 2x - 3$

Worked Example

Solve for x:

a)

$$\left|\frac{2x+3}{2-x}\right| = 5$$

Analytically

Split into two equations

$$\frac{2x+3}{2-x} = \pm 5$$

Solve individually

$\frac{2x+3}{2-x} = 5$

$2x+3 = 10-5x$

$7x = 7$

$x = 1$

$\frac{2x+3}{2-x} = -5$

$2x+3 = 5x-10$

$13 = 3x$

$x = \frac{13}{3}$

Check:

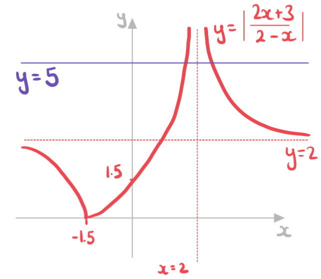
$\left|\frac{2(1)+3}{2-(1)}\right| = 5$ ✓

$\left|\frac{2\left(\frac{13}{3}\right)+3}{2-\left(\frac{13}{3}\right)}\right| = 5$ ✓

$x = 1$ or $x = \frac{13}{3}$

Graphically

Sketch the two graphs



Find the points of intersection

$(1, 5)$ $(4.33, 5)$

Choose the x-coordinates

$x = 1$ or $x = 4.33$ (3sf)

b)

$$|3x-1| = 5x-11.$$

Analytically

Split into two equations

$$3x-1 = \pm(5x-11)$$

Solve individually

$3x-1 = 5x-11$

$10 = 2x$

$x = 5$

$3x-1 = 11-5x$

$8x = 12$

$x = 1.5$

Check:

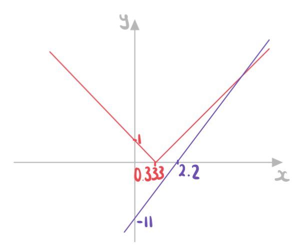
$|3(5)-1| = 14$, $5(5)-11 = 14$ ✓

$|3(1.5)-1| = 3.5$, $5(1.5)-11 = -3.5$ ✗

$x = 5$

Graphically

Sketch the two graphs



Find the points of intersection

$(5, 14)$

Choose the x-coordinates

$x = 5$

Modulus Inequalities

How do I solve modulus inequalities analytically?

- To solve **any** modulus inequality
  - First solve the corresponding modulus equation
    - Remembering to **check whether solutions are valid**
  - Then use a graphical method or a sign table to find the intervals that satisfy the inequality
- Another method is to solve **two pairs of inequalities**
  - For $|f(x)| < g(x)$ solve:
    - $f(x) < g(x)$ when $f(x) \geq 0$
    - $f(x) > -g(x)$ when $f(x) \leq 0$
  - For $|f(x)| > g(x)$ solve:
    - $f(x) > g(x)$ when $f(x) \geq 0$
    - $f(x) < -g(x)$ when $f(x) \leq 0$

Exam Tip

- If a question on this appears on a calculator paper then use the same ideas as solving other inequalities
  - Sketch the graphs and find the intersections

Worked Example

Solve the following inequalities for x.

a)

$|2x-1|<4$

Solve for $2x-1 \geq 0$

For $x \geq \frac{1}{2}$: $2x-1<4 \Rightarrow x<\frac{5}{2}$ $\therefore \frac{1}{2} \leq x < \frac{5}{2}$

Solve for $2x-1 \leq 0$

For $x \leq \frac{1}{2}$: $2x-1>-4 \Rightarrow x>-\frac{3}{2}$ $\therefore -\frac{3}{2} < x \leq \frac{1}{2}$

Combine inequalities $-\frac{3}{2} < x < \frac{5}{2}$

b)

$|x+1|<|2x+3|$

Solve the corresponding equation

$|x+1|=|2x+3| \Rightarrow x+1=\pm(2x+3)$

Solve $x+1=2x+3$ | $x+1=-2x-3$

$x=-2$ | $x=-\frac{4}{3}$

Check $|(-2)+1|=1$, $|2(-2)+3|=1$ ✓ | $|(-\frac{4}{3})+1|=\frac{1}{3}$, $|2(-\frac{4}{3})+3|=\frac{1}{3}$ ✓

Use a sign table

Check $x=-3$	Check $x=-1.5$	Check $x=0$
$\lvert(-3)+1\rvert < \lvert 2(-3)+3\rvert$	$\lvert(-1.5)+1\rvert < \lvert 2(-1.5)+3\rvert$	$\lvert 0+1\rvert < \lvert 2(0)+3\rvert$
$2 < 3$	$0.5 < 0$	$1 < 3$
True ✓	False ✗	True ✓

Write solution $x<-2$ or $x>-\frac{4}{3}$

2.9.4 Reciprocal & Square Transformations

Reciprocal Transformations

What effects do reciprocal transformations have on the graphs?

- The **x-coordinates stay the same**
- The **y-coordinates change**
  - Their values become their **reciprocals**
- The coordinates (x, y) become $\left(x, \frac{1}{y}\right)$ where $y \neq 0$
  - If $y = 0$ then a vertical asymptote goes through the original coordinate
  - Points that lie on the line **$y = 1$** or the line **$y = -1$ stay the same**

How do I sketch the graph of the reciprocal of a function: $y = 1/f(x)$?

- Sketch the **reciprocal transformation** by considering the **different features** of the original graph
- Consider key points on the original graph
  - If (x_1, y_1) is a point on $y = f(x)$ where $y_1 \neq 0$

    $\left(x_1, \frac{1}{y_1}\right)$ is a point on $y = \frac{1}{f(x)}$

    If **$|y_1| < 1$** then the point gets **further away from the x-axis**

    If **$|y_1| > 1$** then the point gets **closer to the x-axis**
  - If $y = f(x)$ has a **y-intercept** at $(0, c)$ where $c \neq 0$

    The reciprocal graph $y = \frac{1}{f(x)}$ has a **y-intercept** at $\left(0, \frac{1}{c}\right)$
  - If $y = f(x)$ has a **root** at $(a, 0)$

    The reciprocal graph $y = \frac{1}{f(x)}$ has a **vertical asymptote** at $x = a$
  - If $y = f(x)$ has a **vertical asymptote** at $x = a$

    The reciprocal graph $y = \frac{1}{f(x)}$ has a **discontinuity** at $(a, 0)$

    The **discontinuity** will look like a **root**
  - If $y = f(x)$ has a **local maximum** at (x_1, y_1) where $y_1 \neq 0$

    The reciprocal graph $y = \frac{1}{f(x)}$ has a **local minimum** at $\left(x_1, \frac{1}{y_1}\right)$
  - If $y = f(x)$ has a **local minimum** at (x_1, y_1) where $y_1 \neq 0$

The reciprocal graph $y = \frac{1}{f(x)}$ has a **local maximum** at $\left(x_1, \frac{1}{y_1}\right)$

- Consider key regions on the original graph
  - If $y = f(x)$ is **positive** then $y = \frac{1}{f(x)}$ is **positive**

    If $y = f(x)$ is **negative** then $y = \frac{1}{f(x)}$ is **negative**
  - If $y = f(x)$ is **increasing** then $y = \frac{1}{f(x)}$ is **decreasing**

    If $y = f(x)$ is **decreasing** then $y = \frac{1}{f(x)}$ is **increasing**
  - If $y = f(x)$ has a **horizontal asymptote** at $y = k$

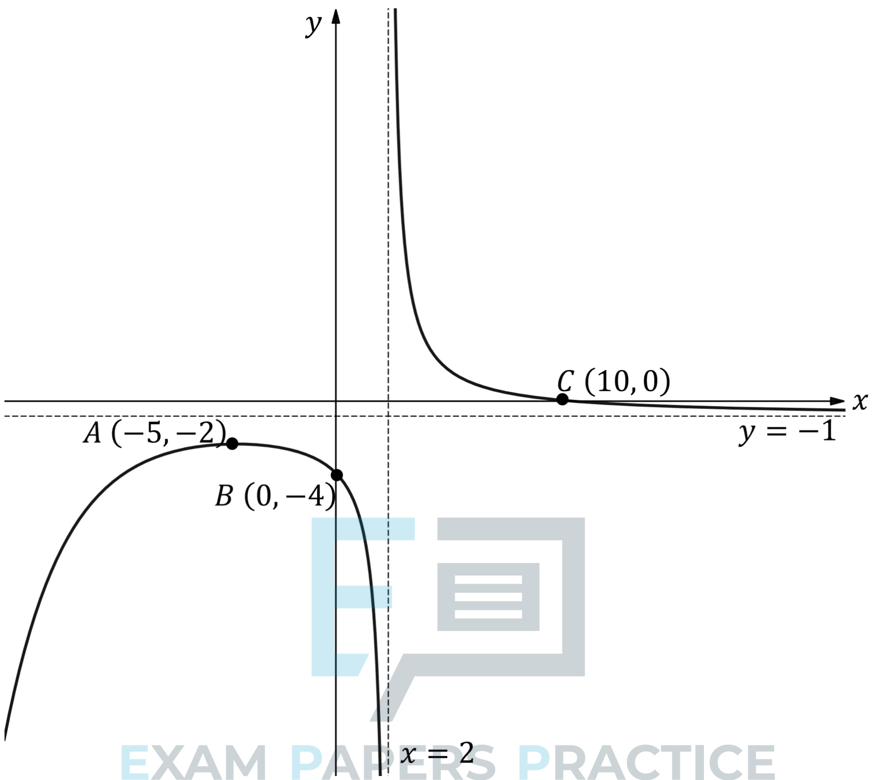
    $y = \frac{1}{f(x)}$ has a **horizontal asymptote** at $y = \frac{1}{k}$ if $k \neq 0$

    $y = \frac{1}{f(x)}$ **tends to ± ∞** if $k = 0$
  - If $y = f(x)$ **tends to ± ∞** as x tends to $+\infty$ or $-\infty$

    $y = \frac{1}{f(x)}$ has a **horizontal asymptote** at $y = 0$

Worked Example

The diagram below shows the graph of $y = f(x)$ which has a local maximum at the point A.



Sketch the graph of $y = \dfrac{1}{f(x)}$.

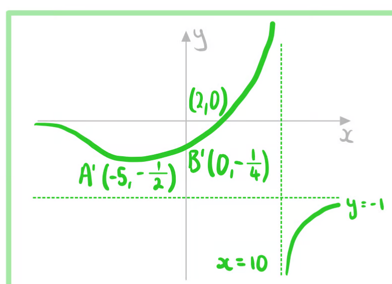
A becomes local minimum $(-5, -\frac{1}{2})$

Vertical asymptote becomes root $(2, 0)$

B becomes $(0, -\frac{1}{4})$

C becomes vertical asymptote $x = 10$

Horizontal asymptote $y = -1$ remains



Square Transformations

What effects do square transformations have on the graphs?

- The effects are **similar to** the transformation ***y* = |*f*(*x*)|**
  - The parts **below the *x*-axis are reflected**
  - The **vertical distance** between a point and the *x*-axis is **squared**
    - This has the effect of **smoothing the curve** at the *x*-axis
- $y = [f(x)]^2$ is **never below the *x*-axis**
- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
  - Their values are **squared**
- The coordinates (x, y) become (x, y^2)
  - Points that lie on the ***x*-axis** or the line ***y* = 1 stay the same**

How do I sketch the graph of the square of a function: y = [*f*(*x*)]²?

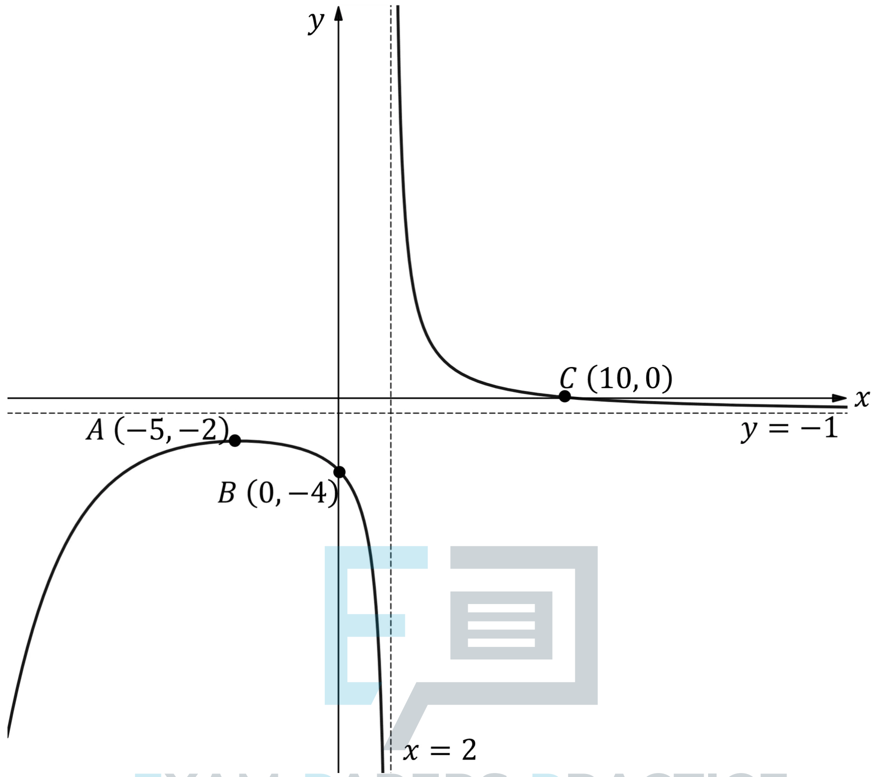
- Sketch the **square transformation** by considering the **different features** of the original graph
- Consider key points on the original graph
  - If (x_1, y_1) is a point on $y = f(x)$
    - (x_1, y_1^2) is a point on $y = [f(x)]^2$
    - If $|y_1| < 1$ then the point gets **closer to the *x*-axis**
    - If $|y_1| > 1$ then the point gets **further away from the *x*-axis**
  - If $y = f(x)$ has a **y-intercept** at $(0, c)$
    - The square graph $y = [f(x)]^2$ has a **y-intercept** at $(0, c^2)$
  - If $y = f(x)$ has a **root** at $(a, 0)$
    - The square graph $y = [f(x)]^2$ has a **root** and **turning point** at $(a, 0)$
  - If $y = f(x)$ has a **vertical asymptote** at $x = a$
    - The square graph $y = [f(x)]^2$ has a **vertical asymptote** at $x = a$
  - If $y = f(x)$ has a **local maximum** at (x_1, y_1)
    - The square graph $y = [f(x)]^2$ has a **local maximum** at (x_1, y_1^2) if $y_1 > 0$
    - The square graph $y = [f(x)]^2$ has a **local minimum** at (x_1, y_1^2) if $y_1 \le 0$
  - If $y = f(x)$ has a **local minimum** at (x_1, y_1)
    - The square graph $y = [f(x)]^2$ has a **local minimum** at (x_1, y_1^2) if $y_1 \ge 0$
    - The square graph $y = [f(x)]^2$ has a **local maximum** at (x_1, y_1^2) if $y_1 < 0$

Exam Tip

- In an exam question when sketching $y = [f(x)]^2$ make it clear that the points where the new graph touches the *x*-axis are smooth
  - This will make it clear to the examiner that you understand the difference between the roots of the graphs $y = |f(x)|$ and $y = [f(x)]^2$

Worked Example

The diagram below shows the graph of $y = f(x)$ which has a local maximum at the point A.



Sketch the graph of $y = [f(x)]^2$.

A becomes local minimum (-5, 4)

Vertical asymptote $x = 2$ remains

B becomes (0, 16)

C becomes local minimum

Horizontal asymptote becomes $y = 1$