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1. **Number & Algebra**

1.4 Simple Proof & Reasoning

MATHS

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EXAM PAPERS PRACTICE

IB Maths DP

1. Number & Algebra

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1.1 Exponentials & Logs

1.1.1 Introduction to Logarithms

Introduction to Logarithms

What are logarithms?

- A logarithm is the inverse of an exponent
  - If $a^x = b$ then $\log_a(b) = x$ where $a > 0$, $b > 0$, $a \neq 1$
    - This is in the formula booklet
    - The number a is called the **base** of the logarithm
    - Your GDC will be able to use this function to solve equations involving exponents
- Try to get used to 'reading' logarithm statements to yourself
  - $\log_a(b) = x$ would be read as "the power that you raise a to, to get b, is x"
  - So $\log_5 125 = 3$ would be read as "the power that you raise 5 to, to get 125, is 3"
- Two important cases are:
  - $\ln x = \log_e(x)$
    - Where e is the mathematical constant 2.718…
    - This is called the **natural logarithm** and will have its own button on your GDC
  - $\log x = \log_{10}(x)$
    - Logarithms of **base 10** are used often and so abbreviated to **log *x***

Why use logarithms?

- Logarithms allow us to solve equations where the exponent is the unknown value
  - We can solve some of these by inspection
    - For example, for the equation $2^x = 8$ we know that x must be 3
  - Logarithms allow use to solve more complicated problems
    - For example, the equation $2^x = 10$ does not have a clear answer
    - Instead, we can use our GDCs to find the value of $\log_2 10$

Exam Tip

- Before going into the exam, make sure you are completely familiar with your GDC and know how to use its logarithm functions

Worked Example

Solve the following equations:

i)
$x = \log_3 27$,

$$x = \log_3 27 \iff 3^x = 27$$

We can see from inspection:

$$3^3 = 27 \iff x = 3$$

$$\boxed{x = 3}$$

OR: use GDC to find answer directly.

ii)
$2^x = 21.4$, giving your answer to 3 s.f.

$2^x = 21.4$ This cannot be seen from inspection:

$$2^x = 21.4 \iff x = \log_2 21.4$$

Use GDC to find answer directly.

$$\log_2 21.4 = 4.4195...$$

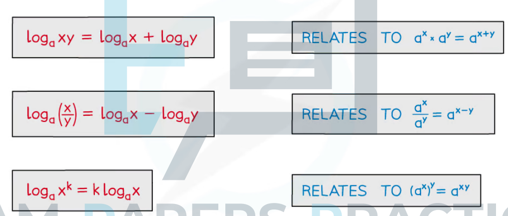
$$\boxed{x = 4.42 \text{ (3 s.f.)}}$$

1.1.2 Laws of Logarithms

Laws of Logarithms

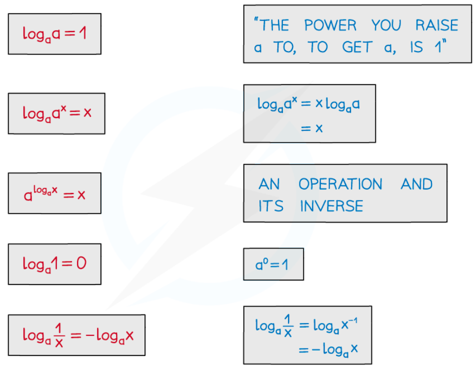
What are the laws of logarithms?

- Laws of logarithms allow you to simplify and manipulate expressions involving logarithms
  - The laws of logarithms are equivalent to the **laws of indices**
- The laws you need to know are, given $a, x, y > 0$:
  - $\log_a xy = \log_a x + \log_a y$
    - This relates to $a^x \times a^y = a^{x+y}$
  - $\log_a \frac{x}{y} = \log_a x - \log_a y$
    - This relates to $a^x \div a^y = a^{x-y}$
  - $\log_a x^m = m\log_a x$
    - This relates to $(a^x)^y = a^{xy}$
- These laws are **in the formula booklet** so you do not need to remember them
  - You must make sure you know how to use them



Useful results from the laws of logarithms

- Given $a > 0$, $a \neq 1$
  - $\log_a 1 = 0$
    - This is equivalent to $a^0 = 1$
- If we substitute *b* for *a* into the given identity in the formula booklet
  - $a^x = b \Leftrightarrow \log_a b = x$ where $a > 0,\ b > 0,\ a \neq 1$
  - $a^x = a \Leftrightarrow \log_a a = x$ gives $a^1 = a \Leftrightarrow \log_a a = 1$
    - This is an important and useful result
- Substituting this into the third law gives the result
  - $\log_a a^k = k$
- Taking the inverse of its operation gives the result
  - $a^{\log_a x} = x$
- From the third law we can also conclude that
  - $\log_a \frac{1}{x} = -\log_a x$



- These useful results are **not in the formula booklet** but can be deduced from the laws that are
- Beware...
  - ...$\log_a(x+y) \neq \log_a x + \log_a y$
- These results apply to $\ln x$ $(\log_e x)$ too
  - Two particularly useful results are
    - $\ln e^x = x$
    - $e^{\ln x} = x$
- Laws of logarithms can be used to ...
  - simplify expressions
  - solve logarithmic equations
  - solve exponential equations

Exam Tip

- Remember to check whether your solutions are valid
  - log (x+k) is only defined if x > -k
  - You will lose marks if you forget to reject invalid solutions

Worked Example

a)
Write the expression $2\log 4 - \log 2$ in the form $\log k$, where $k \in \mathbb{Z}$.

Using the law $\log_a x^m = m\log_a x$

$2\log 4 = \log 4^2 = \log 16$

$2\log 4 - \log 2 = \log 4^2 - \log 2$

$= \log 16 - \log 2$

Using the law $\log_a \frac{x}{y} = \log_a x - \log_a y$

$\log 16 - \log 2 = \log \frac{16}{2} = \log 8$

$2\log 4 - \log 2 = \log 8$

b) Hence, or otherwise, solve $2\log 4 - \log 2 = -\log \frac{1}{x}$.

To solve $2\log 4 - \log 2 = \log \frac{1}{x}$ rewrite as

$\log 8 = -\log \frac{1}{x}$

from part (a)

Use the index law $\frac{1}{x} = x^{-1}$

$\log 8 = -\log x^{-1}$

$\log 8 = \log x$ ($\log_a x^m = m\log_a x$)

$8 = x$

$x = 8$

Change of Base

Why change the base of a logarithm?

- The laws of logarithms can only be used if the logs have the same **base**
  - If a problem involves logarithms with different bases, you can change the base of the logarithm and then apply the laws of logarithms
- **Changing the base** of a logarithm can be particularly useful if you need to evaluate a log problem **without a calculator**
  - Choose the base such that you would know how to solve the problem from the equivalent exponent

How do I change the base of a logarithm?

- The formula for changing the base of a logarithm is

$$\log_a x = \frac{\log_b x}{\log_b a}$$

- This is **in the formula booklet**
- The value you choose for *b* does not matter, however if you do not have a calculator, you can choose *b* such that the problem will be possible to solve

Exam Tip

- Changing the base is a key skill which can help you with many different types of questions, make sure you are confident with it
  - It is a particularly useful skill for examinations where a GDC is not permitted

Worked Example

By choosing a suitable value for *b*, use the change of base law to find the value of $\log_8 32$ without using a calculator.

Change of base law: $\log_a x = \dfrac{\log_b x}{\log_b a}$

$\log_8 32$ ($2^5 = 32$, $2^3 = 8$)

Choose $b = 2$ to allow for a solution by inspection

$$\log_8 32 = \frac{\log_2 32}{\log_2 8} = \frac{5}{3}$$

$$\log_8 32 = 1\tfrac{2}{3}$$

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1.1.3 Solving Exponential Equations

Solving Exponential Equations

What are exponential equations?

- An exponential equation is an equation where the unknown is a power
  - In simple cases the solution can be spotted without the use of a calculator
  - For example,

$$5^{2x} = 125$$
$$2x = 3$$
$$x = \frac{3}{2}$$

- In more complicated cases the laws of logarithms should be used to solve exponential equations
- The **change of base** law can be used to solve some exponential equations without a calculator
  - For example,

$$27^x = 9$$
$$x = \log_{27} 9$$
$$= \frac{\log_3 9}{\log_3 27}$$
$$= \frac{2}{3}$$

How do we use logarithms to solve exponential equations?

- An exponential equation can be solved by taking logarithms of both sides
- The **laws of indices** may be needed to rewrite the equation first
- The **laws of logarithms** can then be used to solve the equation
  - **ln ($\log_e$)** is often used
  - The answer is often written in terms of ln
- A question my ask you to give your answer in a particular form
- Follow these steps to solve exponential equations
  - STEP 1: Take logarithms of both sides
  - STEP 2: Use the laws of logarithms to remove the powers
  - STEP 3: Rearrange to isolate *x*
  - STEP 4: Use logarithms to solve for *x*

What about hidden quadratics?

- Look for hidden squared terms that could be changed to form a quadratic
  - In particular look out for terms such as
    $4^x = (2^2)^x = 2^{2x} = (2^x)^2$

- $e^{2x} = (e^2)^x = (e^x)^2$

Exam Tip

- Always check which form the question asks you to give your answer in, this can help you decide how to solve it
- If the question requires an exact value you may need to leave your answer as a logarithm

Worked Example

Solve the equation $4^x - 3(2^{x+1}) + 9 = 0$. Give your answer correct to three significant figures.

Spot the hidden quadratic: $4^x = (2^2)^x = (2^x)^2$

By the laws of indices $2^{x+1} = 2^x \times 2^1 = 2 \times 2^x$

$(2^x)^2 - 3(2^{x+1}) + 9 = 0$

$(2^x)^2 - 3 \times 2 \times 2^x + 9 = 0$

$(2^x)^2 - 6 \times 2^x + 9 = 0$

Let $u = 2^x$ $\quad u^2 - 6u + 9 = 0$

$(u-3)(u-3) = 0$

$u = 3 \quad \therefore 2^x = 3$

Solve the exponential equation $2^x = 3$

Step 1: Take logarithms of both sides: $\ln(2^x) = \ln(3)$

Step 2: Use the law $\log_a x^m = m\log_a x$ $\quad x\ln 2 = \ln 3$

Step 3: Rearrange to isolate x $\quad x = \frac{\ln 3}{\ln 2}$

Step 4: Solve $\quad x = \frac{\ln 3}{\ln 2} = 1.584...$

$x = 1.58$ (3s.f.)

1.2 Number & Algebra Toolkit

1.2.1 Standard Form

Standard Form

Standard form (sometimes called **scientific notation** or **standard index form**) gives us a way of writing very big and very small numbers using powers of 10.

Why use standard form?

- Some numbers are too big or too small to write easily or for your calculator to display at all
  - Imagine the number 50^{50}, the answer would take 84 digits to write out
  - Try typing 50^{50} into your calculator, you will see it displayed in **standard form**
- Writing very big or very small numbers in standard form allows us to:
  - Write them more neatly
  - Compare them more easily
  - Carry out calculations more easily
- Exam questions could ask for your answer to be written in standard form

How is standard form written?

- In standard form numbers are always written in the form $a \times 10^k$ where a and k satisfy the following conditions:
  - $1 \leq a < 10$
    - So there is one non – zero digit before the decimal point
  - $k \in \mathbb{Z}$
    - So k must be an integer
  - $k > 0$ for large numbers
    - How many times a is multiplied by 10
  - $k < 0$ for small numbers
    - How many times a is divided by 10

How are calculations carried out with standard form?

- Your GDC will display large and small numbers in standard form when it is in normal mode
  - Your GDC may display standard form as *aEn*
    - For example, 2.1×10^{-5} will be displayed as $2.1\text{E}-5$
    - If so, be careful to **rewrite the answer given in the correct form**, you will not get marks for copying directly from your GDC
- Your GDC will be able to carry out calculations in standard form
  - If you put your GDC into scientific mode it will automatically convert numbers into standard form
    - Beware that your GDC may have more than one mode when in scientific mode
    - This relates to the number of significant figures the answer will be displayed in
    - Your GDC may add extra zeros to fill spaces if working with a high number of significant figures, you do not need to write these in your answer
- To add or subtract numbers written in the form $a \times 10^k$ without your GDC you will need to write them in full form first

    - Alternatively you can use 'matching powers of 10', because if the powers of 10 are the same, then the 'number parts' at the start can just be added or subtracted normally
        - For example $(6.3\times10^{14}) + (4.9\times10^{13}) = (6.3\times10^{14}) + (0.49\times10^{14}) = 6.79\times10^{14}$
        - Or $(7.93\times10^{-11}) - (5.2\times10^{-12}) = (7.93\times10^{-11}) - (0.52\times10^{-11}) = 7.41\times10^{-11}$
- To multiply or divide numbers written in the form $a \times 10^k$ without your GDC you can either write them in full form first or use the laws of indices

Exam Tip

- Your GDC will give very big or very small answers in standard form and will have a setting which will allow you to carry out calculations in scientific notation
- Make sure you are familiar with the form that your GDC gives answers in as it may be different to the form you are required to use in the exam

Worked Example

Calculate the following, giving your answer in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

i)
3780×200

Using GDC: Choose scientific mode.
Input directly into GDC as ordinary numbers.

$3780 \times 200 = 7.56 \times 10^5$

GDC will automatically give answer in standard form.

Without GDC:
Calculate the value:

$3780 \times 200 = 756000$

Convert to standard form:

$756000 = 7.56 \times 10^5$

7.56×10^5

ii) $(7 \times 10^5) - (5 \times 10^4)$

Using GDC: Choose scientific mode.

Input directly into GDC

$7 \times 10^5 - 5 \times 10^4 = 6.5 \times 10^5$

This may be displayed as 6.5E5

Without GDC:

Convert to ordinary numbers:

$7 \times 10^5 = 700\,000$

$5 \times 10^4 = 50000$

Carry out the calculation:

$700\,000 - 50000 = 650000$

Convert to standard form:

$650000 = 6.5 \times 10^5$

6.5×10^5

iii)

$(3.6 \times 10^{-3})(1.1 \times 10^{-5})$

Input directly into GDC:
(Choose scientific mode).

$(3.6 \times 10^{-3})(1.1 \times 10^{-5}) = 3.96 \times 10^{-8}$

$$3.96 \times 10^{-8}$$

Note:

$10^{-3} \times 10^{-5} = 10^{-8}$

$(3.6 \times 10^{-3})(1.1 \times 10^{-5}) = 3.96 \times 10^{-8}$

$3.6 \times 1.1 = 3.96$

1.2.2 Laws of Indices

Laws of Indices

What are the laws of indices?

- Laws of indices (or index laws) allow you to simplify and manipulate expressions involving exponents
  - An exponent is a power that a number (called the base) is raised to
  - Laws of indices can be used when the numbers are written with the same base
- The index laws you need to know are:
  - $(xy)^m = x^m y^m$
  - $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$
  - $x^m \times x^n = x^{m+n}$
  - $x^m \div x^n = x^{m-n}$
  - $(x^m)^n = x^{mn}$
  - $x^1 = x$
  - $x^0 = 1$
  - $\frac{1}{x^m} = x^{-m}$
  - $x^{\frac{1}{n}} = \sqrt[n]{x}$
  - $x^{\frac{m}{n}} = \sqrt[n]{x^m}$
- These laws are **not in the formula booklet** so you must remember them

How are laws of indices used?

- You will need to be able to carry out multiple calculations with the laws of indices
  - Take your time and apply each law individually
  - Work with numbers first and then with algebra
- Index laws only work with terms that have the same base, make sure you **change the base** of the term before using any of the index laws
  - Changing the base means rewriting the number as an exponent with the base you need
  - For example, $9^4 = (3^2)^4 = 3^{2\times4} = 3^8$
  - Using the above can them help with problems like $9^4 \div 3^7 = 3^8 \div 3^7 = 3^1 = 3$

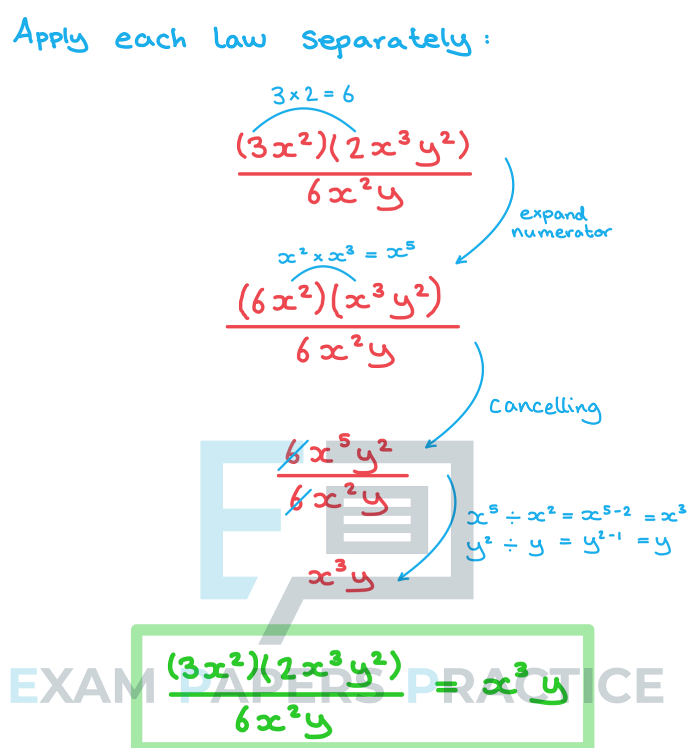
Exam Tip

- Index laws are rarely a question on their own in the exam but are often needed to help you solve other problems, especially when working with logarithms or polynomials
- Look out for times when the laws of indices can be applied to help you solve a problem algebraically

Worked Example

Simplify the following equations:

i)
$$\frac{(3x^2)(2x^3y^2)}{(6x^2y)}.$$



ii)
$$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}.$$

$$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$$

Rewrite as a fraction

$$\frac{(4x^2y^{-4})^3}{(2x^3y^{-1})^2}$$

expand numerator and denominator

$$\frac{64x^6y^{-12}}{4x^6y^{-2}}$$

cancelling

$$\frac{64x^6y^2}{4x^6y^{12}}$$

The negative exponents can be rewritten as their reciprocals

$$16y^{-10}$$

$$\frac{16}{y^{10}}$$

1.2.3 Partial Fractions

Partial Fractions

What are partial fractions?

- Partial fractions allow us to simplify rational expressions into the sum of two or more fractions with constant numerators and linear denominators
  - This allows for integration of rational functions
- The method of partial fractions is essentially the reverse of adding or subtracting fractions
  - When adding fractions, a common denominator is required
  - In partial fractions the common denominator is split into parts (factors)
- If we have a rational function with a quadratic on the denominator partial fractions can be used to rewrite it as the sum of two rational functions with linear denominators
  - This works if the non-linear denominator can be **factorised** into two distinct factors
    - For example: $\frac{ax+b}{(cx+d)(ex+f)} = \frac{A}{cx+d} + \frac{B}{ex+f}$
- If we have a rational function with a linear numerator and denominator partial fractions can be used to rewrite it as the sum of a constant and a fraction with a linear denominator
  - The linear denominator does not need to be factorised
  - For example: $\frac{ax+b}{cx+d} = A + \frac{B}{cx+d}$

How do I find partial fractions if the denominator is a quadratic?

- STEP 1
  Factorise the denominator into the product of two linear factors
  - Check the numerator and cancel out any common factors
    - e.g. $\frac{5x+5}{x^2+x-6} = \frac{5x+5}{(x+3)(x-2)}$
- STEP 2
  Split the fraction into a **sum** of two fractions with single **linear denominators** each having unknown **constant numerators**
  - Use A and B to represent the unknown numerators
    - e.g. $\frac{5x+5}{(x+3)(x-2)} \equiv \frac{A}{x+3} + \frac{B}{x-2}$
- STEP 3
  Multiply through by the denominator to eliminate fractions
  - Eliminate fractions by cancelling all common expressions
    - e.g. $5x+5 \equiv A(x-2) + B(x+3)$
- STEP 4
  Substitute values into the identity and solve for the unknown constants
  - Use the root of each **linear factor** as a value of to find the unknowns
    - e.g. Let $x=2$: $5(2)+5 \equiv A((2)-2) + B((2)+3)$ etc
  - An alternative method is **comparing coefficients**
    - e.g. $5x+5 \equiv (A+B)x + (-2A+3B)$

- STEP 5
  Write the **original** as partial fractions
  - Substitute the values you found for *A* and *B* into your expression from STEP 2
    - e.g. $\frac{5x + 5}{x^2 + x - 6} = \frac{2}{x + 3} + \frac{3}{x - 2}$

How do I find partial fractions if the numerator and denominator are both linear?

- If the denominator is not a quadratic expression you will be given the form in which the partial fractions should be expressed
- For example express $\frac{12x - 2}{3x - 1}$ in the form $A + \frac{B}{3x - 1}$
- STEP 1
  Multiply through by the denominator to eliminate fractions
  - e.g. $12x - 2 \equiv A(3x - 1) + B$
- STEP 2
  Expand the expression on the right-hand side and **compare coefficients**
  - Compare the coefficients of *x* and solve for the first unknown
    - e.g. $12x = 3Ax$
    - therefore $A = 4$
  - Compare the constant coefficients and solve for the second unknown
    - e.g. $-2 = -A + B = -4 + B$
    - therefore $B = 2$
- STEP 3
  Write the **original** as partial fractions
  - $\frac{12x - 2}{3x - 1} = 4 + \frac{2}{3x - 1}$

How do I find partial fractions if the denominator has a squared linear term?

- A **squared linear factor** in the denominator actually represents two factors rather than one
- This must be taken into account when the rational function is split into partial fractions
  - For the squared linear denominator $(ax + b)^2$ there will be two factors: $(ax + b)$ and $(ax + b)^2$
  - So the rational expression $\frac{p}{(ax + b)^2}$ becomes $\frac{A}{ax + b} + \frac{B}{(ax + b)^2}$
- In IB you will be given the form into which you should split the partial fractions
  - Put the rational expression equal to the given form and then continue with the steps above
- There is more than one way of finding the missing values when working with partial fractions
  - Substituting values is usually quickest, however you should look at the number of times a bracket is repeated to help you decide which method to use

Exam Tip

- An exam question will often have partial fractions as part (a) and then integration or using the binomial theorem as part (b)
  - Make sure you use your partial fractions found in part (a) to answer the next part of the question

Worked Example

a)

Express $\frac{2x - 13}{x^2 - x - 2}$ in partial fractions.

$$\frac{2x - 13}{x^2 - x - 2} = \frac{2x - 13}{(x+1)(x-2)}$$

The denominator is a quadratic so factorise first.

$$\frac{2x - 13}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$

Multiply through by the denominator to eliminate fractions

$$2x - 13 \equiv A(x-2) + B(x+1)$$

Choose values of x to substitute into the identity that will eliminate each constant:

Let $x = 2$: $2(2) - 13 = A((2)-2) + B((2)+1)$

$x - 2 = 0$
$x = 2$

$$-9 = 3B$$
$$B = -3$$

Let $x = -1$: $2(-1) - 13 = A((-1)-2) + B((-1)+1)$

$x + 1 = 0$
$x = -1$

$$-15 = -3A$$
$$A = 5$$

$$\frac{2x - 13}{x^2 - x - 2} = \frac{5}{x+1} - \frac{3}{x-2}$$

b)

Express $\frac{x(3x - 13)}{(x+1)(x-3)^2}$ in the form $\frac{A}{(x+1)} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$.

Multiply through by the denominator:

$$\frac{x(3x-13)}{(x+1)(x-3)^2} = \frac{A(x-3)^2 + B(x+1)(x-3) + C(x+1)}{(x+1)(x-3)^2}$$

Eliminate fractions and expand:

$$x(3x-13) = A(x^2-6x+9) + B(x^2-2x-3) + Cx + C$$

$$3x^2 - 13x = (A+B)x^2 + (-6A-2B+C)x + 9A - 3B + C$$

↑ coefficient of x^2 ↑ coefficient of x

Compare coefficients:

$A + B = 3$ ① (coefficients of x^2)

$-6A - 2B + C = -13$ ② (coefficients of x)

$9A - 3B + C = 0$ ③ (constant terms)

Rearrange ① and substitute into ② and ③

$A = 3 - B \Rightarrow -6(3-B) - 2B + C = -13$

$-18 + 6B - 2B + C = -13$

$4B + C = 5$ ②

$\Rightarrow 9(3-B) - 3B + C = 0$

$27 - 9B - 3B + C = 0$

$12B - C = 27$ ③

Solving ② and ③: $4B + C = 5$

$12B - C = 27$

$B = 2, \; C = -3$

Substitute into ①: $A = 3 - B = 3 - 2 = 1$

$$\frac{x(3x-13)}{(x+1)(x-3)^2} = \frac{1}{(x+1)} + \frac{2}{(x-3)} - \frac{3}{(x-3)^2}$$

1.3 Sequences & Series

1.3.1 Language of Sequences & Series

Language of Sequences & Series

What is a sequence?

- A **sequence** is an ordered set of numbers with a rule for finding all of the numbers in the sequence
  - For example 1, 3, 5, 7, 9, … is a sequence with the rule '*start at one and add two to each number*'
- The numbers in a sequence are often called **terms**
- The terms of a sequence are often referred to by letters with a subscript
  - In IB this will be the letter u
  - So in the sequence above, $u_1 = 1$, $u_2 = 3$, $u_3 = 5$ and so on
- Each term in a sequence can be found by **substituting** the term number into **formula for the nth term**

What is a series?

- You get a **series** by summing up the terms in a sequence
  - E.g. For the sequence 1, 3, 5, 7, … the associated series is $1 + 3 + 5 + 7 + \ldots$
- We use the notation S_n to refer to the sum of the first n terms in the series
  - $S_n = u_1 + u_2 + u_3 + \ldots + u_n$
  - So for the series above $S_5 = 1 + 3 + 5 + 7 + 9 = 25$

Worked Example

Determine the first five terms and the value of S_5 in the sequence with terms defined by $u_n = 5 - 2n$.

$u_n = 5 - 2n$

(find the term you want by replacing n with it's value.) (term number)

first term:

$u_1 = 5 - 2(1) = 3$

$u_2 = 5 - 2(2) = 1$

$u_3 = 5 - 2(3) = -1$

$u_4 = 5 - 2(4) = -3$

$u_5 = 5 - 2(5) = -5$

recognise the pattern. −2, −2 ← rule is subtract 2

'start with 3 and subtract 2 from each number'.

$S_5 = 3 + 1 + (-1) + (-3) + (-5) = -5$

(the sum of the first 5 terms)

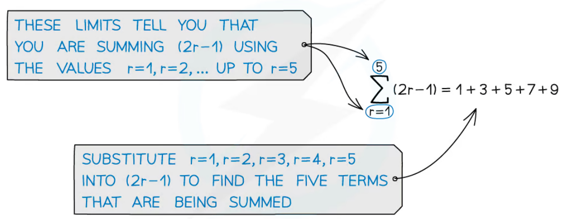
3, 1, −1, −3, −5

$S_5 = -5$

Sigma Notation

What is sigma notation?

- Sigma notation is used to show the sum of a certain number of terms in a sequence
- The symbol Σ is the capital Greek letter sigma
- Σ stands for 'sum'
  - The expression to the right of the Σ tells you what is being summed, and the limits above and below tell you which terms you are summing



- Be careful, the limits don't have to start with 1
  - For example $\sum_{k=0}^{4}(2k+1)$ or $\sum_{k=7}^{14}(2k-13)$
  - r and k are commonly used variables within sigma notation

Exam Tip

- Your GDC will be able to use sigma notation, familiarise yourself with it and practice using it to check your work

Worked Example

A sequence can be defined by $u_n = 2 \times 3^{n-1}$ for $n \in \mathbb{Z}^+$.

a)

Write an expression for $u_1 + u_2 + u_3 + \ldots + u_6$ using sigma notation.

$u_n = 2 \times 3^{n-1},\ n \in \mathbb{Z}^+$ ← n is the set of all positive integers

Using sigma notation

$$u_1 + u_2 + \ldots + u_6 = \sum_{k=1}^{6} u_k$$

$$\sum_{k=1}^{6} (2 \times 3^{k-1})$$

b)

Write an expression for $u_7 + u_8 + u_9 + \ldots + u_{12}$ using sigma notation.

$u_n = 2 \times 3^{n-1},\ n \in \mathbb{Z}^+$ ← n is the set of all positive integers

Using sigma notation

$$u_7 + u_8 + \ldots + u_{12} = \sum_{k=7}^{12} u_k$$

$$\sum_{k=7}^{12} (2 \times 3^{k-1})$$

1.3.2 Arithmetic Sequences & Series

Arithmetic Sequences

What is an arithmetic sequence?

- In an **arithmetic sequence,** the difference between consecutive terms in the sequence is constant
- This **constant difference** is known as the **common difference, *d*,** of the sequence
  - For example, 1, 4, 7, 10, … is an arithmetic sequence with the rule *'start at one and add three to each number'*
    - The **first term, u_1,** is 1
    - The **common difference, *d*,** is 3
  - An arithmetic sequence can be **increasing** (positive common difference) or **decreasing** (negative common difference)
  - Each term of an arithmetic sequence is referred to by the letter *u* with a subscript determining its place in the sequence

How do I find a term in an arithmetic sequence?

- The n^{th} term formula for an arithmetic sequence is given as

$$u_n = u_1 + (n-1)d$$

  - Where u_1 is the first term, and d is the common difference
  - This is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two terms and asked to find both the first term and the common difference
  - Substitute the information into the formula and set up a **system of linear equations**
  - Solve the simultaneous equations
    - You could use your GDC for this

> **Exam Tip**
>
> - Simultaneous equations are often needed within arithmetic sequence questions, make sure you are confident solving them with and without the GDC

Worked Example

The fourth term of an arithmetic sequence is 10 and the ninth term is 25, find the first term and the common difference of the sequence.

$u_4 = 10$, $u_9 = 25$

Formula for n^{th} term of an arithmetic series:

$$u_n = u_1 + (n-1)d$$

Sub in $u_4 = 10$ and $u_9 = 25$

$$u_4 = u_1 + (4-1)d = u_1 + 3d = 10$$

$$u_9 = u_1 + (9-1)d = u_1 + 8d = 25$$

Solve using GDC:

let $u_1 = x$ and $d = y$

$$x + 3y = 10$$
$$x + 8y = 25$$

$x = 1$, $y = 3$

$u_1 = 1$
$d = 3$

Arithmetic Series

How do I find the sum of an arithmetic series?

- An **arithmetic series** is the sum of the terms in an **arithmetic sequence**
  - For the arithmetic sequence 1, 4, 7, 10, ... the arithmetic series is 1 + 4 + 7 + 10 + ...
- Use the following formulae to find the sum of the first *n* terms of the arithmetic series:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) \quad ; \quad S_n = \frac{n}{2}(u_1 + u_n)$$

    - u_1 is the first term
    - d is the common difference
    - u_n is the last term
  - Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - If you know the first term and common difference use the first version
  - If you know the first and last term then the second version is easier to use
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this

Exam Tip

- The formulae you need for arithmetic series are in the formula book, you do not need to remember them
  - Practice finding the formulae so that you can quickly locate them in the exam

Worked Example

The sum of the first 10 terms of an arithmetic sequence is 630.

a)

Find the common difference, *d*, of the sequence if the first term is 18.

$S_{10} = 630$

Formula for the sum of an arithmetic series:

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

Sub in $S_{10} = 630$, $u_1 = 18$

$S_{10} = \frac{10}{2}(2(18) + (10-1)d) = 630$

$5(36 + 9d) = 630$

Solve: $36 + 9d = 126$

$9d = 90$

$d = 10$

$d = 10$

b)

Find the first term of the sequence if the common difference, *d*, is 11.

Sub in $S_{10} = 630$, $d = 11$

$$S_{10} = \frac{10}{2}(2u_1 + (10-1)(11)) = 630$$

$$5(2u_1 + 99) = 630$$

Solve: $2u_1 + 99 = 126$

$$2u_1 = 27$$

$$\boxed{u_1 = 13.5}$$

1.3.3 Geometric Sequences & Series

Geometric Sequences

What is a geometric sequence?

- In a **geometric sequence**, there is a **common ratio, *r*,** between consecutive terms in the sequence
  - For example, 2, 6, 18, 54, 162, ... is a sequence with the rule '*start at two and multiply each number by three*'
    - The **first term, u_1,** is 2
    - The **common ratio, *r*,** is 3
- A geometric sequence can be **increasing** ($r > 1$) or **decreasing** ($0 < r < 1$)
- If the common ratio is a **negative number** the terms will alternate between positive and negative values
  - For example, 1, -4, 16, -64, 256, ... is a sequence with the rule '*start at one and multiply each number by negative four*'
    - The **first term, u_1,** is 1
    - The **common ratio, *r*,** is -4
- Each term of a geometric sequence is referred to by the letter *u* with a subscript determining its place in the sequence

How do I find a term in a geometric sequence?

- The n^{th} term formula for a geometric sequence is given as

$$u_n = u_1 r^{n-1}$$

  - Where u_1 is the first term, and r is the common ratio
  - This formula allows you to find **any term** in the geometric sequence
  - It is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common ratio
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two or more consecutive terms and asked to find both the first term and the common ratio
  - Find the common ratio by dividing a term by the one before it
  - Substitute this and one of the terms into the formula to find the first term
- Sometimes you may be given a term and the formula for the nth term and asked to find the value of *n*
  - You can solve these using **logarithms** on your GDC

Exam Tip

- You will sometimes need to use logarithms to answer geometric sequences questions
  - Make sure you are confident doing this
  - Practice using your GDC for different types of questions

Worked Example

The sixth term, u_6, of a geometric sequence is 486 and the seventh term, u_7, is 1458.

Find,

i)
the common ratio, r, of the sequence,

$u_6 = 486 \quad , \quad u_7 = 1458$

The common ratio, r, is given by

$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \ldots = \frac{u_{n+1}}{u_n}$$

Sub in $u_6 = 486$, $u_7 = 1458$

$$r = \frac{u_7}{u_6} = \frac{1458}{486} = 3$$

$$\boxed{r = 3}$$

ii)
the first term of the sequence, u_1.

Formula for n^{th} term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in $r = 3$ and either $u_6 = 486$ or $u_7 = 1458$

$$u_6 = u_1(3)^{6-1} = 486$$

Solve: $243\, u_1 = 486$

$u_1 = 2$

$$\boxed{u_1 = 2}$$

Geometric Series

How do I find the sum of a geometric series?

- A **geometric series** is the sum of a certain number of terms in a **geometric sequence**
  - For the geometric sequence 2, 6, 18, 54, ... the geometric series is 2 + 6 + 18 + 54 + ...
- The following formulae will let you find the sum of the first n terms of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

    - u_1 is the first term
    - r is the common ratio
  - Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - The first version of the formula is more convenient if $r > 1$ and the second is more convenient if $r < 1$
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term, the common ratio, or the number of terms within the sequence
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this

Exam Tip

- The geometric series formulae are in the formulae booklet, you don't need to memorise them
  - Make sure you can locate them quickly in the formula booklet

Worked Example

A geometric sequence has $u_1 = 25$ and $r = 0.8$. Find the value of u_5 and S_5.

$u_1 = 25, \quad r = 0.8$

Formula for n^{th} term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in $u_1 = 25, \quad r = 0.8$

$$u_5 = 25(0.8)^4 = 10.24$$

Formula for the sum of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

$r < 1$ so this version is easier to use.

Sub in $u_1 = 25, \quad r = 0.8$

$$S_5 = \frac{u_1(1 - r^5)}{1 - r} = \frac{25(1 - 0.8^5)}{1 - 0.8} = 84.04$$

$u_5 = 10.24$

$S_5 = 84.04$

Sum to Infinity

What is the sum to infinity of a geometric series?

- A geometric sequence will either increase or decrease away from zero or the terms will get progressively closer to zero
  - Terms will get closer to zero if the common ratio, *r*, is between 1 and -1
- If the terms are getting closer to zero then the series is said to **converge**
  - This means that the sum of the series will approach a limiting value
  - As the number of terms increase, the sum of the terms will get closer to the limiting value

How do we calculate the sum to infinity?

- If asked to find out if a geometric sequence converges find the value of *r*
  - If $|r| < 1$ then the sequence converges
  - If $|r| \geq 1$ then the sequence does not converge and the sum to infinity cannot be calculated
  - $|r| < 1$ means $-1 < r < 1$
- If $|r| < 1$, then the geometric series **converges** to a finite value given by the formula

$$S_\infty = \frac{u_1}{1-r}, \quad |r| < 1$$

  - u_1 is the first term
  - r is the common ratio
  - This is **in the formula book**, you do not need to remember it

Exam Tip

- Learn and remember the conditions for when a sum to infinity can be calculated

Worked Example

The first three terms of a geometric sequence are $6\,,\ 2\,,\ \frac{2}{3}$. Explain why the series converges and find the sum to infinity.

$u_1 = 6, \quad u_2 = 2, \quad u_3 = \frac{2}{3}$

Find the value of r: $r = \frac{u_2}{u_1}$

$$r = \frac{u_2}{u_1} = \frac{2}{6} = \frac{1}{3}$$

$|r| < 1$ so the series converges

Find the sum to infinity: $S_\infty = \frac{u_1}{1-r}$

$$S_\infty = \frac{u_1}{1-r} = \frac{6}{1-\frac{1}{3}} = \frac{6}{2/3} = 9$$

$S_\infty = 9$

1.3.4 Applications of Sequences & Series

Applications of Arithmetic Sequences & Series

Many real-life situations can be modelled using sequences and series, including but not limited to: patterns made when tiling floors; seating people around a table; the rate of change of a population; the spread of a virus and many more.

What do I need to know about applications of arithmetic sequences and series?

- If a quantity is changing repeatedly by having a fixed amount **added to** or **subtracted from** it then the use of **arithmetic sequences** and **arithmetic series** is appropriate to **model** the situation
  - If a sequence seems to fit the pattern of an arithmetic sequence it can be said to be **modelled** by an arithmetic sequence
  - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of arithmetic sequences and series is **simple interest**
  - Simple interest is when an initial investment is made and then a percentage of the initial investment is added to this amount on a regular basis (usually per year)
- Arithmetic sequences can be used to make estimations about how something will change in the future

Exam Tip

- Exam questions won't always tell you to use sequences and series methods, practice spotting them by looking for clues in the question
- If a given amount is repeated periodically then it is likely the question is on arithmetic sequences or series

Worked Example

Jasper is saving for a new car. He puts USD $100 into his savings account and then each month he puts in USD $10 more than the month before. Jasper needs USD $1200 for the car. Assuming no interest is added, find,

i)

the amount Jasper has saved after four months,

Identify the arithmetic sequence:

$u_1 = 100 \quad , \quad d = 10$

After 4 months Jasper will have saved:

$u_1 + u_2 + u_3 + u_4 = S_4$

Formula for the sum of an arithmetic series:

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$S_4 = \frac{4}{2}(2u_1 + (4-1)d)$

Sub in $u_1 = 100$ and $d = 10$

$S_4 = \frac{4}{2}(2(100) + (4-1)(10))$

$= 2(200 + 30)$

$= 2(230)$

$S_4 = \$460$

ii)

the month in which Jasper reaches his goal of USD $1200.

Sub $S_n = 1200$, $u_1 = 100$, $d = 10$ into formula:

$$1200 = \frac{n}{2}(2(100) + (n-1)(10))$$

Solve using algebraic solver on GDC:

$n = 8.67...$ or $n = -27.67...$

↑ disregard as n cannot be negative.

$\therefore$ $S_8 < 1200$

$S_9 > 1200$ reaches total in 9^{th} month

Jasper will reach USD $1200 in the 9^{th} month.

Applications of Geometric Sequences & Series

What do I need to know about applications of geometric sequences and series?

- If a quantity is changing repeatedly by a fixed **percentage,** or by being **multiplied** repeatedly by a fixed amount, then the use of **geometric sequences** and **geometric series** is appropriate to **model** the situation
  - If a sequence seems to fit the pattern of a geometric sequence it can be said to be **modelled** by a geometric sequence
  - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of geometric sequences and series is **compound interest**
  - Compound interest is when an initial investment is made and then interest is paid on the initial amount **and on the interest already earned** on a regular basis (usually every year)
- Geometric sequences can be used to make estimations about how something will change in the future
- The questions won't always tell you to use sequences and series methods, so be prepared to spot 'hidden' sequences and series questions
  - Look out for questions on savings accounts, salaries, sales commissions, profits, population growth and decay, spread of bacteria etc

Exam Tip

- Exam questions won't always tell you to use sequences and series methods, practice spotting them by looking for clues in the question
- If a given amount is changing by a percentage or multiple then it is likely the question is on geometric sequences or series

Worked Example

A new virus is circulating on a remote island. On day one there were 10 people infected, with the number of new infections increasing at a rate of 40% per day.

a)
Find the expected number of people newly infected on the 7th day.

Identify the geometric sequence:

$u_1 = 10$, $r = 1.4$ ← 40% increase so 140% of the day before

New infections : u_7

Formula for n^{th} term of a geometric series :

$$u_n = u_1 r^{n-1}$$

Sub in $u_1 = 10$, $r = 1.4$

$$u_7 = 10(1.4)^6 = 75.29\ldots$$

Expected number of new infections = 75

b)
Find the expected number of infected people after one week (7 days), assuming no one has recovered yet.

Total infections : S_7

Formula for the sum of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$

← $r > 1$ so this version is easier to use.

Sub in $u_1 = 10$, $r = 1.4$

$$S_7 = \frac{10(1.4^7 - 1)}{1.4 - 1} = 238.53\ldots$$

Expected number of total infections = 239

1.3.5 Compound Interest & Depreciation

Compound Interest

What is compound interest?

- Interest is a small percentage paid by a bank or company that is added on to an initial investment
  - Interest can also refer to an amount paid on a loan or debt, however IB compound interest questions will always refer to interest on **investments**
- **Compound interest** is where interest is paid on **both the initial investment** and any interest that has **already been paid**
  - Make sure you know the difference between compound interest and simple interest
    - Simple interest pays interest only on the initial investment
- The interest paid each time will increase as it is a percentage of a higher number
- Compound interest will be paid in instalments in a given timeframe
  - The interest rate, *r*, will be per annum (per year)
    - This could be written *r* % p.a.
  - Look out for phrases such as **compounding annually** (interest paid yearly) **or compounding monthly** (interest paid monthly)
    - If $\alpha\%$ p.a. (per annum) is paid compounding monthly, then $\frac{\alpha}{12}\%$ will be paid each month
    - The formula for compound interest allows for this so you do not have to compensate separately

How is compound interest calculated?

- The formula for calculating compound interest is:

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

  - Where
    - *FV* is the future value
    - *PV* is the present value
    - *n* is the number of years
    - *k* is the number of compounding periods per year
    - *r* % is the nominal annual rate of interest
  - This formula is **given in the formula booklet**, you do not have to remember it
- Be careful with the *k* value
  - Compounding annually means $k = 1$
  - Compounding half-yearly means $k = 2$
  - Compounding quarterly means $k = 4$
  - Compounding monthly means $k = 12$
- Your GDC will have a finance solver app on it which you can use to find the future value
  - This may also be called the TVM (time value of money) solver
  - You will have to enter the information from the question into your calculator
- Be aware that many questions will be set up such that you will have to use the formula
  - So for compound interest questions it is better to use the formula from your formula booklet than your GDC

Exam Tip

- Your GDC will be able to solve some compound interest problems so it is a good idea to make sure you are confident using it, however you must also familiarise yourself with the formula and make sure you can find it in the formula booklet

Worked Example

Kim invests MYR 2000 (Malaysian Ringgit) in an account that pays a nominal annual interest rate of 2.5% **compounded monthly**. Calculate the amount that Kim will have in her account after 5 years.

Compound interest formula:

$$FV = PV\left(1+\frac{r}{100k}\right)^{kn}$$

- FV: future value
- PV: present value
- r: interest rate
- k: compounding periods
- n: number of years

Substitute values in:

$PV = 2000$ (initial investment)

$k = 12$ (compounding monthly)

$r = 2.5\%$

$n = 5$ (number of years)

$$FV = 2000\left(1+\frac{2.5}{(100)(12)}\right)^{(12\times 5)}$$

$$= 2266.002\ldots$$

$$FV \approx \text{MYR } 2270 \text{ (3sf)}$$

Depreciation

What is depreciation?

- Depreciation is when the **value** of something **falls** over time
- The most common examples of depreciation are the value of cars and technology
- If the depreciation is occurring at a **constant rate** then it is **compound depreciation**

How is compound depreciation calculated?

- The formula for calculating compound depreciation is:

$$FV = PV \times \left(1 - \frac{r}{100}\right)^n$$

  - Where
    - FV is the future value
    - PV is the present value
    - n is the number of years
    - $r\%$ is the rate of depreciation
  - This formula is **not** given in the formula booklet, however it is almost the same as the formula for compound interest but
    - with a **subtraction** instead of an addition
    - the value of k will always be 1
- Your GDC **could** again be used to solve some compound depreciation questions, but watch out for those which are set up such that you will have to use the formula

Exam Tip

- You can use your GDC's "Finance Solver" (TI) or "Compound Interest" (Casio) feature to solve most depreciation questions, by entering the interest rate as a negative value

Worked Example

Kyle buys a new car for AUD $14 999. The value of the car depreciates by 15% each year.

a)

Find the value of the car after 5 years.

Depreciation formula:

$$FV = PV\left(1 - \frac{r}{100}\right)^n$$

where FV = future value, PV = present value, r = rate of depreciation, n = number of years.

Substitute values in:

$PV = 14999$ (initial cost)

$r = 15\%$

$n = 5$ (number of years)

$$FV = 14999\left(1 - \frac{15}{100}\right)^5$$

$$= 6655.13\ldots$$

$FV \approx$ AUD $6660 (3sf)

b)

Find the number of years and months it will take for the value of the car to be approximately AUD $9999.

EXAM PAPERS PRACTICE

$$FV = PV\left(1 - \frac{r}{100}\right)^n$$

$FV \approx 9999$

$PV = 14999$

$r = 15\%$

Substitute values in:

$$9999 \approx 14999\left(1 - \frac{15}{100}\right)^n$$

Use GDC to solve:

$n = 2.495...$

2 years ↗ ↖ 0.495^{th} of a year

Convert to years and months:

2 years + 0.495... × 12 months

≈ 2 years and 6 months

1.4 Simple Proof & Reasoning

1.4.1 Proof

Language of Proof

What is proof?

- Proof is a series of logical steps which show a result is **true** for all specified numbers
  - 'Seeing' that a result works for a few numbers is not enough to **show** that it will work for all numbers
  - Proof allows us to show (usually algebraically) that the result will work for **all values**
- You must be familiar with the notation and language of proof
- LHS and RHS are standard abbreviations for left-hand side and right-hand side
- **Integers** are used frequently in the language of proof
  - The set of **integers** is denoted by $\mathbb{Z}$
  - The set of **positive integers** is denoted by $\mathbb{Z}^+$

How do we prove a statement is true for all values?

- Most of the time you will need to use algebra to show that the left-hand side (LHS) is the same as the right-hand side (RHS)
  - You **must not** move terms from one side to the other
  - Start with one side (usually the LHS) and manipulate it to show that it is the same as the other
- A **mathematical identity** is a statement that is true for all values of x (or θ in trigonometry)
  - The symbol $\equiv$ is used to identify an identity
  - If you see this symbol then you can use proof methods to show it is true
- You can complete your proof by stating that RHS = LHS or writing QED

Exam Tip

- You will need to show each step of your proof clearly and set out your method in a logical manner in the exam
  - Be careful not to skip steps

Worked Example

Prove that $(2x-2)(x-3)+2(x-1)=2(x-2)(x-1)$.

Work with LHS first:

Expand brackets:

LHS: $(2x-2)(x-3)+2(x-1)$ ← FOIL

$2x^2-6x-2x+6+2x-2$

Simplify, take care with negatives:

$2x^2-6x+4$

Factorise the 2:

$2(x^2-3x+2)$

Factorise remaining quadratic:

$2(x-2)(x-1)$ = RHS as required.

$$(2x-2)(x-3)+2(x-1)=2(x-2)(x-1)$$

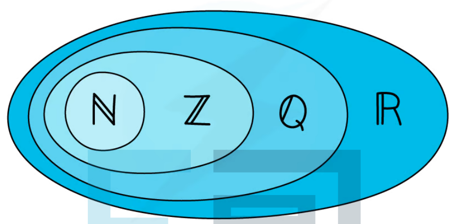
Proof by Deduction

What is proof by deduction?

- A mathematical and logical argument that shows that a result is true

How do we do proof by deduction?

- A proof by deduction question will often involve showing that a result is true for all integers, consecutive integers or even or odd numbers
  - You can begin by letting an integer be n
    - Use conventions for even ($2n$) and odd ($2n - 1$) numbers
- You will need to be familiar with sets of numbers ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$)
  - $\mathbb{N}$ – the set of **natural numbers**
  - $\mathbb{Z}$ – the set of **integers**
  - $\mathbb{Q}$ – the set of **quotients (rational numbers)**
  - $\mathbb{R}$ – the set of **real numbers**



What is proof by exhaustion?

- Proof by exhaustion is a way to show that the desired result works for every allowed value
  - This is a good method when there are only a limited number of cases to check
- Using proof by exhaustion means testing every allowed value not just showing a few examples
  - The allowed values could be specific values
  - They could also be split into cases such as even and odd

Exam Tip

- Try the result you are proving with a few different values
  - Use a sequence of them (eg 1, 2, 3)
  - Try different types of numbers (positive, negative, zero)
- This may help you see a pattern and spot what is going on

Worked Example

Prove that the sum of any two consecutive odd numbers is always even.

Let $2n - 1$ be an odd number

($2n$ must be even)

Let two consecutive odd integers be:

$2n - 1,\ 2n + 1$ (next odd number)

Then their sum is:

$$2n - 1 + 2n + 1 \equiv 4n$$

$$= 2(2n)$$

Any multiple of 2 must be even.

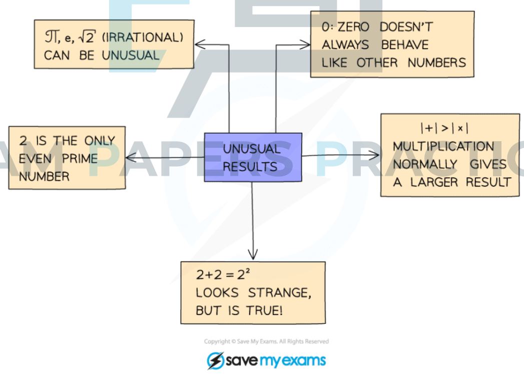
Disproof by Counter Example

What is disproof by counter-example?

- Disproving a result involves finding a value that does **not** work in the result
- That value is called a **counter**-**example**

How do I disprove a result?

- You only need to find **one** value that does not work
- Look out for the set of numbers for which the statement is made, it will often be just integers or natural numbers
- Numbers that have unusual results are often involved
  - It is often a good idea to try the values 0 and 1 first as they often behave in different ways to other numbers
  - The number 2 also behaves differently to other even numbers
    - It is the only even prime number
    - It is the only number that satisfies $n + n = n^n$
  - If it is the set of real numbers consider how rational and irrational numbers behave differently
  - Think about how positive and negative numbers behave differently
    - Particularly when working with inequalities



Exam Tip

- Read the question carefully, looking out for the set of numbers for which you need to prove the result

Worked Example

For each of the following statements, show that they are false by giving a counterexample:

a)
Given $n \in \mathbb{Z}^+$, if n^2 is a multiple of 4, then n is also a multiple of 4.

$n \in \mathbb{Z}^+$ ← set of positive integers only

We are only interested in positive integers so start by trying 1, 2 etc.

$1^2 = 1$ (not a multiple of 4)

$2^2 = 4$ $n^2 = 4$ (multiple of 4)

$n = 2$ (not a multiple of 4)

Let $n = 2$: $n^2 = 4$ (multiple of 4)
$n = 2$ (not a multiple of 4)

b)
Given $x \in \mathbb{Z}$ then $3x$ is always greater than $2x$.

$x \in \mathbb{Z}$ ← Set of integers only

We are interested in both positive and negative integers and zero so consider how each of these groups behave:

Positive integers, e.g. let $x = 1$:

$2x = 2$

$3x = 3 \quad \therefore 3x > 2x$

Zero:

Let $x = 0$

$2x = 0$

$3x = 0 \quad \therefore \quad 3x = 2x$ (this is enough to disprove the result)

Negative integers, e.g. let $x = -1$:

$2x = -2$

$3x = -3 \quad \therefore 2x > 3x$ (Any negative integer can disprove the result)

Let $x = 0$

$2x = 0$

$3x = 0 \quad \therefore \quad 3x \not> 2x$

1.5 Further Proof & Reasoning

1.5.1 Proof by Induction

Proof by Induction

What is proof by induction?

- **Proof by induction** is a way of proving a **result is true for a set of integers** by showing that if it is **true for one integer then it is true for the next integer**
- It can be thought of as dominoes:
  - All dominoes will fall down if:
    - The first domino falls down
    - Each domino falling down causes the next domino to fall down

What are the steps for proof by induction?

- **STEP 1: The basic step**
  - **Show** the result is true for the **base case**
  - This is **normally $n = 1$ or 0** but it could be any integer

    For example: To prove $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ is true for all integers $n \geq 1$ you would first need to show it is true for $n = 1$:

    $\sum_{r=1}^{1} r^2 = \frac{1}{6}(1)((1)+1)(2(1)+1)$
- **STEP 2: The assumption step**
  - **Assume** the result is true for $n = k$ for some integer k

    For example: Assume $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$ is true
  - There is nothing to do for this step apart from writing down the assumption
- **STEP 3: The inductive step**
  - **Using the assumption show** the result is true for $n = k + 1$
  - It can be helpful to simplify LHS & RHS separately and show they are identical
  - The assumption from STEP 2 will be needed at some point

    For example: $LHS = \sum_{r=1}^{k+1} r^2$ and $RHS = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$
- **STEP 4: The conclusion step**
  - **State** the result is true
  - **Explain in words** why the result is true
  - It must include:
    - If true for $n = k$ then it is true for $n = k + 1$
    - Since true for $n = 1$ the statement is true for all $n \in \mathbb{Z}, n \geq 1$ by mathematical induction
  - The sentence will be the same for each proof just change the base case from $n = 1$ if necessary

What type of statements might I be asked to prove by induction?

- **Sums of sequences**
  - If the terms involve factorials then $(k+1)! = (k+1) \times (k!)$ is useful
  - These can be written in the form $\sum_{r=1}^{n} f(r) = g(n)$
  - A useful trick for the inductive step is using $\sum_{r=1}^{k+1} f(r) = f(k+1) + \sum_{r=1}^{k} f(r)$
- **Divisibility** of an expression by an integer
  - These can be written in the form $f(n) = m \times q_n$ where m & q_n are integers
  - A useful trick for the inductive step is using $a^{k+1} = a \times a^k$
- **Complex numbers**
  - You can use proof by induction to prove de Moivre's theorem
- **Derivatives**
  - Such as chain rule, product rule & quotient rule
  - These can be written in the form $f^{(n)}(x) = g(x)$
  - A useful trick for the inductive step is using $f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f^{(k)}(x)\right)$
  - You will have to use the differentiation rules

Exam Tip

- Learn the steps for proof by induction and make sure you can use the method for a number of different types of questions before going into the exam
- The trick to answering these questions well is practicing the pattern of using each step regularly

Worked Example

Prove by induction that $\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n-4)(n+1)$ for $n \in \mathbb{Z}^+$.

Want to prove $\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n-4)(n+1)$

Basic step
Show true for n=1

$LHS = \sum_{r=1}^{1} r(r-3) = (1)(1-3) = -2$

$RHS = \frac{1}{3}(1)(1-4)(1+1) = -2$ ∴ LHS = RHS so true for n=1

Assumption step
Assume true for n=k

Assume $\sum_{r=1}^{k} r(r-3) = \frac{1}{3}k(k-4)(k+1)$

Inductive step
Show true for n=k+1

$RHS = \frac{1}{3}(k+1)((k+1)-4)((k+1)+1) = \frac{1}{3}(k+1)(k-3)(k+2)$

$LHS = \sum_{r=1}^{k+1} r(r-3) = (k+1)((k+1)-3) + \sum_{r=1}^{k} r(r-3)$

$= (k+1)(k-2) + \frac{1}{3}k(k-4)(k+1)$ ← Using assumption

$= \frac{1}{3}(k+1)\left[3(k-2) + k(k-4)\right]$ ← Factorise $\frac{1}{3}(k+1)$

$= \frac{1}{3}(k+1)\left[k^2 - k - 6\right]$

$= \frac{1}{3}(k+1)(k-3)(k+2)$

∴ LHS = RHS so true for n = k+1

Conclusion step
Explain

If true for n=k then true for n=k+1.
Since it is true for n=1, the statement is true for all $n \in \mathbb{Z}^+$

$\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n-4)(n+1)$

1.5.2 Proof by Contradiction

Proof by Contradiction

What is proof by contradiction?

- **Proof by contradiction** is a way of proving a **result is true** by showing that **the negation can not be true**
- It is done by:
  - Assuming the negation (opposite) of the result is true
  - Showing that this then leads to a contradiction

How do I determine the negation of a statement?

- The **negation** of a statement is the **opposite**
  - It is the statement that makes the original statement false
- To negate statements that mention "all", "every", "and" "both":
  - Replace these phrases with "there is at least one", "or" or "there exists" and include the opposite
- To negate statements that mention "there is at least one", "or" or "there exists":
  - Replace these phrases with "all", "every", "and" or "both" and include the opposite
- To negate a statement with "if A occurs then B occurs":
  - Replace with "A occurs and the negation of B occurs"
- Examples include:

Statement	**Negation**
a is rational	*a* is irrational
every even number bigger than 2 can be written as the sum of two primes	there exists an even number bigger than 2 which cannot be written as a sum of two primes
n is even and prime	*n* is not even or *n* is not prime
there is at least one odd perfect number	all perfect numbers are even
n is a multiple of 5 or a multiple of 3	*n* is not a multiple of 5 and *n* is not a multiple of 3
if n^2 is even then *n* is even	n^2 is even and *n* is odd

What are the steps for proof by contradiction?

- **STEP 1: Assume the negation** of the statement is **true**
  - You assume it is true but then try to prove your assumption is wrong

    For example: To prove that there is no smallest positive number you start by assuming there is a smallest positive number called *a*
- **STEP 2:** Find **two results which contradict** each other

- Use algebra to help with this
- Consider how a contradiction might arise
  - For example: ½*a* is positive and it is smaller than *a* which contradicts that *a* was the smallest positive number
- **STEP 3**: **Explain** why the **original statement is true**
  - In your explanation mention:
    - The **negation can't be true** as it led to a contradiction
    - Therefore the **original statement must be true**

What type of statements might I be asked to prove by contradiction?

- **Irrational numbers**
  - To show $\sqrt[n]{p}$ is irrational where *p* is a prime
    - Assume $\sqrt[n]{p} = \frac{a}{b}$ where *a* & *b* are integers with no common factors and $b \neq 0$
    - Use algebra to show that *p* is a factor of both *a* & *b*
  - To show that $\log_p(q)$ is irrational where *p* & *q* are different primes
    - Assume $\log_p(q) = \frac{a}{b}$ where *a* & *b* are integers with no common factors and $b \neq 0$
    - Use algebra to show $q^b = p^a$
  - To show that *a* or *b* must be irrational if their sum or product is irrational
    - Assume *a* & *b* are rational and write as fractions
    - Show that $a + b$ or ab is rational
- **Prime numbers**
  - To show a polynomial is never prime
    - Assume that it is prime
    - Show there is at least one factor that cannot equal 1
  - To show that there is an infinite number of prime numbers
    - Assume there are *n* primes $p_1, p_2, ..., p_n$
    - Show that $p = 1 + p_1 \times p_2 \times ... \times p_n$ is a prime that is bigger than the *n* primes
- **Odds and evens**
  - To show that *n* is even if n^2 is even
    - Assume n^2 is even and *n* is odd
    - Show that n^2 is odd
- **Maximum and minimum values**
  - To show that there is no maximum multiple of 3
    - Assume there is a maximum multiple of 3 called *a*
    - Multiply *a* by 3

Exam Tip

- A question won't always state that you should use proof by contradiction, you will need to recognise that it is the correct method to use
  - There will only be two options (e.g. a number is rational or irrational)
  - Contradiction is often used when no other proof seems reasonable

Worked Example

Prove the following statements by contradiction.

a)
For any integer n, if n^2 is a multiple of 3 then n is a multiple of 3.

Assume the negation is true for a contradiction.

Assume n^2 is a multiple of 3 and n is not a multiple of 3.

Every integer can be written as one of $3k-1, 3k, 3k+1$ for some $k \in \mathbb{Z}$

As n is not a multiple of 3 then $n = 3k+1$ or $n = 3k-1$ for some $k \in \mathbb{Z}$

If $n = 3k+1$: $n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ so not a multiple of 3

If $n = 3k-1$: $n^2 = (3k-1)^2 = 9k^2 - 6k + 1 = 3(3k^2 - 2k) + 1$ so not a multiple of 3

$\therefore n^2$ is not a multiple of 3

This contradicts the statement "n^2 is a multiple of 3".
Therefore the assumption is incorrect.
Therefore if n^2 is a multiple of 3 then n is a multiple of 3.

b)
$\sqrt{3}$ is an irrational number.

Assume the negation is true for a contradiction.

Assume $\sqrt{3}$ is rational so can be written $\sqrt{3} = \frac{a}{b}$ where a and b are integers with no common factors and $b \neq 0$.

Square both sides and rearrange

$3 = \frac{a^2}{b^2} \Rightarrow 3b^2 = a^2 \Rightarrow a^2$ is a multiple of 3 $\Rightarrow a$ is a multiple of 3

Let $a = 3k$ for some $k \in \mathbb{Z}$

$3b^2 = a^2 \Rightarrow 3b^2 = 9k^2 \Rightarrow b^2 = 3k^2 \Rightarrow b^2$ is a multiple of 3

$\therefore b$ and a are multiples of 3

This contradicts the statement "a and b have no common factors".
Therefore the assumption is incorrect.
Therefore $\sqrt{3}$ is irrational.

1.6 Binomial Theorem

1.6.1 Binomial Theorem

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Binomial Theorem

What is the Binomial Theorem?

- The **binomial theorem (**sometimes known as the binomial expansion) gives a method for expanding a **two-term** expression in a bracket raised to a power
  - A **binomial expression** is in fact any two terms inside the bracket, however in IB the expression will usually be linear
- To expand a bracket with a two-term expression in:
  - First choose the most appropriate parts of the expression to assign to *a* and *b*
  - Then use the formula for the binomial theorem:

$$(a+b)^n = a^n + {}^nC_1\, a^{n-1} b + \dots + {}^nC_r\, a^{n-r} b^r + \dots + b^n$$

  - where ${}^nC_r = \frac{n!}{r!(n-r)!}$

    See below for more information on ${}^nC_r$

    You may also see ${}^nC_r$ written as $\binom{n}{r}$ or ${}_nC_r$
- You will usually be asked to find the first three or four terms of an expansion
- Look out for whether you should give your answer in **ascending** or **descending** powers of *x*
  - For **ascending** powers start with the constant term, a^n
  - For **descending** powers start with the term with *x* in

    You may wish to swap *a* and *b* over so that you can follow the general formula given in the formula book
- If you are not writing the full expansion you can either
  - show that the sequence continues by putting an ellipsis (…) after your final term
  - or show that the terms you have found are an approximation of the full sequence by using the sign for approximately equals to (≈)

How do I find the coefficient of a single term?

- Most of the time you will be asked to find the coefficient of a term, rather than carry out the whole expansion
- Use the formula for the general term

$${}^nC_r\, a^{n-r} b^r$$

- The question will give you the power of *x* of the term you are looking for
  - Use this to choose which value of *r* you will need to use in the formula
  - This will depend on where the *x* is in the bracket
  - The laws of indices can help you decide which value of *r* to use:

    For $(a + bx)^n$ to find the coefficient of x^r use $a^{n-r}(bx)^r$

    For $(a + bx^2)^n$ to find the coefficient of x^r use $a^{\frac{n-r}{2}}(bx^2)^{\frac{r}{2}}$

For $\left(a + \frac{b}{x}\right)^n$ look at how the powers will cancel out to decide which value of r to use

So for $\left(3x + \frac{2}{x}\right)^8$ to find the coefficient of x^2 use the term with $r = 3$ and to find the constant term use the term with $r = 4$

There are a lot of variations of this so it is usually easier to see this by inspection of the exponents

- You may also be given the coefficient of a particular term and asked to find an unknown in the brackets
  - Use the laws of indices to choose the correct term and then use the binomial theorem formula to form and solve and equation

Exam Tip

- Binomial expansion questions can get messy, use separate lines to keep your working clear and always put terms in brackets

Worked Example

Find the first three terms, in ascending powers of x, in the expansion of $(3 - 2x)^5$.

$a = 3 \quad b = -2x \quad n = 5$

Substitute values into the formula for $(a+b)^n$

$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \ldots + {}^nC_r a^{n-r} b^r + \ldots + b^n$

Question asks for ascending powers of x, so start with the constant term, a^n.

$(3 - 2x)^5 = 3^5 + {}^5C_1 (3)^{5-1}(-2x) + {}^5C_2 (3)^{5-2}(-2x)^2 + \ldots$

Watch out for the negative

$\approx 243 + 5 \times 81 \times -2x + 10 \times 27 \times 4x^2$

$\approx 243 - 810x + 1080x^2$

$(3 - 2x)^5 \approx 243 - 810x + 1080x^2$

The Binomial Coefficient nCr

What is ${}^{n}C_{r}$?

- If we want to find the number of ways to **choose** *r* items out of *n* different objects we can use the formula for ${}^{n}C_{r}$
  - The formula for *r* **combinations** of *n* items is ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$
  - This formula is given in the formula booklet along with the formula for the binomial theorem
  - The function ${}^{n}C_{r}$ can be written $\begin{pmatrix} n \\ r \end{pmatrix}$ or ${}_{n}C_{r}$ and is often read as '*n choose r*'

    Make sure you can find and use the button on your GDC

How does ${}^{n}C_{r}$ relate to the binomial theorem?

- The formula ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$ is also known as a **binomial** coefficient
- For a binomial expansion $(a + b)^n$ the coefficients of each term will be ${}^{n}C_{0}$, ${}^{n}C_{1}$ and so on up to ${}^{n}C_{n}$
  - The coefficient of the r^{th} term will be ${}^{n}C_{r}$
- ${}^{n}C_{n} = {}^{n}C_{0} = 1$
- The binomial coefficients are symmetrical, so ${}^{n}C_{r} = {}^{n}C_{n-r}$
  - This can be seen by considering the formula for ${}^{n}C_{r}$
  - ${}^{n}C_{n-r} = \dfrac{n!}{(n-r)!(n-(n-r))!} = \dfrac{n!}{r!(n-r)!} = nC_{r}$

Exam Tip

- You will most likely need to use the formula for nCr at some point in your exam
  - Practice using it and don't always rely on your GDC
  - Make sure you can find it easily in the formula booklet

Worked Example

Without using a calculator, find the coefficient of the term in x^3 in the expansion of $(1 + x)^9$.

$n = 9, \quad a = 1, \quad b = x$

Substitute values into the formula for the binomial theorem:

$(a+b)^n = a^n + \dots + {}^nC_r a^{n-r} b^r + \dots + b^n$

where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

$(1+x)^9 = \sum_{r=0}^{9} {}^9C_r (1)^{9-r} (x)^r$ ← Coefficient of x^3 occurs when $r = 3$.

$r = 3$ gives ${}^9C_3 \times (1)^{9-3} (x)^3$

Non-calculator, so work out ${}^nC_r$ separately:

$${}^9C_3 = \frac{9!}{3!(9-3)!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{(3 \times 2)(6 \times 5 \times 4 \times 3 \times 2)}$$

$$= \frac{9 \times 8 \times 7}{6} = 84$$

so the term when $r = 3$ is $84 \times (1)^6 \times x^3$

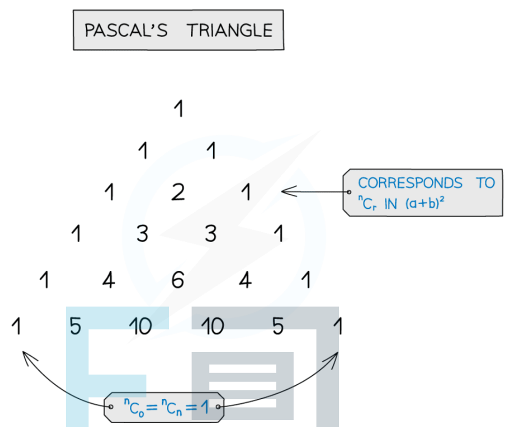
$= 84x^3$

Coefficient of x^3 = 84

Pascal's Triangle

What is Pascal's Triangle?

- Pascal's triangle is a way of arranging the binomial coefficients and neatly shows how they are formed
    - Each term is formed by adding the two terms above it
    - The first row has just the number 1
    - Each row begins and ends with a number 1
    - From the third row the terms in between the 1s are the sum of the two terms above it



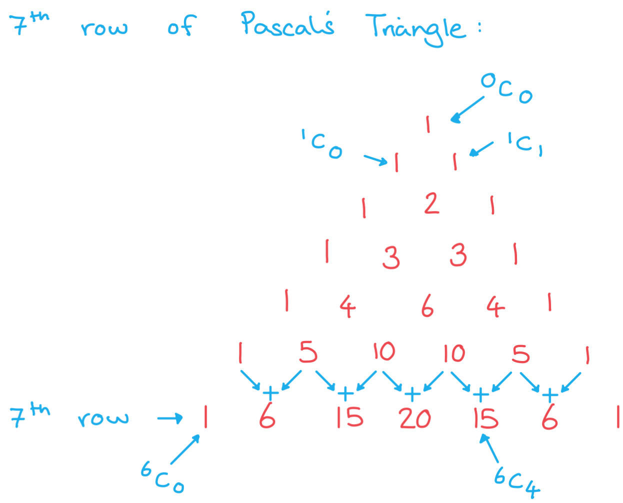
How does Pascal's Triangle relate to the binomial theorem?

- Pascal's triangle is an alternative way of finding the binomial coefficients, ^nC_r
    - It can be useful for finding for smaller values of n without a calculator
    - However for larger values of n it is slow and prone to arithmetic errors
- Taking the first row as zero, $(^0C_0 = 1)$, each row corresponds to the n^{th} row and the term within that row corresponds to the r^{th} term

> **Exam Tip**
> - In the non-calculator exam Pascal's triangle can be helpful if you need to get the coefficients of an expansion quickly, provided the value of *n* is not too big

Worked Example

Write out the 7th row of Pascal's triangle and use it to find the value of ${}^{6}C_{4}$.



7th row of Pascal's Triangle: 1, 6, 15, 20, 15, 6, 1

${}^{6}C_{4} = 15$

EXAM PAPERS PRACTICE

1.6.2 Extension of The Binomial Theorem

Binomial Theorem: Fractional & Negative Indices

How do I use the binomial theorem for fractional and negative indices?

- The formula given in the formula booklet for the binomial theorem applies to positive integers only
  - $(a+b)^n = a^n + {}^nC_1 a^{n-1}b + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n$
  - where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
- For **negative** or **fractional powers** the expression in the brackets must first be changed such that the value for *a* is 1
  - $(a+b)^n = a^n\left(1+\dfrac{b}{a}\right)^n$
  - $(a+b)^n = a^n\left(1+n\left(\dfrac{b}{a}\right)+\dfrac{n(n-1)}{2!}\left(\dfrac{b}{a}\right)^2+\ldots\right),\ n\in\mathbb{Q}$
  - This **is given in the formula booklet**
- If *a* = 1 and *b* = *x* the binomial theorem is simplified to
  - $(1+x)^n = 1+nx+\dfrac{n(n-1)}{2!}x^2+\dfrac{n(n-1)(n-2)}{3!}x^3+\ldots,\ n\in\mathbb{Q},\ \ |x|<1$
  - This is **not** in the formula booklet, you must remember it or be able to derive it from the formula given
- You need to be able to recognise a negative or fractional power
  - The expression may be on the denominator of a fraction

    $\dfrac{1}{(a+b)^n} = (a+b)^{-n}$
  - Or written as a surd

    $\sqrt[n]{(a+b)^m} = (a+b)^{\frac{m}{n}}$
- For $n \notin \mathbb{N}$ the expansion is infinitely long
  - You will usually be asked to find the first three terms
- The expansion is only valid for $|x| < 1$
  - This means $-1 < x < 1$
  - This is known as the **interval of convergence**
  - For an expansion $(a + bx)^n$ the interval of convergence would be $-\dfrac{a}{b} < x < \dfrac{a}{b}$

How do we use the binomial theorem to estimate a value?

- The binomial expansion can be used to form an approximation for a value raised to a power
- Since $|x| < 1$ higher powers of *x* will be very small
  - Usually only the first three or four terms are needed to form an approximation
  - The more terms used the closer the approximation is to the true value
- The following steps may help you use the binomial expansion to approximate a value

- STEP 1: Compare the value you are approximating to the expression being expanded

  e.g. $(1 - x)^{\frac{1}{2}} = 0.96^{\frac{1}{2}}$
- STEP 2: Find the value of x by solving the appropriate equation

  e.g. $1 - x = 0.96$

  $x = 0.04$
- STEP 3: Substitute this value of x into the expansion to find the approximation

  e.g. $1 - \frac{1}{2}(0.04) - \frac{1}{8}(0.04)^2 = 0.9798$

- Check that the value of x is within the **interval of convergence** for the expression
  - If x is outside the interval of convergence then the approximation may not be valid

Exam Tip

- Students often struggle with the extension of the binomial theorem questions in the exam, however the formula is given in the formula booklet
  - Make sure you can locate the formula easily and practice substituting values in
  - Mistakes are often made with negative numbers or by forgetting to use brackets properly
    - Writing one term per line can help with both of these

Worked Example

Consider the binomial expansion of $\dfrac{1}{\sqrt{9-3x}}$.

a)
Write down the first three terms.

Rewrite $\dfrac{1}{\sqrt{9-3x}}$ in the form $k\left(1+\dfrac{x}{a}\right)^n$

$$\frac{1}{\sqrt{9-3x}} = (9-3x)^{-\frac{1}{2}} = 9^{-\frac{1}{2}}\left(1-\frac{3x}{9}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{3}\left(1-\frac{x}{3}\right)^{-\frac{1}{2}}$$

Substitute values into the formula for $(1+x)^n$

$$\frac{1}{3}\left(1-\frac{x}{3}\right)^{-\frac{1}{2}} = \frac{1}{3}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{3}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{3}\right)^2+\ldots\right]$$

$$= \frac{1}{3}\left[1+\frac{x}{6}+\frac{x^2}{24}+\ldots\right]$$

$$= \frac{1}{3}+\frac{x}{18}+\frac{x^2}{72}+\ldots$$

$$\frac{1}{\sqrt{9-3x}} \approx \frac{1}{3}+\frac{x}{18}+\frac{x^2}{72}$$

b)
State the interval of convergence for the complete expansion.

$n \geq 0$ and $n \notin \mathbb{N}$, so the series converges when $|x| < 1$

$\frac{1}{3}\left(1-\frac{x}{3}\right)^{-\frac{1}{2}}$ ← x-term

$\left|-\frac{x}{3}\right| < 1$

$|x| < 3 \Rightarrow -3 < x < 3$

Converges for $-3 < x < 3$

c)

Use the terms found in part (a) to estimate $\dfrac{1}{\sqrt{10}}$. Give your answer as a fraction.

Find the value of x for which $\frac{1}{\sqrt{9-3x}} = \frac{1}{\sqrt{10}}$

$9-3x = 10$

$x = -\frac{1}{3}$ ← $-3 < x < 3$ so can use the expansion

Substitute $x = -\frac{1}{3}$ into the expansion for $\frac{1}{\sqrt{9-3x}}$

$$\frac{1}{\sqrt{9-3(-\frac{1}{3})}} \approx \frac{1}{3} + \frac{(-\frac{1}{3})}{18} + \frac{(-\frac{1}{3})^2}{72}$$

$$\frac{1}{\sqrt{10}} \approx \frac{205}{648}$$

1.7 Permutations & Combinations

1.7.1 Counting Principles

Counting Principles

What is meant by counting principles?

- The fundamental counting principle states that if there are m ways to do one thing and n ways to do another there are $m \times n$ ways to do **both** things
- Applying counting principles allows us to…
  - … analyse patterns and make generalisations about real work situations
  - … find the number of **permutations** of n items
  - … find the number of ways of choosing an item from a list of n items
  - … find the number of ways of choosing r items from n items
  - … find the number of ways of permutating r items from n items
- The topic of counting principles is a particularly interesting part of mathematics that can lead to the development of working with very large numbers
- It is always vital to consider whether objects taken from each list can be repeated or not
  - For example a four digit PIN from ten numbers where each number can be used repeatedly would be $10 \times 10 \times 10 \times 10$
    - There are 10 options for the first and ten options for the second number and so on
  - If the numbers could only be used once then the number of options for each digit would **reduce** with each digit
    - There are 10 options for the first, nine options for the second, eight for the third and so on
    - This concept will be explored further in the **permutations** revision note

How do I choose an item from a list of m items AND another item from a list of n items?

- If a question requires you to choose an item from one list **AND** an item from another list you should **multiply** the number of options in each list
  - In general if you see the word 'AND' you will most likely need to 'MULTIPLY'
- For example if you are choosing a pen and a pencil from 4 pens and 5 pencils:
  - You can choose 1 item from 4 pens AND 1 item from 5 pencils
  - You will have 4×5 different options to choose from

How do I choose an item from a list of m items OR another item from a list of n items?

- If a question requires you to choose an item from one list **OR** an item from another list you should **add** the number of options in each list
  - In general if you see the word 'OR' you will most likely need to 'ADD'
- For example if you are choosing a pen or a pencil from 4 pens and 5 pencils:
  - You can choose 1 item from 4 pens OR 1 item from 5 pencils
  - You will have $4 + 5$ different options to choose from

Exam Tip

- Counting principles and factorials are tightly interlinked with permutations and combinations
- Make sure you fully understand the concepts in this revision note as they will be fundamental to answering perms and combs exam questions

Worked Example

Harry is going to a formal event and is choosing what accessories to add to his outfit. He has seven different ties, four different bow ties and five different pairs of cufflinks. How many different ways can Harry get ready if he chooses:

a)
Either a tie, a bow tie or a pair of cufflinks?

Harry has 7 + 4 + 5 different items to choose from
He wants a tie OR a bow tie OR a pair of cufflinks
OR means ADD $7 + 4 + 5 = 16$

16 different ways

b)
A pair of cufflinks and either a tie or a bow tie?

He wants a tie AND a pair of cufflinks
OR a bow tie AND a pair of cufflinks
AND means MULTIPLY
A tie AND cufflinks = $7 \times 5 = 35$ ways
A bowtie AND cufflinks = $4 \times 5 = 20$ ways
OR means ADD $35 + 20 = 55$ ways

55 different ways

1.7.2 Permutations & Combinations

Permutations

What are Permutations?

- A **permutation** is the number of possible arrangements of a set of objects when the **order** of the arrangements matters
- A permutation can either be finding the number of ways to arrange *n* items or finding the number of ways to arrange *r* out of *n* items

How many ways can *n* different objects be arranged?

- When considering how many ways you can arrange a number of **different** objects in a row consider how many of the objects can go in the first position, how many can go in the second and so on
- For $n = 2$ there are two options for the first position and then there will only be one option to go in the second position so:
  - The first object has **two** places it could go and the second object has **one** place
  - By the fundamental counting principle **both** objects have 2×1 places to go
  - For example to arrange the letters A and B we have
    - AB and BA
- For $n = 3$ there are three options for the first position and then there will be two options for the second position and one for the third position so
  - The first object has **three** places it could go, the second object has **two** places and the third object has **one** place
  - By the fundamental counting principle **the three** objects have $3 \times 2 \times 1$ places to go
  - For example to arrange the letters A, B and C we have
    - ABC, ACB, BAC, BCA, CAB and CBA
- For *n* objects there are *n* options for the first position, *n* - 1 options for the second position and so on until there is only one object left to go in final position
- The number of **permutations** of *n* different objects is *n* factorial (*n*!)
  - Where $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$
  - For 5 **different** items there are $5! = 5 \times 4 \times 3 \times 2 = 120$ permutations
  - For 6 **different** items there are $6! = 6 \times 5 \times 4 \times 3 \times 2 = 720$ permutations
  - It is easy to see how quickly the number of possible permutations of different items can increase
  - For 10 different items there are 10! = 3 628 800 possible permutations

What are factorials?

- Factorials are a type of mathematical operation (just like +, -, ×, ÷)
- The symbol for factorial is !
  - So to take a factorial of any non-negative integer, *n*, it will be written *n*! and pronounced '*n factorial*'
- The factorial function for any positive integer, *n*, is $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$
  - For example, 5 factorial is $5! = 5 \times 4 \times 3 \times 2 \times 1$
- The factorial of a negative number is not defined
  - You cannot arrange a negative number of items

- 0! = 1
  - There are no positive integers less than zero, so zero items can only be arranged once
- Your GDC will have a mode for calculating factorials, make sure you can put yours into the correct mode
- Most normal calculators cannot handle numbers greater than about 70!, experiment with yours to see the greatest value of *x* such that your calculator can handle *x*!

What are the key properties of using factorials?

- Some important relationships to be aware of are:
  - $n! = n \times (n-1)!$

    Therefore $\frac{n!}{(n-1)!} = n$
  - $n! = n \times (n-1) \times (n-2)!$

    Therefore $\frac{n!}{(n-2)!} = n \times (n-1)$
- Expressions with factorials in can be simplified by considering which values cancel out in the fraction
  - Dividing a large factorial by a smaller one allows many values to cancel out

$$\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = 8 \times 7 \times 6$$

How do we find *r* permutations of *n* items?

- If we only want to find the number of ways to arrange a few out of *n* different objects, we should consider how many of the objects can go in the first position, how many can go in the second and so on
- If we wanted to arrange 3 out of 5 different objects, then we would have 3 positions to place the objects in, but we would have 5 options for the first position, 4 for the second and 3 for the third
  - This would be 5 × 4 × 3 ways of permutating 3 out of 5 different objects
  - This is equivalent to $\frac{5!}{2!} = \frac{5!}{(5-3)!}$
- If we wanted to arrange 4 out of 10 different objects, then we would have 4 positions to place the objects in, but we would have 10 options for the first position, 9 for the second, 8 for the third and 7 for the fourth
  - This would be 10 × 9 × 8 × 7 ways of permutating 4 out of 10 different objects
  - This is equivalent to $\frac{10!}{6!} = \frac{10!}{(10-4)!}$
- If we wanted to arrange *r* out of *n* different objects, then we would have *r* positions to place the objects in, but we would have *n* options for the first position, $(n-1)$ for the second, $(n-2)$ for the third and so on until we reach $(n-(r-1))$
  - This would be $n \times (n-1) \times \ldots \times (n-r+1)$ ways of permutating r out of n different objects
  - This is equivalent to $\frac{n!}{(n-r)!}$
- The function $\frac{n!}{(n-r)!}$ can be written as ${}^{n}\mathrm{P}_{r}$

- Make sure you can find and use this button on your calculator
- The same function works if we have *n* spaces into which we want to arrange *r* objects, consider
  - for example arranging five people into a row of ten empty chairs

Permutations when two or more items must be together

- If two or more items must stay together within an arrangement, it is easiest to think of these items as 'stuck' together
- These items will become one within the arrangement
- Arrange this 'one' item with the others as normal
- Arrange the items within this 'one' item separately
- Multiply these two arrangements together

Permutations when two or more items cannot be all together

- If **two** items must be **separated** …
  - consider the number of ways these two items would be together
  - subtract this from the total number of arrangements without restrictions
- If **more than two** items must be separated…
  - consider whether all of them must be **completely separate** (none can be next to each other) or whether they **cannot all be together** (but two could still be next to each other)
  - If they **cannot all be together** then we can treat it the same way as separating two items and subtract the number of ways they would all be together from the total number of permutations of the items, the final answer will include all permutations where two items are still together
  - If the items must all be **completely separate** then
    - lay out the rest of the items in a line with a space in between each of them where one of the items which cannot be together could go
    - remember that this could also include the space before the first and after the last item
    - You would then be able to fit the items which cannot be together into any of these spaces, using the *r* permutations of *n* items rule $\left({}^{n}\mathrm{P}_{r}\right)$
    - You do not need to fill every space

Permutations when two or more items must be in specific places

- Most commonly this would be arranging a word where specific letters would go in the first and last place
- Or arranging objects where specific items have to be at the ends/in the middle
  - Imagine these specific items are stuck in place, then you can find the number of ways to arrange the rest of the items around these 'stuck' items
- Sometimes the items must be grouped
  - for example all vowels must be before the consonants
  - Or all the red objects must be on one side and the blue objects must be on the other
  - Find the number of permutations within each group separately and multiply them together
  - Be careful to check whether the groups could be in either place

e.g. the vowels on one side and consonants on the other
or if they must be in specific places (the vowels **before** the consonants)

- If the groups could be in either place than your answer would be multiplied by two
- If there were *n* groups that could be in any order then you're answer would be multiplied by *n*!

Exam Tip

- The wording is very important in permutations questions, just one word can change how you answer the question
- Look out for specific details such as whether three items must all be separated or just cannot be all together (there is a difference)
- Pay attention to whether items must be in alternating order (e.g. red and blue items must alternate, either RBRB… or BRBR…) or whether a particular item must come first (red **then** blue and so on)
- If items should be at the ends, look out for whether they can be at either end or whether one must be at the beginning and the other at the end

Worked Example

Find the number of ways nine different tasks can be carried out given that two particular tasks must not be carried out consecutively.

Start by considering the tasks that <u>can</u> be carried out consecutively:

$T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6 \quad T_7$

There are 7! ways of carrying out the other seven tasks.

Then consider the tasks with restrictions:

$\times\, T_1 \,\times\, T_2 \,\times\, T_3 \,\times\, T_4 \,\times\, T_5 \,\times\, T_6 \,\times\, T_7 \,\times$ ← positions where T_8 or T_9 could go

There are 8 positions in which the two tasks could go but only 2 tasks to fill the spaces $= 8 \times 7$ options

$= 8P2$

(8 options for T_8 and 7 options for T_9)

Total $= 7! \times 8 \times 7 = 282240$ ways

Alternative method:

Put the two tasks together and then subtract from the total.

$T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6 \quad T_7 \quad (T_8 \quad T_9)$ ← The two tasks are 'stuck' together.

$9! - (8! \times 2!)$

- $9!$: number of ways without restrictions
- $8!$: 7 'free' tasks plus the two tasks 'stuck' together.
- $2!$: There are 2! ways of carrying out the two 'stuck' tasks

Combinations

What is the difference between permutations and combinations?

- A **combination** is the number of possible arrangements of a set of objects when the **order** of the arrangements **does not matter**
  - On the other hand a **permutation** is when the order of arrangement **does matter**
- A combination will be finding the number of ways to **choose** *r* out of *n* items
  - The order in which the *r* items are chosen is not important
  - For example if we are choosing two letters from the word CAB, AB and BA would be considered the same combination but different permutations

How do we find *r* combinations of *n* items?

- If we want to find the number of ways to **choose** 2 out of 3 different objects, but we don't mind the order in which they are chosen, then we could find the number of **permutations of 2 items from 3** and then divide by the number of ways of arranging each combination
  - For example if we want to choose 2 letters from *A, B* and *C*

    There are 6 permutations of 2 letters:

    AB, BA, AC, CA, BC, CB

    For each combination of 2 letters there are 2 (2 × 1) ways of arranging them (for example, *AB* and *BA*)

    So divide the total number of permutations (6) by the number of ways of arranging each combination (2) to get 3 combinations

- If we want to find the number of ways to **choose** 3 out of 5 different objects, but we don't mind the order in which they are chosen, then we could find the number of **permutations of 3 items from 5** and then divide by the number of ways of arranging each combination
  - For example if we want to choose 3 letters from *A, B, C, D* and *E*

    There are 60 permutations of 3 letters:

    ABC, ACB, BAC, BCA, CAB, CBA, ABD, ADB, etc

    For each combination of 3 letters there are 6 (3 × 2 ×1) ways of arranging them (for example, *ABC, ACB, BAC, BCA, CAB* and *CBA*)

    So divide the total number of permutations (60) by the number of ways of arranging each combination (3! = 6) to get 10 combinations

- If we want to find the number of ways to **choose** *r* items out of *n* different objects, but we don't mind the order in which they are chosen, then we could find the number of **permutations of *r* items from *n*** and then divide by the number of ways of arranging each combination
- Recall that the formula for *r* permutations of *n* items is
  - $^{n}P_{r} = \dfrac{n!}{(n-r)!}$
- This would include *r*! ways of repeating each combination
- The formula for *r* **combinations** of *n* items is
  - $\dfrac{^{n}P_{r}}{r!} = \dfrac{n!}{(n-r)!\,r!}$

- The function $\frac{n!}{(n-r)!\,r!}$ can be written as ${}^{n}C_{r}$ or $\binom{n}{r}$ and is often read as '*n choose r*'
  - Make sure you can find and use this button on your calculator
- The formulae for permutations and combinations satisfy the following relationship:
  - ${}^{n}C_{r} = \frac{{}^{n}P_{r}}{r!}$
- The formula ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$ can be found in the formula booklet

What do I need to know about combinations?

- The formula ${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$ is also known as a **binomial** coefficient
- ${}^{n}C_{n} = {}^{n}C_{0} = 1$
  - It is easy to see that there is only one way of arranging *n* objects out of *n* and also there can only be one way of arranging 0 objects out of *n*
  - By considering the formula for this, it reinforces the fact that 0! Must equal 1
- The binomial coefficients are symmetrical, so ${}^{n}C_{r} = {}^{n}C_{n-r}$
  - This can be seen by considering the formula for ${}^{n}C_{r}$
  - ${}^{n}C_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!} = {}^{n}C_{r}$

How do I know when to multiply or add?

- Many questions will ask you to find combinations of a group of different items from a bigger group of a specified number of those different items
  - For example, find the number of ways five questions could be chosen from a bank of twenty different pure and ten different statistics questions
  - The hint in this example is the word 'chosen', this tells you that the order in which the questions are chosen doesn't matter
- Sometimes questions will have restrictions,
  - For example there should be three pure and two statistics chosen from the bank of questions,
  - Or there must be at least two pure questions within the group
- If unsure about whether to add or multiply your options, ask yourself if A **and** B are both needed, or if A **or** B is needed
  - Always **multiply** if the answer is **and,** and **add** if the answer is **or**
  - For example if we needed exactly three pure **and** two statistics questions we would find the amount of each and multiply them
  - If we could have either five statistics **or** five pure questions we would find them separately and add the answers

Exam Tip

- It is really important that you can tell whether a question is about permutations or combinations
  - Look out for key words such as **arrange** (for permutations) or **choose** or **select** (for combinations)
- Don't be confused if a question asks for the number of **ways**, this could be for either a permutations or a combinations question
  - Look out for other clues

Worked Example

Oscar has to choose four books from a reading list to take home over the summer. There are four fantasy books, five historical fiction books and two classics available for him to choose from. In how many ways can Oscar choose four books if he decides to have:

i)

Two fantasy books and two historical fictions?

Choosing two fantasy from four: $^4C_2$

Choosing two historical fiction from five: $^5C_2$

Total $= {^4C_2} \times {^5C_2} = 6 \times 10$

60 options

ii)

At least one of each type of book?

To choose two of one and one of each of the others:

Let F represent a fantasy book, H represent historical fiction and C represent a classic:

F	H	C
4	5	2

(these are the numbers of options Oscar can choose from)

these are the choices he can make:

F	H	C		
1	1	2	$^4C_1 \times {^5C_1} \times {^2C_2}$	$= 4 \times 5 \times 1$
1	2	1	$^4C_1 \times {^5C_2} \times {^2C_1}$	$= 4 \times 10 \times 2$
2	1	1	$^4C_2 \times {^5C_1} \times {^2C_1}$	$= 6 \times 5 \times 2$

'and' so multiply.

Oscar can choose from one of these options so we add them:

$20 + 80 + 60$

'or' so add

160 options

iii)

At least two fantasy books?

Oscar can choose two, three or four fantasy books. It does not matter whether the other books are classics or historical fiction.

This is just 'Not Fantasy' (5H + 2C)

F	NF	
2	2	$4C2 \times 7C2 = 6 \times 21 = 126$
3	1	$4C3 \times 7C1 = 4 \times 7 = 28$
4	0	$4C4 \times 7C0 = 1 \times 1 = 1$

Use a system to make sure you don't forget any. Check that all the options add up to 4.

Total number of options = 126 + 28 + 1

155 Options

1.8 Complex Numbers

1.8.1 Intro to Complex Numbers

Cartesian Form

What is an imaginary number?

- Up until now, when we have encountered an equation such as $x^2 = -1$ we would have stated that there are "no real solutions"
  - The solutions are $x = \pm\sqrt{-1}$ which are not real numbers
- To solve this issue, mathematicians have defined one of the square roots of negative one as i; an imaginary number
  - $\sqrt{-1} = \mathrm{i}$
  - $\mathrm{i}^2 = -1$
- The square roots of other negative numbers can be found by rewriting them as a multiple of $\sqrt{-1}$
  - using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

What is a complex number?

- Complex numbers have both a real part and an imaginary part
  - For example: $3 + 4\mathrm{i}$
  - The real part is 3 and the imaginary part is 4i
- Complex numbers are often denoted by z
  - We refer to the real and imaginary parts respectively using $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$
- Two complex numbers are equal if, and only if, both the real and imaginary parts are identical.
  - For example, $3 + 2\mathrm{i}$ and $3 + 3\mathrm{i}$ are **not equal**
- The set of all complex numbers is given the symbol $\mathbb{C}$

What is Cartesian Form?

- There are a number of different forms that complex numbers can be written in
- The form $z = a + bi$ is known as **Cartesian Form**
  - $a, b \in \mathbb{R}$
  - This is the first form given in the formula booklet
- In general, for $z = a + bi$
  - $\mathrm{Re}(z) = a$
  - $\mathrm{Im}(z) = b$
- A complex number can be easily represented geometrically when it is in Cartesian Form
- Your GDC may call this **rectangular form**
  - When your GDC is set in rectangular settings it will give answers in Cartesian Form
  - If your GDC is **not** set in a complex mode it will not give any output in complex number form
  - Make sure you can find the settings for using complex numbers in Cartesian Form and practice inputting problems
- Cartesian form is the easiest form for adding and subtracting complex numbers

Exam Tip

- Remember that complex numbers have both a real part and an imaginary part
  - 1 is purely real (its imaginary part is zero)
  - i is purely imaginary (its real part is zero)
  - 1 + i is a complex number (both the real and imaginary parts are equal to 1)

Worked Example

a)
Solve the equation $x^2 = -9$

$x^2 = -9$

$x = \pm\sqrt{-9}$

Using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ $\quad x = \pm\sqrt{9}\sqrt{-1}$

$x = \pm 3i$

b)
Solve the equation $(x+7)^2 = -16$, giving your answers in Cartesian form.

$(x+7)^2 = -16$

$x + 7 = \pm\sqrt{-16}$

$x + 7 = \pm\sqrt{16}\sqrt{-1}$

$x + 7 = \pm 4i$ ← Using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

Rearrange answer into Cartesian form:

$x = -7 \pm 4i$

Complex Addition, Subtraction & Multiplication

How do I add and subtract complex numbers in Cartesian Form?

- Adding and subtracting complex numbers should be done when they are in **Cartesian form**
- When adding and subtracting complex numbers, simplify the real and imaginary parts separately
  - Just like you would when collecting like terms in algebra and surds, or dealing with different components in vectors
  - $(a + bi) + (c + di) = (a + c) + (b + d)i$
  - $(a + bi) - (c + di) = (a - c) + (b - d)i$

How do I multiply complex numbers in Cartesian Form?

- Complex numbers can be multiplied by a constant in the same way as algebraic expressions:
  - $k(a + bi) = ka + kbi$
- Multiplying two complex numbers in Cartesian form is done in the same way as multiplying two linear expressions:
  - $(a + bi)(c + di) = ac + (ad + bc)i + bdi^2 = ac + (ad + bc)i - bd$
  - This is a complex number with real part $ac - bd$ and imaginary part $ad + bc$
  - The most important thing when multiplying complex numbers is that $i^2 = -1$
- Your GDC will be able to multiply complex numbers in Cartesian form
  - Practise doing this and use it to check your answers
- It is easy to see that multiplying more than two complex numbers together in Cartesian form becomes a lengthy process prone to errors
  - It is easier to multiply complex numbers when they are in different forms and usually it makes sense to convert them from Cartesian form to either Polar form or Euler's form first
- Sometimes when a question describes multiple complex numbers, the notation $z_1, z_2, \ldots$ is used to represent each complex number

How do I deal with higher powers of i?

- Because $i^2 = -1$ this can lead to some interesting results for higher powers of i
  - $\mathbf{i}^3 = \mathbf{i}^2 \times \mathbf{i} = -\mathbf{i}$
  - $\mathbf{i}^4 = (\mathbf{i}^2)^2 = (-1)^2 = 1$
  - $\mathbf{i}^5 = (\mathbf{i}^2)^2 \times \mathbf{i} = \mathbf{i}$
  - $\mathbf{i}^6 = (\mathbf{i}^2)^3 = (-1)^3 = -1$
- We can use this same approach of using i^2 to deal with much higher powers
  - $\mathbf{i}^{23} = (\mathbf{i}^2)^{11} \times \mathbf{i} = (-1)^{11} \times \mathbf{i} = -\mathbf{i}$
  - Just remember that -1 raised to an even power is 1 and raised to an odd power is -1

Exam Tip

- When revising for your exams, practice using your GDC to check any calculations you do with complex numbers by hand
  - This will speed up using your GDC in rectangular form whilst also giving you lots of practice of carrying out calculations by hand

Worked Example

a)
Simplify the expression $2(8-6\text{i})-5(3+4\text{i})$.

Expand the brackets

$2(8-6i)-5(3+4i) = 16-12i-15-20i$

Collect the real and imaginary parts

$16-15-12i-20i$

Simplify

$1-32i$

b)
Given two complex numbers $z_1 = 3+4\text{i}$ and $z_2 = 6+7\text{i}$, find $z_1 \times z_2$.

Expand the brackets

$(3+4i)(6+7i) = 18+21i+24i+28i^2$

$= 18+21i+24i+(28)(-1)$ ← Using $i^2=-1$

Collect the real and imaginary parts

$18+21i+24i-28 = 18-28+(21+24)i$

Simplify

$-10+45i$

Complex Conjugation & Division

When **dividing** complex numbers, the **complex conjugate** is used to change the denominator to a real number.

What is a complex conjugate?

- For a given complex number $z = a + b\text{i}$, **the complex conjugate of z is denoted as** z^*, where $z^* = a - b\text{i}$
- If $z = a - b\text{i}$ then $z^* = a + b\text{i}$
- You will find that:
  - $z + z^*$ is always real because $(a + b\text{i}) + (a - b\text{i}) = 2a$
    For example: $(6 + 5\text{i}) + (6 - 5\text{i}) = 6 + 6 + 5\text{i} - 5\text{i} = 12$
  - $z - z^*$ is always imaginary because $(a + b\text{i}) - (a - b\text{i}) = 2b\text{i}$
    For example: $(6 + 5\text{i}) - (6 - 5\text{i}) = 6 - 6 + 5\text{i} - (-5\text{i}) = 10\text{i}$
  - $z \times z^*$ is always real because $(a + b\text{i})(a - b\text{i}) = a^2 + ab\text{i} - ab\text{i} - b^2\text{i}^2 = a^2 + b^2$ (as $\text{i}^2 = -1$)
    For example: $(6 + 5\text{i})(6 - 5\text{i}) = 36 + 30\text{i} - 30\text{i} - 25\text{i}^2 = 36 - 25(-1) = 61$

How do I divide complex numbers?

- To divide two complex numbers:
  - STEP 1: Express the calculation in the form of a fraction
  - STEP 2: Multiply **the top and bottom by the conjugate of the denominator**:
    $$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$$
    This ensures we are multiplying by 1; so not affecting the overall value
  - STEP 3: Multiply out and simplify your answer
    This should have a real number as the denominator
  - STEP 4: Write your answer in Cartesian form as two terms, simplifying each term if needed
    OR convert into the required form if needed
- Your GDC will be able to divide two complex numbers in Cartesian form
  - Practise doing this and use it to check your answers if you can

Exam Tip

- We can speed up the process for finding zz^* by using the basic pattern of $(x + a)(x - a) = x^2 - a^2$
- We can apply this to complex numbers: $(a + b\text{i})(a - b\text{i}) = a^2 - b^2\text{i}^2 = a^2 + b^2$ (using the fact that $\text{i}^2 = -1$)
  - So $3 + 4\text{i}$ multiplied by its conjugate would be $3^2 + 4^2 = 25$

Worked Example

Find the value of $(1+7i) \div (3-i)$.

Rewrite as a fraction: $\frac{1+7i}{3-i}$ complex conjugate of $3-i$ is $3+i$

Multiply top and bottom of the fraction by the complex conjugate of the denominator.

$$\frac{1+7i}{3-i} \times \frac{3+i}{3+i} = \frac{(1+7i)(3+i)}{(3-i)(3+i)}$$

$$= \frac{3+i+21i+7i^2}{9+3i-3i-i^2}$$

$i^2 = -1$

the imaginary parts eliminate each other

$$= \frac{3+22i+(-7)}{9-(-1)}$$

Simplify $= \frac{-4+22i}{10}$

Write in Cartesian form $= -\frac{4}{10} + \frac{22}{10}i$

$$\boxed{-\frac{2}{5} + \frac{11}{5}i}$$

Simplify final answer.

1.8.2 Modulus & Argument

Modulus & Argument

How do I find the modulus of a complex number?

- The modulus of a complex number is its **distance** from the origin when plotted on an Argand diagram
- The modulus of z is written $|z|$
- If $z = x + \mathrm{i}y$, then we can use **Pythagoras** to show…
  - $|z| = \sqrt{x^2 + y^2}$
- A modulus is **never negative**

What features should I know about the modulus of a complex number?

- the modulus is related to the complex **conjugate** by…
  - $zz^* = z^*z = |z|^2$
  - This is because $zz^* = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2$
- In general, $|z_1 + z_2| \neq |z_1| + |z_2|$
  - e.g. both $z_1 = 3 + 4\mathrm{i}$ and $z_2 = -3 + 4\mathrm{i}$ have a modulus of 5, but $z_1 + z_2$ simplifies to 8**i** which has a modulus of 8

How do I find the argument of a complex number?

- The argument of a complex number is the **angle** that it makes on an **Argand diagram**
  - The angle must be taken from the **positive real axis**
  - The angle must be in a **counter-clockwise** direction
- Arguments are measured in **radians**
  - They can be given exact in terms of π
- The argument of z is written $\arg z$
- Arguments can be calculated using right-angled **trigonometry**
  - This involves using the tan ratio plus a sketch to decide whether it is positive/negative and acute/obtuse

What features should I know about the argument of a complex number?

- Arguments are usually given in the range $-\pi < \arg z \leq \pi$
  - Negative arguments are for complex numbers in the third and fourth quadrants
  - Occasionally you could be asked to give arguments in the range $0 < \arg z \leq 2\pi$
    - The question will make it clear which range to use
- The argument of zero, $\arg 0$ is undefined (no angle can be drawn)

What are the rules for moduli and arguments under multiplication and division?

- When two complex numbers, z_1 and z_2, are **multiplied** to give $z_1 z_2$, their **moduli** are also **multiplied**
  - $|z_1 z_2| = |z_1||z_2|$

- When two complex numbers, z_1 and z_2, are **divided** to give $\frac{z_1}{z_2}$, their **moduli** are also **divided**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- When two complex numbers, z_1 and z_2, are **multiplied** to give $z_1 z_2$, their **arguments** are **added**
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- When two complex numbers, z_1 and z_2, are **divided** to give $\frac{z_1}{z_2}$, their **arguments** are **subtracted**

Exam Tip

- Always draw a quick sketch to help you see what quadrant the complex number lies in when working out an argument
- Look for the range of values within which you should give your argument
  - If it is $-\pi < \arg z \leq \pi$ then you may need to measure it in the negative direction
  - If it is $0 < \arg z \leq 2\pi$ then you will always measure in the positive direction (counter - clockwise)

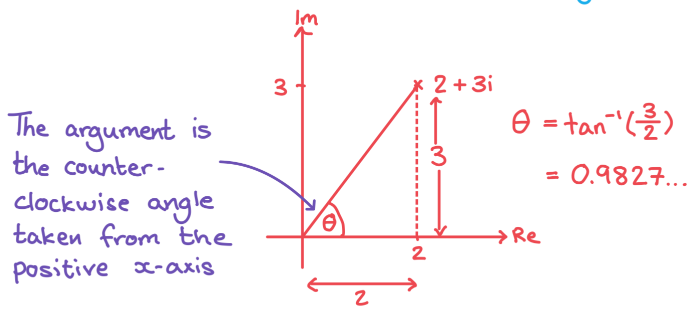
Worked Example

a)

Find the modulus and argument of $z = 2 + 3\mathrm{i}$

$|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$

Draw a sketch to help find the argument:



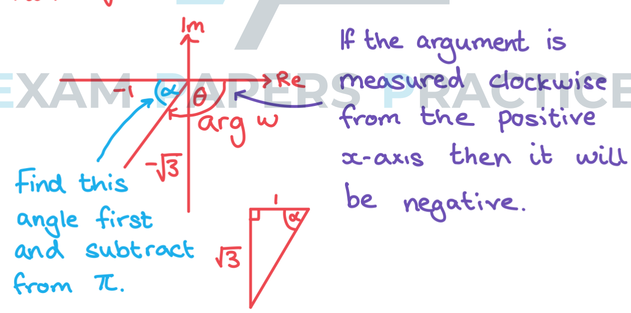
$\text{Mod } z = |z| = \sqrt{13}$

$\arg z = \theta = 0.983 \text{ (3sf)}$

b)

Find the modulus and argument of $w = -1 - \sqrt{3}\mathrm{i}$

$|w| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{4}$



$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\text{Mod } z = |z| = 2$

$\arg z = -\theta = -\frac{2\pi}{3}$

1.8.3 Introduction to Argand Diagrams

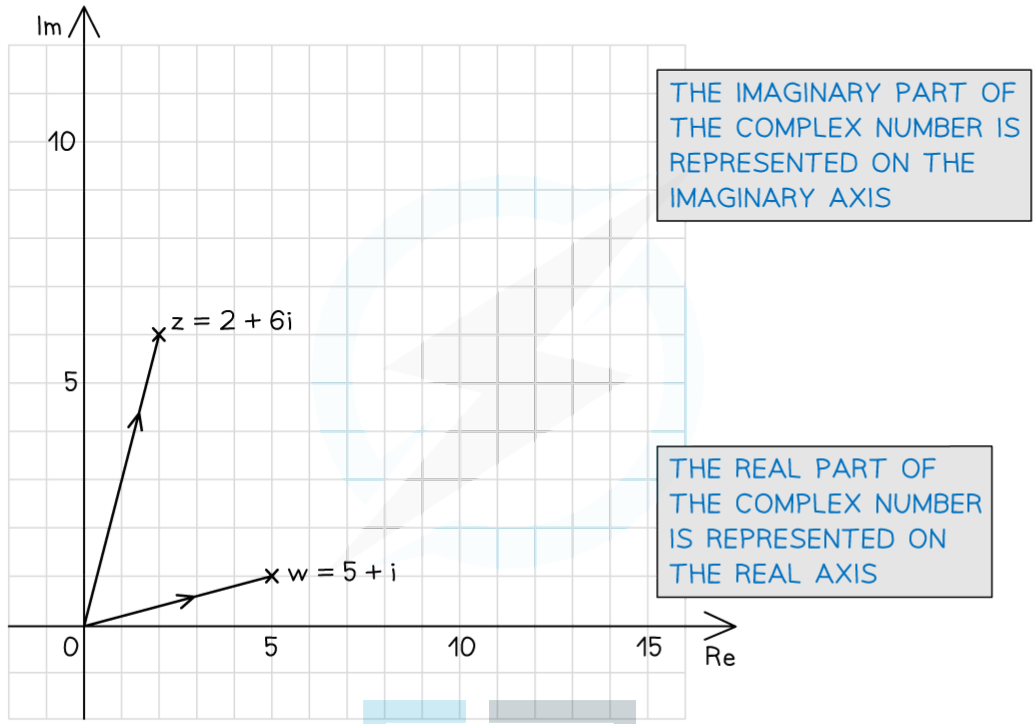
Argand Diagrams

What is the complex plane?

- The complex plane, sometimes also known as the Argand plane, is a two-dimensional plane on which complex numbers can be represented geometrically
- It is similar to a two-dimensional Cartesian coordinate grid
  - The x-axis is known as the **real** axis (Re)
  - The y-axis is known as the **imaginary** axis (Im)
- The complex plane emphasises the fact that a complex number is two dimensional
  - i.e it has two parts, a real and imaginary part
  - Whereas a real number only has one dimension represented on a number line (the x-axis only)

What is an Argand diagram?

- An Argand diagram is a geometrical representation of complex numbers on a **complex plane**
  - A complex number can be represented as either a point or a vector
- The complex number $x + yi$ is represented by the point with cartesian coordinate (x, y)
  - The **real** part is represented by the point on the **real** (x-) axis
  - The **imaginary** part is represented by the point on the **imaginary** (y-) axis
- Complex numbers are often represented as **vectors**
  - A line segment is drawn from the origin to the cartesian coordinate point
  - An arrow is added in the direction away from the origin
  - This allows for geometrical representations of complex numbers



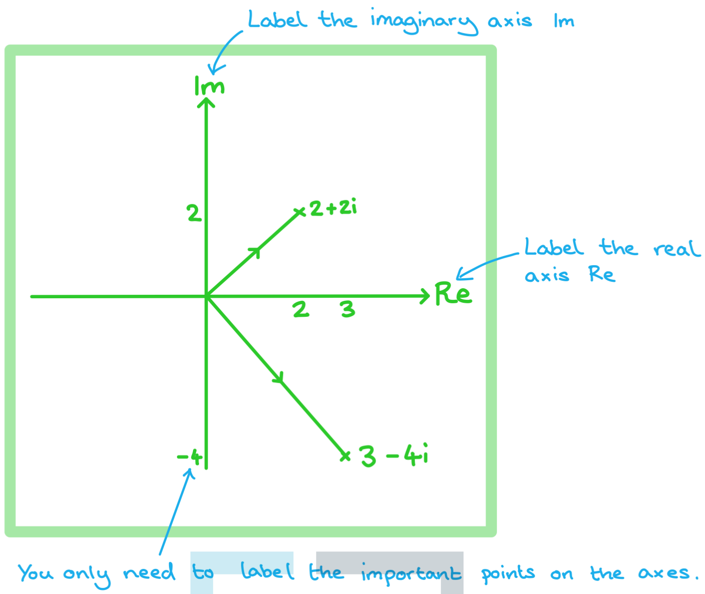
Exam Tip

- When setting up an Argand diagram you do not need to draw fully scaled axes, you only need the essential information for the points you want to show, this will save a lot of time

Worked Example

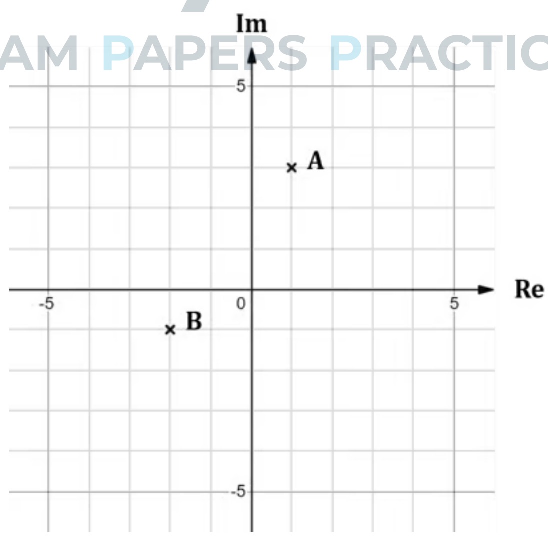
a)

Plot the complex numbers $z_1 = 2 + 2i$ and $z_2 = 3 - 4i$ as points on an Argand diagram.



b)

Write down the complex numbers represented by the points A and B on the Argand diagram below.



A: 1 + 3i
B: -2 - i

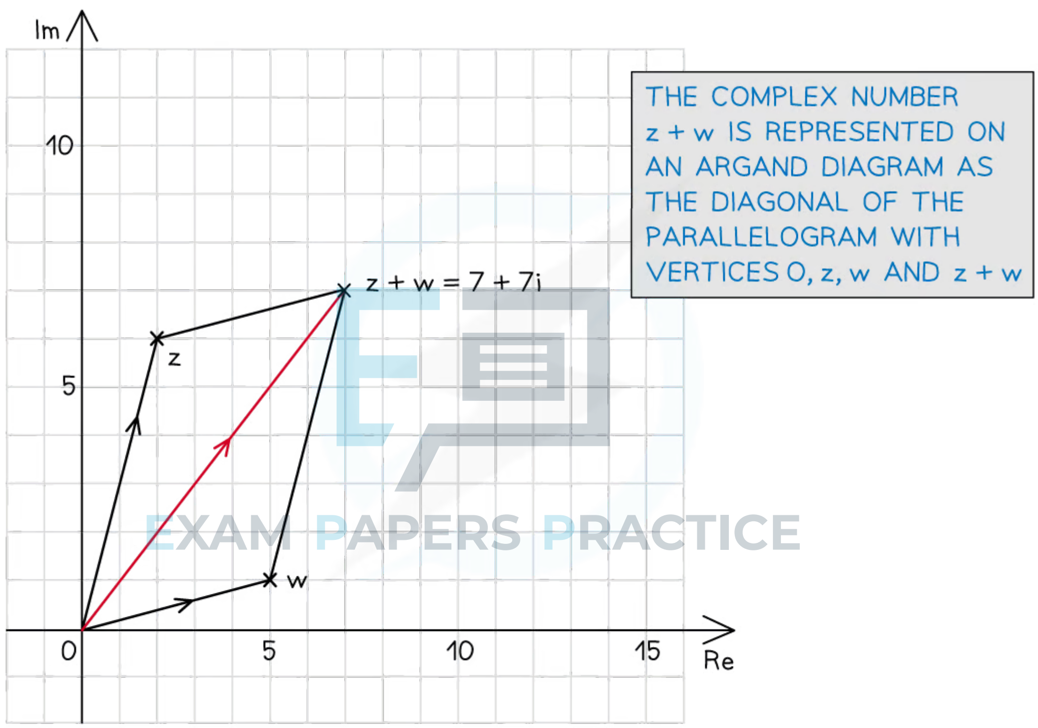
1.9 Further Complex Numbers

1.9.1 Geometry of Complex Numbers

Geometry of Complex Addition & Subtraction

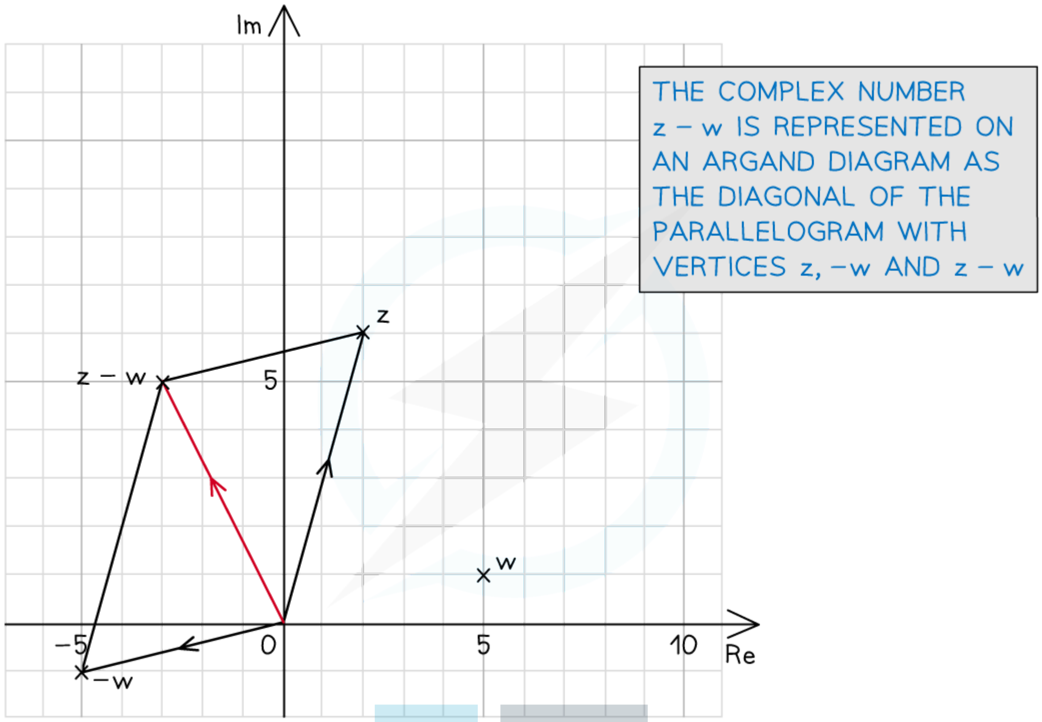
What does addition look like on an Argand diagram?

- In Cartesian form two complex numbers are added by adding the real and imaginary parts
- When plotted on an Argand diagram the complex number $z_1 + z_2$ is the longer diagonal of the parallelogram with vertices at the origin, z_1, z_2 and $z_1 + z_2$



What does subtraction look like on an Argand diagram?

- In Cartesian form the difference of two complex numbers is found by subtracting the real and imaginary parts
- When plotted on an Argand diagram the complex number $z_1 - z_2$ is the shorter diagonal of the parallelogram with vertices at the origin, z_1, $-z_2$ and $z_1 - z_2$



REMEMBER TO PLOT THE POINT −w BEFORE DRAWING THE PARALLELOGRAM

What are the geometrical representations of complex addition and subtraction?

- Let *w* be a given complex number with **real part *a*** and **imaginary part *b***
  - $w = a + bi$
- Let *z* be any complex number represented on an Argand diagram
- **Adding *w* to *z*** results in *z* being:
  - **Translated** by vector $\begin{pmatrix} a \\ b \end{pmatrix}$
- **Subtracting *w* from *z*** results in *z* being:
  - **Translated** by vector $\begin{pmatrix} -a \\ -b \end{pmatrix}$

Exam Tip

- Take extra care when representing a subtraction of a complex number geometrically
  - Remember that your answer will be a translation of the shorter diagonal of the parallelogram made up by the two complex numbers

Worked Example

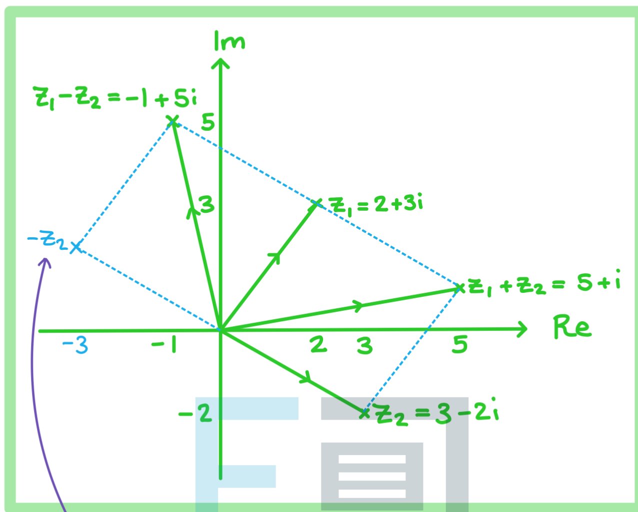
Consider the complex numbers $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$.

On an Argand diagram represent the complex numbers $z_1, z_2, z_1 + z_2$ and $z_1 - z_2$.

First find $z_1 + z_2$ and $z_1 - z_2$:

$z_1 + z_2 = (2 + 3i) + (3 - 2i) = 5 + i$

$z_1 - z_2 = (2 + 3i) - (3 - 2i) = -1 + 5i$



The geometrical properties can be seen by adding in $-z_2 = -3 + 2i$

Geometry of Complex Multiplication & Division

What do multiplication and division look like on an Argand diagram?

- The geometrical effect of multiplying a complex number by a real number, a, will be an enlargement of the vector by scale factor a
  - For positive values of a the direction of the vector will not change but the distance of the point from the origin will increase by scale factor a
  - For negative values of a the direction of the vector will change and the distance of the point from the origin will increase by scale factor a
- The geometrical effect of dividing a complex number by a real number, a, will be an enlargement of the vector by scale factor $1/a$
  - For positive values of a the direction of the vector will not change but the distance of the point from the origin will increase by scale factor $1/a$
  - For negative values of a the direction of the vector will change and the distance of the point from the origin will increase by scale factor $1/a$
- The geometrical effect of multiplying a complex number by i will be a rotation of the vector 90° counter-clockwise
  - $\mathrm{i}(x + y\mathrm{i}) = -y + x\mathrm{i}$
- The geometrical effect of multiplying a complex number by an imaginary number, $a\mathrm{i}$, will be a rotation 90° counter-clockwise and an enlargement by scale factor a
  - $a\mathrm{i}(x + y\mathrm{i}) = -ay + ax\mathrm{i}$
- The geometrical effect of multiplying or dividing a complex number by a complex number will be an enlargement and a rotation
  - The direction of the vector will change
    - The angle of rotation is the **argument**
  - The distance of the point from the origin will change
    - The scale factor is the **modulus**

What does complex conjugation look like on an Argand diagram?

- The geometrical effect of plotting a **complex conjugate** on an Argand diagram is a reflection in the real axis
  - The **real** part of the complex number will stay the same and the **imaginary** part will change sign

> **Exam Tip**
>
> - Make sure you remember the transformations that different operations have on complex numbers, this could help you check your calculations in an exam

Worked Example

Consider the complex number $z = 2 - \mathrm{i}$.

On an Argand diagram represent the complex numbers z, 3z, iz, z^* and zz^*.

First find 3z, iz and z^*

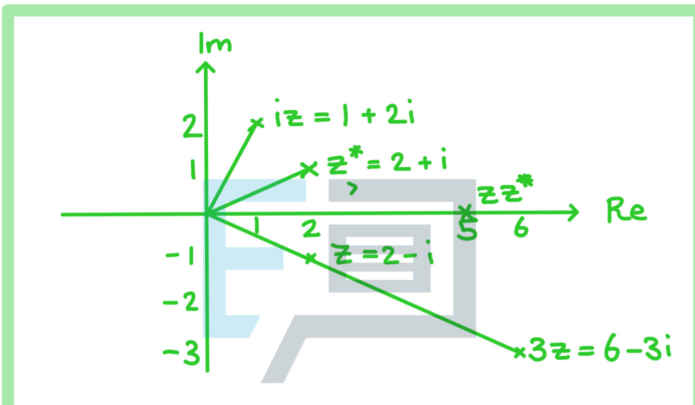
$z = 2 - i$

$3z = 3(2 - i) = 6 - 3i$

$iz = i(2 - i) = 2i - i^2 = 2i - (-1) = 1 + 2i$

$z^* = 2 + i$

$zz^* = (2 - i)(2 + i) = 4 - i^2 = 4 - (-1) = 5$



1.9.2 Forms of Complex Numbers

Modulus-Argument (Polar) Form

How do I write a complex number in modulus-argument (polar) form?

- The **Cartesian form** of a complex number, $z = x + \mathrm{i}y$, is written in terms of its real part, x, and its imaginary part, y
- If we let $r = |z|$ and $\theta = \arg z$, then it is possible to write a complex number in terms of its modulus, r, and its argument, θ, called the **modulus-argument (polar) form**, given by...
  - $z = r(\cos\theta + \mathrm{i}\sin\theta)$
  - This is often written as $z = r\,\mathrm{cis}\,\theta$
  - This is given in the formula book under Modulus-argument (polar) form and exponential (Euler) form
- It is usual to give arguments in the range $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$
  - Negative arguments should be shown clearly
  - e.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right) = 2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$
    - without simplifying $\cos\left(-\frac{\pi}{3}\right)$ to either $\cos\left(\frac{\pi}{3}\right)$ or $\frac{1}{2}$
- The **complex conjugate** of $r\,\mathrm{cis}\,\theta$ is $r\,\mathrm{cis}\,(-\theta)$
- If a complex number is given in the form $z = r(\cos\theta - \mathrm{i}\sin\theta)$, then it is not in modulus-argument (polar) form due to the minus sign
  - It can be converted by considering transformations of trigonometric functions
    - $-\sin\theta = \sin(-\theta)$ and $\cos\theta = \cos(-\theta)$
  - So $z = r(\cos\theta - \mathrm{i}\sin\theta) = z = r(\cos(-\theta) + \mathrm{i}\sin(-\theta)) = r\,\mathrm{cis}\,(-\theta)$
- To convert from modulus-argument (polar) form back to Cartesian form, evaluate the real and imaginary parts
  - E.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ becomes $z = 2\left(\frac{1}{2} + \mathrm{i}\left(-\frac{\sqrt{3}}{2}\right)\right) = 1 - \sqrt{3}\,\mathrm{i}$

How do I multiply complex numbers in modulus-argument (polar) form?

- The main benefit of writing complex numbers in modulus-argument (polar) form is that they multiply and divide very easily
- To **multiply** two complex numbers in modulus-argument (polar) form we **multiply their moduli** and **add their arguments**
  - $|z_1 z_2| = |z_1||z_2|$
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- So if $z_1 = r_1\,\mathrm{cis}\,(\theta_1)$ and $z_2 = r_2\,\mathrm{cis}\,(\theta_2)$
  - $z_1 z_2 = r_1 r_2\,\mathrm{cis}\,(\theta_1 + \theta_2)$
- Sometimes the new argument, $\theta_1 + \theta_2$, does not lie in the range $-\pi < \theta \leq \pi$ (or $0 \leq \theta < 2\pi$ if this is being used)
  - An out-of-range argument can be adjusted by either **adding or subtracting** 2π
  - E.g. If $\theta_1 = \frac{2\pi}{3}$ and $\theta_2 = \frac{\pi}{2}$ then $\theta_1 + \theta_2 = \frac{7\pi}{6}$

- This is currently not in the range $-\pi < \theta \le \pi$
- Subtracting 2π from $\frac{7\pi}{6}$ to give $-\frac{5\pi}{6}$, a new argument is formed

  This lies in the correct range and represents the same angle on an Argand diagram
- The rules of **multiplying the moduli** and **adding the arguments** can also be applied when…
  - …multiplying three complex numbers together, $z_1 z_2 z_3$, or more
  - …finding powers of a complex number (e.g. z^2 can be written as zz)
- The rules for multiplication can be proved algebraically by multiplying $z_1 = r_1 \text{ cis } (\theta_1)$ by $z_2 = r_2 \text{ cis } (\theta_2)$, expanding the brackets and using compound angle formulae

How do I divide complex numbers in modulus-argument (polar) form?

- To **divide** two complex numbers in modulus-argument (polar) form, we **divide their moduli** and **subtract their arguments**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
  - $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
- So if $z_1 = r_1 \text{ cis } (\theta_1)$ and $z_2 = r_2 \text{ cis } (\theta_2)$ then
  - $\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}\left(\theta_1 - \theta_2\right)$
- Sometimes the new argument, $\theta_1 - \theta_2$, can lie out of the range $-\pi < \theta \le \pi$ (or the range $0 < \theta \le 2\pi$ if this is being used)
  - You can **add or subtract** 2π to bring out-of-range arguments back in range
- The rules for division can be proved algebraically by dividing $z_1 = r_1 \text{ cis } (\theta_1)$ by $z_2 = r_2 \text{ cis } (\theta_2)$ using **complex division** and the compound angle formulae

Exam Tip

- Remember that $r \text{ cis } \theta$ only refers to $r(\cos\theta + i\sin\theta)$
  - If you see a complex number written in the form $z = r(\cos\theta - i\sin\theta)$ then you will need to convert it to the correct form first
  - Make sure you are confident with basic trig identities to help you do this

Worked Example

Let $z_1 = 4\sqrt{2}\ \text{cis}\ \dfrac{3\pi}{4}$ and $z_2 = \sqrt{8}\left(\cos\left(\dfrac{\pi}{2}\right) - \text{i}\sin\left(\dfrac{\pi}{2}\right)\right)$

a)

Find $z_1 z_2$, giving your answer in the form $r(\cos\theta + \text{i}\sin\theta)$ where $0 \le \theta < 2\pi$

$z_1 = 4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$, $z_2 = \sqrt{8}\left(\cos\left(-\frac{\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{2}\right)\right) = 2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)$

For $z_1 z_2$, multiply the moduli and add the arguments.

$z_1 z_2 = \left(4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)\right)\left(2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)\right)$

$= (4\sqrt{2})(2\sqrt{2})\ \text{cis}\left(\frac{3\pi}{4} + \left(-\frac{\pi}{2}\right)\right)$

$= 16\ \text{cis}\left(\frac{\pi}{4}\right)$

$$z_1 z_2 = 16\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$$

b)

Find $\dfrac{z_1}{z_2}$, giving your answer in the form $r(\cos\theta + \text{i}\sin\theta)$ where $-\pi \le \theta < \pi$

For $\frac{z_1}{z_2}$, divide the moduli and subtract the arguments

$\frac{z_1}{z_2} = \frac{4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)}{2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)} = \frac{4\sqrt{2}}{2\sqrt{2}}\ \text{cis}\left(\frac{3\pi}{4} - \left(-\frac{\pi}{2}\right)\right)$

$= 2\ \text{cis}\left(\frac{5\pi}{4}\right)$

$= 2\ \text{cis}\left(\frac{5\pi}{4} - 2\pi\right)$

$\frac{5\pi}{4}$ is not in the range $-\pi \le \theta \le \pi$ so subtract 2π to bring it into range

$$\frac{z_1}{z_2} = 2\left(\cos\left(\frac{-3\pi}{4}\right) + \text{i}\sin\left(\frac{-3\pi}{4}\right)\right)$$

Exponential (Euler's) Form

How do we write a complex number in Euler's (exponential) form?

- A complex number can be written in Euler's form as $z = r\mathrm{e}^{\mathrm{i}\theta}$
  - This relates to the modulus-argument (polar) form as $z = r\mathrm{e}^{\mathrm{i}\theta} = r\,\mathrm{cis}\,\theta$
  - This shows a clear link between exponential functions and trigonometric functions
  - This is given in the formula booklet under 'Modulus-argument (polar) form and exponential (Euler) form'
- The argument is normally given in the range $0 \le \theta < 2\pi$
  - However in exponential form other arguments can be used and the same convention of adding or subtracting 2π can be applied

How do we multiply and divide complex numbers in Euler's form?

- Euler's form allows for quick and easy multiplication and division of complex numbers
- If $z_1 = r_1\mathrm{e}^{\mathrm{i}\theta_1}$ and $z_2 = r_2\mathrm{e}^{\mathrm{i}\theta_2}$ then
  - $z_1 \times z_2 = r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1 + \theta_2)}$

    Multiply the moduli and add the arguments
  - $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\theta_1 - \theta_2)}$

    Divide the moduli and subtract the arguments
- Using these rules makes multiplying and dividing more than two complex numbers much easier than in Cartesian form
- When a complex number is written in Euler's form it is easy to raise that complex number to a power
  - If $z = r\mathrm{e}^{\mathrm{i}\theta}$, $z^2 = r^2\mathrm{e}^{2\mathrm{i}\theta}$ and $z^n = r^n\mathrm{e}^{n\mathrm{i}\theta}$

What are some common numbers in exponential form?

- As $\cos(2\pi) = 1$ and $\sin(2\pi) = 0$ you can write:
  - $1 = \mathrm{e}^{2\pi\mathrm{i}}$
- Using the same idea you can write:
  - $1 = \mathrm{e}^0 = \mathrm{e}^{2\pi\mathrm{i}} = \mathrm{e}^{4\pi\mathrm{i}} = \mathrm{e}^{6\pi\mathrm{i}} = \mathrm{e}^{2k\pi\mathrm{i}}$
  - where k is any integer
- As $\cos(\pi) = -1$ and $\sin(\pi) = 0$ you can write:
  - $\mathrm{e}^{\pi\mathrm{i}} = -1$
  - Or more commonly written as $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$

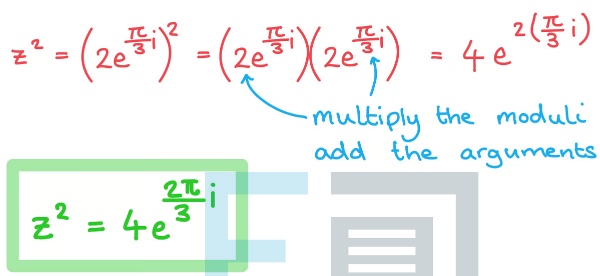
    This is known as Euler's identity and is considered by some mathematicians as the most beautiful equation
- As $\cos\left(\dfrac{\pi}{2}\right) = 0$ and $\sin\left(\dfrac{\pi}{2}\right) = 1$ you can write:
  - $\mathrm{i} = \mathrm{e}^{\frac{\pi}{2}\mathrm{i}}$

Exam Tip

- Euler's form allows for easy manipulation of complex numbers, in an exam it is often worth the time converting a complex number into Euler's form if further calculations need to be carried out
    - Familiarise yourself with which calculations are easier in which form, for example multiplication and division are easiest in Euler's form but adding and subtracting are easiest in Cartesian form

Worked Example

Consider the complex number $z = 2e^{\frac{\pi}{3}i}$. Calculate z^2 giving your answer in the form $re^{i\theta}$.



Conversion of Forms

Converting from Cartesian form to modulus-argument (polar) form or exponential (Euler's) form.

- To convert from Cartesian form to modulus-argument (polar) form or exponential (Euler) form use
  - $r = |z| = \sqrt{x^2 + y^2}$
- and
  - $\theta = \arg z$

Converting from modulus-argument (polar) form or exponential (Euler's) form to Cartesian form.

- To convert from modulus-argument (polar) form to Cartesian form
  - Write $z = r(\cos\theta + i\sin\theta)$ as $z = r\cos\theta + (r\sin\theta)i$
  - Find the values of the trigonometric ratios $r\sin\theta$ and $r\cos\theta$
    - You may need to use your knowledge of trig exact values
  - Rewrite as $z = a + bi$ where
    - $a = r\cos\theta$ and $b = r\sin\theta$
- To convert from exponential (Euler's) form to Cartesian form first rewrite $z = re^{i\theta}$ in the form $z = r\cos\theta + (r\sin\theta)i$ and then follow the steps above

Exam Tip

- When converting from Cartesian form into Polar or Euler's form, always leave your modulus and argument as an exact value
  - Rounding values too early may result in inaccuracies later on

Worked Example

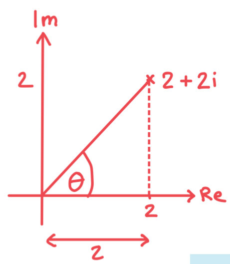
Two complex numbers are given by $z_1 = 2 + 2\mathrm{i}$ and $z_2 = 3\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$.

a)
Write z_1 in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

$z_1 = 2 + 2\mathrm{i}$

Find the modulus: $|z_1| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Draw a sketch to help find the argument:



$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1)$
$= \frac{\pi}{4}$

$r = 2\sqrt{2}, \quad \theta = \frac{\pi}{4}$

$z_1 = 2\sqrt{2}\,\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}$

b)
Write z_2 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$ and then convert it to Cartesian form.

$z_2 = 3\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}} = 3\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$
$= 3\left(-\frac{1}{2} + \mathrm{i}\left(\frac{\sqrt{3}}{2}\right)\right)$

$z_2 = \frac{3}{2}\left(-1 + \sqrt{3}\,\mathrm{i}\right)$

1.9.3 Complex Roots of Polynomials

Complex Roots of Quadratics

What are complex roots?

- A quadratic equation can either have two real roots (zeros), a repeated real root or no real roots
  - This depends on the location of the graph of the quadratic with respect to the x-axis
- If a quadratic equation has no real roots we would previously have stated that it has **no real solutions**
  - The quadratic equation will have a **negative discriminant**
  - This means taking the square root of a negative number
- Complex numbers provide solutions for quadratic equations that have **no real roots**

How do we solve a quadratic equation when it has complex roots?

- If a quadratic equation takes the form $ax^2 + bx + c = 0$ it can be solved by either using the quadratic formula or completing the square
- If a quadratic equation takes the form $ax^2 + b = 0$ it can be solved by rearranging
- The property $\mathrm{i} = \sqrt{-1}$ is used
  - $\sqrt{-a} = \sqrt{a \times -1} = \sqrt{a} \times \sqrt{-1}$
- If the coefficients of the quadratic are real then the complex roots will occur in complex conjugate pairs
  - If $\boldsymbol{z = m + n\mathrm{i}\ (n \neq 0)}$ is a root of a quadratic with real coefficients then $\boldsymbol{z^* = m - n\mathrm{i}}$ is also a root
- The **real part** of the solutions will have the same value as the x coordinate of the turning point on the graph of the quadratic

- When the coefficients of the quadratic equation are **non-real**, the solutions will **not** be complex conjugates
  - To solve these you can use the quadratic formula

How do we factorise a quadratic equation if it has complex roots?

- If we are given a quadratic equation in the form $az^2 + bz + c = 0$, where a, b, and $c \in \mathbb{R}$, $a \neq 0$ we can use its complex roots to write it in **factorised form**
  - Use the quadratic formula to find the two roots, $z = p + q\mathrm{i}$ and $z^* = p - q\mathrm{i}$
  - This means that $z - (p + q\mathrm{i})$ and $z - (p - q\mathrm{i})$ must both be factors of the quadratic equation
  - Therefore we can write $az^2 + bz + c = (z - (p + q\mathrm{i}))(z - (p - q\mathrm{i}))$
  - This can be rearranged into the form $(z - p - q\mathrm{i})(z - p + q\mathrm{i})$

How do we find a quadratic equation when given a complex root?

- If we are given a complex root in the form $z = p + q\mathrm{i}$ we can find the quadratic equation in the form $az^2 + bz + c = 0$, where a, b, and $c \in \mathbb{R}$, $a \neq 0$
  - We know that the second root must be $z^* = p - q\mathrm{i}$
  - This means that $z - (p + q\mathrm{i})$ and $z - (p - q\mathrm{i})$ must both be factors of the quadratic equation

- Therefore we can write $az^2 + bz + c = (z - (p + q\text{i}))(z - (p - q\text{i}))$
- Rewriting this as $((z - p) - q\text{i})((z - p) + q\text{i})$ makes expanding easier
- Expanding this gives the quadratic equation $z^2 - 2pz + (p^2 + q^2)$

  $a = 1$

  $b = -2p$

  $c = p^2 + q^2$

- This demonstrates the important property $(x - z)(x - z^*) = x^2 - 2\text{Re}(z)x + |z|^2$

Exam Tip

- Once you have your final answers you can check your roots are correct by substituting your solutions back into the original equation
  - You should get 0 if correct! [Note: 0 is equivalent to $0 + 0\text{i}$]

Worked Example

a)

Solve the quadratic equation $z^2 - 2z + 5 = 0$ and hence, factorise $z^2 - 2z + 5$.

Use the quadratic formula or completing the square to find the solutions.

Solutions of a quadratic equation	$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ a \neq 0$

$a = 1$
$b = -2$
$c = 5$

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm \sqrt{16}\sqrt{-1}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$z_1 = 1 + 2i \qquad z_2 = 1 - 2i$$

If the solutions are $z_1 = 1 + 2i$ and $z_2 = 1 - 2i$ then the factors must be $z - (1+2i)$ and $z - (1-2i)$

$$z^2 - 2z + 5 = (z - (1 + 2i))(z - (1 - 2i))$$

$$(z - 1 - 2i)(z - 1 + 2i)$$

b)

Given that one root of a quadratic equation is $z = 2 - 3i$, find the quadratic equation in the form $az^2 + bz + c = 0$, where a, b, and $c \in \mathbb{R}$, $a \neq 0$.

If $2-3i$ is one root then $2+3i$ must be the other root and the two factors must be $z-(2-3i)$ and $z-(2+3i)$.

$$(z-(2-3i))(z-(2+3i))=0$$

$$((z-2)+3i)((z-2)-3i)=0$$

$$(z-2)^2-(3i)^2=0$$

$$z^2-4z+4-9i^2=0$$

$i^2=-1 \therefore -9i^2=9$

$$z^2-4z+13=0$$

Complex Roots of Polynomials

How many roots should a polynomial have?

- We know that every **quadratic** equation has **two roots** (not necessarily distinct or real)
- This is a particular case of a more general rule:
  - Every polynomial equation, with real coefficients, of degree *n* has *n* roots
  - The *n* roots are not necessarily all **distinct** and therefore we need to count any **repeated** roots that may occur individually
- From the above rule we can state the following:
  - A **cubic** equation of the form $ax^3 + bx^2 + cx + d = 0$ can have either:
    - 3 **real** roots
    - Or 1 **real** root and a complex **conjugate** pair
- A **quartic** equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ will have one of the following cases for roots:
  - 4 **real** roots
  - 2 **real** and 2 **nonreal** (a complex conjugate pair)
  - 4 **nonreal** (two complex conjugate pairs)
- When a real polynomial of any degree has one complex root it will always also have the complex conjugate as a root

How do we solve a cubic equation with complex roots?

- Steps to solve a cubic equation with complex roots
  - If we are told that $p + qi$ is a root, then we know $p - qi$ is also a root
  - This means that $z - (p + qi)$ and $z - (p - qi)$ must both be factors of the cubic equation
  - Multiplying the above factors together gives us a quadratic factor of the form $(Az^2 + Bz + C)$
  - We need to find the third factor $(z - \alpha)$
  - **Multiply** the factors and **equate** to our original equation to get
    - $(Az^2 + Bz + C)(z - \alpha) = ax^3 + bx^2 + cx + d$
  - From there either
    - **Expand** and **compare coefficients** to find
    - Or use **polynomial division** to find the factor $(z - \alpha)$
  - Finally, write your **three roots** clearly

How do we solve a polynomial of any degree with complex roots?

- When asked to find the roots of any polynomial when we are given one, we use almost the same method as for a cubic equation
  - State the initial root and its conjugate and write their factors as a quadratic factor (as above) we will have two unknown roots to find, write these as factors $(z - \alpha)$ and $(z - \beta)$
  - The unknown factors also form a quadratic factor $(z - \alpha)(z - \beta)$
  - Then continue with the steps from above, either **comparing coefficients** or using **polynomial division**
    - If using polynomial division, then solve the quadratic factor you get to find the roots α and β

How do we solve polynomial equations with unknown coefficients?

- Steps to find unknown variables in a given equation when given a root:
  - **Substitute** the given root $p + qi$ into the equation $f(z) = 0$
  - **Expand** and **group** together the **real** and **imaginary** parts (these expressions will contain our unknown values)
  - **Solve** as simultaneous equations to find the unknowns
  - **Substitute** the values into the **original** equation
  - From here continue using the previously described methods for finding other roots for the polynomial

How do we factorise a polynomial when given a complex root?

- If we are given a root of a polynomial of any degree in the form $z = p + qi$
  - We know that the complex conjugate, $z^* = p - qi$ is another root
  - We can write $(z - (p + qi))$ and $(z - (p - qi))$ as two linear factors
    - Or rearrange into one quadratic factor
  - This can be multiplied out with another factor to find further factors of the polynomial
- For higher order polynomials more than one root may be given
  - If the further given root is complex then its complex conjugate will also be a root
  - This will allow you to find further factors

Exam Tip

- You can speed up multiplying two complex conjugate factors together by
  - rewrite $(z - (p + qi))(z - (p - qi))$ as $((z - p) - qi))((z - p) + qi))$
  - Then $((z - p) - qi))((z - p) + qi)) = (z - p)^2 - (qi)^2 = (z - p)^2 + q^2$
- If you are working on a calculator paper read the question carefully to see how much of the working needs to be shown but always remember to use your GDC to check your working where you can

Worked Example

Given that one root of a polynomial $p(x) = z^3 + z^2 - 7z + 65$ is $2 - 3i$, find the other roots.

If $2-3i$ is one root then $2+3i$ must be the other root and two of the factors must be $z-(2-3i)$ and $z-(2+3i)$

Therefore a quadratic factor is $z^2 - 4z + 13$

There must exist a linear factor $(az+b)$

$\therefore (az+b)(z^2-4z+13) = z^3 + z^2 - 7z + 65$

Compare coefficients: $az^3 = z^3$ coefficient of z^3

$a = 1$

$13b = 65$ constant coefficient

$b = 5$

Therefore the factors are $z-(2-3i)$, $z-(2+3i)$ and $(z+5)$

$(z-(2-3i))(z-(2+3i))(z+5) = 0$

$z = (2 \pm 3i)$ and $z = -5$

EXAM PAPERS PRACTICE

1.9.4 De Moivre's Theorem

De Moivre's Theorem

What is De Moivre's Theorem?

- De Moivre's theorem can be used to find powers of complex numbers
- It states that for $z = r\,\text{cis}\,\theta$, $z^n = [r(\cos\theta + \text{i}\sin\theta)]^n = r^n(\cos n\theta + \text{i}\sin n\theta)$
  - Where
    - $z \neq 0$
    - r is the modulus, $|z|$, $r \in \mathbb{R}^+$
    - θ is the argument, $\arg z$, $\theta \in \mathbb{R}$
    - $n \in \mathbb{R}$
- In Euler's form this is simply:
  - $(r\text{e}^{\text{i}\theta})^n = r^n\text{e}^{\text{i}n\theta}$
- In words de Moivre's theorem tells us to raise the modulus by the power of n and multiply the argument by n
- In the formula booklet de Moivre's theorem is given in both polar and Euler's form:
  - $[r(\cos\theta + \text{i}\sin\theta)]^n = r^n(\cos n\theta + \text{i}\sin n\theta) = r^n\text{e}^{\text{i}n\theta} = r^n\ \text{cis}\ n\theta$

How do I use de Moivre's Theorem to raise a complex number to a power?

- If a complex number is in Cartesian form you will need to convert it to either modulus-argument (polar) form or exponential (Euler's) form first
  - This allows de Moivre's theorem to be used on the complex number
- You may need to convert it back to Cartesian form afterwards
- If a complex number is in the form $z = r(\cos(\theta) - \text{i}\sin(\theta))$ then you will need to rewrite it as $z = r(\cos(-\theta) + \text{i}\sin(-\theta))$ before applying de Moivre's theorem
- A useful case of de Moivre's theorem allows us to easily find the reciprocal of a complex number:
  - $\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + \text{i}\sin(-\theta)) = \frac{1}{r}\text{e}^{-\text{i}\theta}$
  - Using the trig identities $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$ gives
  - $\frac{1}{z} = z^{-1} = r^{-1}[\cos(\theta) - \text{i}\sin(\theta)] = \frac{1}{r}[\cos(\theta) - \text{i}\sin(\theta)]$
- In general
  - $z^{-n} = r^{-n}[\cos(-n\theta) + \text{i}\sin(-n\theta)] = r^{-n}[\cos(n\theta) - \text{i}\sin(n\theta)]$

Exam Tip

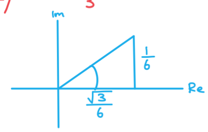
- You may be asked to find all the powers of a complex number, this means there will be a repeating pattern
  - This can happen if the modulus of the complex number is 1
  - Keep an eye on your answers and look for the point at which they begin to repeat themselves

Worked Example

Find the value of $\left(\frac{\sqrt{3}}{6}+\frac{1}{6}\mathrm{i}\right)^{-3}$, giving your answer in the form $a+b\mathrm{i}$.

Write in Polar form: $r=\sqrt{\left(\frac{\sqrt{3}}{6}\right)^2+\left(\frac{1}{6}\right)^2}=\frac{1}{3}$

$\theta=\tan^{-1}\left(\frac{\frac{1}{6}}{\frac{\sqrt{3}}{6}}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$



$\left(\frac{\sqrt{3}}{6}+\frac{1}{6}\mathrm{i}\right)^{-3}=\left(\frac{1}{3}\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^{-3}$

Apply DeMoivre's Theorem: $\left(\frac{1}{3}\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^{-3}=\left(\frac{1}{3}\right)^{-3}\operatorname{cis}\left(\frac{-3\pi}{6}\right)$

Convert back to Cartesian form:

$27\operatorname{cis}\left(-\frac{\pi}{2}\right)=27\left(\cos\left(-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$

$=27(0-\mathrm{i})$

$-27\mathrm{i}$

Proof of De Moivre's Theorem

How is de Moivre's Theorem proved?

- When written in Euler's form the proof of de Moivre's theorem is easy to see:
  - Using the index law of brackets: $(re^{i\theta})^n = r^n e^{in\theta}$
- However Euler's form cannot be used to prove de Moivre's Theorem when it is in modulus-argument (polar) form
- **Proof by induction** can be used to prove de Moivre's Theorem for positive integers:
  - To prove de Moivre's Theorem for all positive integers, n
  - $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$
- STEP 1: Prove it is true for $n = 1$
  - $[r(\cos\theta + i\sin\theta)]^1 = r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$
  - So de Moivre's Theorem is true for $n = 1$
- STEP 2: Assume it is true for $n = k$
  - $[r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$
- STEP 3: Show it is true for $n = k + 1$
  - $[r(\cos\theta + i\sin\theta)]^{k+1} = ([r(\cos\theta + i\sin\theta)]^k)([r(\cos\theta + i\sin\theta)]^1)$
  - According to the assumption this is equal to
    $$(r^k(\cos k\theta + i\sin k\theta))\ (r(\cos\theta + i\sin\theta))$$
  - Using laws of indices and multiplying out the brackets:
    $$= r^{k+1}[\cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta]$$
  - Letting $i^2 = -1$ and collecting the real and imaginary parts gives:
    $$= r^{k+1}[\cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)]$$
  - Recognising that the real part is equivalent to $\cos(k\theta + \theta)$ and the imaginary part is equivalent to $\sin(k\theta + \theta)$ gives
    $$(r\,\mathrm{cis}\,\theta)^{k+1} = r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$$
  - So de Moivre's Theorem is true for $n = k + 1$
- STEP 4: Write a conclusion to complete the proof
  - The statement is true for $n = 1$, and if it is true for $n = k$ it is also true for $n = k + 1$
  - Therefore, by the principle of mathematical induction, the result is true for all positive integers, n
- De Moivre's Theorem works for all real values of n
  - However you could only be asked to prove it is true for positive integers

Exam Tip

- Learning the standard proof for de Moivre's theorem will also help you to memorise the steps for proof by induction, another important topic for your AA HL exam

Worked Example

Show, using proof by mathematical induction, that for a complex number $z = r\,\text{cis}\,\theta$ and for all positive integers, n,

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Step 1: Prove it is true for $n = 1$

$z^1 = [r(\cos\theta + i\sin\theta)]^1 = r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$

Step 2: Assume it is true for $n = k$

$z^k = [r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$

Step 3: Show it is true for $n = k+1$

$z^{k+1} = [r(\cos\theta + i\sin\theta)]^{k+1}$ Addition law of indices: $a^k a^1 = a^{k+1}$

$= ([r(\cos\theta + i\sin\theta)]^k)([r(\cos\theta + i\sin\theta)]^1)$ using the assumption

$r^k \times r^1 = r^{k+1}$

$= [r^k(\cos k\theta + i\sin k\theta)][r(\cos\theta + i\sin\theta)]$

$= r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$i^2 = -1$

$= r^{k+1}[\cos k\theta\cos\theta + \cos k\theta(i\sin\theta) + \cos\theta(i\sin k\theta) + i^2\sin k\theta\sin\theta]$

$= r^{k+1}[\cos k\theta\cos\theta + i(\cos k\theta\sin\theta + \cos\theta\sin k\theta) - \sin k\theta\sin\theta]$ collect Re and Im parts

$= r^{k+1}[\cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)]$

$\cos k\theta\cos\theta - \sin k\theta\sin\theta = \cos(k\theta + \theta)$, $\quad \cos k\theta\sin\theta + \sin k\theta\cos\theta = \sin(k\theta + \theta)$

$= r^{k+1}[\cos(k\theta + \theta) + i\sin(k\theta + \theta)]$

$= r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$ So true for $n = k+1$

Step 4: Write a conclusion:

> De Moivre's theorem is true for $n = 1$, and if it is true for $n = k$ it is also true for $n = k+1$.
> Therefore it is true for all $n \in \mathbb{Z}^+$

1.9.5 Roots of Complex Numbers

Roots of Complex Numbers

How do I find the square root of a complex number?

- The square roots of a complex number will themselves be complex:
  - i.e. if $z^2 = a + b\text{i}$ then $z = c + d\text{i}$
- We can then square $(c + d\text{i})$ and equate it to the original complex number $(a + b\text{i})$, as they both describe z^2:
  - $a + b\text{i} = (c + d\text{i})^2$
- Then expand and simplify:
  - $a + b\text{i} = c^2 + 2cd\text{i} + d^2\text{i}^2$
  - $a + b\text{i} = c^2 + 2cd\text{i} - d^2$
- As both sides are equal we are able to equate real and imaginary parts:
  - Equating the real components: $a = c^2 - d^2$ (1)
  - Equating the imaginary components: $b = 2cd$ (2)
- These equations can then be solved simultaneously to find the real and imaginary components of the square root
  - In general, we can rearrange (2) to make $\dfrac{b}{2d} = c$ and then substitute into (1)
  - This will lead to a quartic equation in terms of d; which can be solved by making a substitution to turn it into a quadratic
- The values of d can then be used to find the corresponding values of c, so we now have both components of both square roots $(c + d\text{i})$
- Note that one root will be the negative of the other root
  - g. $c + d\text{i}$ and $-c - d\text{i}$

How do I use de Moivre's Theorem to find roots of a complex number?

- De Moivre's Theorem states that a complex number in modulus-argument form can be raised to the power of *n* by
  - Raising the modulus to the power of *n* and multiplying the argument by *n*
- When in modulus-argument (polar) form de Moivre's Theorem can then be used to find the roots of a complex number by
  - Taking the *n*th root of the modulus and dividing the argument by *n*
  - If $z = r(\cos\theta + \text{i}\sin\theta)$ then $\sqrt[n]{z} = [r(\cos(\theta + 2\pi k) + \text{i}\sin(\theta + 2\pi k))]^{\frac{1}{n}}$

    $k = 0,\ 1,\ 2,\ \ldots\ ,\ n - 1$

    Recall that adding 2*π* to the argument of a complex number does not change the complex number

    Therefore we must consider how different arguments will give the same result
  - This can be rewritten as $\sqrt[n]{z} = r^{\frac{1}{n}}\left(\cos\left(\dfrac{\theta + 2\pi k}{n}\right) + \text{i}\sin\left(\dfrac{\theta + 2\pi k}{n}\right)\right)$
- This can be written in exponential (Euler's) form as
  - For $z^n = r\text{e}^{\text{i}\theta}$, $z = \sqrt[n]{r}\,\text{e}^{\frac{\theta + 2\pi k}{n}\text{i}}$

- The *n*th root of complex number will have *n* roots with the properties:
  - The modulus is $\sqrt[n]{r}$ for all roots
  - There will be *n* different arguments spaced at equal intervals on the unit circle
  - This creates some geometrically beautiful results:
    - The five roots of a complex number raised to the power 5 will create a regular pentagon on an Argand diagram
    - The eight roots of a complex number raised to the power 8 will create a regular octagon on an Argand diagram
    - The *n* roots of a complex number raised to the power *n* will create a regular *n*-sided polygon on an Argand diagram
- Sometimes you may need to use your GDC to find the roots of a complex number
  - Using your GDC's store function will help when entering complicated modulus and arguments
  - Make sure you choose the correct form to enter your complex number in
  - Your GDC should be able to give you the answer in your preferred form

Exam Tip

- De Moivre's theorem makes finding roots of complex numbers very easy, but you must be confident converting from Cartesian form into Polar and Euler's form first
  - If you are in a calculator exam your GDC will be able to do this for you but you must clearly show how you got to your answer
  - You must also be prepared to do this by hand in a non-calculator paper

Worked Example

a)

Find the square roots of $5 + 12i$, giving your answers in the form $a + bi$.

Let $z^2 = 5 + 12i$, then $z = a + bi$

$z^2 = a^2 + 2abi + b^2i^2$ ($i^2 = -1$)

$= a^2 + 2abi - b^2$

Therefore $5 + 12i = (a^2 - b^2) + 2abi$

Equate the real components: $a^2 - b^2 = 5$ ①

Equate the imaginary components: $2ab = 12$ ②

Solve the simultaneous equations: $a = \frac{6}{b} \Rightarrow \left(\frac{6}{b}\right)^2 - b^2 = 5$

$b^4 + 5b^2 - 36 = 0$

$(b^2 + 9)(b^2 - 4) = 0$

$b^2 = -9$ (no real solutions) or $b^2 = 4$

$b = \pm 2$

$a = \pm 3$

$z_1 = 3 + 2i$, $z_2 = -3 - 2i$

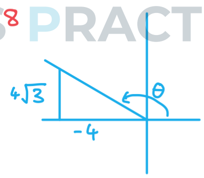
b)

Solve the equation $z^3 = -4 + 4\sqrt{3}\,i$ giving your answers in the form $r \operatorname{cis} \theta$.

Convert $-4 + 4\sqrt{3}i$ to Polar form:

$r = \sqrt{(-4)^2 + (4\sqrt{3})^2} = \sqrt{64} = 8$

$\theta = \pi - \left(\tan^{-1}\left(\frac{4\sqrt{3}}{4}\right)\right) = \frac{2\pi}{3}$



$\therefore \quad -4 + 4\sqrt{3}i = 8 \operatorname{cis}\left(\frac{2\pi}{3}\right)$

$z^3 = -4 + 4\sqrt{3}i$

$z = \sqrt[3]{-4 + 4\sqrt{3}i} = \left(8 \operatorname{cis}\left(\frac{2\pi}{3}\right)\right)^{\frac{1}{3}}$

$= \left(8^{\frac{1}{3}}\right) \operatorname{cis}\left(\frac{\frac{2\pi}{3} + 2\pi k}{3}\right)$

Order 3 so there are 3 roots, use $k = 0, 1, 2$:

$z = 2 \operatorname{cis}\left(\frac{2\pi}{9}\right), \quad 2 \operatorname{cis}\left(\frac{8\pi}{9}\right), \quad 2 \operatorname{cis}\left(\frac{14\pi}{9}\right)$

1.10 Systems of Linear Equations

1.10.1 Systems of Linear Equations

Introduction to Systems of Linear Equations

What are systems of linear equations?

- A linear equation is an equation of the first order (**degree 1**)
  - This means that the **maximum degree** of each term is 1
  - These are examples of linear equations:

    $2x + 3y = 5$ & $5x - y = 10 + 5z$
  - These are examples of non-linear equations:

    $x^2 + 5x + 3 = 0$ & $3x + 2xy - 5y = 0$

    The terms x^2 and xy have degree 2
- A system of linear equations is where **two or more linear equations** work together
  - These are also called **simultaneous equations**
- If there are ***n* variables** then you will need **at least *n* equations** in order to solve it
  - For your exam n will be 2 or 3
- A **2×2 system** of linear equations can be written as
  - $$a_1x + b_1y = c_1$$ $$a_2x + b_2y = c_2$$
- A **3×3 system** of linear equations can be written as
  - $$a_1x + b_1y + c_1z = d_1$$ $$a_2x + b_2y + c_2z = d_2$$ $$a_3x + b_3y + c_3z = d_3$$

What do systems of linear equations represent?

- The most common application of systems of linear equations is in **geometry**
- For a **2×2** system
  - Each equation will represent a **straight line in 2D**
  - The solution (if it exists and is unique) will correspond to the **coordinates** of the point where the **two lines intersect**
- For a **3×3** system
  - Each equation will represent a **plane in 3D**
  - The solution (if it exists and is unique) will correspond to the **coordinates** of the point where the **three planes intersect**

Systems of Linear Equations

How do I set up a system of linear equations?

- Not all questions will have the equations written out for you
- There will be **bits of information** given about the variables
  - **Two bits** of information for a **2×2 system**
  - **Three bits** of information for a **3×3 system**
  - Look out for clues such as 'assuming a linear relationship'
- Choose to assign ***x*, *y* & *z*** to the given variables
  - This will be helpful if using a GDC to solve
- Or you can choose to use more meaningful variables if you prefer
  - Such as c for the number of cats and *d* for the number of dogs

How do I use my GDC to solve a system of linear equations?

- You can use your **GDC to solve** the system on the **calculator papers (paper 2 & paper 3)**
- Your GDC will have a function within the algebra menu to solve a system of linear equations
- You will need to choose the number of equations
  - For two equations the variables will be x and y
  - For three equations the variables will be x, y and z
- If required, write the equations in the given form
  - $ax + by = c$
  - $ax + by + cz = d$
- Your GDC will display the values of x and y (or x, y, and z)

Exam Tip

- Make sure that you are familiar with how to use your GDC to solve a system of linear equations because even if you are asked to use an algebraic method and show your working, you can use your GDC to check your final answer
- If a systems of linear equations question is asked on a non-calculator paper, make sure you check your final answer by inputting the values into all original equations to ensure that they satisfy the equations

Worked Example

On a mobile phone game, a player can purchase one of three power-ups (fire, ice, electricity) using their points.

- Adam buys 5 fire, 3 ice and 2 electricity power-ups costing a total of 1275 points.
- Alice buys 2 fire, 1 ice and 7 electricity power-ups costing a total of 1795 points.
- Alex buys 1 fire and 1 ice power-ups which in total costs 5 points less than a single electricity power up.

Find the cost of each power-up.

Let x be the cost of a fire power-up
Let y be the cost of an ice power-up
Let z be the cost of an electricity power-up

Form 3 equations

$5x + 3y + 2z = 1275$

$2x + y + 7z = 1795$

$x + y = z - 5 \quad \rightarrow \quad x + y - z = -5$

Write in form $ax + by + cz = d$

Type the 3 equations into the GDC and solve

$x = 120, \ y = 85, \ z = 210$

Fire costs 120 points
Ice costs 85 points
Electricity costs 210 points

1.10.2 Algebraic Solutions

Row Reduction

How can I write a system of linear equations?

- To save space we can just write the **coefficients without the variables**
- For 2 variables: $\begin{matrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{matrix}$ can be written shorthand as $\left[\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right]$
- For 3 variables: $\begin{matrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{matrix}$ can be written shorthand as $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$

What is a row reduced system of linear equations?

- A system of linear equations is in row reduced form if it is written as:
  - $\left[\begin{array}{ccc|c} A_1 & B_1 & C_1 & D_1 \\ 0 & B_2 & C_2 & D_2 \\ 0 & 0 & C_3 & D_3 \end{array}\right]$ which corresponds to $\begin{matrix} A_1x + B_1y + C_1z = D_1 \\ B_2y + C_2z = D_2 \\ C_3z = D_3 \end{matrix}$

    It is very helpful if the values of A_1, B_2, C_3 are **equal to 1**

What are row operations?

- **Row operations** are used to make the linear equations **simpler to solve**
  - They **do not affect the solution**
- You can **switch any two rows**
  - $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ can be written as $\left[\begin{array}{ccc|c} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \end{array}\right]$ using $r_1 \leftrightarrow r_3$

    This is useful for getting zeros to the bottom

    Or getting a one to the top
- You can **multiply any row by a (non-zero) constant**
  - $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ can be written as $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ ka_2 & kb_2 & kc_2 & kd_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ using $k \times r_2 \to r_2$

    This is useful for getting a 1 as the first non-zero value in a row
- You can **add multiples of a row to another row**
  - $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ can be written as $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2+5a_3 & b_2+5b_3 & c_2+5c_3 & d_2+5d_3 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ using $r_2 + 5r_3 \to r_2$

    This is useful for creating zeros under a 1

How can I row reduce a system of linear equations?

- **STEP 1**: Get a **1 in the top left corner**
  - You can do this by **dividing the row** by the current value in its place
  - If the current value is 0 or an awkward number then you can **swap rows first**

$$\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ * & * & * & * \\ * & * & * & * \end{array}\right]$$

- **STEP 2**: Get **0's in the entries below the 1**
  - You can do this by **adding/subtracting a multiple of the first row** to each row

$$\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right]$$

- **STEP 3**: **Ignore the first row and column** as they are now complete
  - **Repeat STEPS 1 – 2** to the remaining section

Get a 1: $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & * & * & * \end{array}\right]$

Then 0 underneath: $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & 0 & * & * \end{array}\right]$

- **STEP 4**: Get a **1 in the third row**
  - Using the same idea as **STEP 1**

$$\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & 0 & 1 & D_3 \end{array}\right]$$

How do I solve a system of linear equations once it is in row reduced form?

- Once you row reduced the equations you can then **convert back to the variables**
  - $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & 0 & 1 & D_3 \end{array}\right]$ corresponds to $\begin{aligned} x + B_1 y + C_1 z &= D_1 \\ y + C_2 z &= D_2 \\ z &= D_3 \end{aligned}$
- Solve the equations **starting at the bottom**
  - You have the value for z from the third equation
  - Substitute z into the second equation and solve for y
  - Substitute z and y into the first each and solve for x

Exam Tip

- To reduce the number of operations you do whilst solving a system of operations, you can do a couple of things:
  - You can set up your original matrix with the equations in any order, so if one of the equations already has a 1 in the top left corner, put that one first
  - You do not need to make every equation so that it only has a single variable with a value of 1, you just need to do that for 1 of the equations and use that result to work out the others

EXAM PAPERS PRACTICE

Worked Example

Solve the following system of linear equations using algebra.

$$2x - 3y + 4z = 14$$
$$x + 2y - 2z = -2$$
$$3x - y - 2z = 10$$

Write without the variables

$$\left[\begin{array}{ccc|c} 2 & -3 & 4 & 14 \\ 1 & 2 & -2 & -2 \\ 3 & -1 & -2 & 10 \end{array}\right]$$

Swap rows to get 1 in top left corner

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 2 & -3 & 4 & 14 \\ 3 & -1 & -2 & 10 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

Add multiples of R_1 to R_2 and R_3 to get zeros under the 1

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & -7 & 8 & 18 \\ 0 & -7 & 4 & 16 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 3R_1 \rightarrow R_3 \end{array}$$

Multiple the second row to get a 1

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -\frac{8}{7} & -\frac{18}{7} \\ 0 & -7 & 4 & 16 \end{array}\right] \quad R_2 \times -\frac{1}{7} \rightarrow R_2$$

Repeat the steps

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -\frac{8}{7} & -\frac{18}{7} \\ 0 & 0 & -4 & -2 \end{array}\right] \quad R_3 + 7R_2 \rightarrow R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -\frac{8}{7} & -\frac{18}{7} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \quad R_3 \times -\frac{1}{4} \rightarrow R_3$$

Write out the equations starting at the bottom

$z = \frac{1}{2}$

$y - \frac{8}{7}z = -\frac{18}{7} \Rightarrow y - \frac{4}{7} = -\frac{18}{7} \Rightarrow y = -\frac{14}{7} = -2$

$x + 2y - 2z = -2 \Rightarrow x - 4 - 1 = -2 \Rightarrow x = 3$

$x = 3, \quad y = -2, \quad z = \frac{1}{2}$

Number of Solutions to a System

How many solutions can a system of linear equations have?

- There could be
  - 1 **unique solution**
  - **No solutions**
  - An **infinite number** of solutions
- You can determine the case by looking at the row reduced form

How do I know if the system of linear equations has no solutions?

- Systems with **no solutions** are called **inconsistent**
- When trying to solve the system after using the row reduction method you will end up with a **mathematical statement which is never true**:
  - Such as: 0 = 1
- The **row reduced system will contain**:
  - **At least one row** where the entries to the **left of the line are zero** and the entry on the **right of the line is non-zero**
    - Such a row is called **inconsistent**
  - For example:

Row 2 is inconsistent $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 0 & 0 & D_2 \\ 0 & 0 & 1 & D_3 \end{array}\right]$ if D_2 is non-zero

How do I know if the system of linear equations has infinite solutions?

- Systems with **at least one solution** are called **consistent**
  - The solution could be unique or there could be an infinite number of solutions
- When trying to solve the system after using the row reduction method you will end up with a **mathematical statement which is always true**
  - Such as: 0 = 0
- The **row reduced system will contain**:
  - **At least one row** where **all the entries are zero**
  - **No inconsistent rows**
  - For example:

$$\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

How do I find the general solution of a dependent system?

- A **dependent system** of linear equations is one where there are **infinite number of solutions**
- The general solution will depend on **one or two parameters**
- In the case where **two rows are zero**
  - Let the **variables corresponding** to the **zero rows** be equal to the **parameters** λ & μ
    - For example: If the first and second rows are zero rows then let $x = \lambda$ & $y = \mu$

- Find the **third** variable in terms of the two parameters using the equation from the third row
  - For example: $z = 4\lambda - 5\mu + 6$
- In the case where **only one row is zero**
  - Let the **variable corresponding** to the **zero row** be equal to the **parameter** λ
    - For example: If the first row is a zero row then let $x = \lambda$
  - Find the **remaining two variables in terms of the parameter** using the equations formed by the other two rows
    - For example: $y = 3\lambda - 5$ & $z = 7 - 2\lambda$

Exam Tip

- Common questions that pop up in an IB exam include questions with equations of lines
- Being able to recognise whether there are no solutions, 1 solution or infinite solutions is really useful for identifying if lines are coincident, skew or intersect!

Worked Example

$$x + 2y - z = 3$$
$$3x + 7y + z = 4$$
$$x - 9z = k$$

a)
Given that the system of linear equations has an infinite number of equations, find the value of k.

Write without the variables

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & 7 & 1 & 4 \\ 1 & 0 & -9 & k \end{array}\right]$$

Use the row reduction method

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & k-3 \end{array}\right] \begin{array}{l} \\ r_2 - 3r_1 \to r_2 \\ r_3 - r_1 \to r_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & k-13 \end{array}\right] \begin{array}{l} \\ \\ r_3 + 2r_2 \to r_3 \end{array}$$

There are an infinite number of solutions if a row is zero

$k - 13 = 0$

$k = 13$

b)
Find a general solution to the system.

The third row is zero so let the third variable (z) equal a parameter

$z = \lambda$

Use equations to find expressions for the other variables

$y + 4z = -5 \Rightarrow y + 4\lambda = -5 \Rightarrow y = -4\lambda - 5$

$x + 2y - 1 = 3 \Rightarrow x - 8\lambda - 10 - \lambda = 3 \Rightarrow x = 9\lambda + 13$

$x = 9\lambda + 13,\ y = -4\lambda - 5,\ z = \lambda$ for $\lambda \in \mathbb{R}$