AS LEVEL SPECIFICATION

FURTHER MATHEMATICS A

MARK SCHEME 2

1) a) i) $\arg(z_1) = -\pi/4$ B1

ii) $\arg(z_2) = \pi/6$ B1

iii) $\arg(z_1 z_2) = -\pi/4 + \pi/6 = -\pi/12$ B1ft

b) $z_1 + z_2 = (1 + 3\sqrt{3}) + 2i$

so $|z_1 + z_2| = \sqrt{(1+3\sqrt{3})^2 + 2^2}$ M1

$= \sqrt{28 + 6\sqrt{3} + 4}$

$= \sqrt{32 + 6\sqrt{3}}$ A1

2) a) $\det M = 1\begin{vmatrix}1 & 4\\ 0 & 1\end{vmatrix} - 5\begin{vmatrix}q & 4\\ 0 & 1\end{vmatrix} + p\begin{vmatrix}q & 1\\ 0 & 0\end{vmatrix}$

$= 1 - 5q$ M1

so invertible for all $p \in \mathbb{R}$, and for $q \neq \frac{1}{5}$.

A1 A1

b) $M = \begin{pmatrix}1 & 5 & p\\ q & 1 & 4\\ 0 & 0 & 1\end{pmatrix}$

$$\text{Minors} = \begin{pmatrix}\begin{vmatrix}1 & 4\\ 0 & 1\end{vmatrix} & \begin{vmatrix}q & 4\\ 0 & 1\end{vmatrix} & \begin{vmatrix}q & 1\\ 0 & 0\end{vmatrix}\\ \begin{vmatrix}5 & p\\ 0 & 1\end{vmatrix} & \begin{vmatrix}1 & p\\ 0 & 1\end{vmatrix} & \begin{vmatrix}1 & 5\\ 0 & 0\end{vmatrix}\\ \begin{vmatrix}5 & p\\ 1 & 4\end{vmatrix} & \begin{vmatrix}1 & p\\ q & 4\end{vmatrix} & \begin{vmatrix}1 & 5\\ q & 1\end{vmatrix}\end{pmatrix}$$

$$= \begin{pmatrix}1 & q & 0\\ 5 & 1 & 0\\ 20-p & 4-pq & 1-5q\end{pmatrix} \quad \text{M1}$$

$$C = \begin{pmatrix}1 & -q & 0\\ -5 & 1 & 0\\ 20-p & pq-4 & 1-5q\end{pmatrix} \quad \text{M1}$$

$$C^T = \begin{pmatrix}1 & -5 & 20-p\\ -q & 1 & pq-4\\ 0 & 0 & 1-5q\end{pmatrix}$$

$$\text{so } M^{-1} = \frac{1}{1-5q}\begin{pmatrix}1 & -5 & 20-p\\ -q & 1 & pq-4\\ 0 & 0 & 1-5q\end{pmatrix} \quad \text{A1}$$

c) $\det M^{-1} = \frac{1}{\det M} = \frac{1}{1-5q}$ B1

d) Area scale factor $= \frac{24}{4} = 6 \Rightarrow 1-5q = 6 \Rightarrow q = -1.$ M1 A1

$$\begin{pmatrix}1 & 5 & p\\ -1 & 1 & 4\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}8\\ -4\\ 2\end{pmatrix} = \begin{pmatrix}8\\ -4\\ 2\end{pmatrix}$$

$\Rightarrow 8 - 20 + 2p = 8$ M1

$\Rightarrow p = 10$ A1

3) a) $\frac{dy}{dx} = 5\sinh^2 x + 5\cosh^2 x - 24\overset{\sinh}{\cancel{\cosh}}\,2x = 0$ M1

$\Rightarrow 5\cosh 2x - 24\sinh 2x = 0$

$\Rightarrow \tanh 2x = \frac{5}{24}$ M1

$\Rightarrow 2x = \frac{1}{2}\ln\left(\frac{1+\frac{5}{24}}{1-5/24}\right)$

$\Rightarrow x = \frac{1}{4}\ln\left(\frac{29}{19}\right)$ A1

b) $\frac{d^2y}{dx^2} = 10\sinh 2x - 48\cosh 2x$

$= 5(e^{2x} - e^{-2x}) - 24(e^{2x} + e^{-2x})$ M1

$= -19e^{2x} - 29e^{-2x} < 0$ for all x, so maximum turning point.

A1

4) a) $\dfrac{1-2x}{x^4-2x^3+x^2} = \dfrac{1-2x}{x^2(x-1)^2} \equiv \dfrac{Ax+B}{x^2} + \dfrac{Cx+D}{(x-1)^2}$

M1

$\Rightarrow 1-2x \equiv (Ax+B)(x-1)^2 + (Cx+D)x^2$ M1

~~Setting $x=0$, $1=B$~~

~~Setting $x=1$, $-1=D-C$~~

~~Setting $x=-1$, $3=$~~

$\Rightarrow (1-2x) \equiv (Ax+B)(x^2-2x+1) + (Cx+D)x^2$

$\equiv x^3(A+C) + x^2(-2A+B+D) + x(A-2B) + B$

$\Rightarrow B=1$

$A-2B=-2$

$\Rightarrow A=0$

$-2A+B+D=0$

$\Rightarrow D=-1$

$A+C=0$ A2

$\Rightarrow C=0$

So $= \dfrac{1}{x^2} - \dfrac{1}{(x-1)^2}$

~~b) $= -\sum_{n=3}^{10} \frac{1}{n^2} - \frac{1}{(n-1)^2} = -\left(\frac{1}{10^2} - \frac{1}{3^2}\right)$ M1~~

~~$= -\left(\frac{1}{100} - \frac{1}{9}\right)$~~

~~$\Rightarrow \frac{91}{900}$ A1~~

b) $= -\sum_{r=k+1}^{n} \frac{1}{r^2} - \frac{1}{(r-1)^2} = -\left(\frac{1}{n^2} - \frac{1}{k^2}\right)$ M1

$= \dfrac{n^2-k^2}{n^2k^2}$ A1

c) $\dfrac{n^2-25}{25n^2} > 0.039$ M1 for k=5

$n^2-25 > 0.975n^2$

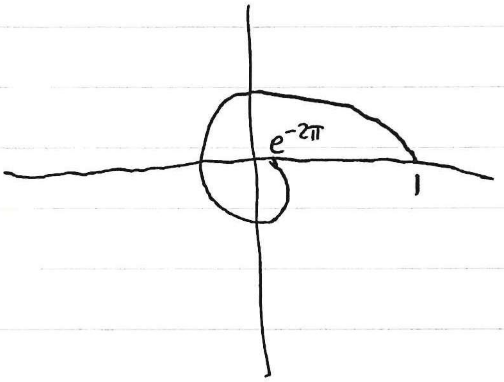
$0.025n^2 > 25$ M1

$n^2 > 1000$

$n > 31.6$

so $n=32$ A1

5) i) $r = e^{-\theta}$



B1

ii) $x = r\cos\theta$

$= \cos\theta e^{-\theta}$ M1

$\frac{dx}{d\theta} = e^{-\theta}(-\cos\theta - \sin\theta) = 0$ M1

$\Rightarrow -\cos\theta = \sin\theta$

$\Rightarrow \tan\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ A1

so $(0.0948, 3\pi/4)$ and $(0.00410, 7\pi/4)$ A1

6) a) $x^2 = \frac{1}{1+(y-2)^2}$

$$\text{so } V = \pi \int_0^3 \frac{1}{1+(y-2)^2}\, dy \qquad M1$$

$$= \pi \left[\arctan(y-2)\right]_0^3 \qquad M1$$

$$= \pi\left(\arctan 1 - \arctan(-2)\right)$$

$$= 5.946 \qquad A1$$

b) $x = \frac{1}{\sqrt{1+(y-2)^2}}$

$$\text{so area} = \int_0^3 \frac{1}{\sqrt{1+(y-2)^2}}\, dy \qquad M1$$

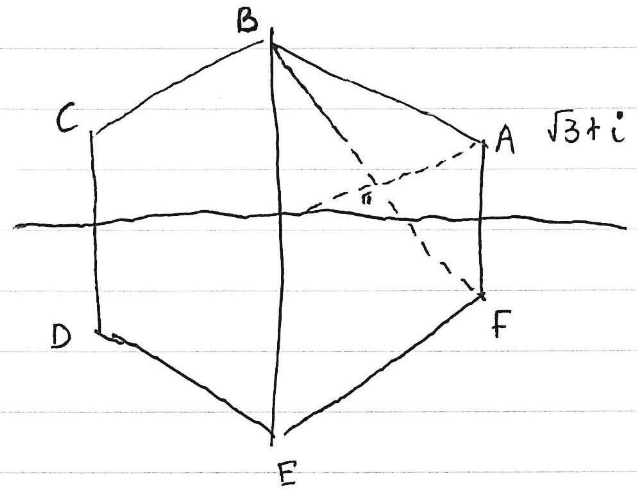
$$= \left[\operatorname{arsinh}(y-2)\right]_0^3$$

$$= \operatorname{arsinh} 1 - \operatorname{arsinh}(-2) \qquad M1$$

$$= \ln(1+\sqrt{2}) - \cancel{\operatorname{arsinh}}\ \ln(-2+\sqrt{5})$$

$$= \ln\left(\frac{1+\sqrt{2}}{-2+\sqrt{5}}\right) \qquad A1$$

7)



$|z| = |z-\sqrt{3}-i|$ is the perpendicular bisector of O and A.
ABOF is a parallelogram so this is line BF. M1

The area of the whole hexagon is

$$6 \times \frac{1}{2} \times 2^2 \times \sin\frac{\pi}{3}$$

$$= 6\sqrt{3}.$$ A1

The area of triangle ABF is

$$\frac{1}{2} \times 2^2 \times \sin\left(\frac{2\pi}{3}\right)$$ M1

$$= \sqrt{3}.$$ A1

So the area of the larger region is $5\sqrt{3}$ M1

So the ratio is $5:1$. A1

8) a) $\underline{n}_1 = \begin{pmatrix} -5 \\ 6 \\ 1 \end{pmatrix}$

$\underline{n}_2 = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$

$\underline{n}_1 \cdot \underline{n}_2 = |\underline{n}_1||\underline{n}_2| \cos\theta$

$\Rightarrow -15 - 24 + 2 = \sqrt{5^2+6^2+1^2}\sqrt{3^2+4^2+2^2}\cos\theta$ M1

$\Rightarrow \cos\theta = \frac{-37}{\sqrt{62}\sqrt{29}}$ M1

$\Rightarrow \theta = 150.8^0$

so acute angle is $29.2°$ A1

b) Line in both planes is perpendicular to $\underline{n}_1$ and $\underline{n}_2$

$\begin{vmatrix} -5 & 6 & 1 \\ 3 & -4 & 2 \\ i & j & k \end{vmatrix} = 16i + 13j + 2k$ M1 A1

so $\underline{r} = \underline{r}_0 + \lambda \begin{pmatrix} 16 \\ 13 \\ 2 \end{pmatrix}$.

Picking (e.g.) $\underline{r}_0 = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$, and substituting into Π_1 and Π_2,

$6y + z = -52$
$-4y + 2z = -24$
$\Rightarrow y = -8,\ z = -4$ M1

so $\underline{r} = \begin{pmatrix} 0 \\ -8 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 16 \\ 13 \\ 2 \end{pmatrix}$ A1

c) $|\vec{OA}|^2 = (16\lambda)^2 + (-8 + 13\lambda)^2 + (-4 + 2\lambda)^2$
$= 256\lambda^2 + 169\lambda^2 - 208\lambda + 64 + 4\lambda^2 - 16\lambda + 16$
$= 429\lambda^2 - 224\lambda + 80$ M1

$\frac{d}{d\lambda} = 858\lambda - 224 = 0$ M1
$\Rightarrow \lambda = \frac{224}{858} = \frac{112}{429}.$ A1

So $A = \begin{pmatrix} 0 \\ -8 \\ -4 \end{pmatrix} + \frac{112}{429}\begin{pmatrix} 16 \\ 13 \\ 2 \end{pmatrix}$

$= \frac{1}{429}\begin{pmatrix} 1792 \\ -1976 \\ -1492 \end{pmatrix}$ A1

9. a) $m^2 + 6m + 9 = 0$

$\Rightarrow (m+3)^2 = 0$ M1

$\Rightarrow x = (A + Bt)e^{-3t}$ A1

PI $x = \lambda t^2 e^{-3t}$ M1

$\frac{dx}{dt} = e^{-3t}(2\lambda t - 3\lambda t^2)$

$\frac{d^2x}{dt^2} = e^{-3t}(2\lambda - 6\lambda t - 6\lambda t + 9\lambda t^2)$

$= e^{-3t}(9\lambda t^2 - 12\lambda t + 2\lambda)$ M1

so $e^{-3t}(9\lambda t^2 - 12\lambda t + 2\lambda) + 6e^{-3t}(2\lambda t - 3\lambda t^2) + 9\lambda t^2 e^{-3t} = -4e^{-3t}$

$\Rightarrow t^2(9\lambda - 18\lambda + 9\lambda) + t(-12\lambda + 12\lambda) + 2\lambda = -4$

$\Rightarrow \lambda = -2$ A1

So $x = (A + Bt)e^{-3t} - 2t^2 e^{-3t}$ A1

b) $x(0) = 5/3 \Rightarrow A = 5/3$ A1

$\frac{dx}{dt} = e^{-3t}(B - 4t - 3A - 3Bt + 6t^2)$ M1

so $\dot{x}(0) = 0 \Rightarrow B - 3A = 0$

$\Rightarrow B = 5$ A1

So $x = e^{-3t}\left(\frac{5}{3} + 5t - 2t^2\right)$ A1

so $\dot{x} = e^{-3t}(6t^2 - 19t) = 0$ M1

$\Rightarrow t = 0, \frac{19}{6}$.

so $t = \frac{19}{6}$ A1

$$\lim_{k\to\infty}\int_0^k e^{-3t}\left[\frac{5}{3}+5t-2t^2\right]\,dt \qquad u\ \tfrac{5}{3}+5t-2t^2 \quad v'\ e^{-3t}$$

$$u'\ 5-4t \quad v\ -\tfrac{1}{3}e^{-3t} \qquad M1$$

$$=\lim_{k\to\infty}\left[\left(\frac{5}{3}+5t-2t^2\right)\left(-\frac{1}{3}e^{-3t}\right)\right]_0^k+\lim_{k\to\infty}\int_0^k (5-4t)\left(+\frac{1}{3}e^{-3t}\right)\,dt \quad M1$$

$$=\frac{5}{9}-\lim_{k\to\infty}\left[(5-4t)\left(\frac{1}{9}e^{-3t}\right)\right]_0^k - \lim_{k\to\infty}\int_0^k \frac{4}{9}e^{-3t}\,dt \qquad u\ 5-4t \quad v'\ \tfrac{1}{3}e^{-3t} \qquad u'\ -4 \quad v\ -\tfrac{1}{9}e^{-3t} \quad M1$$

M1

$$=\left(\frac{5}{9}+\frac{5}{9}\right)-\lim_{k\to\infty}\left[\frac{-4}{27}e^{-3t}\right]_0^k \quad M1$$

$$=\frac{5}{9}+\frac{5}{9}-\frac{4}{27}$$

$$=\frac{26}{27} \qquad A1$$