EXAM PAPERS PRACTICE

AS LEVEL SPECIFICATION

FURTHER MATHEMATICS A

MARK SCHEME 1

1) $\alpha = -3+2i$ B1

$\beta = -3-2i$

$\alpha\beta = 13$ B1

$\alpha + \beta = -6$

$z^3 + pz^2 - 5z + q = 0$

so $\alpha\beta + \beta\gamma + \gamma\alpha = -5$ M1

$\Rightarrow \alpha\beta + (\alpha+\beta)\gamma = -5$

so $13 - 6\gamma = -5$ M1

$\Rightarrow \gamma = 3$ A1

$p = -(\alpha+\beta+\gamma)$ M1

$= -(-6+3)$

$= 3$ A1

$q = -\alpha\beta\gamma$

$= -13 \times 3$

$= -39.$ A1

2) For $n=1$,

$4\times(-3)^1+5^1$

$= -12+5$

$=-7$, So true for $n=1$. B1

~~Assume true for n=k, so and~~,

~~u_k~~

For $n=2$,

$4\times(-3)^2+5^2$

$= 36+25$

$= 61$, so true for $n=2$. B1

Assume true for $n=k$ and $n=k+1$, so

$u_k = 4(-3)^k + 5^k$ and $u_{k+1} = 4(-3)^{k+1} + 5^{k+1}$ M1

Then

$$\begin{aligned}
u_{k+2} &= 2u_{k+1} + 15u_k \\
&= 8(-3)^{k+1} + 2(5)^{k+1} + 60(-3)^k + 15(5)^k \quad \text{M1} \\
&= 8(-3)^{k+1} + 2(5)^{k+1} + -20(-3)^{k+1} + 3(5)^k \\
&= -12(-3)^{k+1} + 5(5)^{k+1} \\
&= 4(-3)^{k+2} + 5^{k+2} \quad \text{A1}
\end{aligned}$$

So true for $n=k+2$.

True for $n=1$ and $n=2$, and if true for $n=k$ and $n=k+1$, then true for $n=k+2$. So true $\forall k \in \mathbb{Z}$ by induction A1

3) i) $\lim\limits_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx$

$= \lim\limits_{t\to\infty}\left[\arctan x\right]_0^t$ M1

$\cancel{= \lim\limits_{t\to\infty} \arctan} = \lim\limits_{t\to\infty} \arctan t - \arctan 0$

$= \pi/2$ A1

So it is convergent.

(ii) $\lim\limits_{t\to 0}\int_{-1}^{t} \frac{1}{x}\,dx + \lim\limits_{u\to 0}\int_u^1 \frac{1}{x}\,dx$ M1

$= \lim\limits_{t\to 0}\left[\ln|x|\right]_{-1}^{t} + \lim\limits_{u\to 0}\left[\ln|x|\right]_u^1$

$= \lim\limits_{t\to 0} \ln|t| - 0 \ \cancel{\lim\limits_{u\to 0}}\ 0 + \lim\limits_{u\to 0} \ln|u|$ A1

which does not exist, so the integral is not convergent. A1

$\cancel{iii)\ \lim\limits_{t\to 0}\int_{-1}^{t} x^{-1/3}\,dx + \lim\limits_{u\to 0}\int_a^1 x^{-1/3}\,dx}$

$\cancel{= \lim\limits_{t\to 0}\left[\tfrac{3}{2}\sqrt{x}\right]_{-1}^{t} + \lim\limits_{u\to 0}\left[2\sqrt{x}\right]_a^1}$

$\cancel{= \lim\limits_{t\to 0}}$

$\cancel{= \lim\limits_{t\to 0}\left[\tfrac{3}{2}x^{2/3}\right]_{-1}^{t}} +$

iii) $\lim\limits_{t\to 0}\int_t^8 x^{-1/3}\,dx$

$= \lim\limits_{t\to 0}\left[\frac{3}{2}x^{2/3}\right]_0^8$ M1

$= \frac{3}{2}\cdot 4 - \lim\limits_{t\to 0}\frac{3}{2}x^{2/3}$

$= 6$ A1

So it is convergent.

4) i) $\cosh t \dfrac{dN}{dt} = \sinh t(\sinh^3 t\cosh t - N)$

$\Rightarrow \cosh t\dfrac{dN}{dt} + \sinh t\, N = \sinh^4 t\cosh t$

$\Rightarrow \dfrac{dN}{dt} + \tanh t\, N = \sinh^4 t$ M1

$IF = e^{\int \tanh t\, dt} = \cosh t$ B1

so $N\cosh t = \int \sinh^4 t\cosh t\, dt$

$= \frac{1}{5}\sinh^5 t + c$

so $N = \frac{1}{5}\sinh^4 t\tanh t + c\,\text{sech}\, t$ A1

$N(0) = 0.5 \Rightarrow c = 0.5$ M1

so $N = \frac{1}{5}\sinh^4 t\tanh t + 0.5\,\text{sech}\, t$

So $N(0.75) = \frac{1}{5}\sinh^4 0.75\tanh 0.75 + 0.5\,\text{sech}\, 0.75$ M1

~~= 0.647~~ $= 0.444$

~~≈ 647,000 bacteria.~~ A1 $= 440,000$ bacteria.

ii) The model predicts the population will grow without bound. B1

5 i) $\cos(6\theta) = \mathrm{Re}\left[(\cos\theta + i\sin\theta)^6\right]$ M1

$= \cos^6\theta - \binom{6}{2}\cos^4\theta\sin^2\theta + \binom{6}{4}\cos^2\theta\sin^4\theta - \sin^6\theta$ A1

$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$ M1

$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta + \cos^6\theta - 3\cos^4\theta$

$+ 3\cos^2\theta - 1$ M1

$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$ A1

ii) $\frac{1}{2}(32x^6 - 48x^4 + 18x^2 - 1) = \frac{1}{4}$ M1

set $x = \cos\theta$

$\Rightarrow \cos 6\theta = \frac{1}{2}$ A1

$\Rightarrow 6\theta = \pi/3,\ 5\pi/3,\ 7\pi/3,\ 11\pi/3,\ 13\pi/3,\ 17\pi/3$ M1

$\theta = \pi/18,\ 5\pi/18,\ 7\pi/18,\ 11\pi/18,\ 13\pi/18,\ 17\pi/18$ M1

$x = \cos\theta = 0.985,\ 0.643,\ 0.342,\ -0.342,\ -0.643,\ -0.985$ A1

6) $2 + \cos 6\theta = 2$ M1

$\Rightarrow \cos 6\theta = 0$

$\Rightarrow 6\theta = \pi/2,\ 3\pi/2,\ ----$

$\Rightarrow \theta = \pi/12,\ 3\pi/12$ A1

$$\text{total area} = 6 \times \frac{1}{2}\left[\int_{\pi/12}^{3\pi/12} 2^2 \cancel{d\theta - \int_{\pi/12}^{3\pi/12}} - (2+\cos 6\theta)^2 \, d\theta\right]$$ M1

$$= 3\int_{\pi/12}^{3\pi/12} -\cos^2 6\theta - 4\cos 6\theta \, d\theta$$

$$= 3\int_{\pi/12}^{3\pi/12} -\frac{1}{2} - \frac{1}{2}\cos 12\theta - 4\cos 6\theta \, d\theta$$ M1 A1

$$= 3\left[-\frac{1}{2}\theta - \frac{1}{24}\sin 12\theta - \frac{2}{3}\sin 6\theta\right]_{\pi/12}^{3\pi/12}$$

$$= 3\left[\left(-\frac{3\pi}{24} - 0 + \frac{2}{3}\right) - \left(-\frac{\pi}{24} - 0 - \frac{2}{3}\right)\right]$$ A1

$$= 3\left(\frac{4}{3} - \frac{\pi}{12}\right)$$

$$= 4 - \frac{\pi}{4}.$$ A1

7. i) $f(x) = e^{-x^2}$

$f'(x) = -2xe^{-x^2}$ M1

$f''(x) = e^{-x^2}(4x^2 - 2)$

$\cancel{f'''(x) = e^{-x^2}(4x^2 - 2 + 8x)}$

$f'''(x) = e^{-x^2}(8x - 2x(4x^2 - 2))$ A1

$= e^{-x^2}(-8x^3 + 12x)$

$f^{(4)}(x) = e^{-x^2}(-2x(-8x^3 + 12x) - 24x^2 + 12)$

$= e^{-x^2}(16x^4 - 48x^2 + 12)$

$f(0) = 1$

$f'(0) = 0$

$f''(0) = -2$

$f'''(0) = 0$

$f^{(4)}(0) = 12$

so $f(x) = 1 - \dfrac{2x^2}{2!} + \dfrac{12x^4}{4!} + \dots$

$= 1 - x^2 + \dfrac{x^4}{2} + \dots$ M1 A1

ii) $\displaystyle\int_0^1 e^{-x^2}\,dx \approx \int_0^1 1 - x^2 + \frac{x^4}{2}\,dx$ M1

$= \left[x - \frac{1}{3}x^3 + \frac{x^5}{10}\right]_0^1$

$= 1 - \frac{1}{3} + \frac{1}{10}$

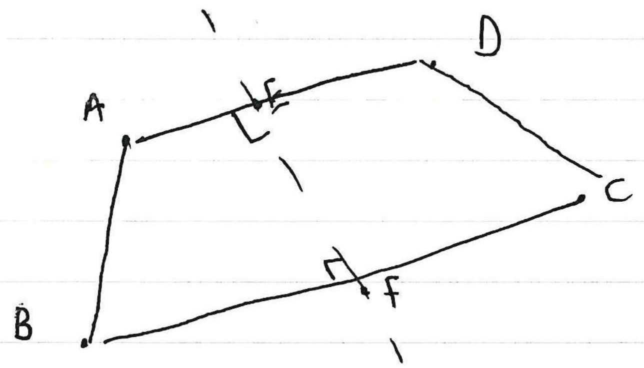
$= \dfrac{23}{30}$ A1

iii) $\displaystyle\int_0^1 e^{-x^2}\,dx = 0.7468\dots$

% error $= 2.66\%$ B1

iv) The approximation is only valid for small values of x.

8. $\begin{pmatrix}2\\-3\\1\end{pmatrix}$ is perpendicular to plane.



$\cancel{\left[\begin{pmatrix}-3\\2\\5\end{pmatrix}+\lambda\begin{pmatrix}-3\\2\\5\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix}=2}$

$\cancel{\Rightarrow}$

$\begin{pmatrix}1+2\lambda\\5-3\lambda\\1+\lambda\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix}=2$ M1

$\Rightarrow 2+4\lambda-15+9\lambda+1+\lambda=2$

$14\lambda=14$

$\Rightarrow \lambda=1$ M1 meets plane at $E=\begin{pmatrix}3\\2\\2\end{pmatrix}$

so $D=\begin{pmatrix}1\\5\\1\end{pmatrix}+2\begin{pmatrix}2\\-3\\1\end{pmatrix}=\begin{pmatrix}5\\-1\\3\end{pmatrix}$. A1

$\begin{pmatrix}-3+2\lambda\\10-3\lambda\\-4+\lambda\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix}=2$

$\cancel{\Rightarrow 2a+4\lambda+9+9\lambda}$

$\Rightarrow -6+4\lambda-30+9\lambda-4+\lambda=2$ M1

$\Rightarrow 14\lambda=42$

$\Rightarrow \lambda=3$ meets plane at $F=\begin{pmatrix}3\\1\\-1\end{pmatrix}$

so $C=\begin{pmatrix}-3\\10\\-4\end{pmatrix}+6\begin{pmatrix}2\\-3\\1\end{pmatrix}=\begin{pmatrix}9\\-8\\2\end{pmatrix}$. A1

so $|AD|=\sqrt{4^2+6^2+2^2}$
$=\sqrt{56}$
$=2\sqrt{14}$

$|BC|=\sqrt{12^2+18^2+6^2}$
$=\sqrt{504}$
$=6\sqrt{14}$

M1 A1

$|EF|=\sqrt{0^2+1^2+3^2}$ M1
$=\sqrt{10}$

So area $=\frac{1}{2}(2\sqrt{14}+6\sqrt{14})\sqrt{10}$ M1
$=4\sqrt{140}$
$=8\sqrt{35}$ A1

9) i) $\frac{dx}{dt} = 3x - 15y \Rightarrow y = \frac{1}{5}x - \frac{1}{15}\frac{dx}{dt}$

$\frac{dy}{dt} = 6x - 3y \Rightarrow y = \cancel{2x \ldots \tfrac{1}{3}}$

$\frac{d^2x}{dt^2} = 3\frac{dx}{dt} - 15\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{5}\frac{dx}{dt} - \frac{1}{15}\frac{d^2x}{dt^2}$ M1

so $\frac{1}{5}\frac{dx}{dt} - \frac{1}{15}\frac{d^2x}{dt^2} = 6x - \frac{3}{5}x + \frac{1}{5}\frac{dx}{dt}$ M1

$\Rightarrow 3\frac{dx}{dt} - \frac{d^2x}{dt^2} = 90x - 9x + 3\frac{dx}{dt}$

$\Rightarrow \frac{d^2x}{dt^2} + 81x = 0$ A1

$\Rightarrow x = A\cos 9t + B\sin 9t$ A1

$y = \frac{1}{5}x - \frac{1}{15}\frac{dx}{dt}$

$= \frac{A}{5}\cos 9t + \frac{B}{5}\sin 9t + \frac{9}{15}A\sin 9t - \frac{9}{15}B\cos 9t$ M1

$= \left(\frac{A-3B}{5}\right)\cos 9t + \left(\frac{B+3A}{5}\right)\sin 9t$ A1

So both particles undergo SHM.

ii) $x(0) = 0 \Rightarrow A = 0$

$\dot{x}(0) = 90 \Rightarrow 9B = 90 \Rightarrow B = 10$ M1

so $x = 10\sin 9t$

$y = -6\cos 9t + 2\sin 9t$ A1

$10\sin 9t = -6\cos 9t + 2\sin 9t$

$\Rightarrow -6\cos 9t = 8\sin 9t$

$\Rightarrow \tan 9t = -3/4$ M1

$\Rightarrow 9t = 2.498\ldots$

$\Rightarrow t = 0.278$ seconds. A1

~~$\tan 9t = -3/4 \Rightarrow \sin 9t = 3/5$ so $x = y = 10 \times 3/5 = 6$cm from the origin in the positive direction.~~

$\tan 9t = -3/4 \Rightarrow \sin 9t = 3/5$

So $x = y = 10 \times \frac{3}{5}$ M1

$= 6$ cm from the origin in the positive direction. A1