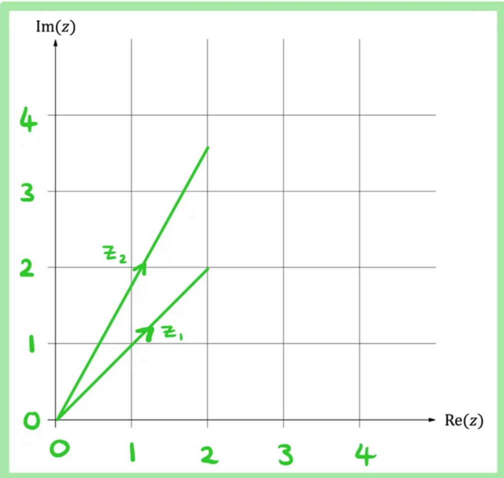
Question 1a

Consider the complex numbers $z_1 = 2 + 2i$ and $z_2 = 2 + 2\sqrt{3}i$.

(a) Sketch z_1 and z_2 on the Argand diagram below, be sure to include an appropriate scale.

(a)



[2]

(b) Find the modulus of z_1 and z_2.

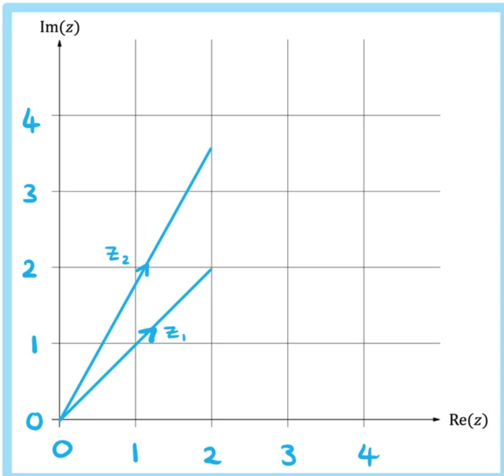
[3]

(c) Find the argument of z_1 and z_2.

[3]

Consider the complex numbers $z_1 = 2 + 2i$ and $z_2 = 2 + 2\sqrt{3}i$.

(a) Sketch z_1 and z_2 on the Argand diagram below, be sure to include an appropriate scale.



[2]

(b) Find the modulus of z_1 and z_2.

[3]

(c) Find the argument of z_1 and z_2.

[3]

(b) $|z_1| = \sqrt{2^2 + 2^2} = \sqrt{8}$

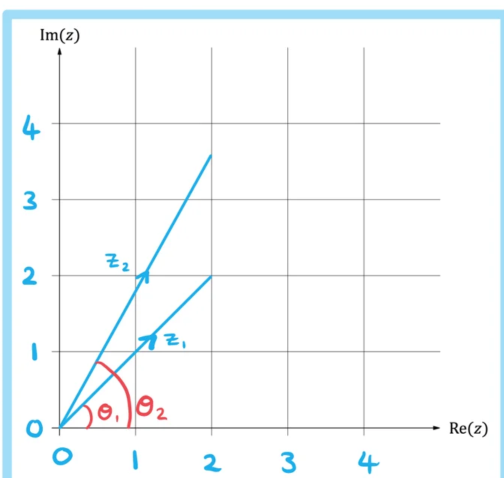
$\boxed{|z_1| = 2\sqrt{2}}$

$|z_2| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{16}$

$\boxed{|z_2| = 4}$

Consider the complex numbers $z_1 = 2 + 2i$ and $z_2 = 2 + 2\sqrt{3}i$.

(a) Sketch z_1 and z_2 on the Argand diagram below, be sure to include an appropriate scale.



[2]

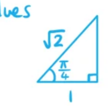
(b) Find the modulus of z_1 and z_2.

[3]

(c) Find the argument of z_1 and z_2.

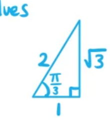
[3]

(c) $\tan\theta_1 = \frac{2}{2} = 1$ ← Trig exact values $\tan\frac{\pi}{4} = 1$



$\boxed{\theta_1 = \frac{\pi}{4}}$

$\tan\theta_2 = \frac{2\sqrt{3}}{2} = \sqrt{3}$ ← Trig exact values $\tan\frac{\pi}{3} = \sqrt{3}$



$\boxed{\theta_2 = \frac{\pi}{3}}$

EXAM PAPERS PRACTICE

Question 2

Solve the following equations for x

(i) $x^2 + 4x + 5 = 0$

(ii) $x^2 = -625$

(iii) $x^4 = 24 - 2x^2$.

[7]

(i) $x = \dfrac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)}$

$= \dfrac{-4 \pm \sqrt{-4}}{2}$

$= \dfrac{-4 \pm 2\sqrt{-1}}{2}$

$\boxed{x = -2 \pm i}$

(ii) $x = \sqrt{-625}$

$= \pm 25\sqrt{-1}$

$\boxed{x = \pm 25 i}$

(iii) $x^4 + 2x^2 - 24 = 0$

Treat like a quadratic

$(x^2)^2 + 2(x^2) - 24 = 0$

$(x^2 - 4)(x^2 + 6) = 0$

$x^2 = 4$

$\boxed{x = \pm 2}$

$x^2 = -6$

$x = \pm\sqrt{-6}$

$\boxed{x = \pm\sqrt{6}\, i}$

Question 3

Let $w_1 = z_1 z_2$, where $z_1 = 5 + i$ and $z_2 = 1 + 2i$.

(a) Express w in the form $w = a + bi$.

[2]

(b) Find the modulus and argument for w.

[4]

(a) $w = (5 + i)(1 + 2i)$

$= 5 + i + 10i + 2i^2$

$= 5 + 11i - 2$

$\boxed{w = 3 + 11i}$

Let $w_1 = z_1 z_2$, where $z_1 = 5 + i$ and $z_2 = 1 + 2i$.

(a) Express w in the form $w = a + bi$. [2]

$\boxed{w = 3 + 11i}$

(b) Find the modulus and argument for w. [4]

(b) $|w| = \sqrt{3^2 + 11^2}$

$\boxed{|w| = \sqrt{130}}$

$\tan\theta_w = \frac{11}{3}$

$\boxed{\theta_w = 1.30 \text{ rad}}$

Question 4

Let $z = \frac{w_1}{w_2}$, where $w_1 = 4 - i$ and $w_2 = 1 - 2i$.

(a) Express z in the form $z = a + bi$. [3]

(a) $z = \frac{4-i}{1-2i} \times \frac{1+2i}{1+2i}$ Multiply by the conjugate to cancel out i in the denominator

$= \frac{4 - i + 8i - 2i^2}{1 - 2i + 2i - 4i^2}$

$= \frac{4 + 7i + 2}{1 + 4}$

$= \frac{6 + 7i}{5}$

$\boxed{z = \frac{6}{5} + \frac{7}{5}i}$

(b) Find the modulus and argument for z. [4]

Let $z = \frac{w_1}{w_2}$, where $w_1 = 4 - i$ and $w_2 = 1 - 2i$.

(a) Express z in the form $z = a + bi$. [3]

$\boxed{z = \frac{6}{5} + \frac{7}{5}i}$

(b) Find the modulus and argument for z. [4]

(b) $|z| = \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{7}{5}\right)^2}$

$\boxed{|z| = \frac{\sqrt{85}}{5}}$

$\tan\theta_z = \frac{\left(\frac{7}{5}\right)}{\left(\frac{6}{5}\right)}$

$\boxed{\theta_z = 0.862 \text{ rad}}$

Question 5a

Consider the complex numbers $z = 3 - 4i$ and $w = 7 - 2i$.

(a) Find

(i) $z + w$

(ii) $w - z$.

[2]

Let z^* and w^* represent the complex conjugates of z and w, respectively.

(b) Write down z^* and w^*, giving your answers in the form $a + bi$.

[2]

(c) Find

(i) $z^* w$

(ii) $\frac{w^*}{z}$.

[4]

(a) (i) $z + w = (3 - 4i) + (7 - 2i)$

$\boxed{z + w = 10 - 6i}$

(ii) $w - z = (7 - 2i) - (3 - 4i)$

$\boxed{w - z = 4 + 2i}$

Question 5b

Consider the complex numbers $z = 3 - 4i$ and $w = 7 - 2i$.

(a) Find

(i) $z + w$

(ii) $w - z$.

[2]

Let z^* and w^* represent the complex conjugates of z and w, respectively.

(b) Write down z^* and w^*, giving your answers in the form $a + bi$.

[2]

(c) Find

(i) $z^* w$

(ii) $\frac{w^*}{z}$.

[4]

(b) $\boxed{\begin{array}{l} z^* = 3 + 4i \\ w^* = 7 + 2i \end{array}}$

EXAM PAPERS PRACTICE

Question 5c

Consider the complex numbers $z = 3 - 4i$ and $w = 7 - 2i$.

(a) Find

(i) $z + w$

(ii) $w - z$.

[2]

Let z^* and w^* represent the complex conjugates of z and w, respectively.

(b) Write down z^* and w^*, giving your answers in the form $a + bi$.

$$\boxed{\begin{aligned} z^* &= 3 + 4i \\ w^* &= 7 + 2i \end{aligned}}$$

[2]

(c) Find

(i) $z^* w$

(ii) $\frac{w^*}{z}$.

[4]

(c) (i) $z^* w = (3 + 4i)(7 - 2i)$

$= 21 + 28i - 6i + 8$

$$\boxed{z^* w = 29 + 22i}$$

(ii) $\frac{w^*}{z} = \frac{7 + 2i}{3 - 4i} \times \frac{3 + 4i}{3 + 4i}$ ← Multiply by the conjugate to cancel out i in the denominator

$= \frac{21 + 6i + 28i - 8}{9 - 12i + 12i + 16}$

$= \frac{13 + 34i}{25}$

$$\boxed{\frac{w^*}{z} = \frac{13}{25} + \frac{34}{25}i}$$

Question 6

Find all possible real values for a and b such that

(i) $a + bi = 8i$

(ii) $(2 + 3i)(a + bi) = 13$

(iii) $(a + i)(2 + bi) = -6 + 22i$.

[7]

(i) $\boxed{a = 0 \quad b = 8}$

(ii) $(2+3i)(a+bi) = 13$

$2a + 3ai + 2bi - 3b = 13$

$\underbrace{(2a - 3b)}_{\text{Real parts must sum to 13}} + \underbrace{(3a + 2b)}_{\text{Imaginary parts must sum to 0}}i = 13$

$2a - 3b = 13$

$3a + 2b = 0$

Solve equations simultaneously to find a and b

$2a - 3b = 13 \rightarrow 6a - 9b = 39$

$3a + 2b = 0 \rightarrow 6a + 4b = 0$ $-$

$-13b = 39$

$\boxed{b = -3}$

$3a + 2(-3) = 0$

$3a - 6 = 0$

$\boxed{a = 2}$

(iii) $(a+i)(2+bi) = -6 + 22i$

$2a + 2i + abi - b = -6 + 22i$

$(2a - b) + (2 + ab)i = -6 + 22i$

Equate the real parts and the imaginary parts

$2a - b = -6 \rightarrow b = 2a + 6$

$2 + ab = 22 \rightarrow ab = 20$

Solve equations simultaneously to find a and b

$a(2a + 6) = 20$

$2a^2 + 6a - 20 = 0$

$a^2 + 3a - 10 = 0$

$(a+5)(a-2) = 0$

$a = -5$

$b = 2(-5) + 6 = -4$

$a = 2$

$b = 2(2) + 6 = 10$

$\boxed{a = -5,\ b = -4 \quad \text{or} \quad a = 2,\ b = 10}$

Question 7

Consider the complex numbers $w = iz$ and $w + 2z = 7 + 6i$.

Find

(i) $\mathrm{Re}(w)$

(ii) $\mathrm{Im}(w)$

(iii) $\mathrm{Re}(z)$

(iv) $\mathrm{Im}(z)$.

[7]

Write z and w in terms of two unknown variables x and y

$$z = x + yi \rightarrow w = i(x + yi)$$

Replace z and w in the equation $w + 2z = 7 + 6i$ to find the values of x and y

$$i(x + yi) + 2(x + yi) = 7 + 6i$$

$$xi - y + 2x + 2yi = 7 + 6i$$

$$(2x - y) + (x + 2y)i = 7 + 6i$$

Equate the real parts and the imaginary parts from both sides, then solve simultaneously to find x and y

$$2x - y = 7$$

$$x + 2y = 6 \rightarrow 2x + 4y = 12 \quad -$$

$$-5y = -5$$

$$y = 1$$

$$x + 2(1) = 6$$

$$x = 4$$

Substitute the values for x and y back into the equations for w and z

$$z = 4 + i \qquad w = i(4 + i) = -1 + 4i$$

(i) $\boxed{\mathrm{Re}(w) = -1}$ (ii) $\boxed{\mathrm{Im}(w) = 4i}$

(iii) $\boxed{\mathrm{Re}(z) = 4}$ (iv) $\boxed{\mathrm{Im}(z) = i}$

Question 8

It is given that $z_1 = 3 + 4i$ and $z_2 = -2 + 2i$.

Find

(i) $iz_1 + z_2$

(ii) $\frac{z_1}{iz_2}$

(iii) $i(z_1 z_2)$.

[7]

(i) $iz_1 + z_2 = i(3+4i) + (-2+2i)$

$= 3i - 4 - 2 + 2i$

$\boxed{iz_1 + z_2 = -6 + 5i}$

(ii) $\frac{z_1}{iz_2} = \frac{3+4i}{i(-2+2i)}$

$= \frac{3+4i}{-2-2i}$

$= \frac{3+4i}{-2-2i} \times \frac{-2+2i}{-2+2i}$

Multiply by the conjugate to cancel out i in the denominator

$= \frac{-6-8i+6i-8}{4+2i-2i+4}$

$= \frac{-14-2i}{8}$

$\boxed{\frac{z_1}{iz_2} = -\frac{7}{4} - \frac{1}{4}i}$

(iii) $i(z_1 z_2) = i(3+4i)(-2+2i)$

$= i(-6-8i+6i-8)$

$= -6i + 8 - 6 - 8i$

$\boxed{i(z_1 z_2) = 2 - 14i}$

Question 9

Find the complex numbers z and w such that

$$2z - iw^* = 5 + 7i$$

$$w + iz^* = 5 + 16i$$

[8]

Write z in terms of two unknown variables x and y and write w in terms of two unknown variables a and b

$$z = x + yi \qquad w = a + bi$$

Replace z and w and their conjugates in to both equations and equate the real parts and the imaginary parts

1st equation: $2(x + yi) - i(a - bi) = 5 + 7i$

$$2x + 2yi - ai - b = 5 + 7i$$

$$(2x - b) + (2y - a)i = 5 + 7i$$

$$\Rightarrow \quad 2x - b = 5, \quad 2y - a = 7$$

2nd equation: $(a + bi) + i(x - yi) = 5 + 16i$

$$a + bi + xi + y = 5 + 16i$$

$$(y + a) + (x + b)i = 5 + 16i$$

$$\Rightarrow \quad y + a = 5, \quad x + b = 16$$

Solve both pairs of simultaneous equations for x, y, a and b

$$\begin{aligned} 2x - b &= 5 \\ x + b &= 16 \quad + \\ \hline 3x &= 21 \\ x &= 7 \end{aligned} \qquad \begin{aligned} 2y - a &= 7 \\ y + a &= 5 \quad + \\ \hline 3y &= 12 \\ y &= 4 \end{aligned}$$

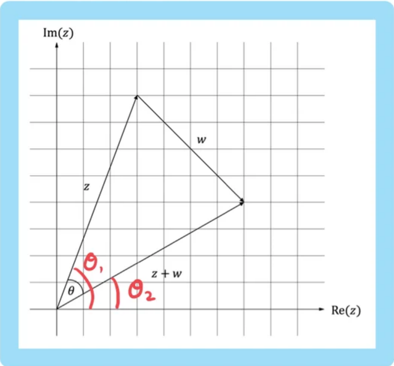
$$\begin{aligned} (7) + b &= 16 \\ b &= 9 \end{aligned} \qquad \begin{aligned} (4) + a &= 5 \\ a &= 1 \end{aligned}$$

$$\boxed{z = 7 + 4i} \qquad \boxed{w = 1 + 9i}$$

Question 10

Let $z = 3 + 8i$ and $w = 4 - 4i$.

(a) Find θ, the angle shown on the diagram below.



[5]

(b) Find the area of the triangle formed in the diagram above.

[3]

(a) Find the argument of z, θ_1

$\tan\theta_1 = \frac{8}{3}$

$\theta_1 = 1.21202565...$

Find $z + w$ and its argument, θ_2

$z + w = 7 + 4i$

$\tan\theta_2 = \frac{4}{7}$

$\theta_2 = 0.519146114...$

Find θ

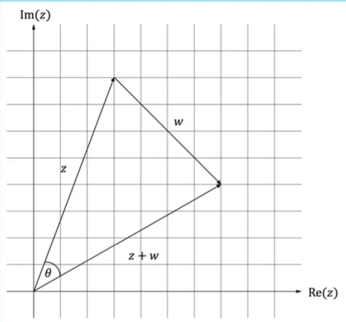
$\theta_1 - \theta_2 = 1.21202565... - 0.519146114...$

$\boxed{\theta = 0.693 \text{ rad}}$

Let $z = 3 + 8i$ and $w = 4 - 4i$.

$\boxed{z + w = 7 + 4i}$

(a) Find θ, the angle shown on the diagram below.



[5]

(b) Find the area of the triangle formed in the diagram above.

$\boxed{\theta = 0.693 \text{ rad}}$

[3]

(b) Find $|z|$ and $|z+w|$

$|z| = \sqrt{3^2 + 8^2}$

$= \sqrt{73}$

$|z+w| = \sqrt{7^2 + 4^2}$

$= \sqrt{65}$

$$A = \frac{1}{2}ab\sin C$$ Formula booklet

$A = \frac{1}{2}\sqrt{73}\sqrt{65}\sin 0.6928795\ldots$

$= 21.9999999\ldots$

$\boxed{A = 22.0 \text{ units}^2}$

Question 11

Let $z = -1 - 3i$ and $w = 1 + i$.

(a) Find zw.

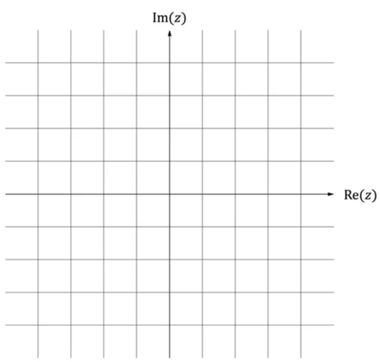
(a) $zw = (-1-3i)(1+i)$

$= -1 - 3i - i + 3$

$\boxed{zw = 2 - 4i}$

[2]

(b) Sketch z, w and zw on the Argand diagram below.



[3]

Let θ be the angle between z and zw and ϕ be the angle between w and zw.

(c) Find the angles θ and ϕ, giving your answers in degrees.

[4]

Let $z = -1 - 3i$ and $w = 1 + i$.

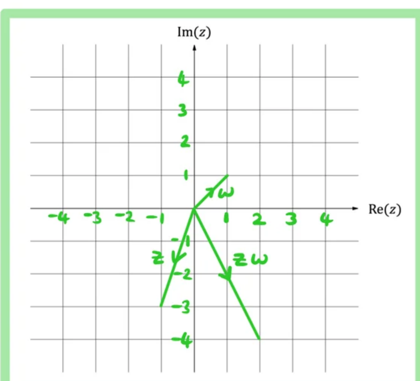
(a) Find zw.

$\boxed{zw = 2 - 4i}$

[2]

(b) Sketch z, w and zw on the Argand diagram below.

(b)



[3]

Let θ be the angle between z and zw and ϕ be the angle between w and zw.

(c) Find the angles θ and ϕ, giving your answers in degrees.

[4]

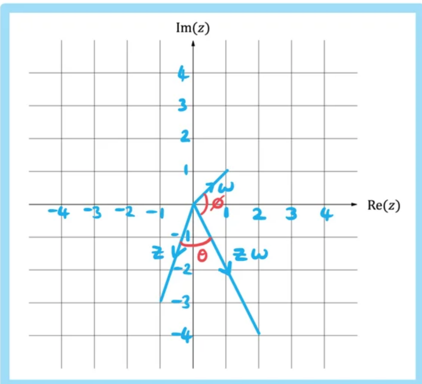
Question 11c

Let $z = -1 - 3i$ and $w = 1 + i$.

(a) Find zw.

[2]

(b) Sketch z, w and zw on the Argand diagram below.



[3]

Let θ be the angle between z and zw and ϕ be the angle between w and zw.

(c) Find the angles θ and ϕ, giving your answers in degrees.

[4]

(c) Find θ

$$\theta = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{2}{4}\right)$$

$$\boxed{\theta = 45°}$$

Find ϕ

$$\phi = \tan^{-1}\left(\frac{1}{1}\right) + \tan^{-1}\left(\frac{4}{2}\right)$$

$$\boxed{\phi = 108°}$$

EXAM PAPERS PRACTICE

Question 12

Let $w = \frac{z+1}{z^*+1}$, where $z = a + bi$, $a, b \in \mathbb{R}$.

(a) Write w in the form $x + yi$, $x, y \in \mathbb{R}$.

[4]

(b) Determine the conditions under which w is purely imaginary.

[3]

(a) Replace z and its conjugate in terms of a and b

$$w = \frac{(a+bi)+1}{(a-bi)+1}$$

Group the real parts together

$$= \frac{(a+1)+bi}{(a+1)-bi}$$

$$= \frac{(a+1)+bi}{(a+1)-bi} \times \frac{(a+1)+bi}{(a+1)+bi}$$

Multiply by the conjugate to cancel out i in the denominator

$$= \frac{(a+1)^2 + 2(a+1)bi - b^2}{(a+1)^2 + b^2}$$

$$\boxed{w = \left(\frac{(a+1)^2 - b^2}{(a+1)^2 + b^2}\right) + \left(\frac{2(a+1)b}{(a+1)^2 + b^2}\right)i}$$

Let $w = \frac{z+1}{z^*+1}$, where $z = a + bi$, $a, b \in \mathbb{R}$.

(a) Write w in the form $x + yi$, $x, y \in \mathbb{R}$.

$$\boxed{w = \left(\frac{(a+1)^2 - b^2}{(a+1)^2 + b^2}\right) + \left(\frac{2(a+1)b}{(a+1)^2 + b^2}\right)i}$$

[4]

(b) Determine the conditions under which w is purely imaginary.

[3]

(b) If w is imaginary the real part must be equal to 0

$$\frac{(a+1)^2 - b^2}{(a+1)^2 + b^2} = 0$$

$$(a+1)^2 - b^2 = 0$$

$$(a+1)^2 = b^2$$

$$\pm(a+1) = b$$

So b must be 1 more than a if a is positive and 1 less than a if a is negative

Looking at the real part of the original equation, you can see that $a \neq -1$ or $b = 0$

$$\boxed{\pm(a+1) = b \quad \text{and} \quad a \neq -1, \; b \neq 0}$$